CS53000 - Spring 2018

Introduction to Scientific Visualization

Grids and Data Reconstruction

January 16, 2018

Outline

- Mesh types
- Interpolation
- Spatial queries
- Common pre-processing tasks

Input Data

- Discrete positions (called *vertices*)
- *N* dimensions, *N*= *1*, *2*, *3*, ...
- With or without connectivity information
  - Structured
  - Unstructured
  - Scattered

grid topology

Input Data

- Discrete positions (called *vertices*)
- *N* dimensions, *N*= *1*, *2*, *3*, ...
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  - Scattered

grid topology

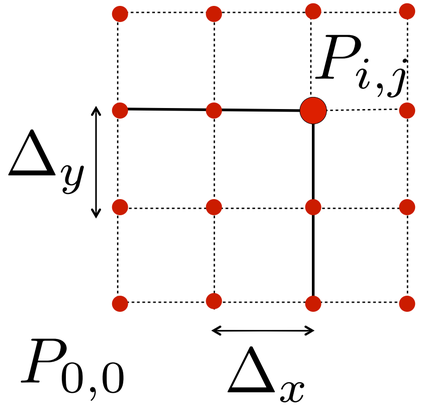
Mesh Classification

- Geometry
  - Position of vertices in space
  - Structured (follows a pattern) / unstructured
- Topology
  - Cells
  - Connectivity information
  - Neighborhood definition
  - Structured / unstructured

Mesh Geometry

Uniform

- implicit relationship between points
- positions can be computed (procedural)

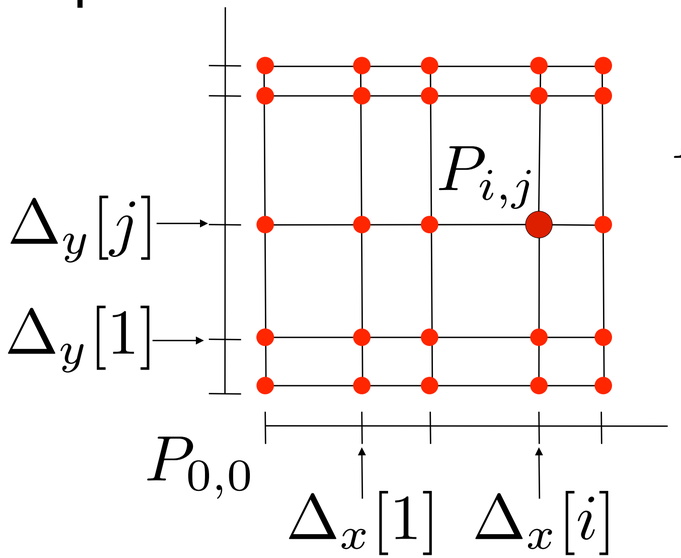


$$P_{i,j,k} = P_{0,0} + i\Delta_x\vec{e}_x + j\Delta_y\vec{e}_y + k\Delta_z\vec{e}_z$$

Mesh Geometry

Structured

- implicit relationship between points
- positions can be computed (procedural)

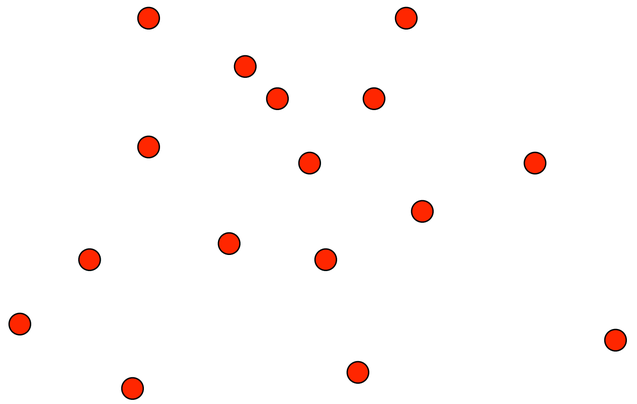


$$P_{i,j,k} = P_{0,0} + \Delta_x[i]\vec{e}_x + \Delta_y[j]\vec{e}_y + \Delta_z[k]\vec{e}_z$$

Mesh Geometry

Unstructured

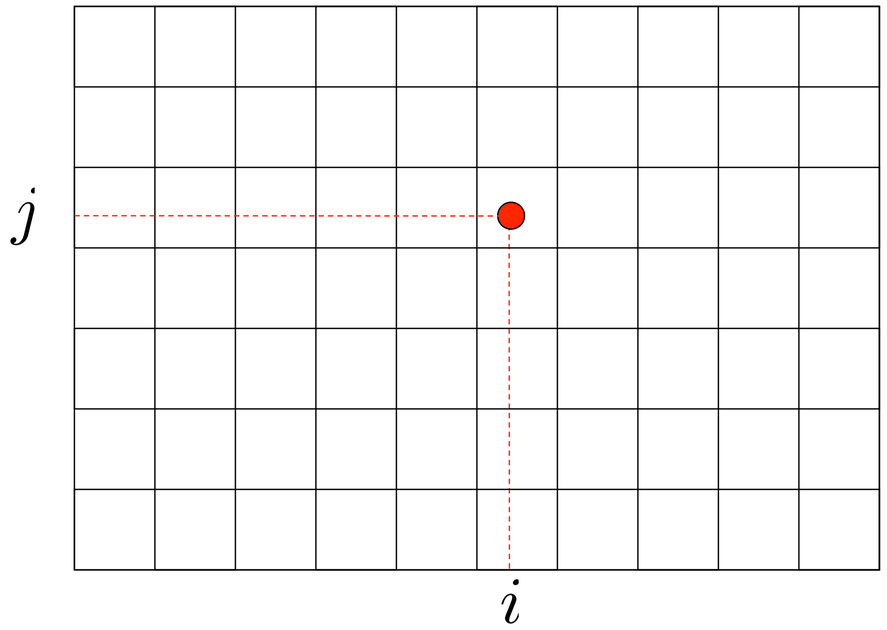
- No underlying structure
- Requires explicit knowledge of every vertex's position: (x_0, y_0, z_0), (x_1, y_1, z_1), ...,



Mesh Topology

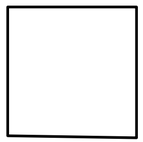
Structured *(cf. quadrilateral / hexahedron)*

- Implicit connectivity between vertices
- Implicit cell definition



When associated with
structured geometry:

very efficient point location

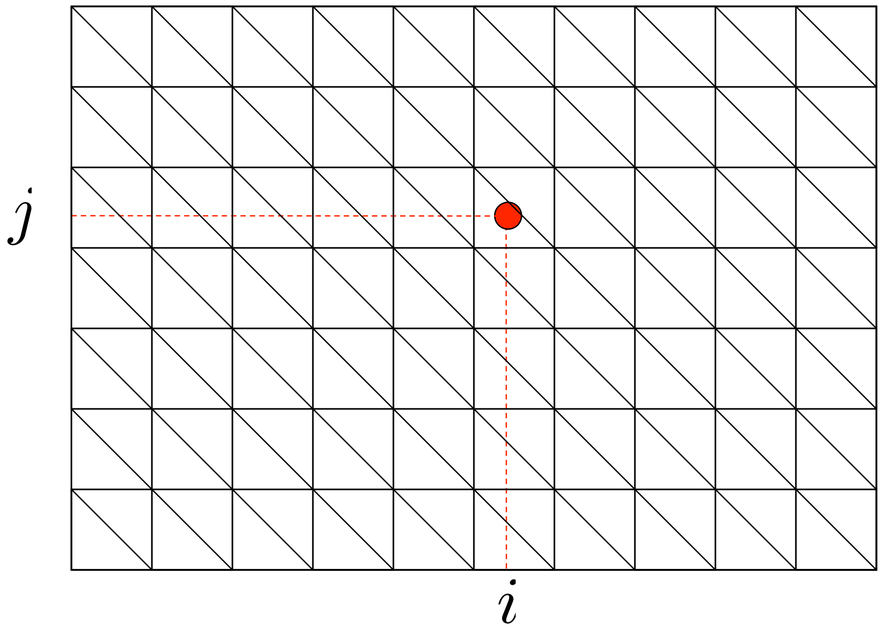


Pixel

Mesh Topology

Structured *(cf. quadrilateral / hexahedron)*

- Implicit connectivity between vertices
- Implicit cell definition



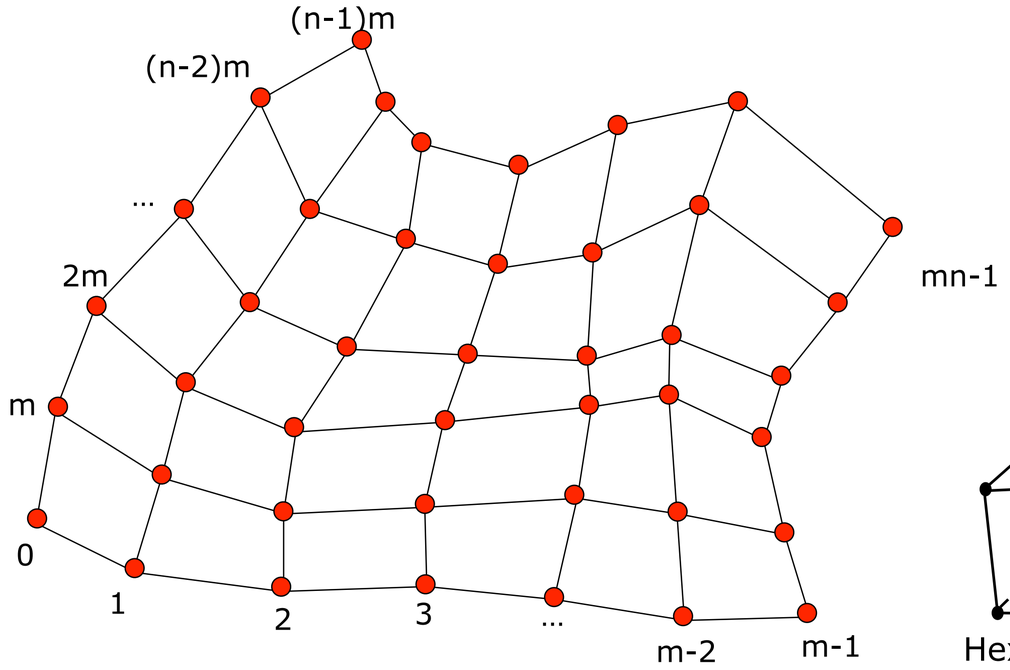
When associated with
structured geometry:

- very efficient position location
- implicit triangulation

Mesh Topology

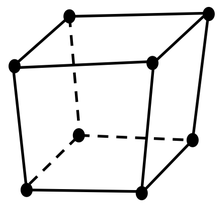
Structured *(quadrilateral / hexahedron)*

- Implicit connectivity between vertices
- Implicit cell definition

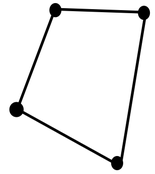


When associated with
unstructured geometry:

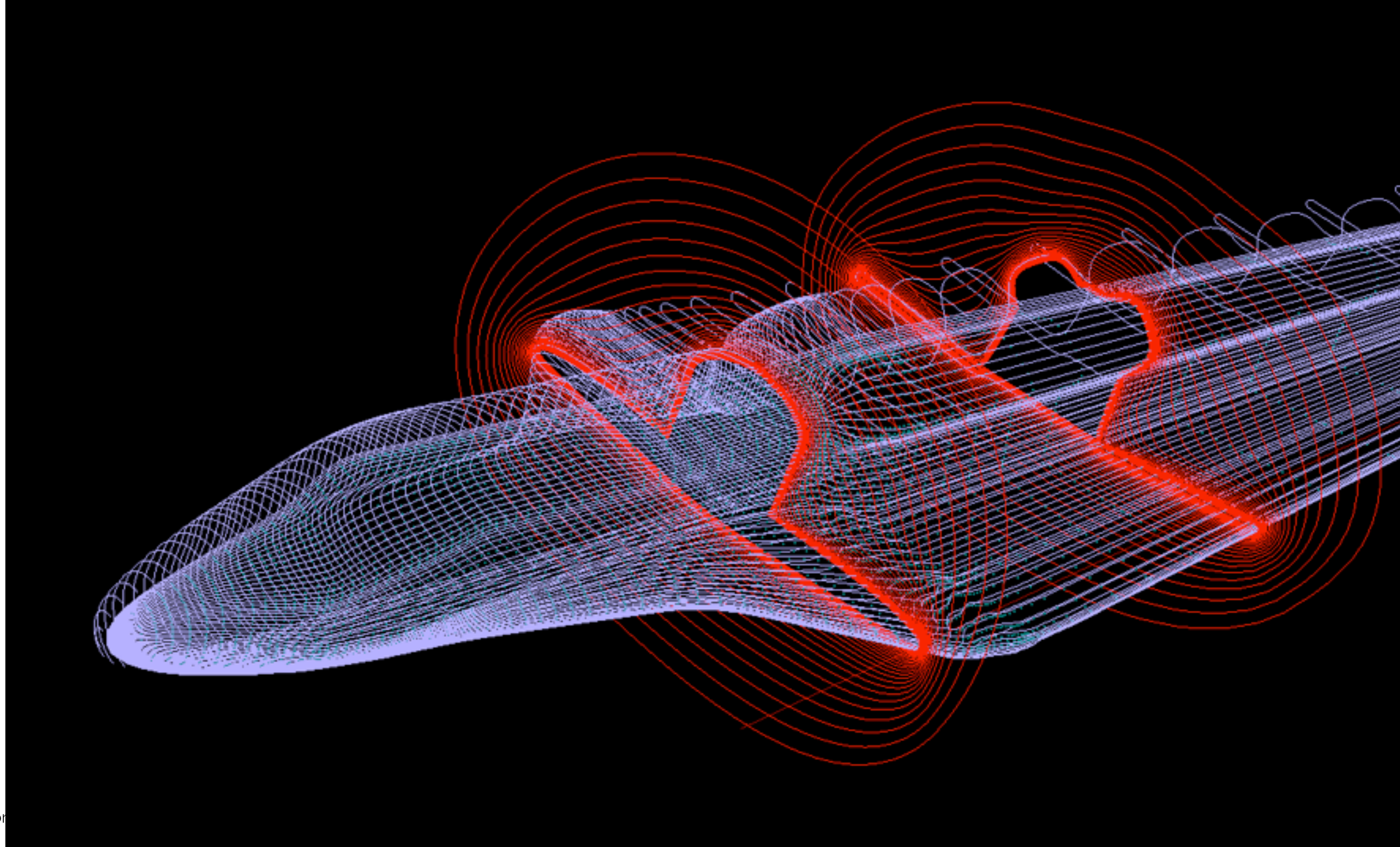
curvilinear grid



Hexahedron



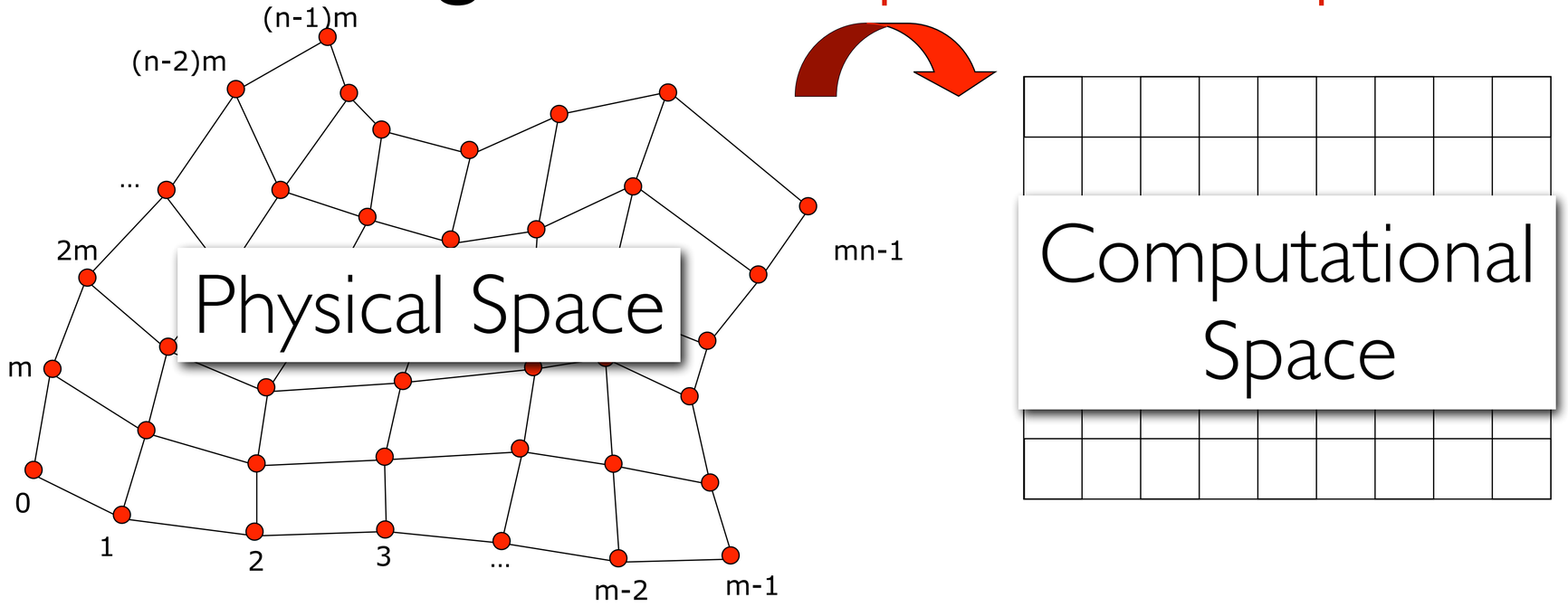
Quadrilateral



Mesh Topology

Structured *(quadrilateral / hexahedron)*

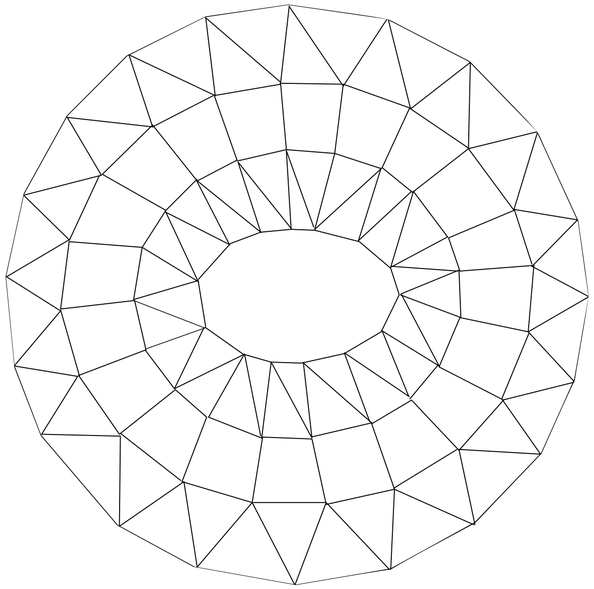
- Implicit connectivity between vertices
- Implicit cell definition
- **Uniform grid** in computational space

