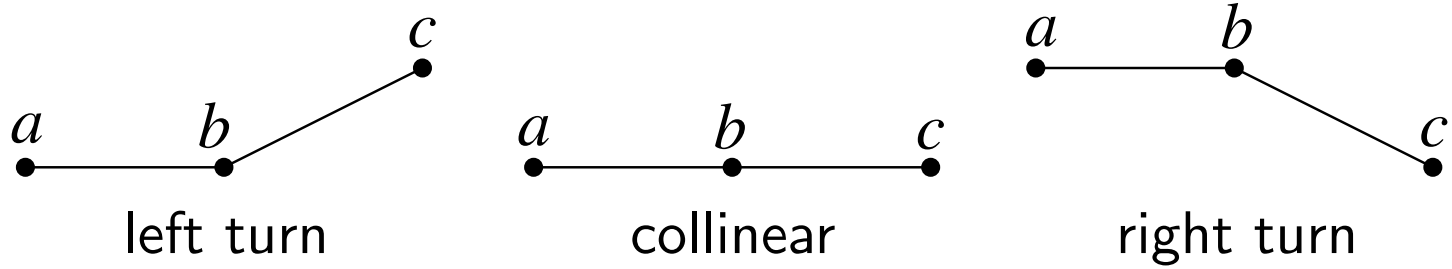
Computational Geometry

Algorithmic study of combinatorial geometry.

- many simple elements (points, lines, triangles).
- queries and constructions.
- optimal algorithms and lower bounds.

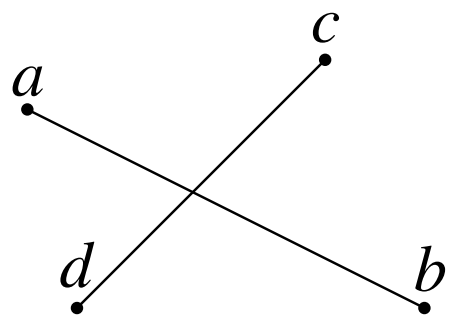
Parameters and Predicates



- Geometry is modeled with numerical parameters.
- Predicates are signs of polynomials in these parameters.
- Geometric properties are expressed as predicates.
- Example: path abc is a left turn if

$$\mathrm{circ}(a, b, c) = (c_x - b_x)(a_y - b_y) - (c_y - b_y)(a_x - b_x) > 0.$$

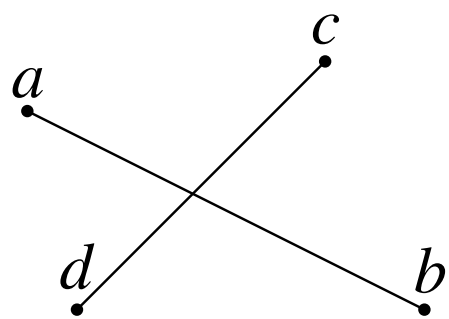
Line Segment Intersection



When do segments ab and cd intersect?

- Can test if the line intersection point is on both segments.
- But the intersection point is not needed.

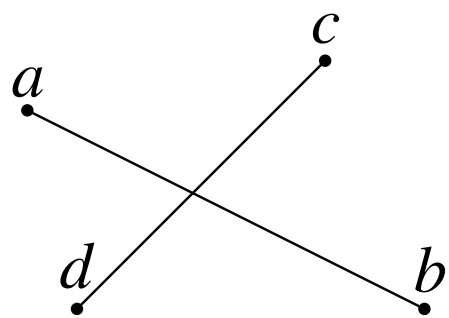
Line Segment Intersection



When do segments ab and cd intersect?

- Can test if the line intersection point is on both segments.
- But the intersection point is not needed.
- Check if c and d are on opposite sides of the ab line, and if a and b are on opposite sides of the cd line.

