Q1. $P(H \mid \text{biased}) = 0.6 \quad P(H \mid \text{fair}) = 0.5$

$P(\text{biased}) = 0.5$

$$P(\text{biased} \mid H_{twice}) = \frac{P(H_{twice} \mid \text{biased})\, P(\text{biased})}{P(H_{twice})}$$

$$= \frac{(0.6 \times 0.6) \times 0.5}{P(H_{twice})}$$

$$P(H_{twice}) = P(H_{twice} / \text{biased})\, P(\text{biased}) + P(H_{twice} \mid \text{fair})\, P(\text{fair})$$

$$= 0.5\,(0.6^2 + 0.5^2)$$

$\Rightarrow P(\text{biased} \mid H_{twice}) = 0.59$ Ans

Q2. $P(B) + P(G) = 1$ & $P(G) = 0.9 \Rightarrow P(B) = 0.1$

$P(SB \mid B) = 0.75 \quad P(SG \mid G) = 0.25$

$P(SG \mid G) = 0.75 \quad P(SB \mid G) = 0.25$

Events

$B \rightarrow$ Car is blue

$G \rightarrow$ Car is green

$SB \rightarrow$ Car is seen as blue

$SG \rightarrow$ Car is seen as green

$$P(B \mid SB) = \frac{P(SB \mid B)\, P(B)}{P(SB)}$$

$$= \frac{0.75 \times 0.1}{0.75 \times 0.1 + 0.25 \times 0.9}$$

$= 0.25$ Ans

$$p\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{-1/2}} \exp\left(-\frac{1}{2}\left(\begin{bmatrix}x\\y\end{bmatrix}-\begin{bmatrix}\mu_x\\\mu_y\end{bmatrix}\right)^T \begin{bmatrix}\gamma_{xx} & \gamma_{xy}\\ \gamma_{yx} & \gamma_{yy}\end{bmatrix}\left(\begin{bmatrix}x\\y\end{bmatrix}-\begin{bmatrix}\mu_x\\\mu_y\end{bmatrix}\right)\right)$$

where $\begin{bmatrix}\gamma_{xx} & \gamma_{xy}\\ \gamma_{yx} & \gamma_{yy}\end{bmatrix}^{-1} = \begin{bmatrix}\Sigma_{xx} & \Sigma_{xy}\\ \Sigma_{yx} & \Sigma_{yy}\end{bmatrix}$

$$p(x|y=y_0) \propto p(x, y=y_0)$$

$$\propto \exp\Big(-\frac{1}{2}(x-\mu_x)^T\gamma_{xx}(x-\mu_x) - (x-\mu_x)^T\gamma_{xy}(y_0-\mu_y) - \frac{1}{2}(y_0-\mu_y)^T\gamma_{yy}(y_0-\mu_y)\Big)$$

$$\propto \exp\Big(-\frac{1}{2}(x-\mu_x)^T\gamma_{xx}(x-\mu_x) - (x-\mu_x)^T\gamma_{xy}(y_0-\mu_y)\Big)$$

$$\propto \exp\Big(-\frac{1}{2}(x-\mu_x)^T\gamma_{xx}(x-\mu_x) - (x-\mu_x)^T\gamma_{xx}\gamma_{xx}^{-1}\gamma_{xy}(y_0-\mu_y) - \frac{1}{2}\left[(y_0-\mu_y)\gamma_{yx}\gamma_{xx}^{-1}\right]^T\gamma_{xx}\gamma_{xx}^{-1}\gamma_{xy}(y_0-\mu_y) + \frac{1}{2}\left[(y_0-\mu_y)\gamma_{yx}\gamma_{xx}^{-1}\right]^T\gamma_{xx}\gamma_{xx}^{-1}\gamma_{xy}(y_0-\mu_y)\Big)$$

$$\propto \exp\Big(-\frac{1}{2}\left(x-\mu_x+\gamma_{xx}^{-1}\gamma_{yx}(y_0-\mu_y)\right)^T\gamma_{xx}\left(x-\mu_x+\gamma_{xx}^{-1}\gamma_{yx}(y_0-\mu_y)\right)\Big)$$

$$\cdot \exp\Big(\frac{1}{2}(y_0-\mu_y)\gamma_{yx}\gamma_{xx}^{-1}\gamma_{xx}\gamma_{xx}^{-1}\gamma_{xy}(y_0-\mu_y)\Big)$$

$$\propto \exp\Big(-\frac{1}{2}\left(x-\mu_x+\gamma_{xx}^{-1}\gamma_{xy}(y_0-\mu_y)\right)^T\gamma_{xx}\left(x-\mu_x+\gamma_{xx}^{-1}\gamma_{yx}(y_0-\mu_y)\right)\Big)$$