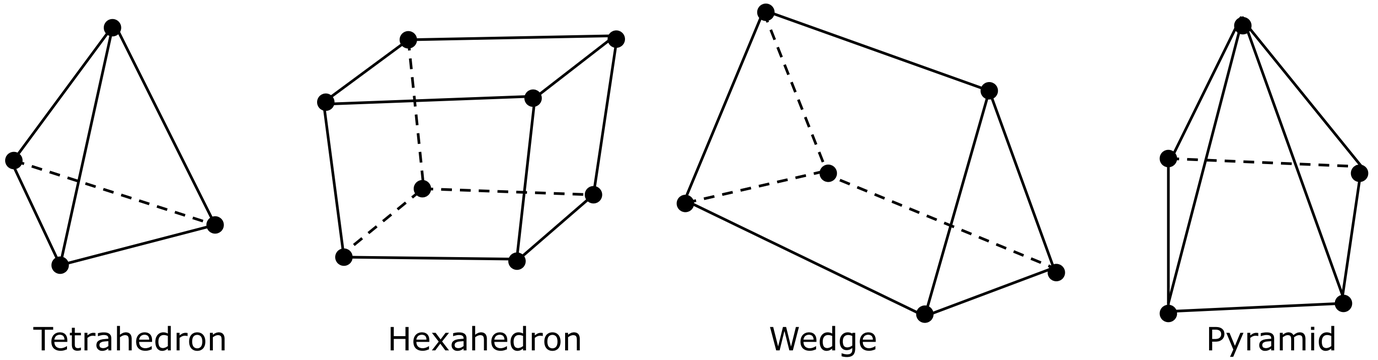


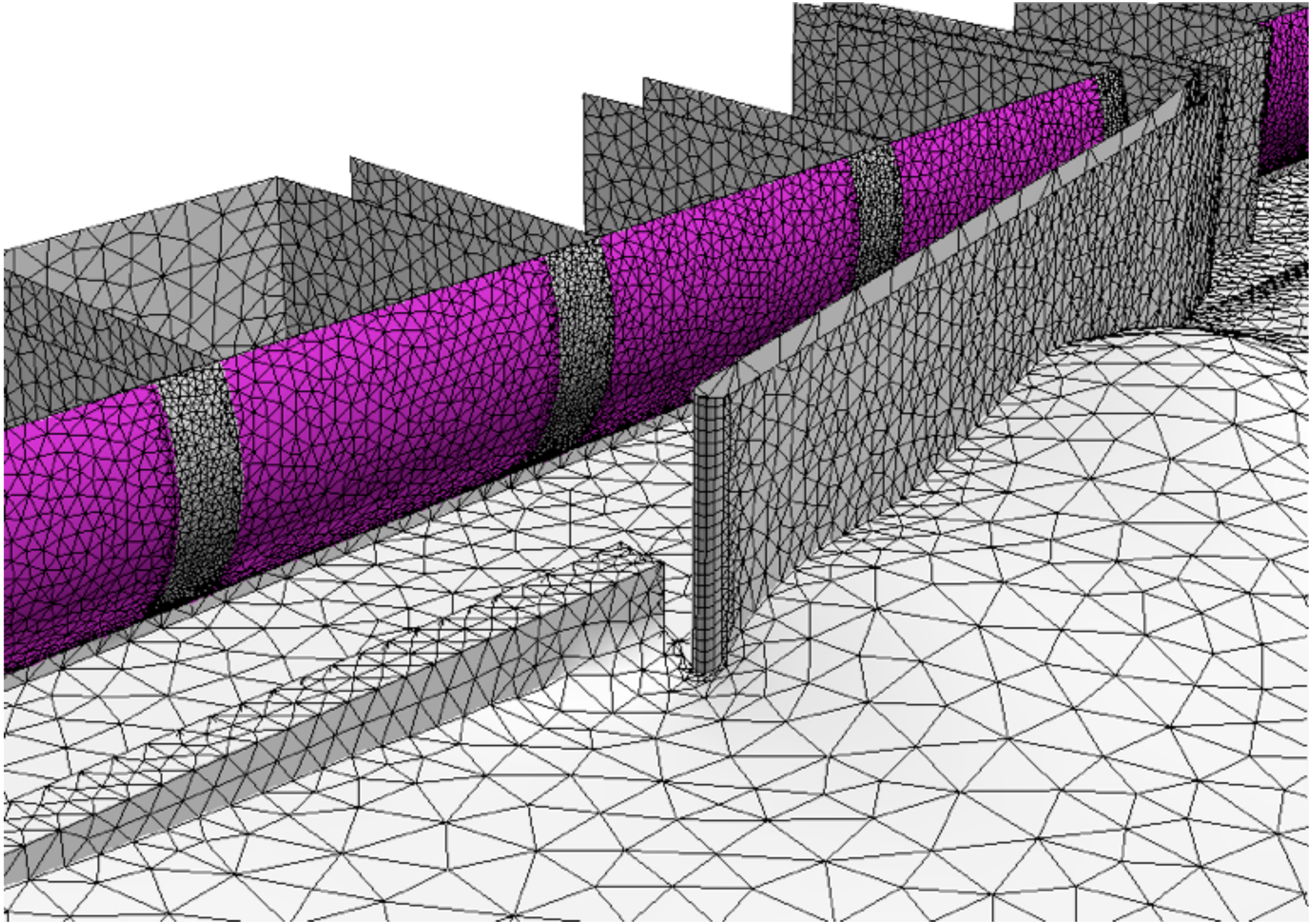
Mesh Topology

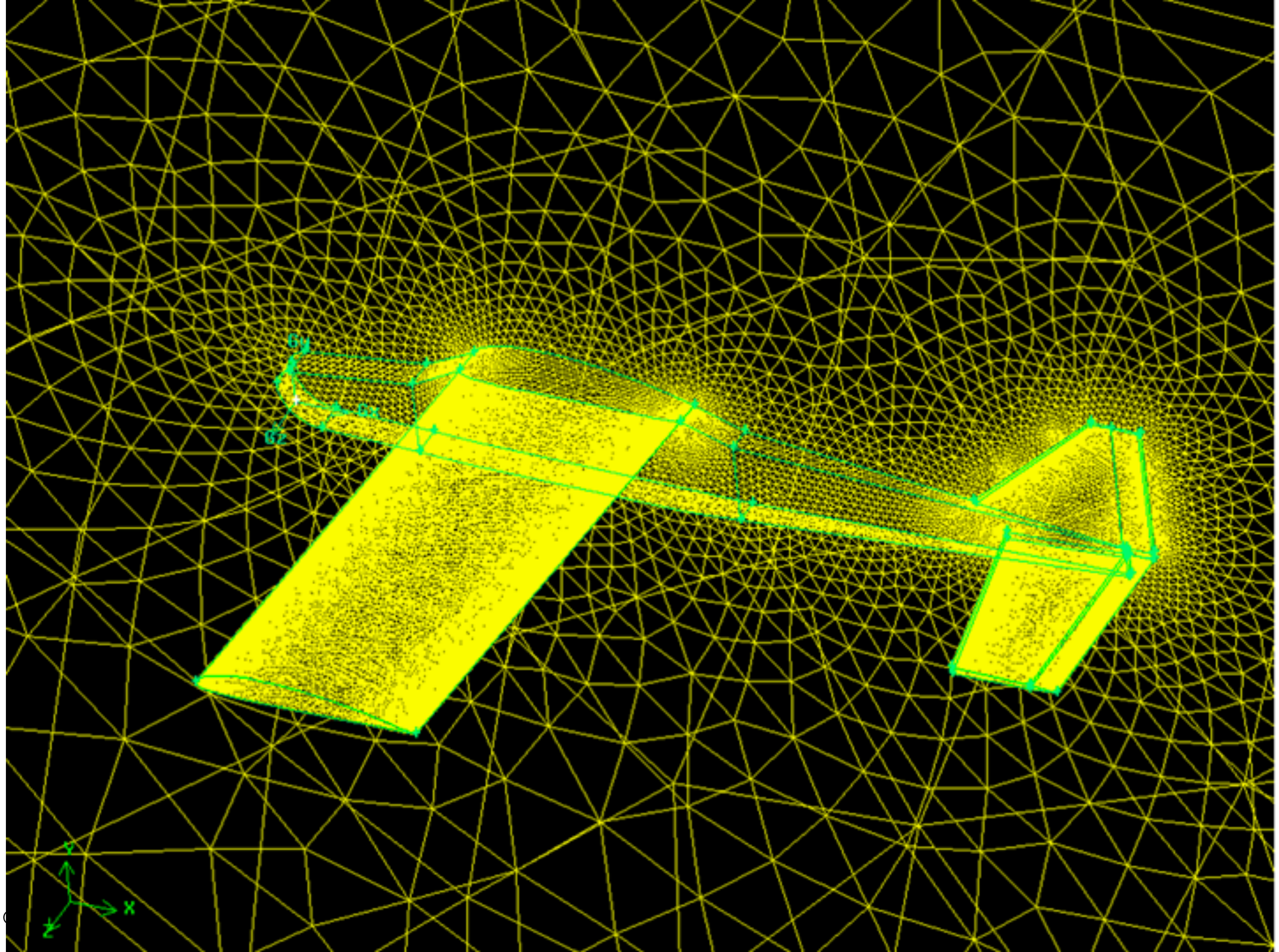
Unstructured *(any cell type)*

- Explicit cell definition
  - Types
  - Vertices








Cy
Cx
Cz
Y
X
Z

Mesh Types Summary

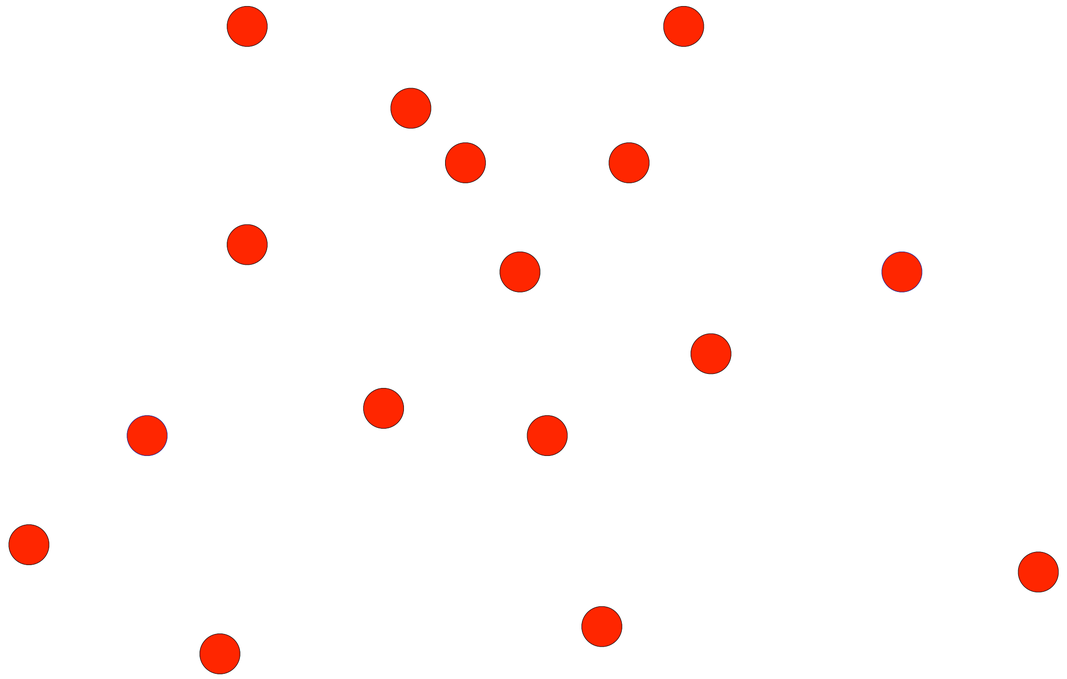
Topology →

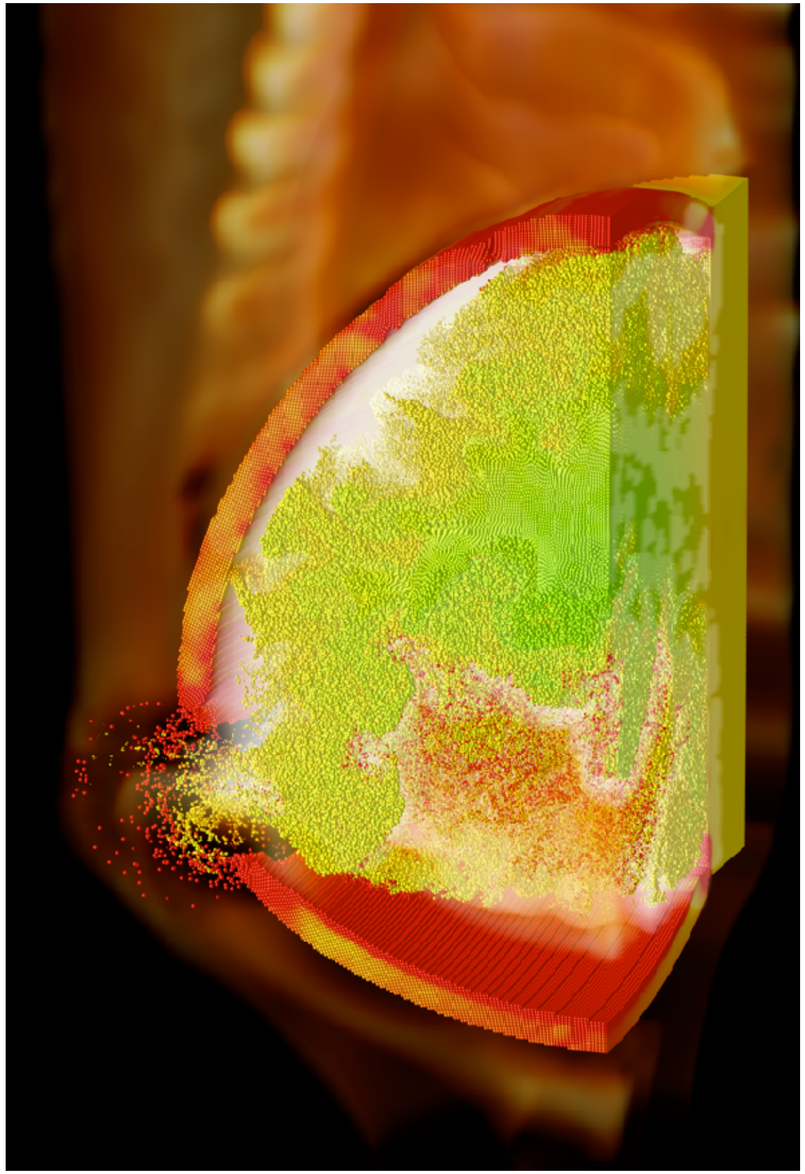
Geometry ↕

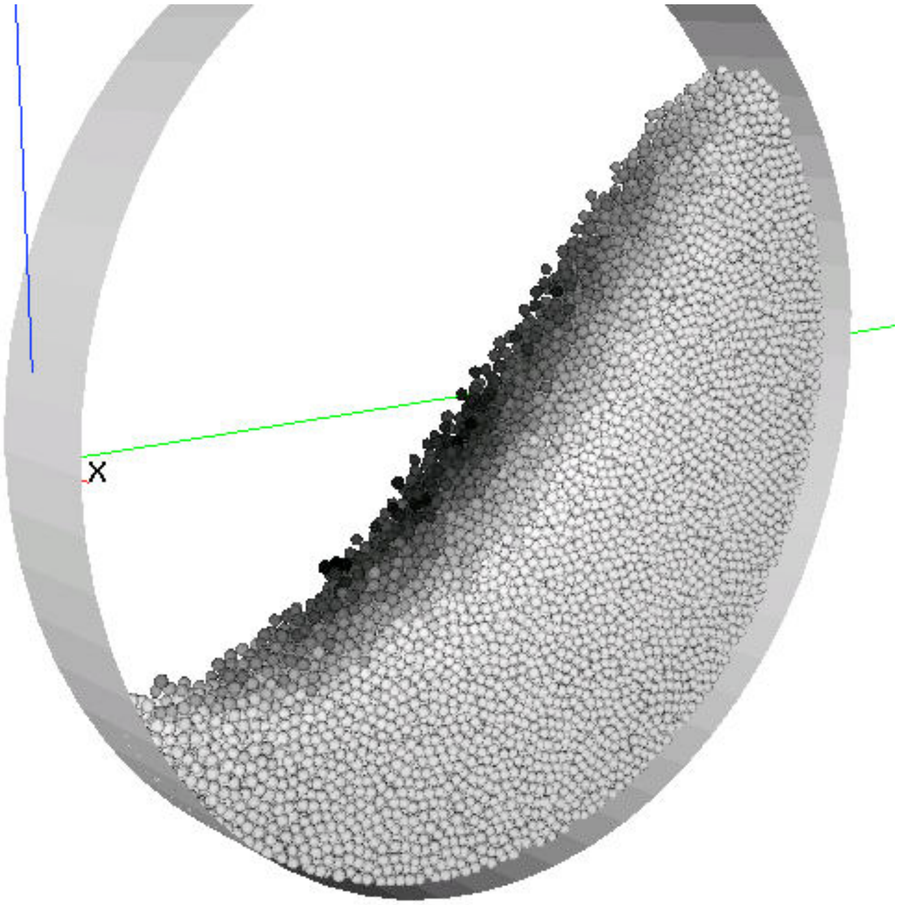
	Structured	Unstructured
Uniform	***Image***	***Unstructured***
Structured	***Rectilinear***	***Unstructured***
Unstructured	***Curvilinear***	***Unstructured***