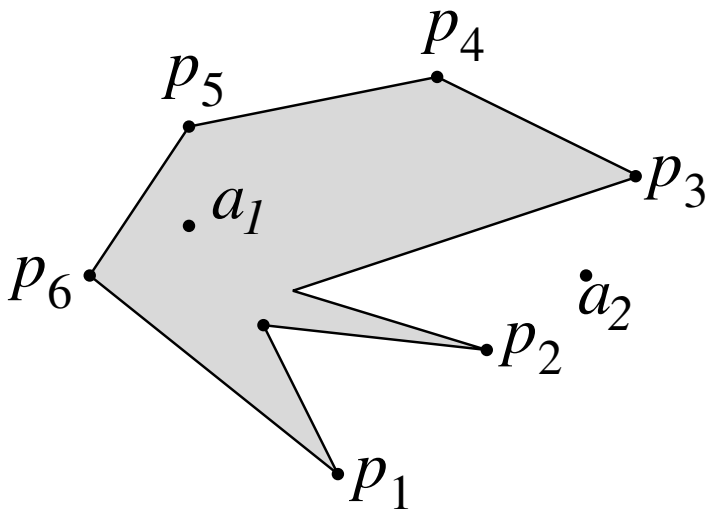
Line Segment Intersection



When do segments ab and cd intersect?

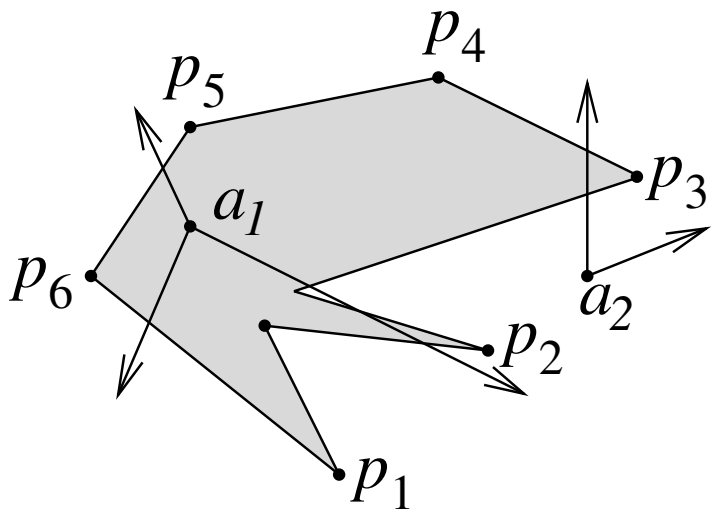
- Can test if the line intersection point is on both segments.
- But the intersection point is not needed.
- Check if c and d are on opposite sides of the ab line, and if a and b are on opposite sides of the cd line.
- Intersection test:
  $\mathrm{circ}(a, b, c)\mathrm{circ}(a, b, d) < 0$ and $\mathrm{circ}(c, d, a)\mathrm{circ}(c, d, b) < 0$.

Point in Polygon



When is a point inside a polygon?

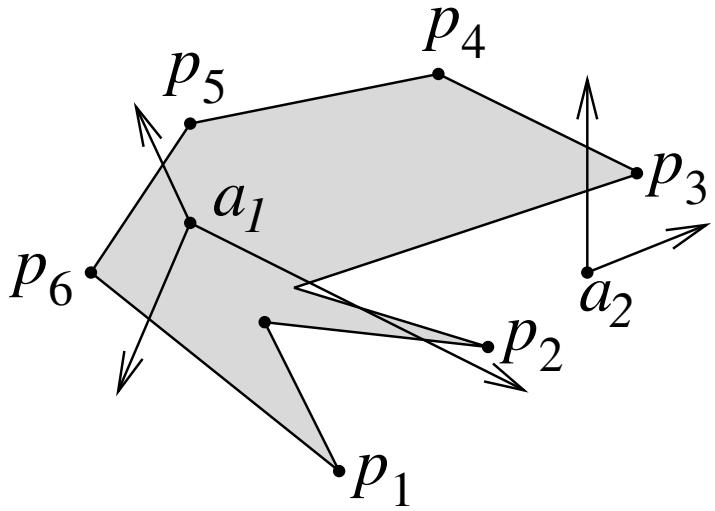
Point in Polygon



When is a point inside a polygon?