$\therefore$ from form of exponential in gaussian

for $p(x|y=y_0)$ $\mu' = \mu_x - \gamma_{xx}^{-1}\gamma_{xy}(y_0-\mu_y)$; $\Sigma' = \gamma_{xx}$

from properties of inversion of block matrices $\begin{bmatrix}A & B\\ C & D\end{bmatrix}$

$$\mu' = \mu_x - \Sigma_{xy}\Sigma_{yy}^{-1}(y_0-\mu_y) \Rightarrow \mu_* = \mu_x - \frac{\sigma_{xy}}{\sigma_y^2}(y-\mu_y)$$

$$\Sigma' = \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}$$

$\longrightarrow$ when x, y scalar variables

$$\Rightarrow \sigma_*^2 = \sigma_{xx}^2 - \frac{\sigma_{xy}^2}{\sigma_{yy}}$$

Q4 from Table 2,

since on day 4, sensor says rainy,

∴ day 4 is indeed rainy

$X_d \rightarrow$ state of day d

$Z_d \rightarrow$ measurement "

R – Rainy

C – Cloudy

S - Sunny

$a = P(X_4 = R | Z_4 = R)$

$$= P(Z_4 = R | X_4 = R) \underbrace{\bar{P}(X_4 = R)\eta}_{k}$$

($\bar{P}(X_4=R)$ → prior belief)

$$= 1 . k$$

$b = P(X_4 = S | Z_4 = R)$

$$= P(Z_4 = R | X_4 = S) \times k$$

$$= 0. k = 0$$

Similarly $c = P(X_4 = C | Z_4 = R) = 0$

$a + b + c = 1 \Rightarrow k = 1$

$\Rightarrow P(X_4 = R | Z_4 = R) = 1$

↓ posterior belief for day 4

for day 5

$$\bar{P}(X_5 = S) = \Sigma P(X_5 | X_4) P(X_4)$$

$$\Rightarrow \bar{P}(X_5 = S) = 0.8 \times 0 + 0.4 \times 0 + 0.2 \times 1 = 0.2$$

$$\bar{P}(X_5 = C) = 0.6$$

$$\bar{P}(X_5 = R) = 0.2$$

posterior ← $P(X_5) = \eta \, P(Z_5 | X_5) \bar{P}(X_5)$

$$\Rightarrow P(X_5 = S) = \eta \times 0.6 \times 0.2 = 0.12\eta$$

$$P(X_5 = C) = 0.18\eta$$

$$P(X_5 = R) = 0$$

$$\Sigma P(X_5) = 1 \Rightarrow \eta = \frac{10}{3}$$

∴ $P(X_5 = S) = 0.4$ Ans

↓

prob it is indeed sunny

Q 6 a) $x_t = ut - \frac{1}{2}gt^2$

$x_{t-1} = u(t-dt) - \frac{1}{2}g(t-dt)^2$

$\Rightarrow \quad x_t = x_{t-1} + \dot{x}_{t-1}dt - \frac{g}{2}dt^2$

$\dot{x}_t = u - gt$

$\dot{x}_{t-1} = u - g(t-dt)$

$\Rightarrow \dot{x}_t = \dot{x}_{t-1} + (-g)dt$

$$\Rightarrow \begin{bmatrix} x \\ \dot{x}_t \end{bmatrix} = \begin{bmatrix} 1 & dt \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \dot{x}_{t-1} \end{bmatrix} + \begin{bmatrix} -g/2\, dt^2 \\ -g\,dt \end{bmatrix}$$

$dt = 0.1\,s$