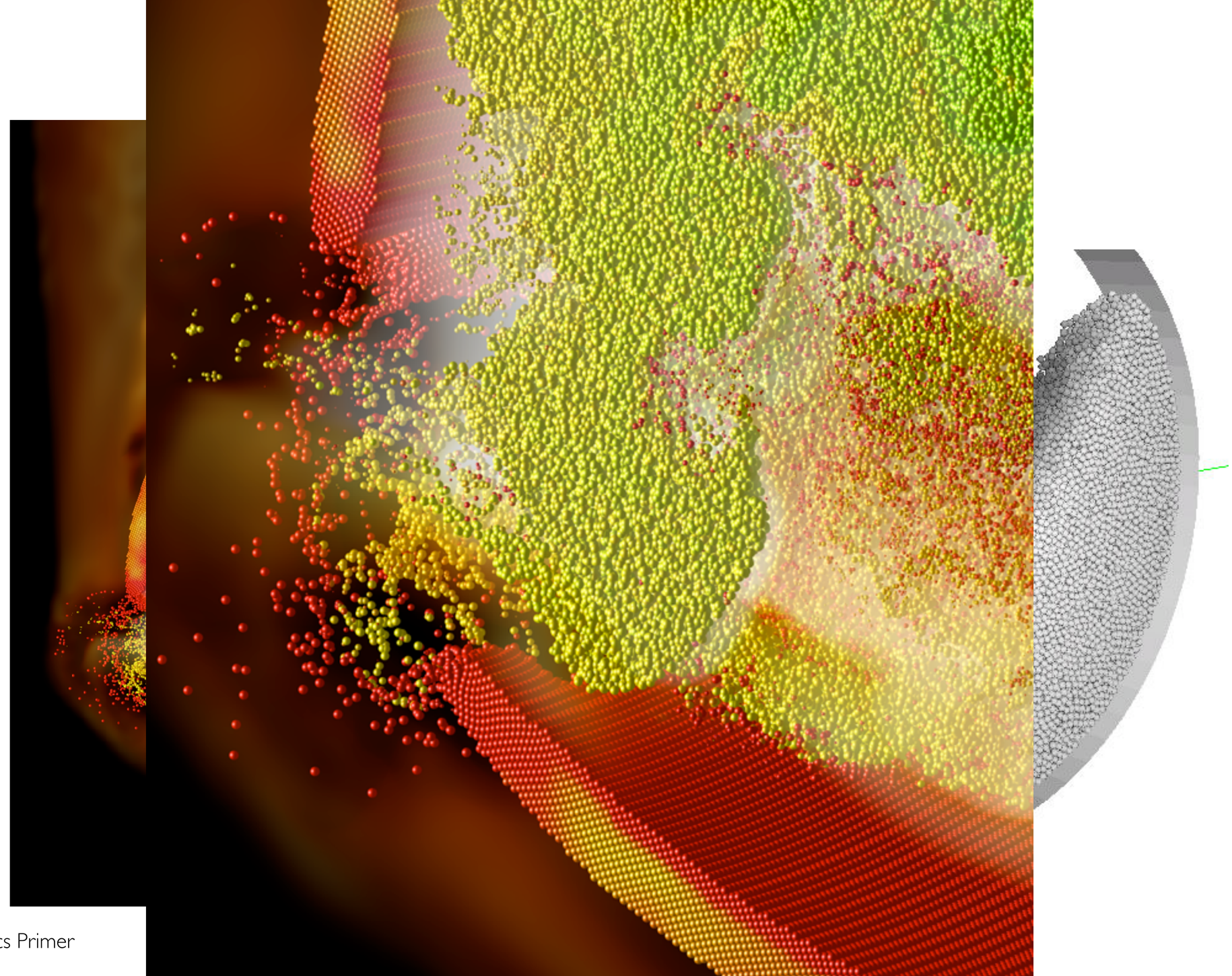
Mesh Topology

- **Mesh-free** (no connectivity)






X

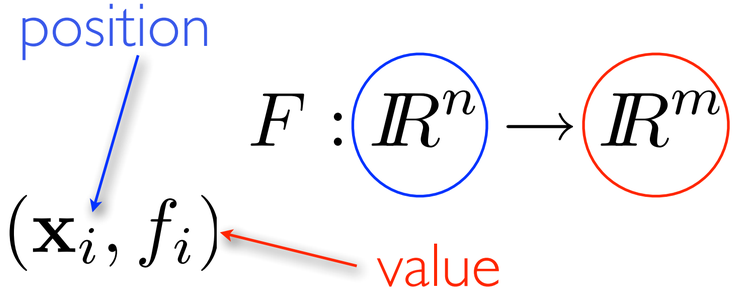


Outline

- Mesh types
- Interpolation
- Spatial queries
- Common pre-processing tasks

Interpolation

- Continuous reconstruction of discrete input data



$$\forall i \in \{1, .., n\}, F(\mathbf{x}_i) = f_i$$

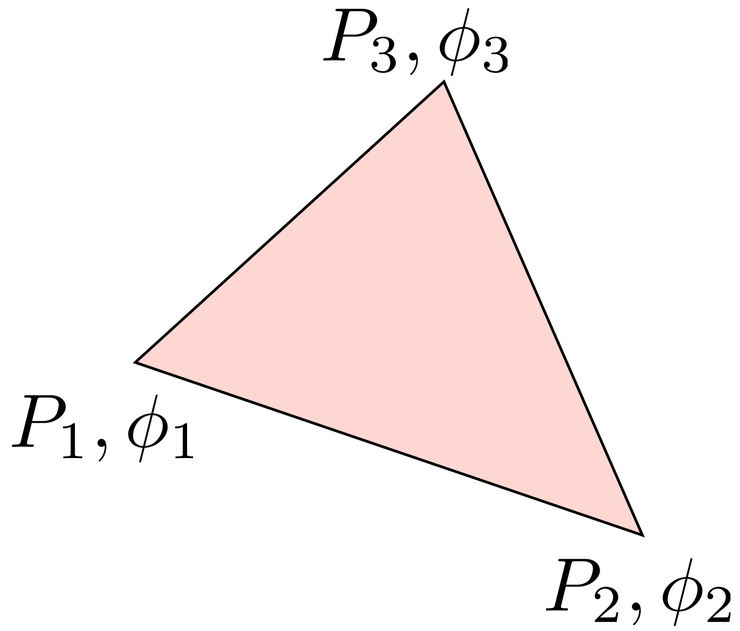
- Data is discrete but it represents a continuous phenomenon.
- Depends on mesh structure (if available)
- Interpolation vs. approximation

Linear Interpolation

- In **triangle** (2D **simplex**)

$$\phi(x, y) = a + bx + cy$$

$$\forall i, \phi(P_i) = \phi_i$$

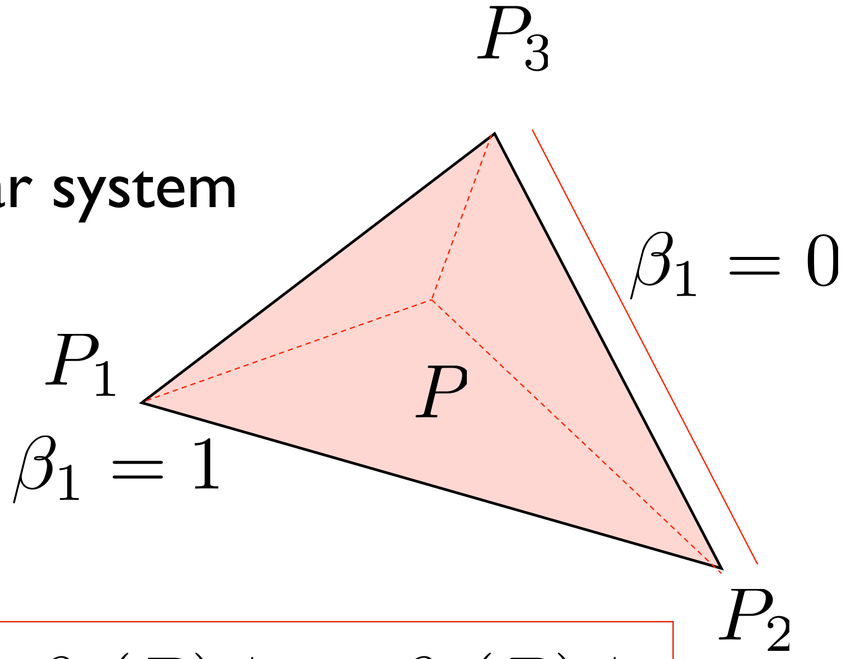


Linear Interpolation

- **Barycentric coordinates**

$$\begin{cases} P = \sum_{i=1}^{3} \beta_i P_i \\ \sum_{i=1}^{3} \beta_i = 1 \end{cases}$$

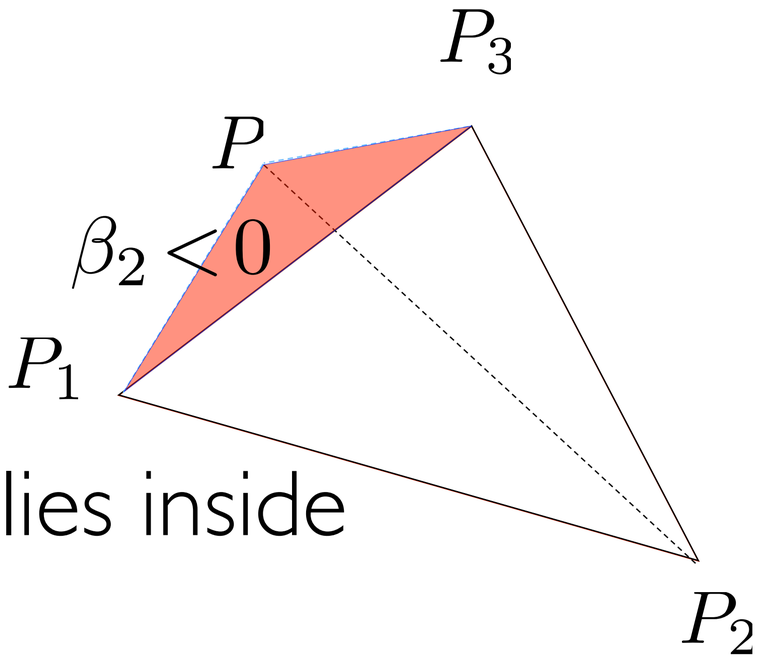
Linear system



$$\phi(P) = \beta_1(P)\phi_1 + \beta_2(P)\phi_2 + \beta_3(P)\phi_3$$

Linear Interpolation

- **Barycentric coordinates**
  - P lies outside triangle ⇔ at least one β_i is negative



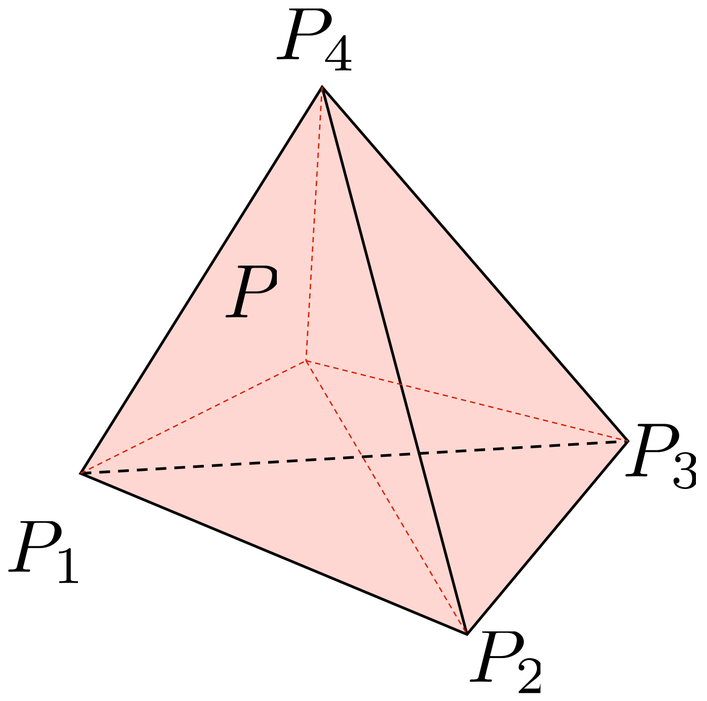
  - Easy way to check if a position lies inside a triangle

Linear Interpolation

- In **tetrahedron** (3D simplex)
  - Barycentric coordinates

$$P = \sum_{i=1}^{4} \beta_i P_i, \sum_{i=1}^{4} \beta_i = 1$$

$$\phi(P) = \sum_{i=1}^{4} \beta_i(P)\phi_i$$



  - Same inclusion test

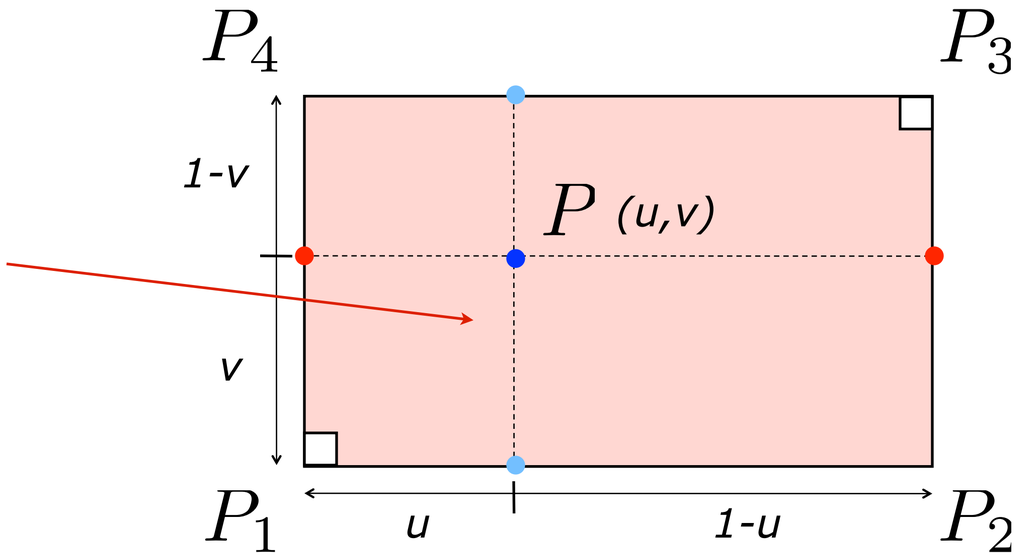
Bilinear Interpolation

- In quadrilateral cell

$$\phi(x, y) = axy + bx + cy + d$$

$$\forall i, \phi(P_i) = \phi_i$$

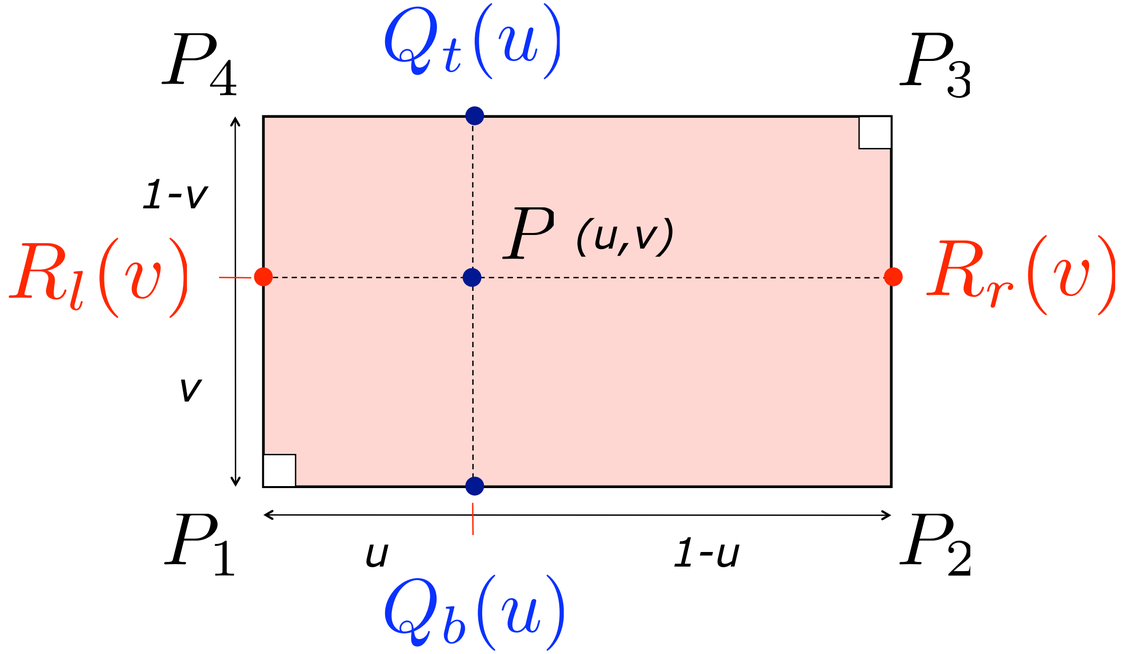
Combination of two consecutive linear interpolations



Bilinear Interpolation

- In **rectangle**

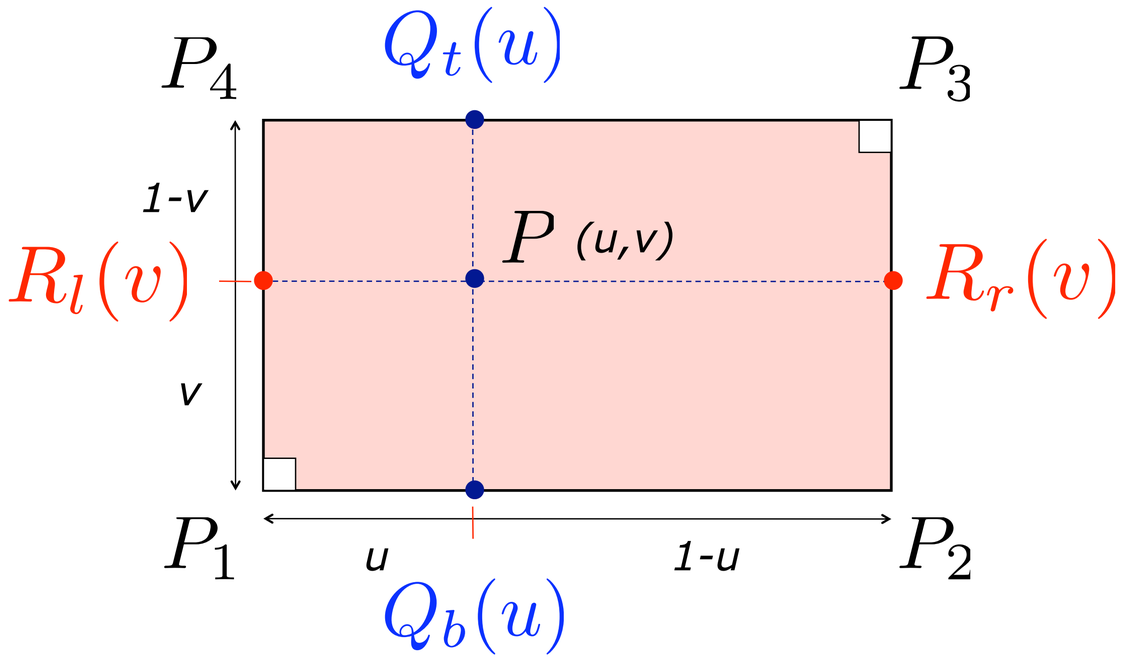
$$P = (1-v)Q_b(u) + vQ_t(u)$$
$$= (1-u)R_l(v) + uR_r(v)$$