- Any ray based at a intersects p an odd number of times.

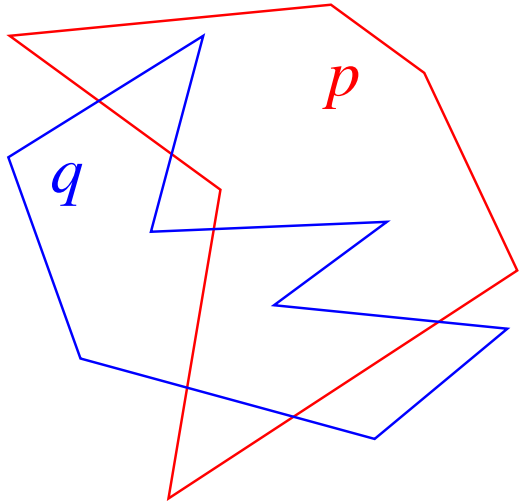
Point in Polygon

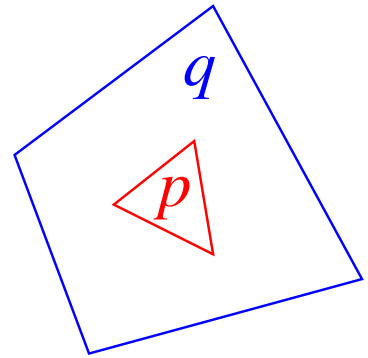


When is a point inside a polygon?

- Any ray based at a intersects p an odd number of times.
- What about special cases?

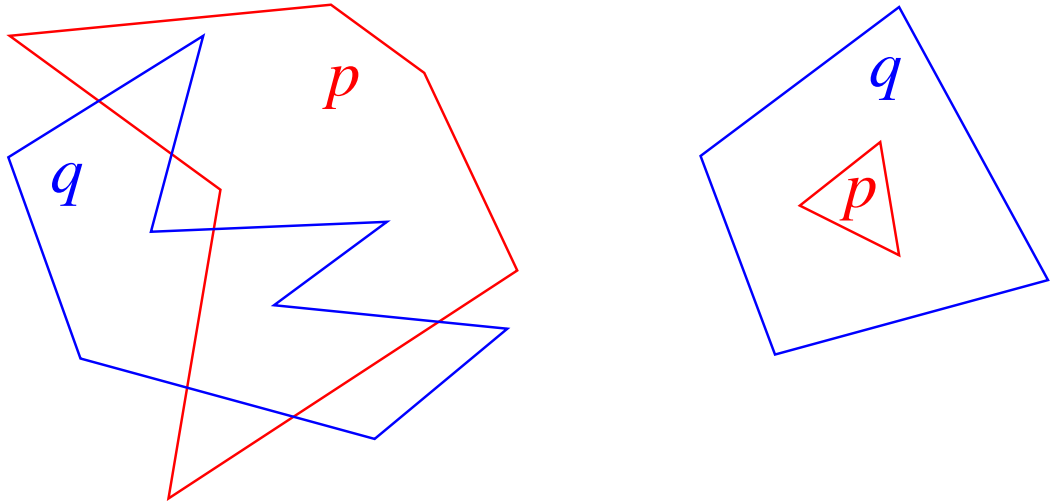
Polygon Intersection





When do polygons p and q intersect?