Bilinear Interpolation

- In **rectangle**

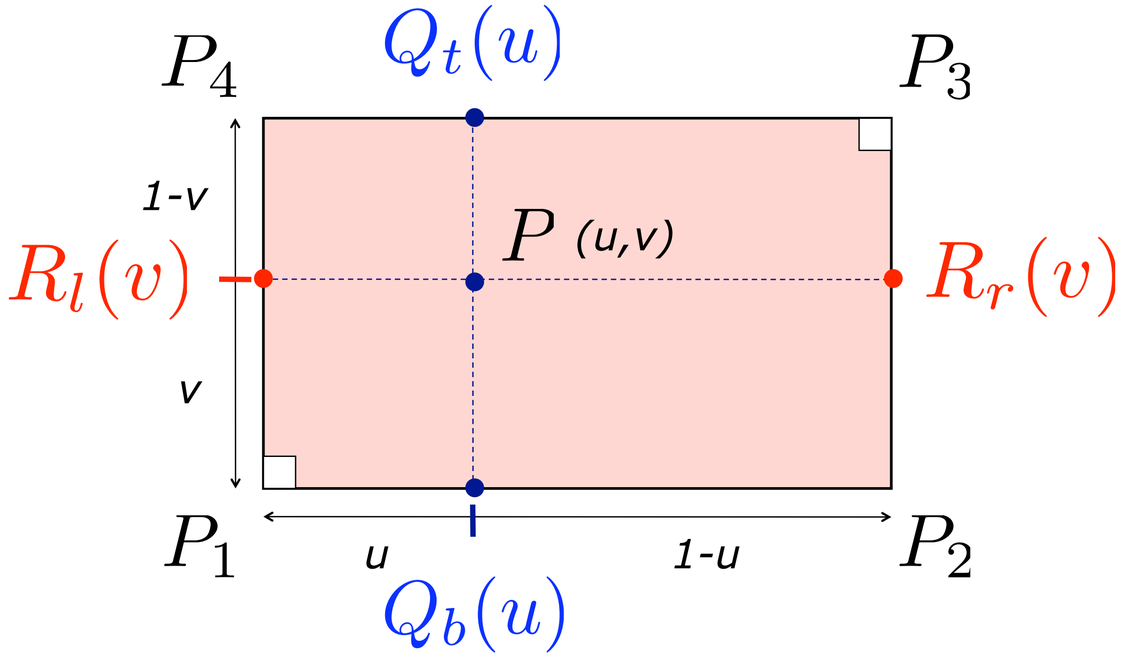
$$P = P_1 + u(P_2 - P_1) + v(P_4 - P_1) + uv(P_1 - P_2 + P_3 - P_4)$$



Bilinear Interpolation

- In **rectangle**

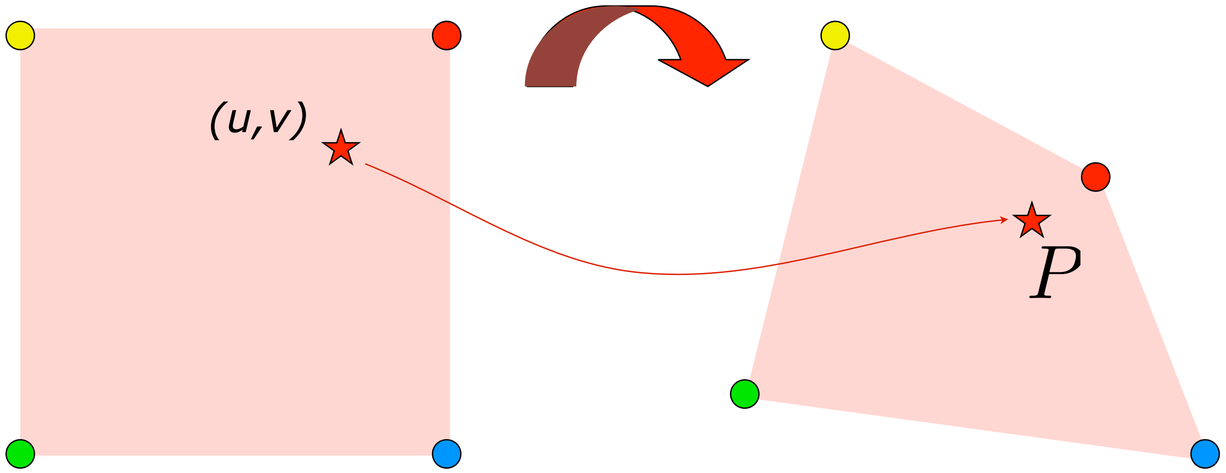
$$\phi(P) = \phi_1 + u(\phi_2 - \phi_1) + v(\phi_4 - \phi_1) + uv(\phi_1 - \phi_2 + \phi_3 - \phi_4)$$



Bilinear Interpolation

- In arbitrary **quadrilateral**

i. If we know coordinates (u,v) in computational space: apply previous formula to obtain position in physical space (*easy*)



Bilinear Interpolation

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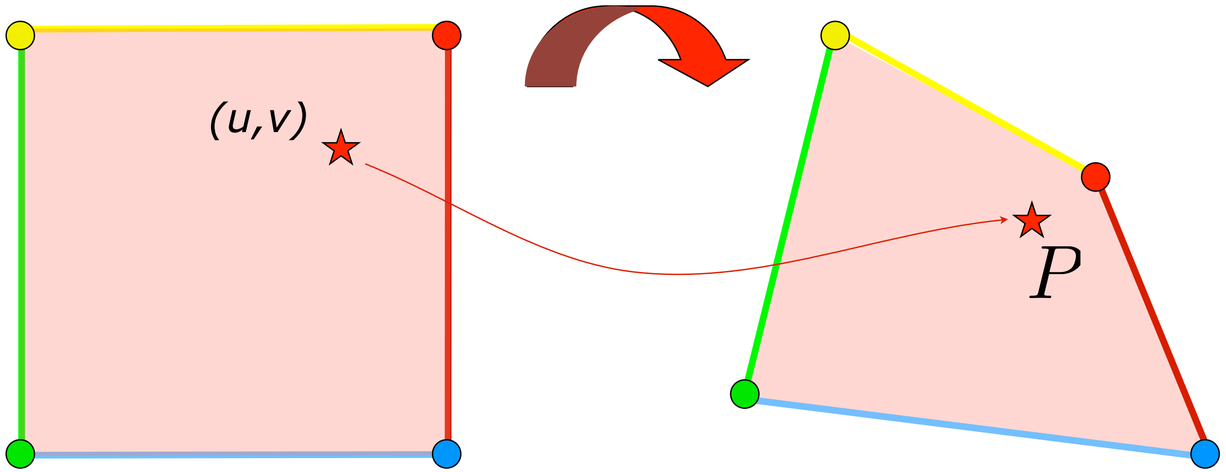


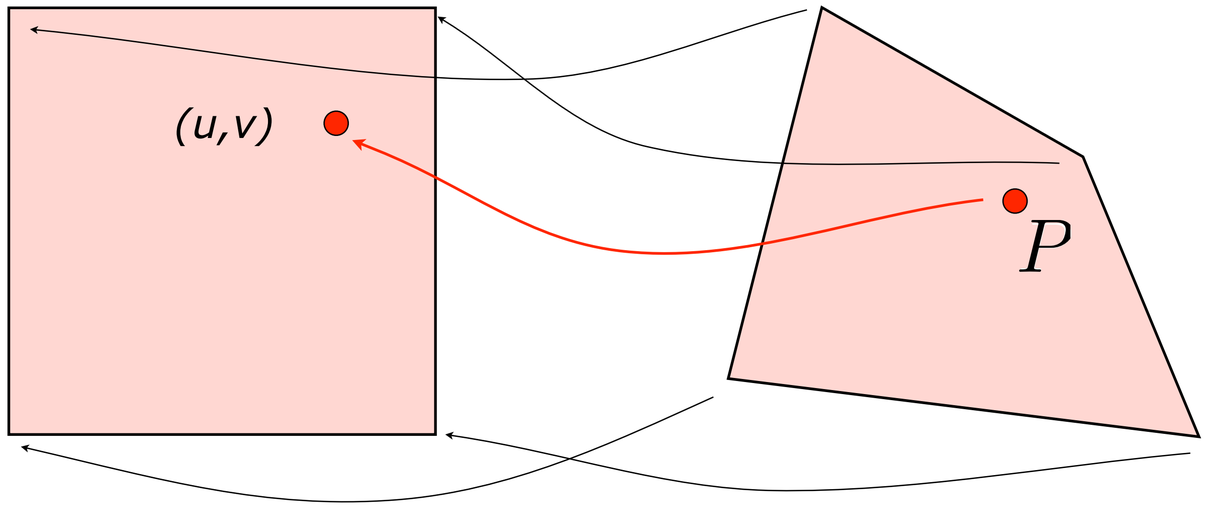
Important: axis-parallel straight lines are mapped to straight lines

Bilinear Interpolation

- In arbitrary **quadrilateral**

ii. If we only know coordinates (x,y) in physical space

1. Compute (inverse) mapping from physical space to computational space (unit square): quadratic equations
2. Apply bilinear formula



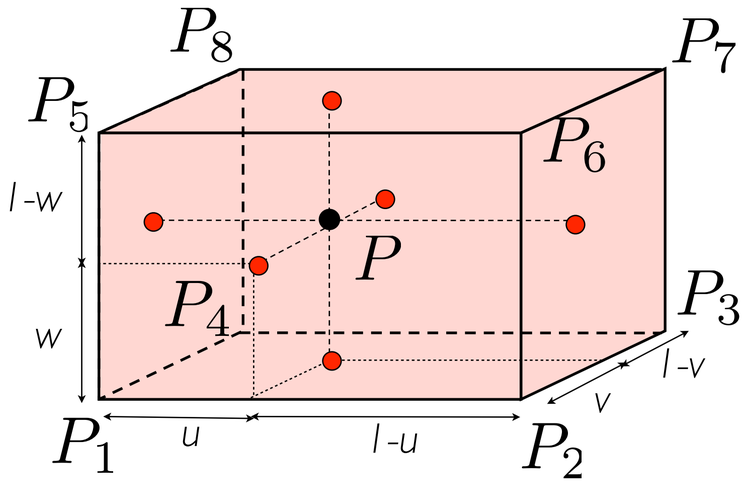
Trilinear Interpolation

- In a **cuboid** (axis parallel)
  - general polynomial expression

$$\phi(x, y, z) = axyz + bxy + cxz + dyz + ex + fy + gz + h$$

  - expressed in local coordinates

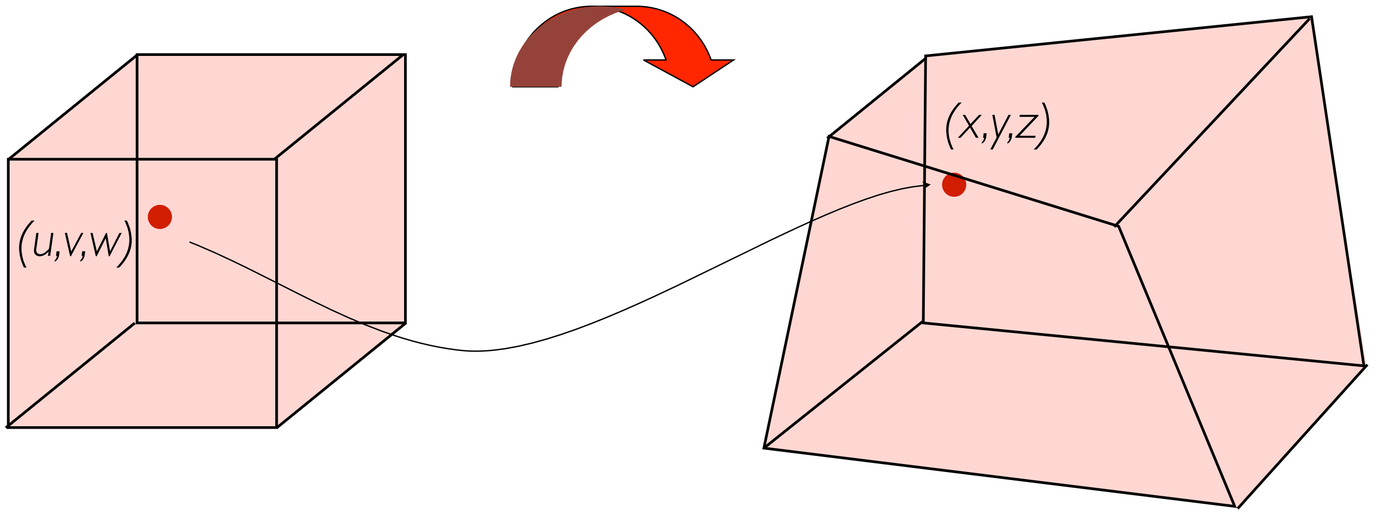
$$\begin{aligned} P \quad = \quad & P_1 + u\,(P_2 - P_1) + \\ & v\,(P_4 - P_1) + w\,(P_5 - P_1) + \\ & uv\,(P_1 - P_2 + P_3 - P_4) + \\ & uw\,(P_1 - P_2 - P_5 + P_6) + \\ & vw\,(P_1 - P_4 - P_5 + P_8) + \\ & uvw\,(-P_1 + P_2 - P_3 + P_4 - P_5 + P_6 - P_7 + P_8) \end{aligned}$$



Trilinear Interpolation

- In arbitrary **hexahedron**

i. If we know coordinates (u,v,w) in computational space, apply previous formula (*easy*)



Trilinear Interpolation

- In arbitrary **hexahedron**

ii. If we only know coordinates (x,y,z) in physical space:

1. compute inverse mapping from physical space to computational space (axis aligned unit cube)

   i. nonlinear problem

$$(x, y, z)^T = F(u, v, w)^T$$

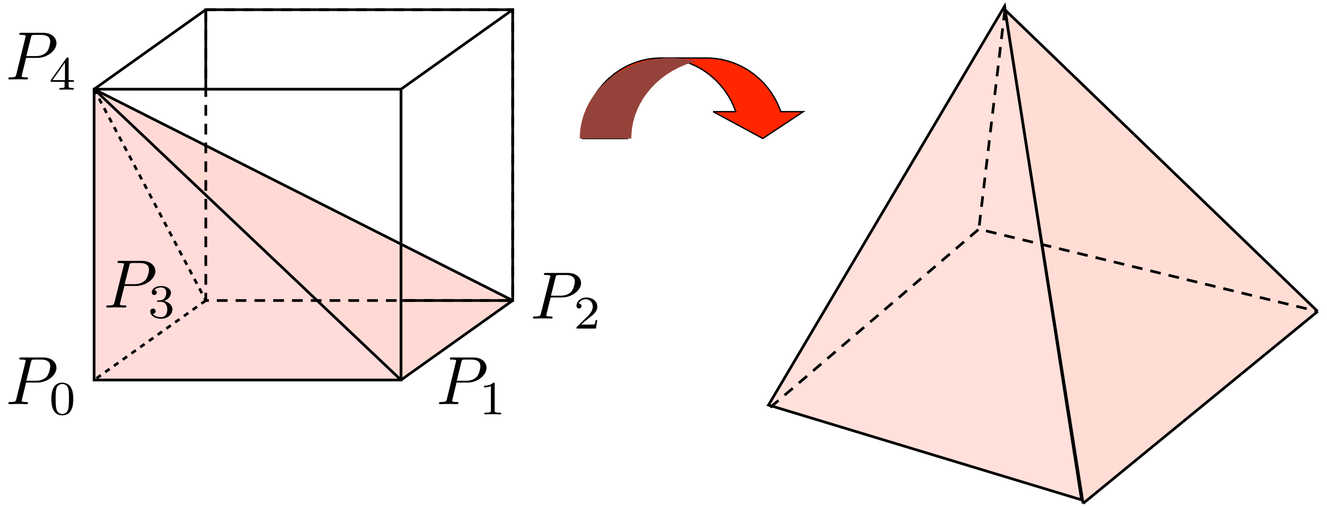
   ii. solved numerically with iterative scheme

$$F(\mathbf{u} + \vec{\delta}) = F(\mathbf{u}) + J\vec{\delta} + \ldots$$
$$J\vec{\delta} = -F(\mathbf{u})$$

2. goto i.

Trilinear Interpolation

- In **pyramids** (special case of trilinear int.)

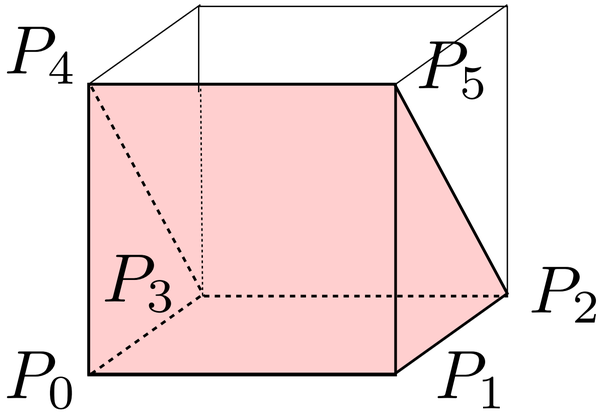


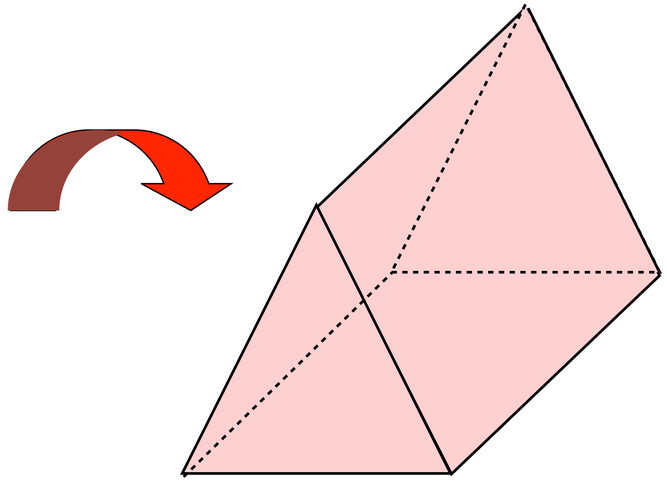
$$P(u,v,w) = (1-u)(1-v)(1-w)P_0$$
$$+u(1-v)(1-w)P_1$$
$$+uv(1-w)P_2$$
$$+(1-u)v(1-w)P_3 + wP_4$$

Note: Set $P_5 = P_6 = P_7 = P_4$ in formula on s. 31

Trilinear Interpolation

- In **prisms** (special case of trilinear int.)



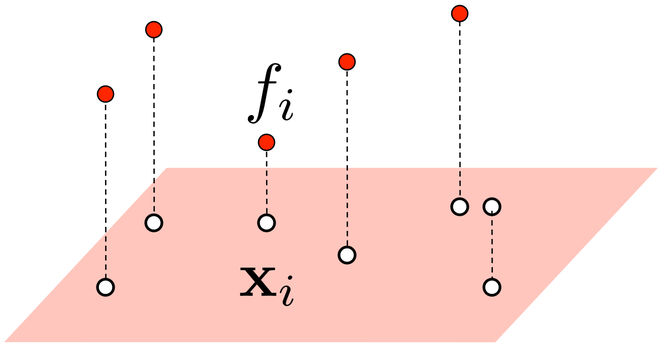


$$P(u,v,w) = (1-u)(1-v)(1-w)P_0$$
$$+u(1-v)(1-w)P_1$$
$$+uv(1-w)P_2$$
$$+(1-u)v(1-w)P_3$$
$$+(1-u)wP_4 + uwP_5$$

Note: Set $P_6 := P_5$ and $P_7 := P_4$ in formula on s. 31

Scattered Data Interpolation

- **Shepard methods**

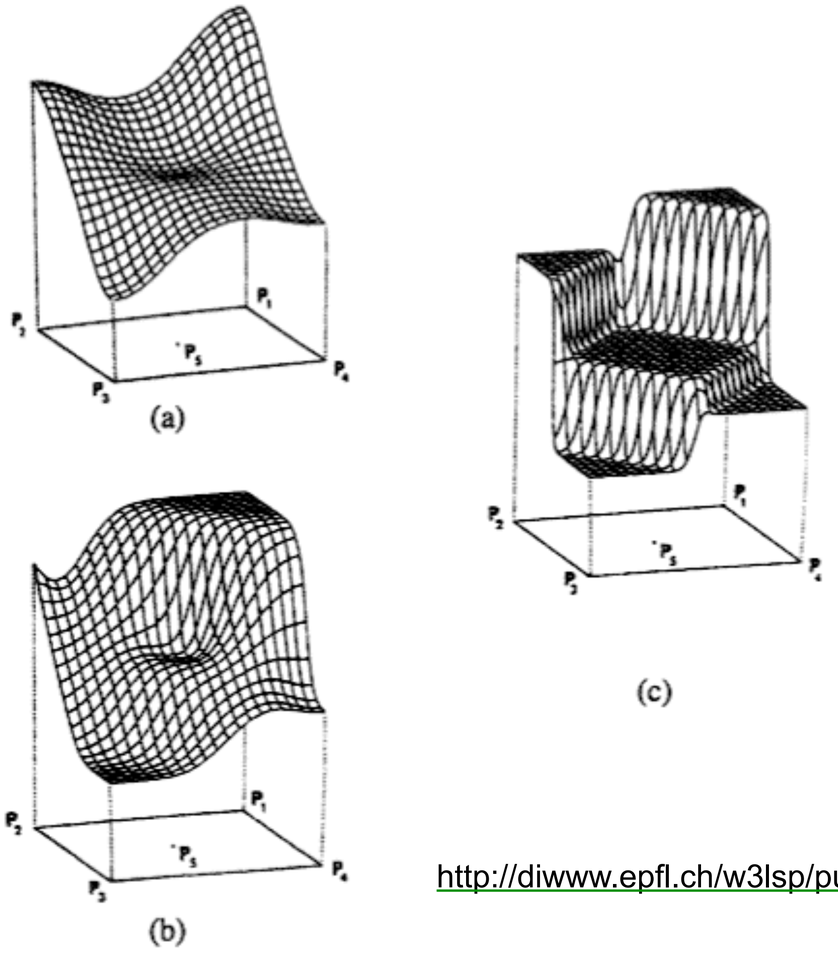


$$\sigma_i(\mathbf{x}) = \frac{1}{||\mathbf{x} - \mathbf{x}_i||^k}$$

Original Shepard
- flat spots
- global

Scattered Data Interpolation

- **Shepard methods**



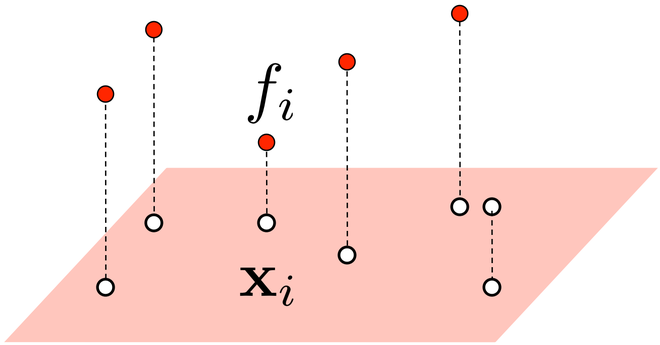
http://diwww.epfl.ch/w3lsp/publications/other/sdimfeisas.pdf

Scattered Data Interpolation

- **Shepard methods**

$$F(\mathbf{x}) = \sum_{i=1}^{N} \omega_i(\mathbf{x}) f_i$$

$$\omega_i(\mathbf{x}) = \frac{\sigma_i(\mathbf{x})}{\sum_{j=1}^{N} \sigma_j(\mathbf{x})}$$



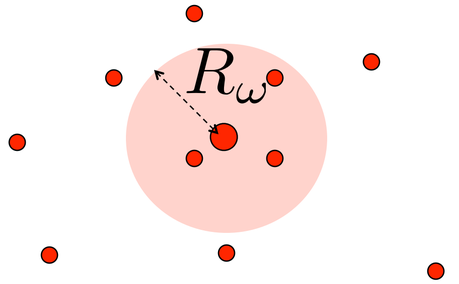
$$\sigma_i(\mathbf{x}) = \exp -(\alpha_i(x - x_i)^2 + \beta_i(y - y_i)^2 + \gamma_i(z - z_i)^2)$$

Scattered Data Interpolation

- **Shepard methods**

$$F(\mathbf{x}) = \sum_{i=1}^{N} \omega_i(\mathbf{x}) Q_i(\mathbf{x})$$

$$\omega_i(\mathbf{x}) = \frac{\sigma_i(\mathbf{x})}{\sum_{j=1}^{N} \sigma_j(\mathbf{x})}$$



$$\sigma_i(\mathbf{x}) = \frac{1}{||\mathbf{x} - \mathbf{x}_i||} \left(1 - \frac{||\mathbf{x} - \mathbf{x}_i||}{R_\omega}\right)_+^2$$

Franke, Nielson
quadratic fit
local

Radial Basis Functions

- Set of points: $X = (\mathbf{x}_i)_{i=1}^N \subset \mathbb{R}^d$
- Associated values: $(f_i)_{i=1}^N \subset \mathbb{R}$
- Find coefficients $(c_i)_{i=1}^N \subset \mathbb{R}$ satisfying

$$P_f(\mathbf{x}) = \sum_{i=1}^{N} c_i \phi(||\mathbf{x} - \mathbf{x}_i||)$$

- System of equations: $\mathbf{Ac} = \mathbf{f}$ with $A_{ij} = \phi(||\mathbf{x}_i - \mathbf{x}_j||)$

radial basis function

Radial Basis Functions

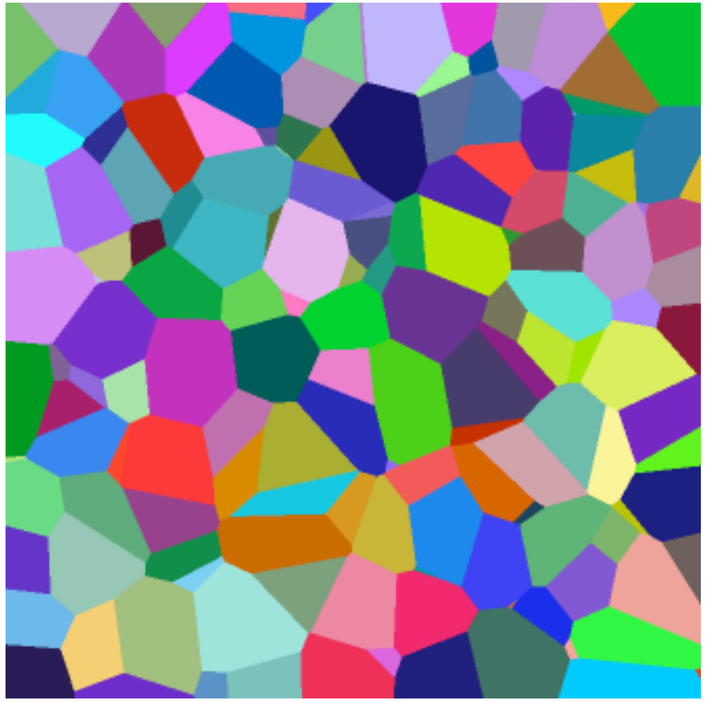
- Hardy's inverse multiquadrics

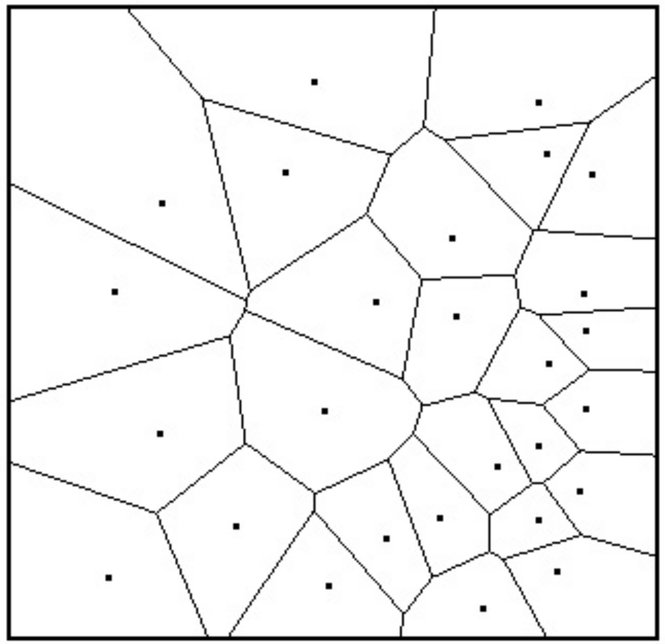
$$f(r) = \frac{1}{\sqrt{r^2 + c}}$$

- Other radial functions used in practice
  - Thin plate splines $f(r) = r^2 \ln r$
  - Truncated gaussians $f(r) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{r^2}{2\sigma^2}}$
  - Polyharmonic splines $f(r) = r^\beta$

But Also...

- Nearest Neighbor interpolation

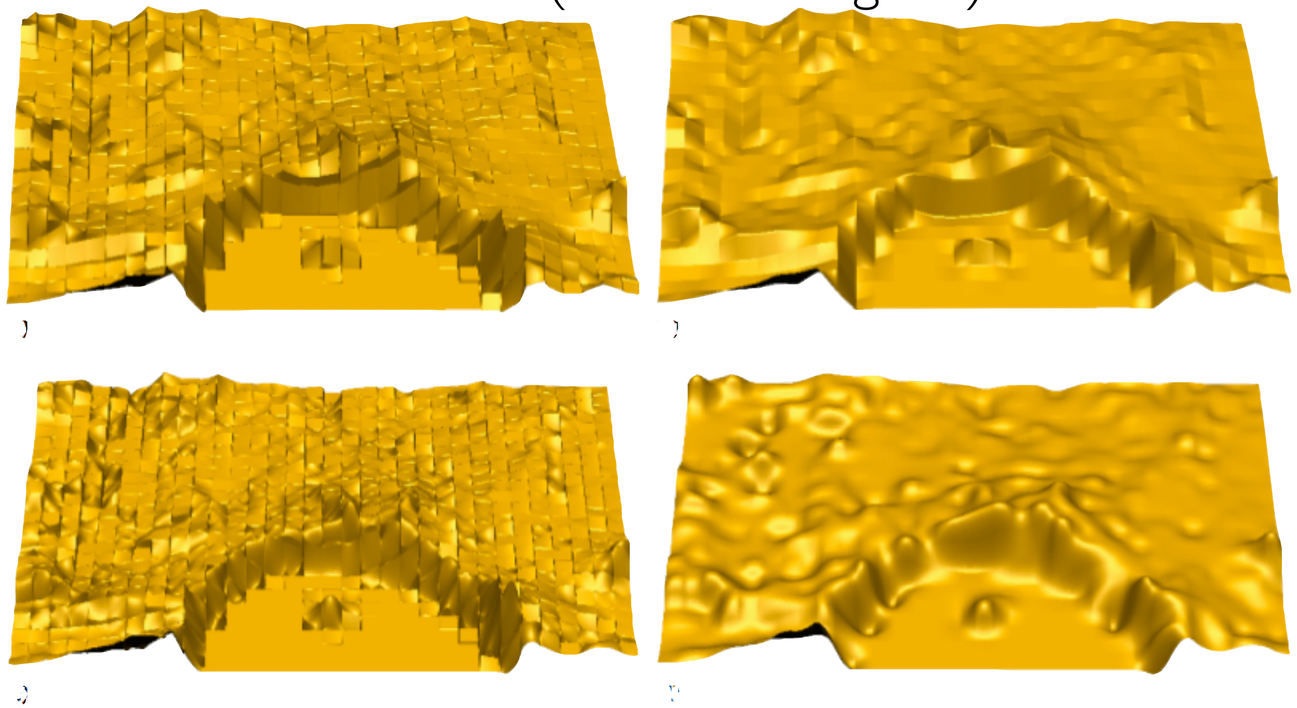




Voronoi diagram

But Also...

- Higher-order interpolation schemes
  - splines, local polynomial fit (interpolation, least squares, ...)
  - smooth reconstruction kernels (on uniform grids)



Outline

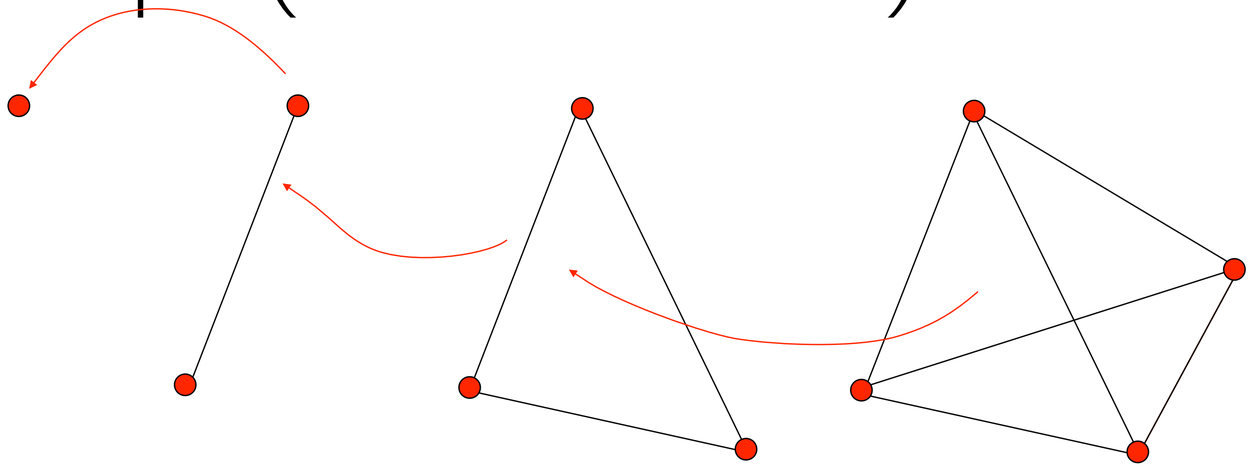
- Mesh types
- Interpolation
- Spatial queries
- Common pre-processing tasks

Context

- Queries in large, unstructured grids
  - arbitrary probing and interpolation
  - resampling (e.g., on regular grid)
- Goal: speed up computation
- Expensive: avoid it when you can
  - iteration (loop over grid vertices)
  - use neighborhood information: leverage correlation between consecutive queries (e.g., streamline, isosurface)

Neighborhood Data

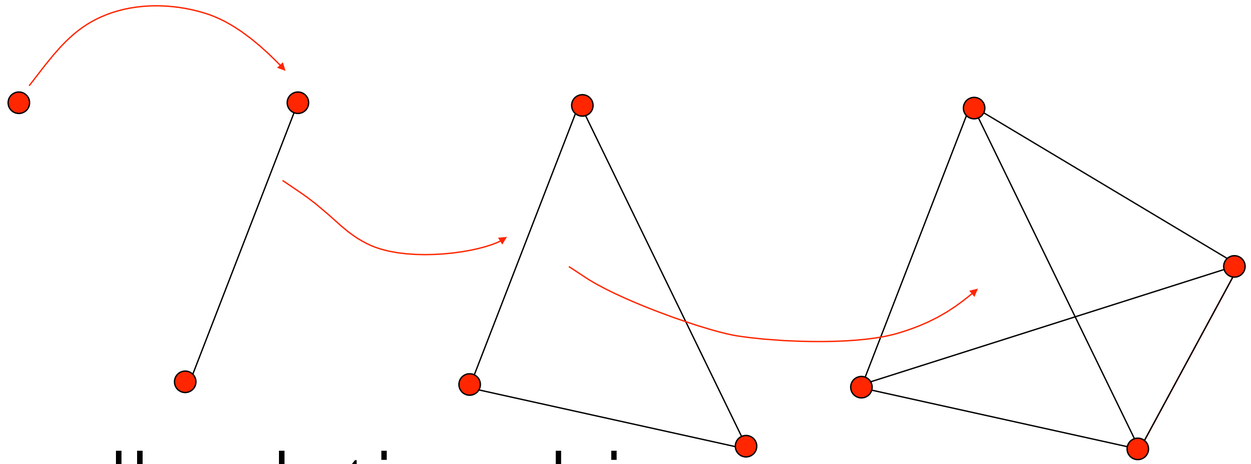
- Data structures store top-down relationships (cell to vertex)



- No direct support for neighborhood information (vertex to cell, cell to cell)

Neighborhood Data

- Bottom-up relationships obtained by computation



- Cell to cell relationship:

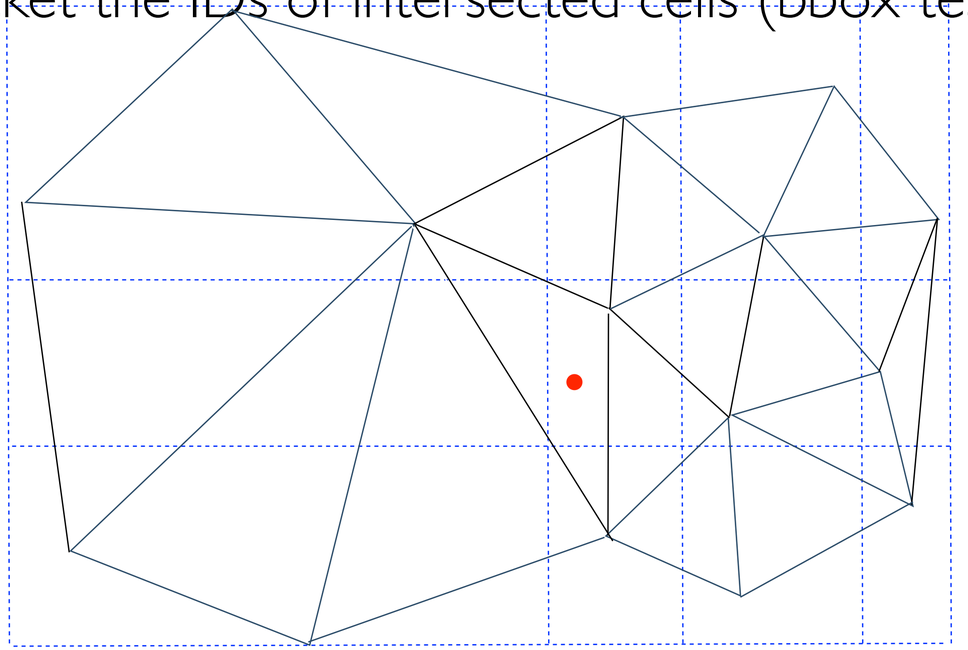
  cell → vertices → cells

Top-down

Bottom-up

Regular Space Subdivision

- Overlay uniform/rectilinear grid over unstructured domain
  - store in each bucket the IDs of intersected cells (bbox test)