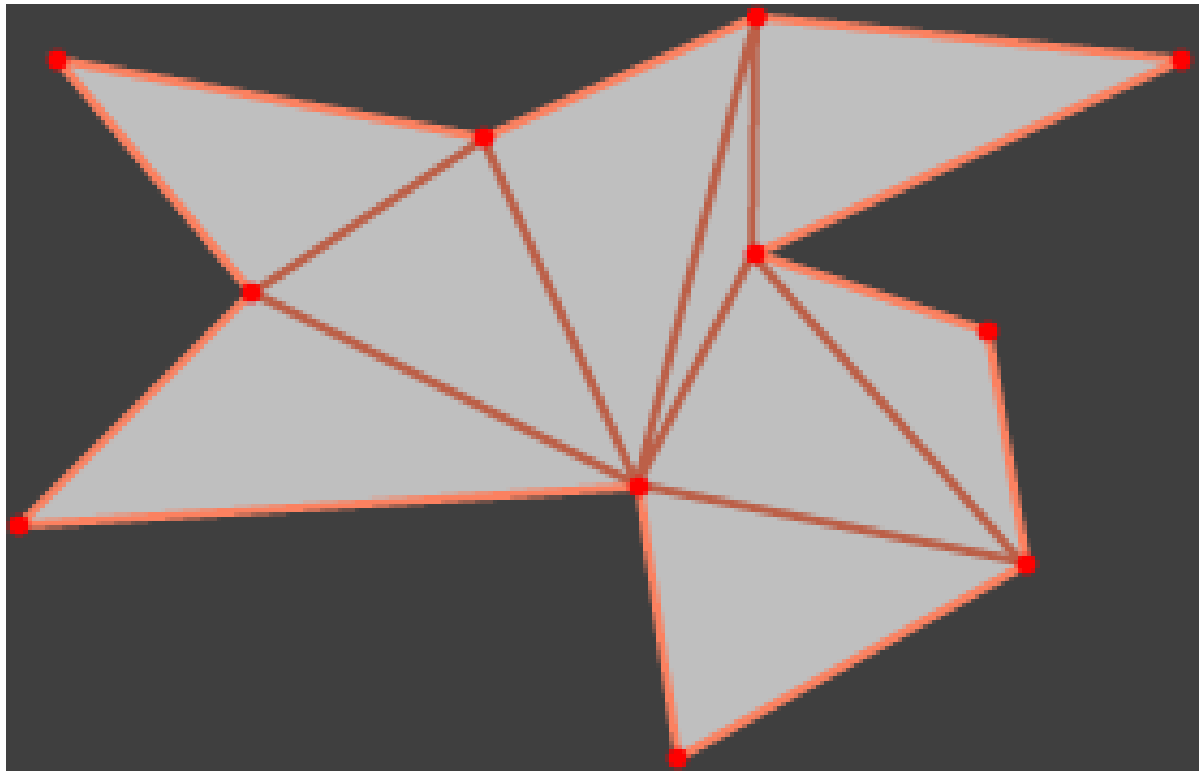
Polygon Intersection



When do polygons p and q intersect?

- Two edges intersect, p is inside q, or q is inside p.
- We know how to test for edge intersection.
- There is a faster algorithm called a line sweep.
- If no edges intersect, p is inside q if any vertex of p is inside q.

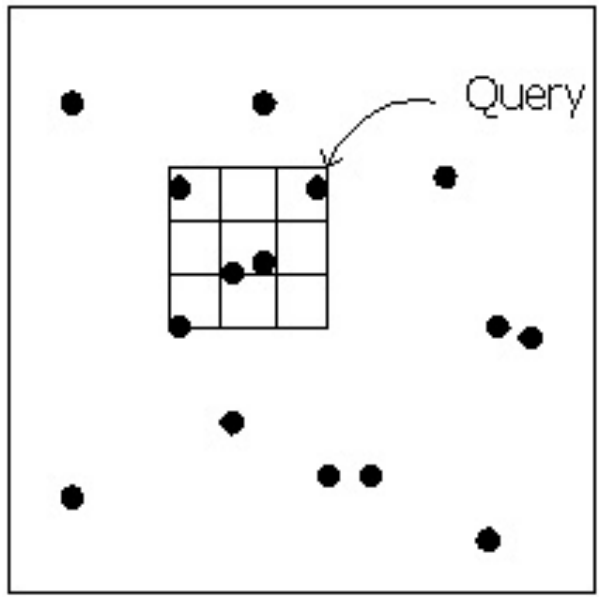
Polygon Triangulation



Decompose polygonal region into triangles.