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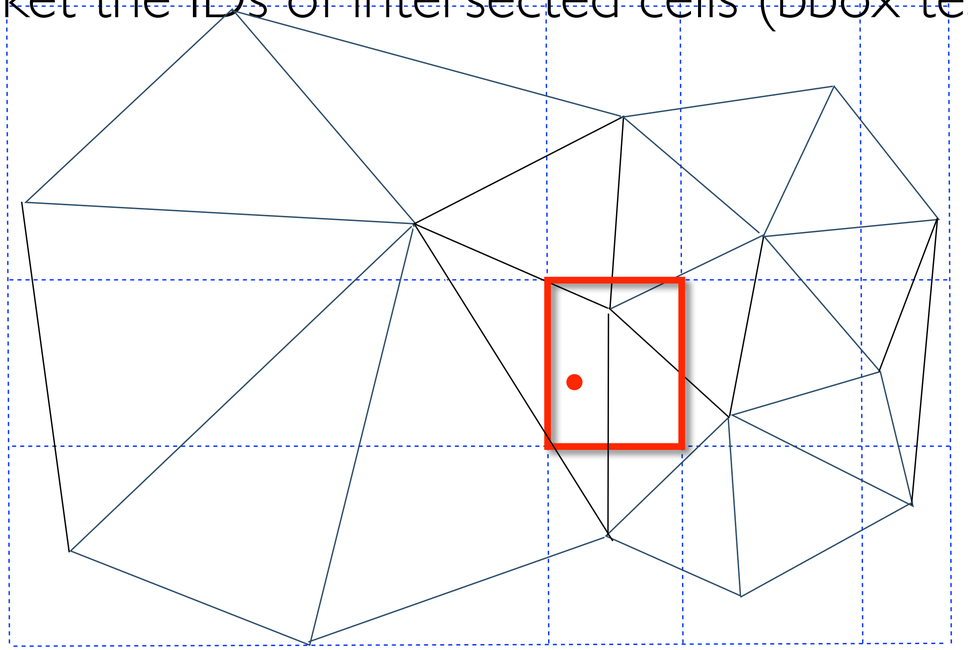

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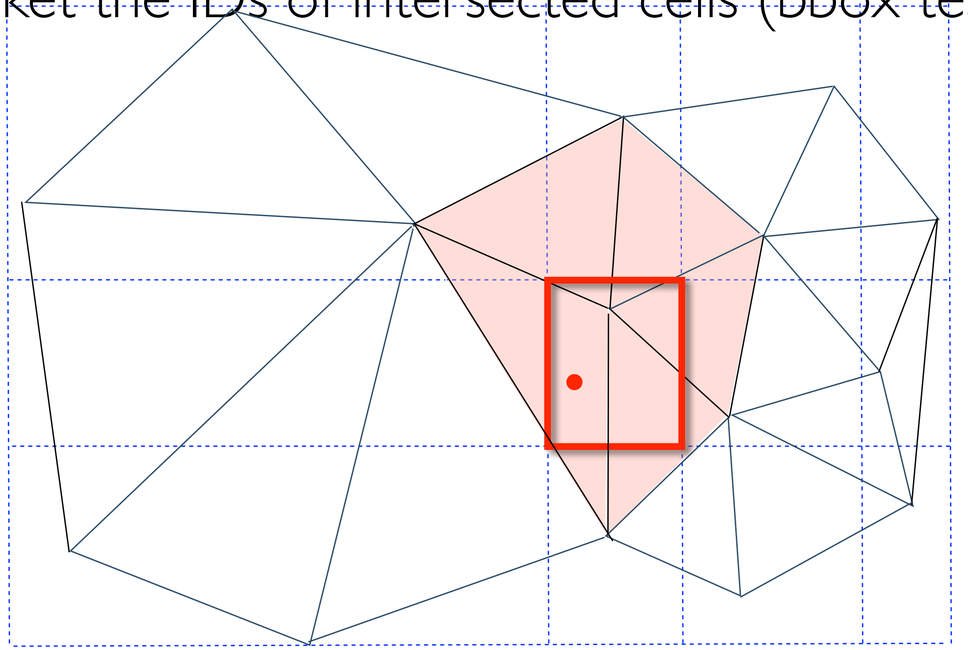

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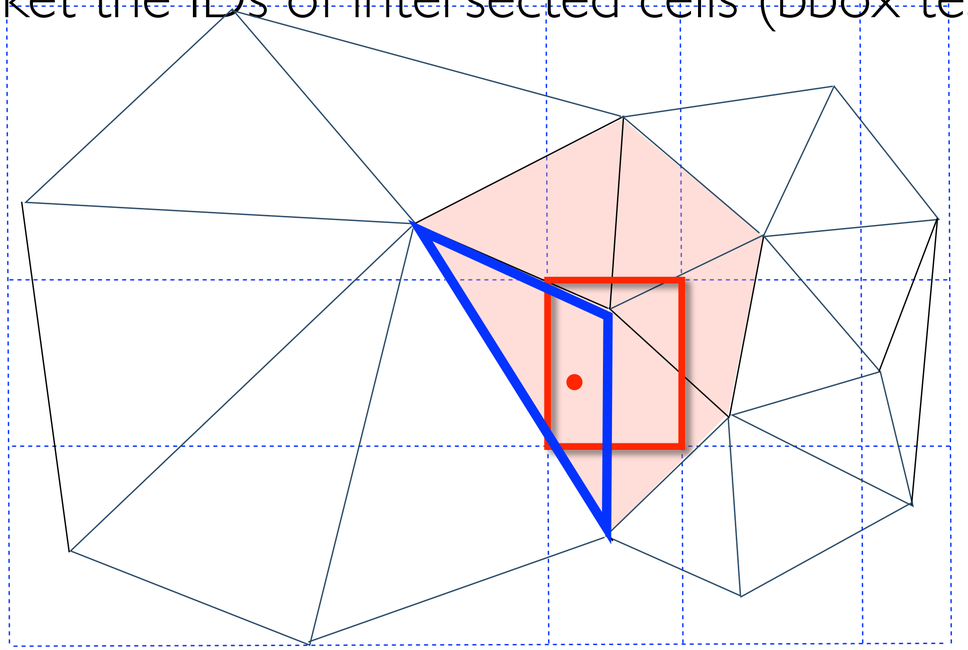


Regular Space Subdivision

✓ Easy to implement

✓ Works well if cell distribution is fairly uniform

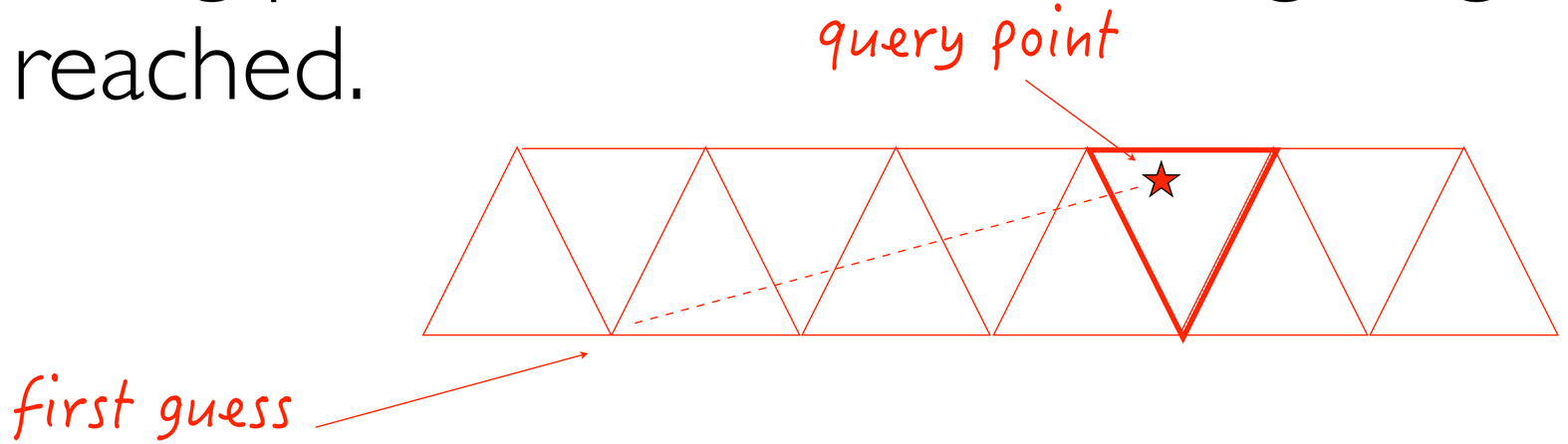
- Extremely slow for highly unstructured meshes

➡ many cells per bucket

➡ many buckets per cell

"*Jump & Walk*" Approach

- Use data structure to quickly determine "good" starting point
- Using cell information, walk across cells from starting point until cell containing target point is reached.



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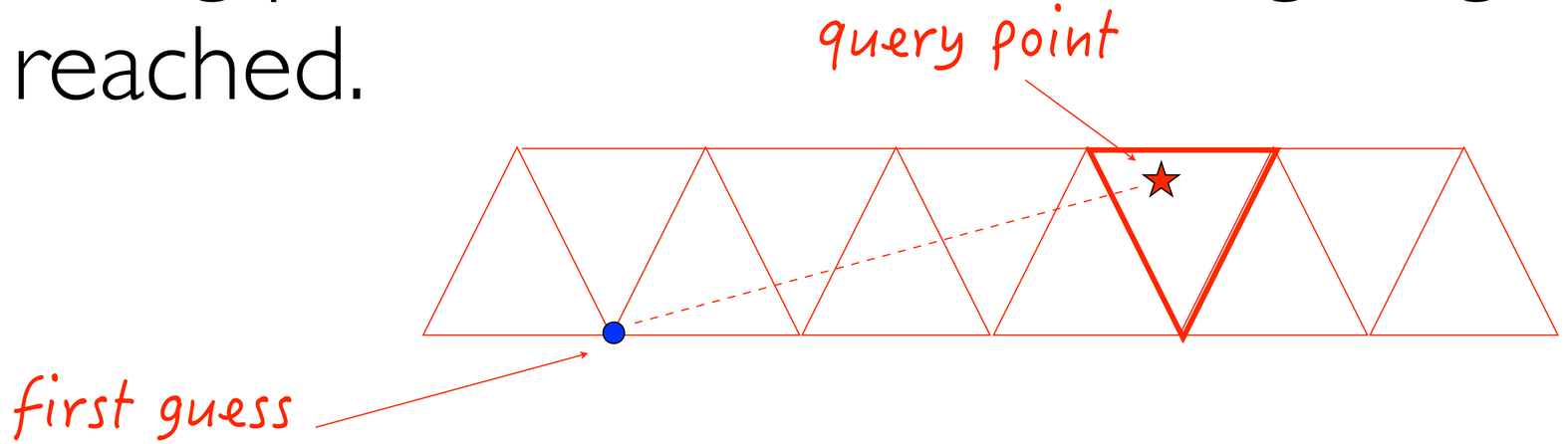

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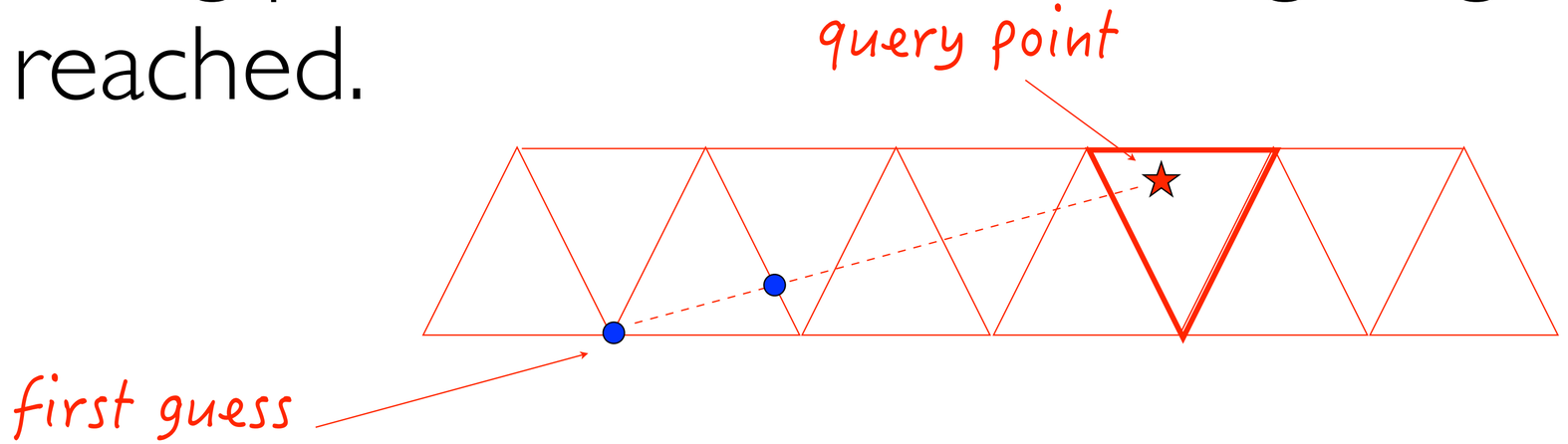

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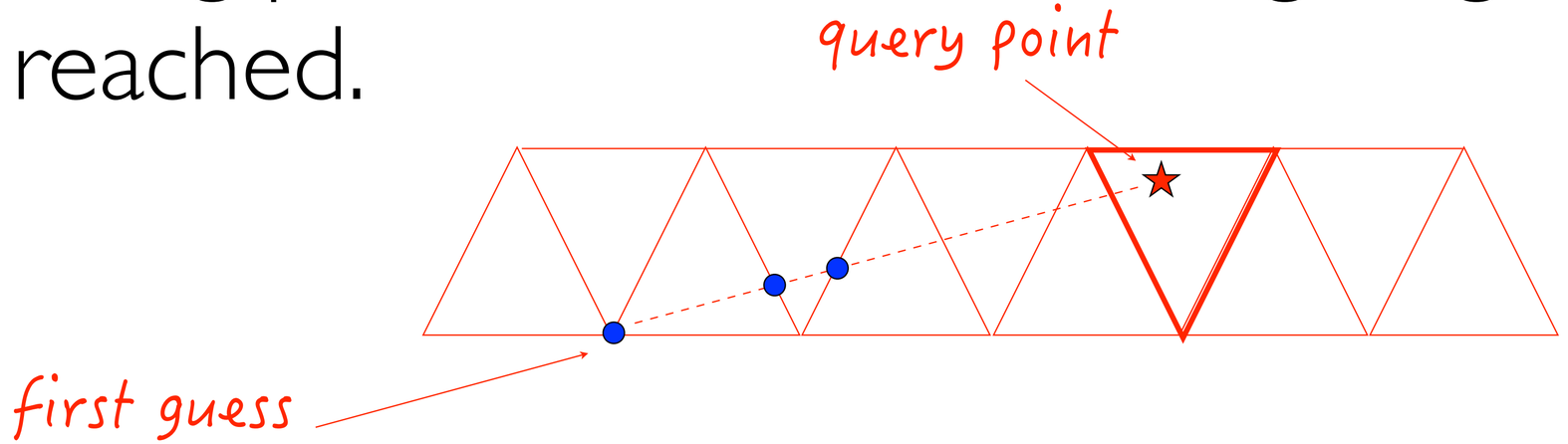

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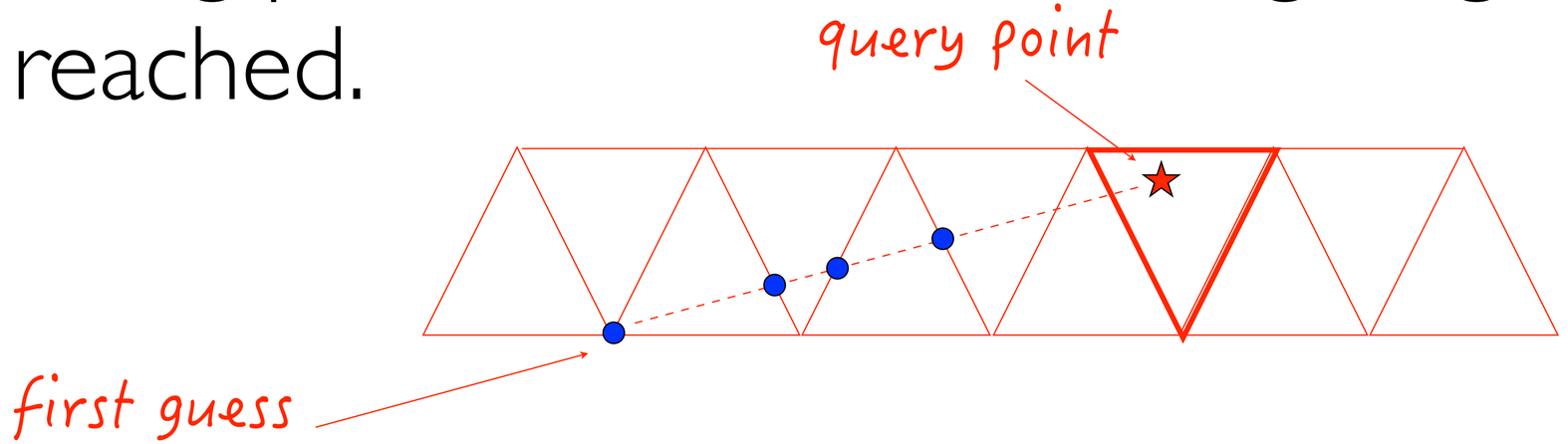

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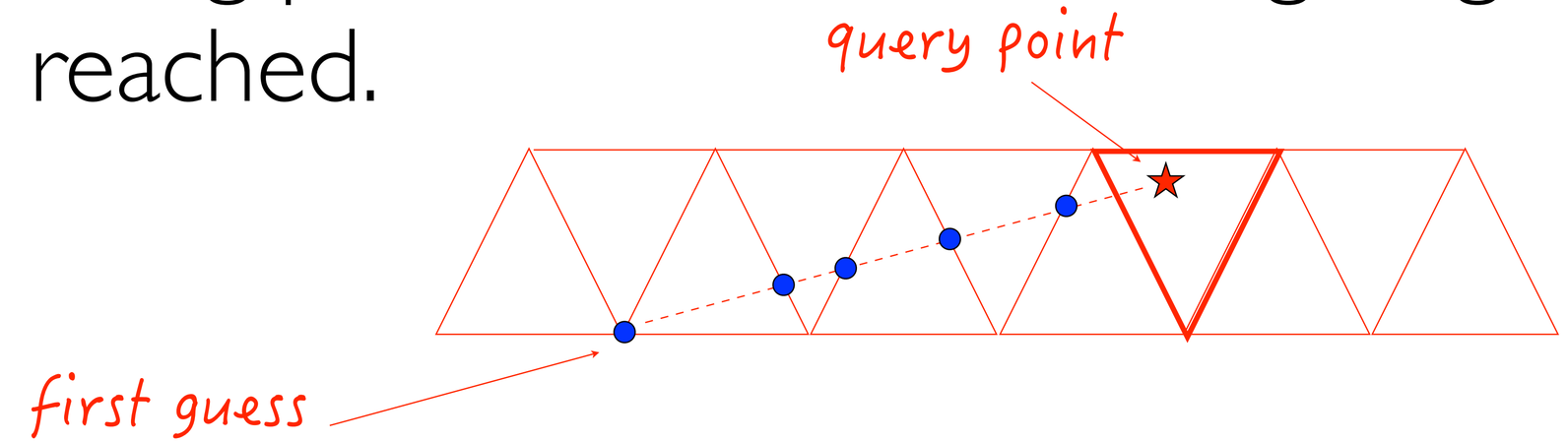

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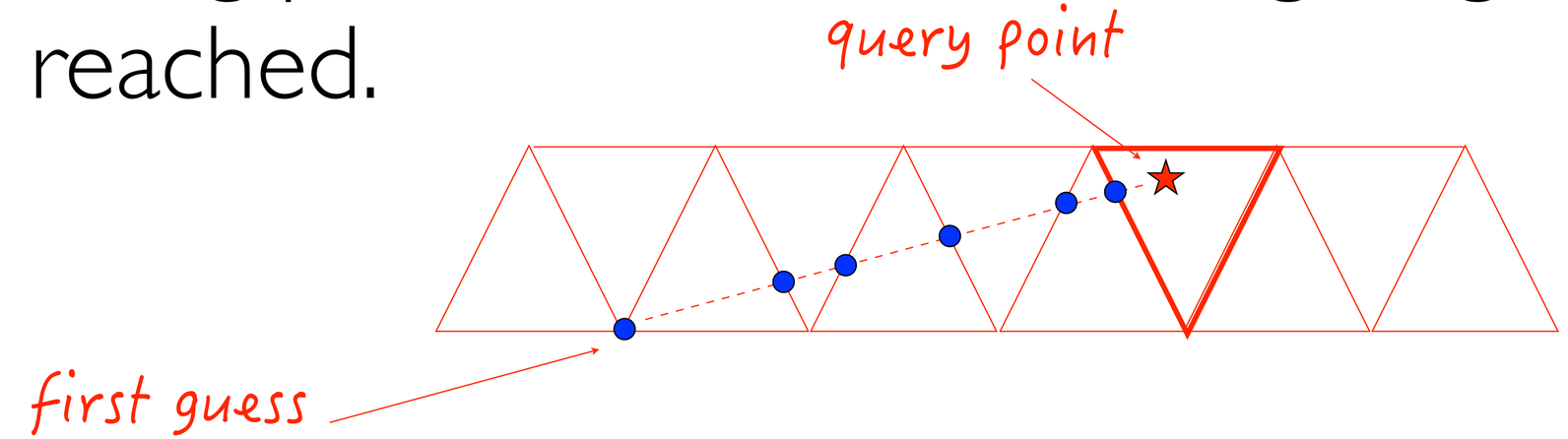

Octree

- Octrees (resp. quadtrees) regularly subdivide space along each coordinate axis at each level

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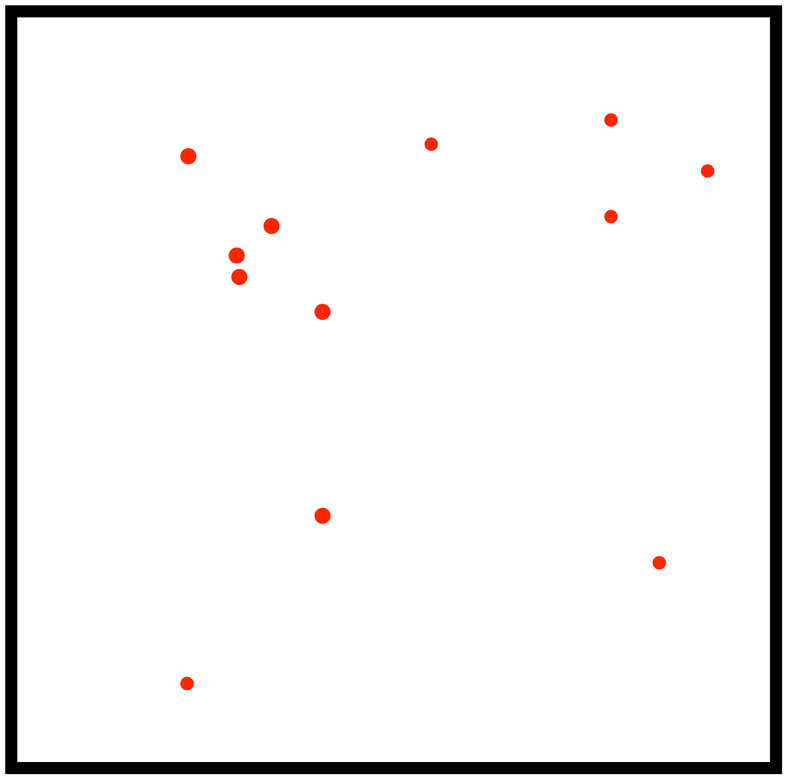

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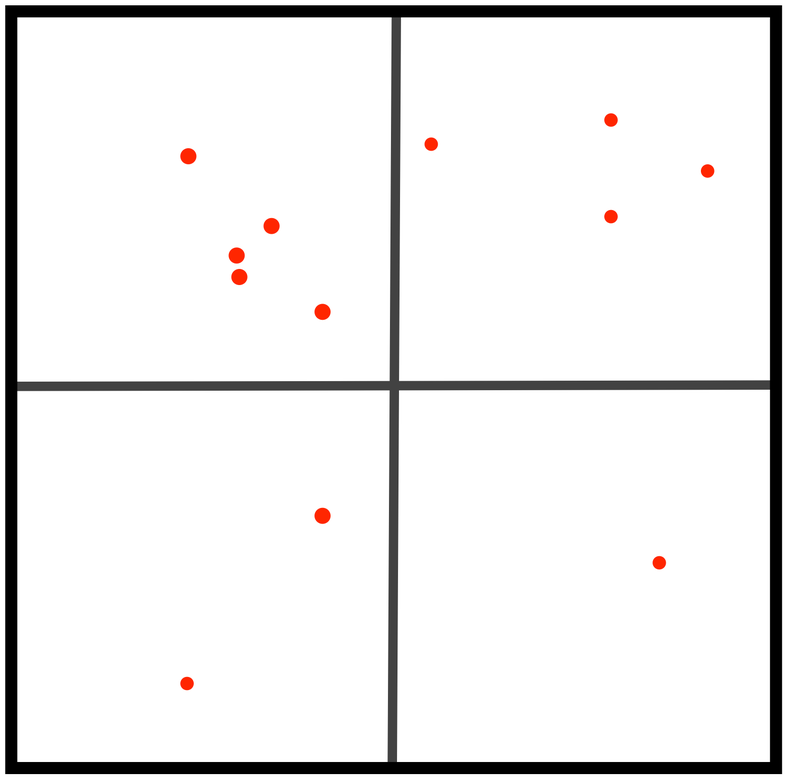

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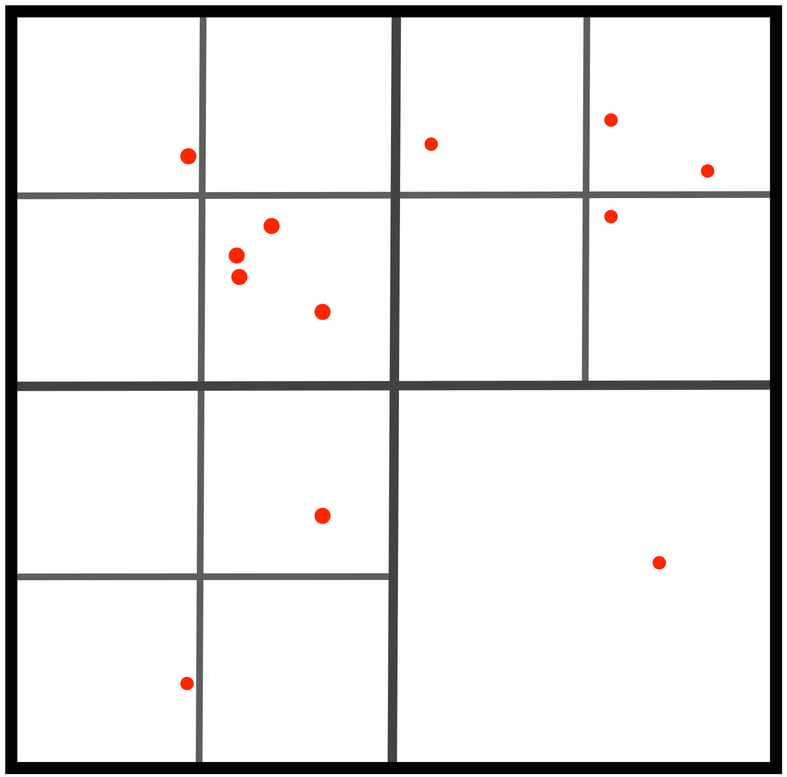

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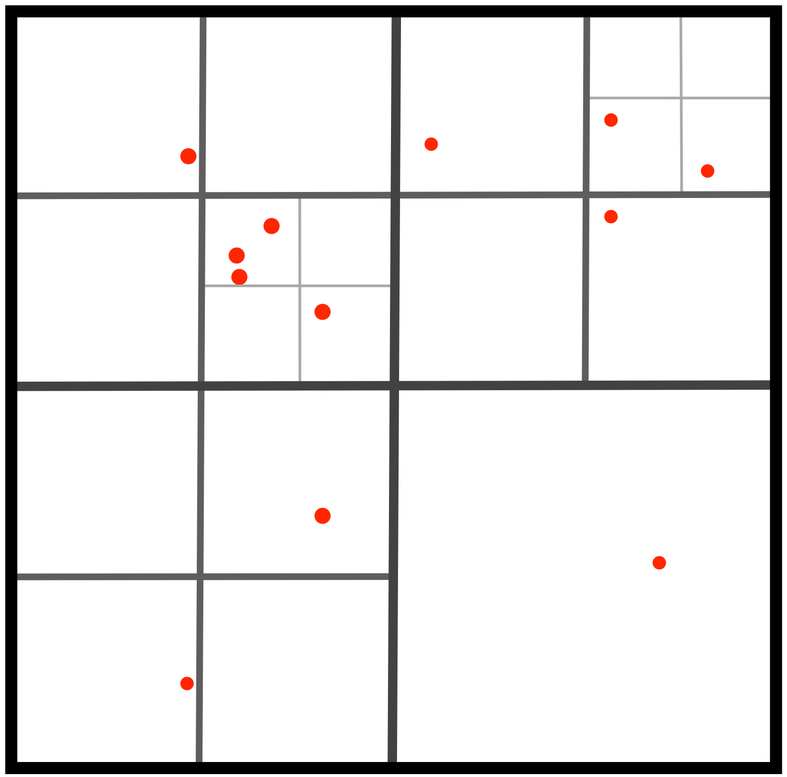

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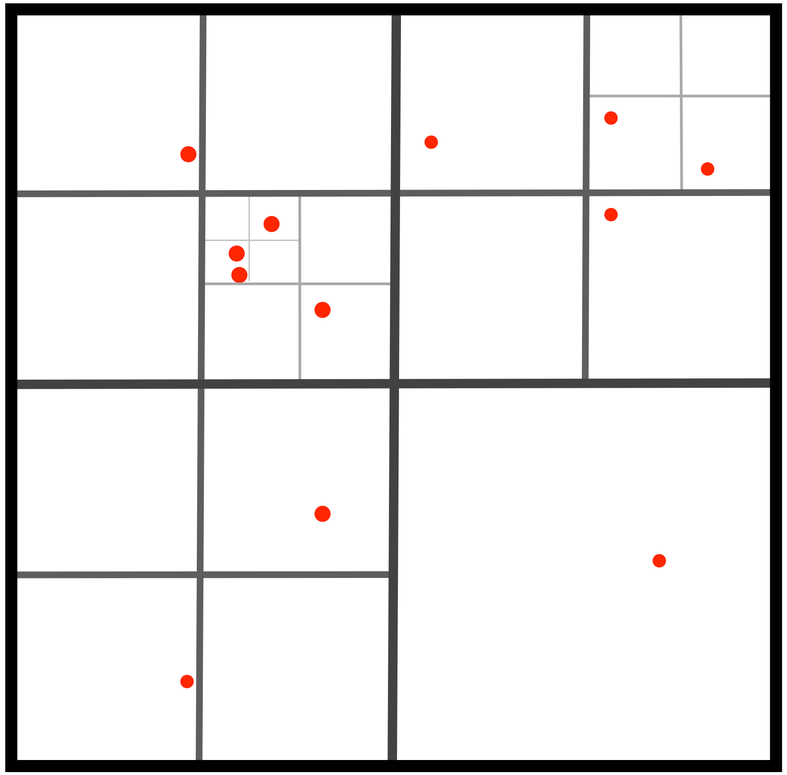

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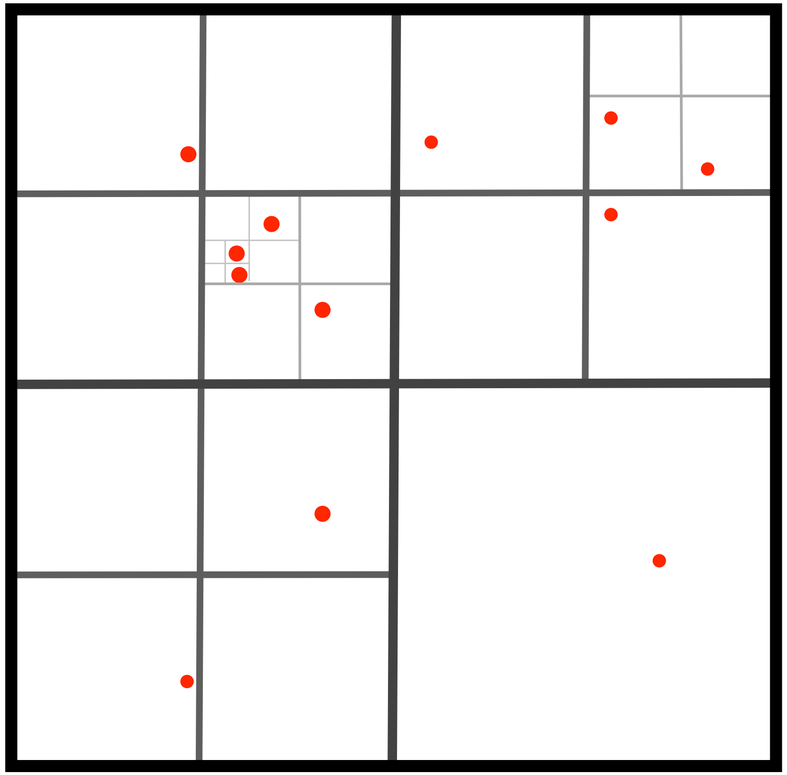

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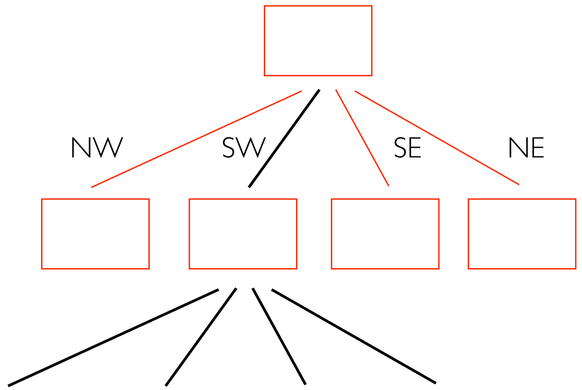

tree structure

Octree

✓ Widely used in practice

✓ Easy to implement

✓ Allows for local refinement

- Inefficient when cells are skinny
  - large depth value required
  - too many cells per bucket
  - many empty buckets

k-D Tree

- Split spatial domain along successive axis-parallel hyperplanes at data point
  - Equal number of points in each half space
  - balanced tree (optimize overall height)

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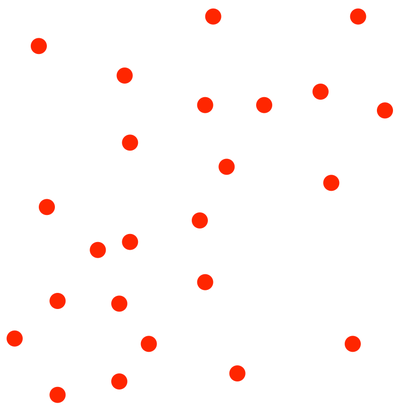

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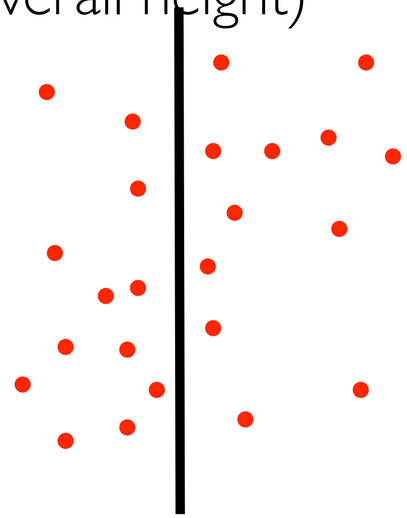

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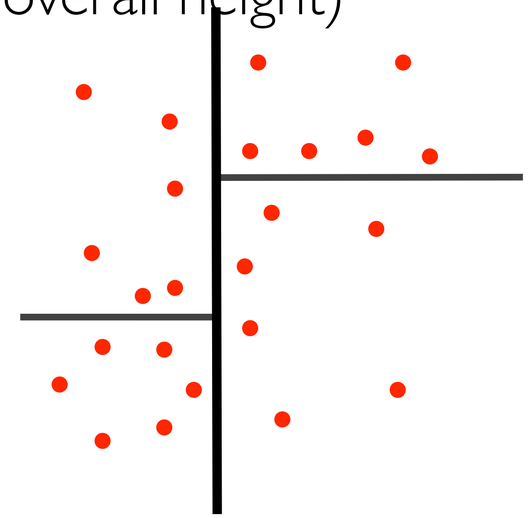

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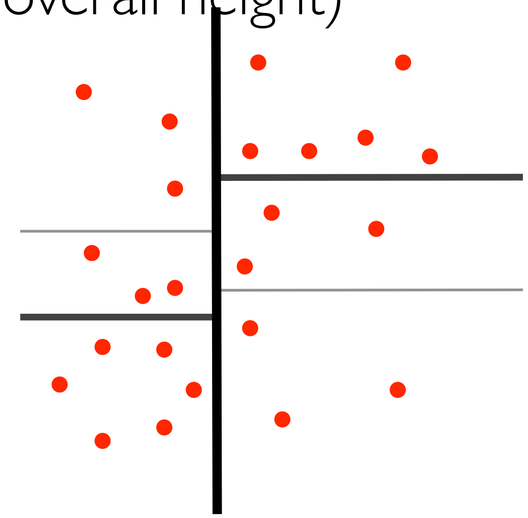