- 2D: $n \log n$ for n vertices.
- 3D: $nr + r^2 \log r$ for $r = \mathrm{O}(n^2)$ reflex vertices.

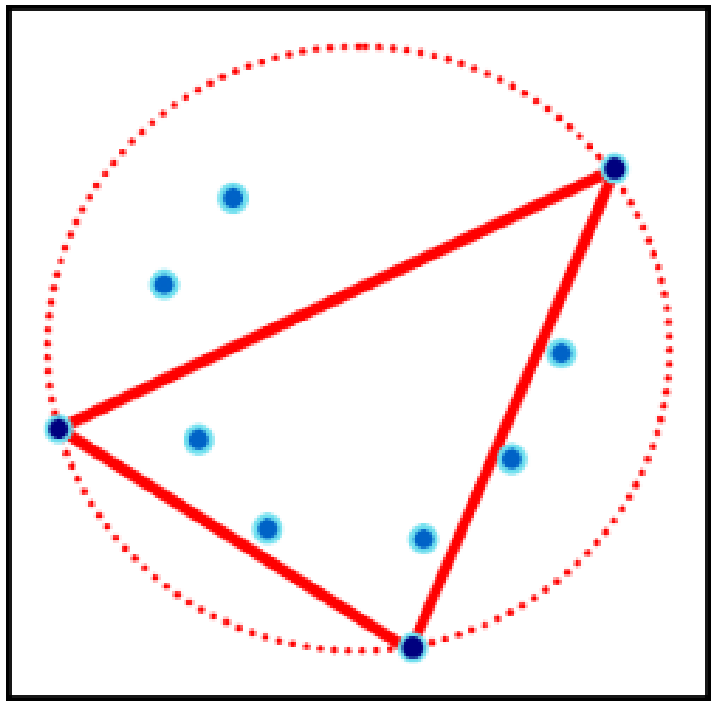
Range Search



Find points in axis-aligned box.

- Input size is n; output size is k.
- 2D: $k + \log n$ query; $n \log n$ preprocessing.
- 3D: $k + \log^2 n$ query; $n \log^2 n$ preprocessing.
- Octrees and bsp trees: $k + n^2$ and $k + n^3$.

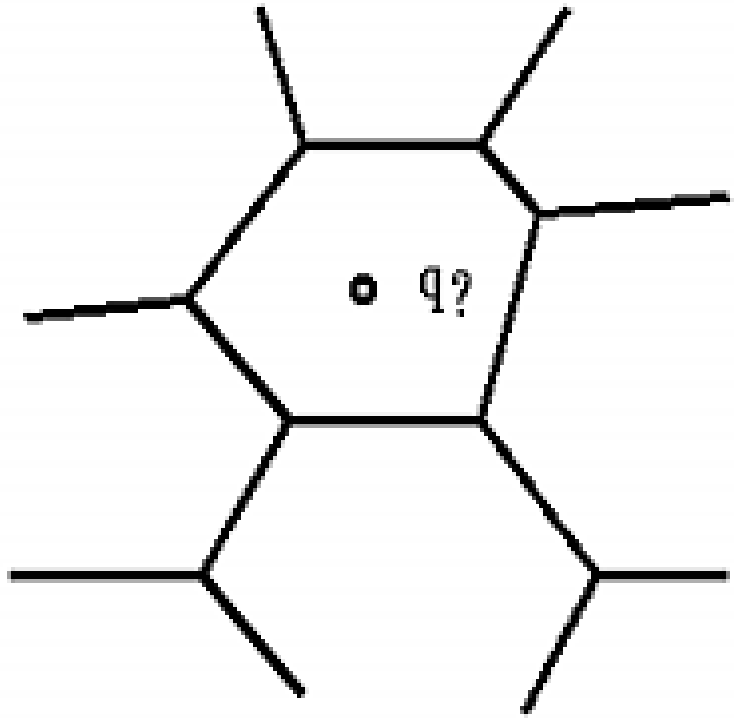
Simplex Search



Find points in triangle.