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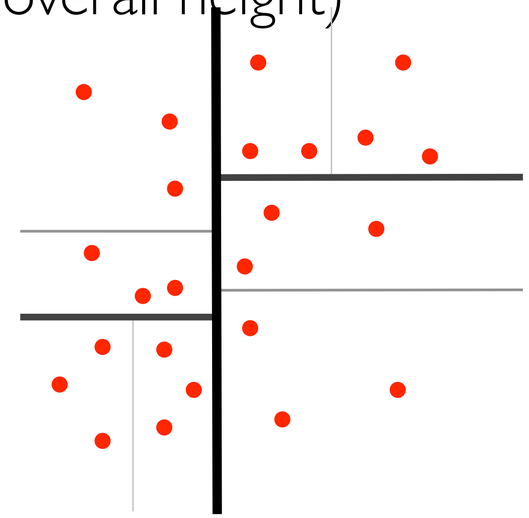

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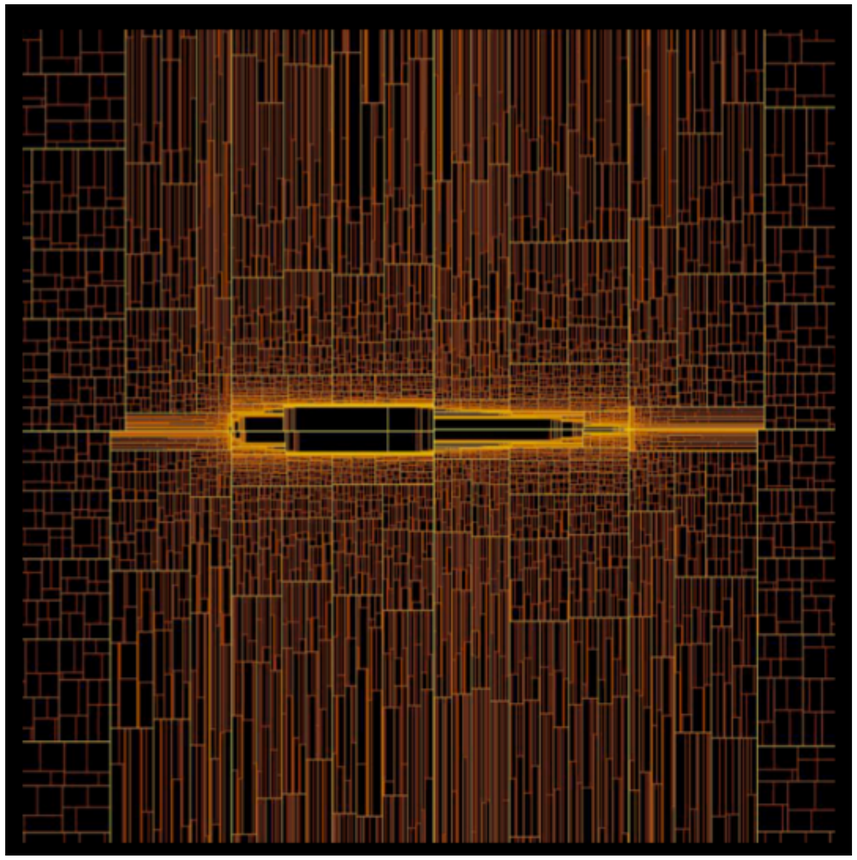
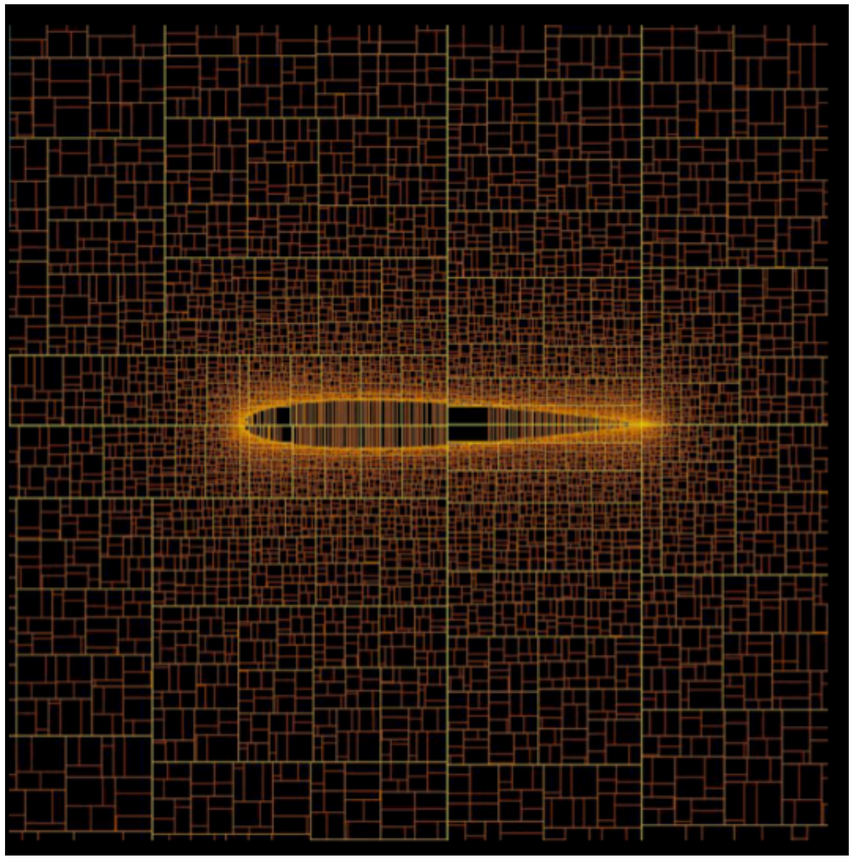


Alternating split



Adaptive split

Outline

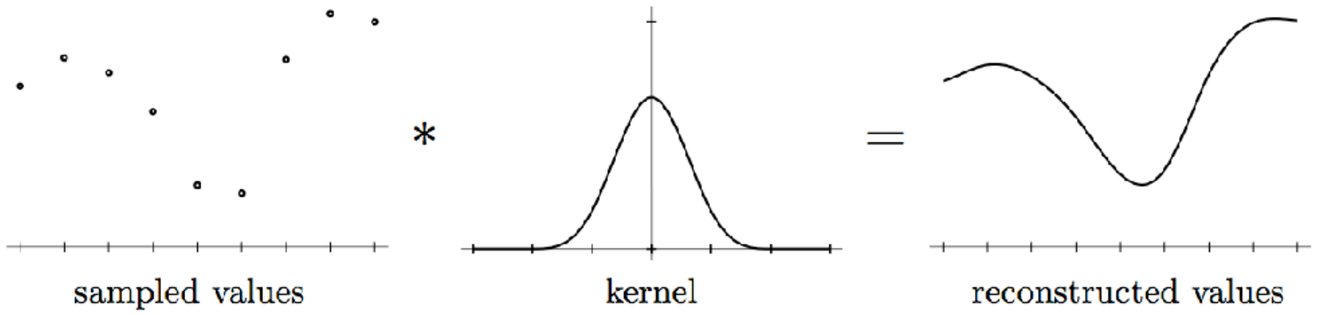
- Mesh types
- Interpolation
- Spatial queries
- Common pre-processing tasks

Smoothing

- Implemented as weighted average over a neighborhood
- Smoothing is useful to:
  - reduce noise in data
  - filter out small scale structures (visual clutter)

Smoothing

- If grid is uniform or rectilinear, smooth convolution kernels from image processing are applicable:
  - interpolation
  - approximation



from G. Kindlmann's PhD thesis

Smoothing

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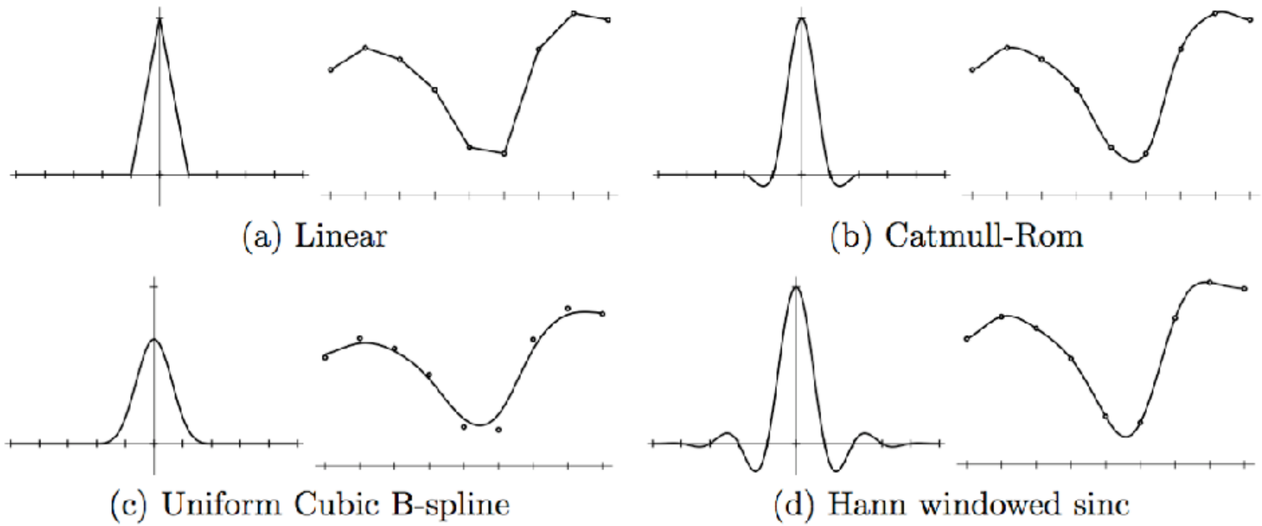
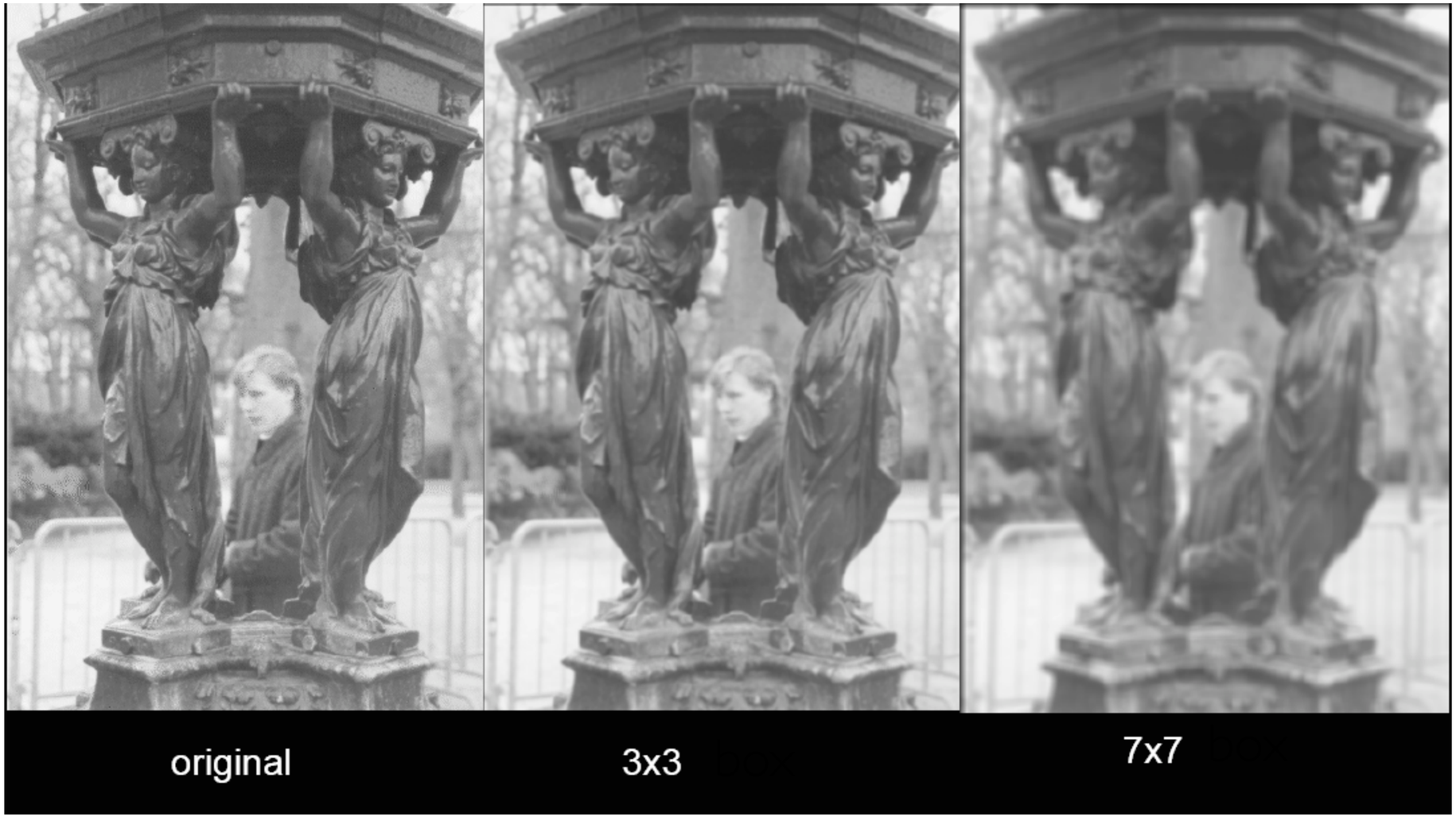


(a) Linear (b) Catmull-Rom

(c) Uniform Cubic B-spline (d) Hann windowed sinc

from G. Kindlmann's PhD thesis

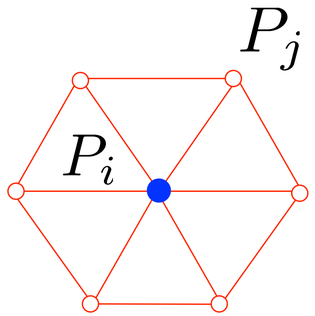
Smoothing



Smoothing

Smoothing

Unstructured grids



- *Umbrella* operator:
  - Local averaging of surrounding values
  - Applied component-wise for multidimensional data

$$f_{k+1}(P_i) = (1-\omega) f_k(P_i) + \frac{\omega}{|N_1(P_i)|} \sum_{j \in N_1(P_i)} f_k(P_j)$$

average value of direct neighbors

  - Isotropic
  - Iterative
  - Fast!

Smoothing

Unstructured grids

- Take proximity of vertices into account (geometric similarity): closer = more weight

$$f_{k+1}(P_i) = (1-\omega) f_k(P_i) + \frac{\omega}{\sum_{j \in N_1(P_i)} \omega_{ij}} \sum_{j \in N_1(P_i)} \omega_{ij} f_k(P_j)$$

$$\omega_{ij} = \omega(||P_i P_j||)$$

- Non-uniform point distribution, edges of embedded surfaces

Smoothing

Unstructured grids

- Take difference between values into account (data similarity)

$$f_{k+1}(P_i) = (1-\omega) f_k(P_i) + \frac{\omega}{\sum_{j \in N_1(P_i)} \omega_{ij}} \sum_{j \in N_1(P_i)} \omega_{ij} f_k(P_j)$$

$$\omega_{ij} = \psi(|f_k(P_i) - f_k(P_j)|)$$

- Preserve strong gradients

Smoothing

Unstructured grids

- Combine both criteria (cf. robust statistics)

$$f_{k+1}(P_i) = (1-\omega) f_k(P_i) + \frac{\omega}{\sum_{j \in N_1(P_i)} \omega_{ij}} \sum_{j \in N_1(P_i)} \omega_{ij} f_k(P_j)$$

$$\omega_{ij} = \phi(||P_i P_j||)\psi(|f_k(P_i) - f_k(P_j)|)$$

- Preserve strong gradients

Derivative Computation

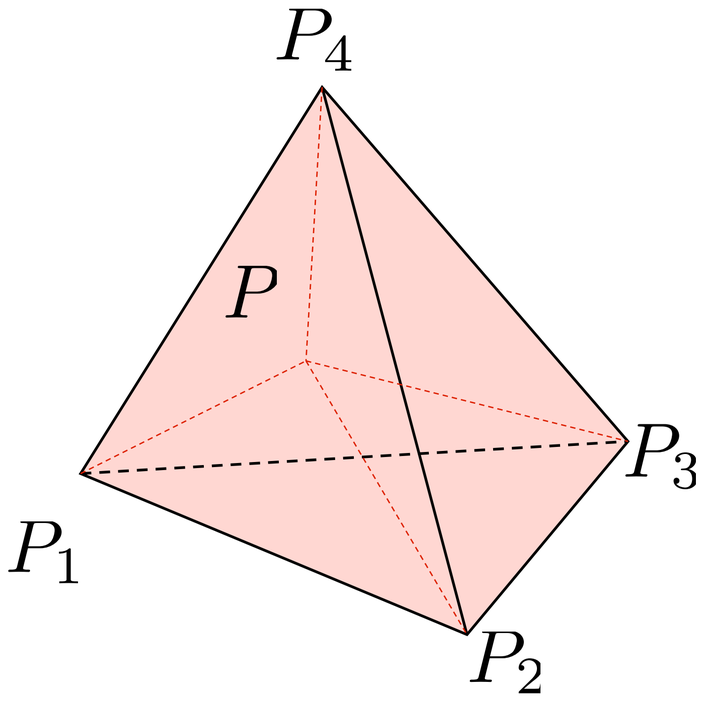
Cell-wise: derive interpolating function

- Linear interpolation (tri's, tet's): constant value

$$\phi(P) = \sum_{i=1}^{4} \beta_i(P)\phi_i$$

$$\nabla\phi(P) = \sum_{i=1}^{4} \nabla\beta_i\phi_i$$

cell-wise constant



Derivative Computation

Cell-wise: derive interpolating function

- Bilinear (quads): value depends on local coordinates
- Trilinear (hexahedra, wedges, pyramids): same

Derivative Computation

Point-wise

- Weighted combination of surrounding cell-wise derivatives
- Can be used iteratively for higher-order derivatives (smoothing effect)
- Analytical derivatives of polynomial fit of surrounding values (Taylor expansion)

Resampling

- 2D slice out of 3D volume
- Approximate original data over more convenient structure
- Regular (structured) grid:
  - Lower memory requirements
  - Higher reconstruction quality (derivation, convolution)
  - Suitable for GPU-based algorithms
  - Allows for multi-resolution representation

Resampling

- Challenging with unstructured grids
  - Subsampling (small cells): aliasing
  - Upsampling (big cells): interpolation artifacts
- Subsample after smoothing original data
- Upsample smooth interpolation of the original data
  - convolution kernels on rectilinear grids
  - scattered data interpolation on unstructured grids