- 2D: $k + \log n$ query time; n preprocessing time.
- 3D: $k + \log n$ query time; n^2 preprocessing.
- complicated algorithms.

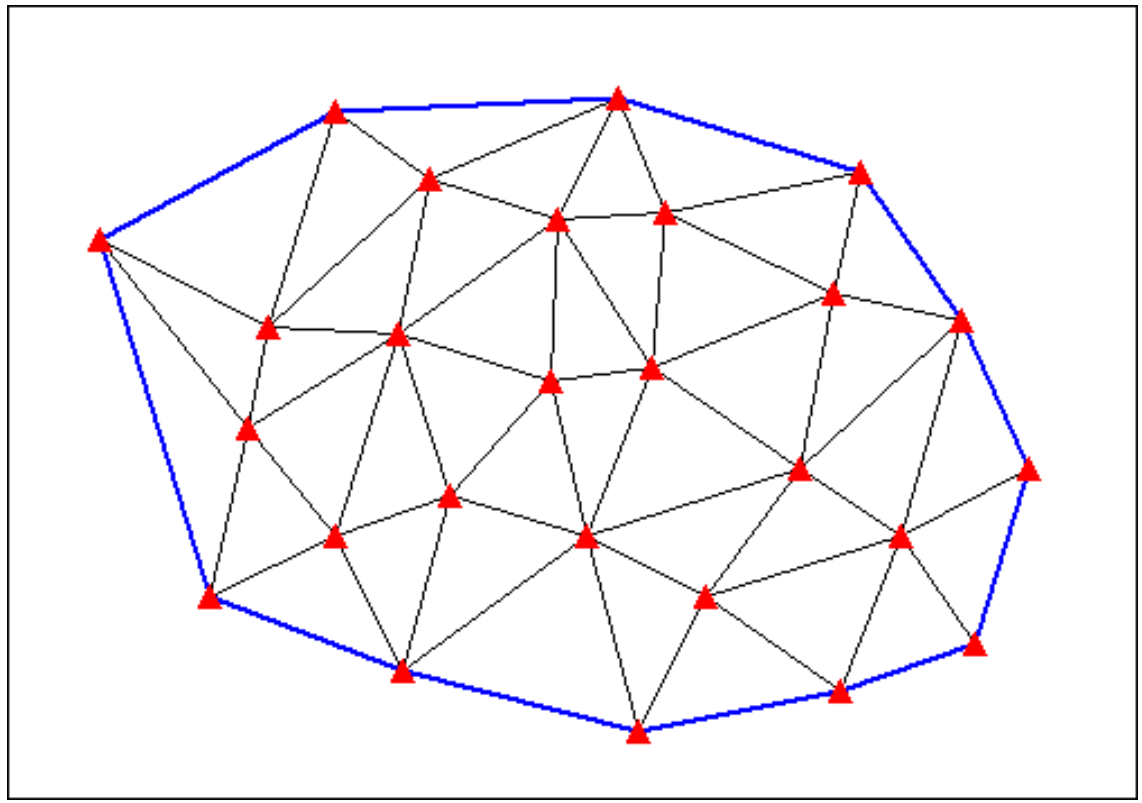
Point Location



Locate the cell of a mesh that contains a point.

- 2D: $\log n$ query; $n \log n$ preprocessing for mesh of n triangles.
- 3D: open problem!

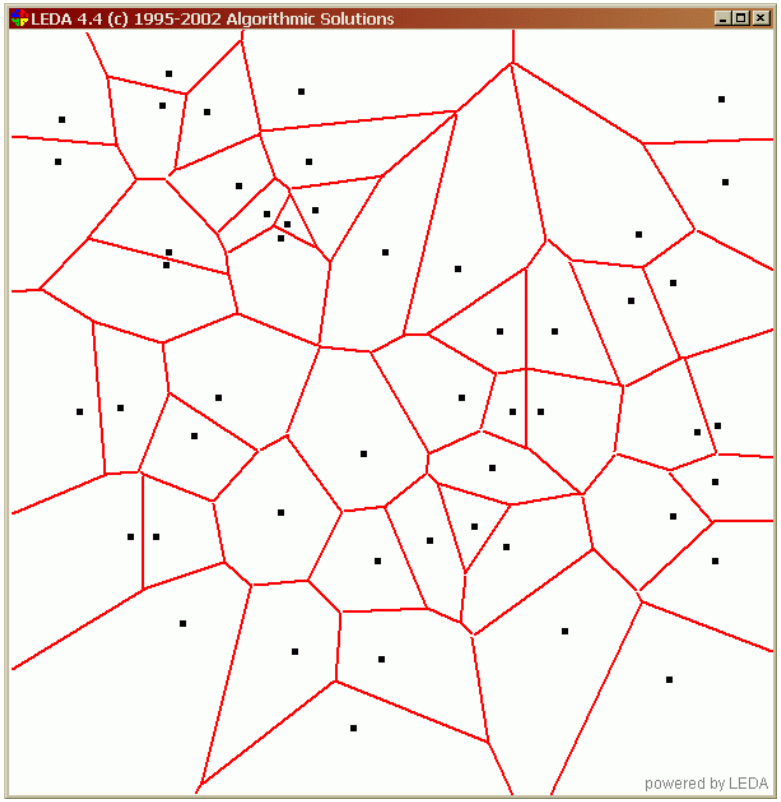
Convex Hull



Smallest convex region containing points.

- 2D: $n \log n$ for n points.
- 3D: $n \log n$.

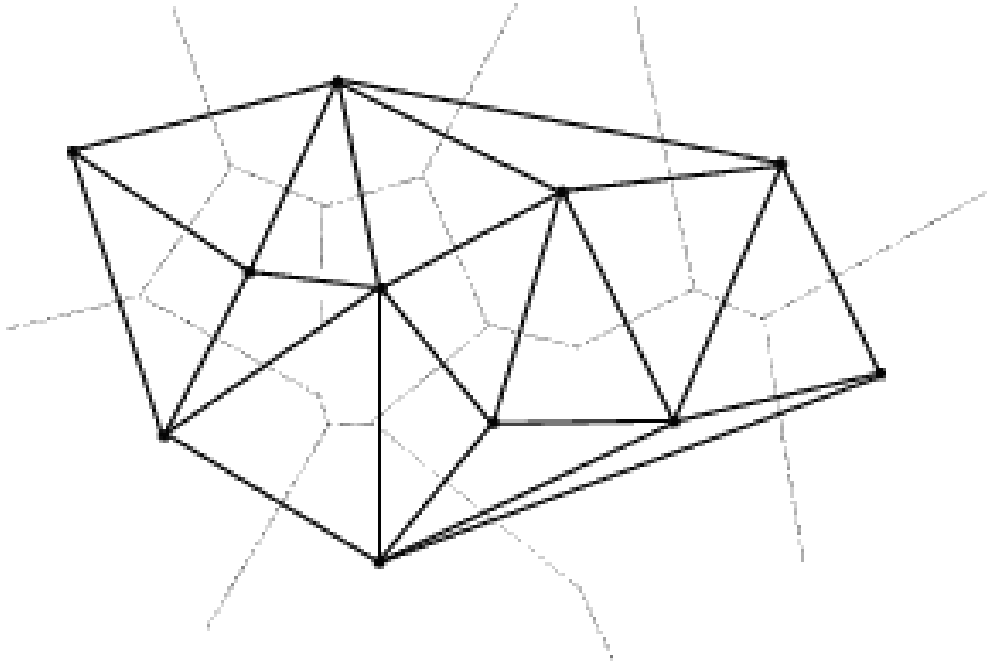
Voronoi Diagram



Compute the region that is closest to each site.

- 2D: $n \log n$ for n sites.
- 3D: $k + n \log n$ for output size $k = O(n^2)$.

Delaunay Triangulation



Triangulation with maximal minimum angle.

- Equivalent to Voronoi diagram.
- convex hull in dimension d gives Delaunay triangulation in dimension $d-1$.