Multivariable Control

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Some of the techniques of state-space were introduced in Chapter 80. The emphasis there was principally on the analysis of the behaviour of multivariable systems. This chapter introduces the use of matrix techniques, and diagonalisation in particular, for design purposes. A case study based upon a blending system is used as a realistic, but simple, example to demonstrate the principles. The approach used is known as internal model control because the inverse of the model of the plant to be controlled is embedded in the design of the controller. This gives an insight to the problems of the design of large multivariable control systems which provides a basis for considering related techniques such as sensitivity analysis in Chapter 111, state feedback in Chapter 112 and model predictive control in Chapter 117.

81.1 Case Study

An in-line blending system is depicted in Figure 81.1.

In summary, two streams f_1 and f_2 are blended to produce a third stream f_0 of concentration $c^\#$. Its dynamics are fairly fast because the blending system is in-line. Also it is a highly interactive system. Any change in either flow f_1 or f_2 will affect both f_0 and $c^\#$. Note that changes in flow are propagated throughout the system instantaneously, whereas changes in concentration have an associated time delay.

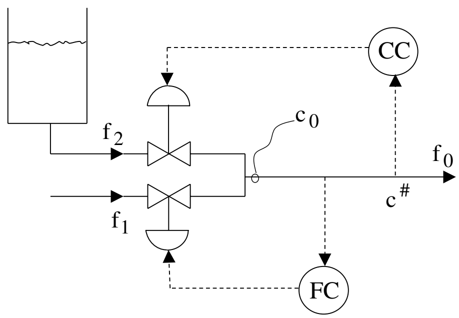

Fig. 81.1 An in line blending system

A model of this system in matrix form is developed in Chapter 86 using deviation variables:

$$\begin{bmatrix} f_0(s) \\ c^\#(s) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -K_1.e^{-Ls} & K_2.e^{-Ls} \end{bmatrix} \begin{bmatrix} f_1(s) \\ f_2(s) \end{bmatrix} \quad (81.1)$$

which may be denoted:

$$\underline{x}(s) = P(s).\underline{f}(s)$$

This is depicted in block diagram form in Figure 81.2.

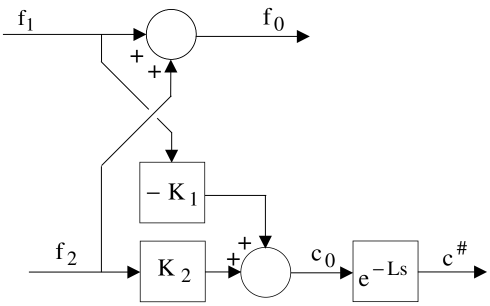

Fig. 81.2 Model of in line blending process

From a control point of view, this is a MIMO system with two inputs and two outputs. Suppose that both the flow rate and concentration of the dilute product stream are to be controlled simultaneously. A scheme consisting of two conventional feedback control loops as depicted in Figure 81.1 will work. The block diagram for this is depicted in Figure 81.3.

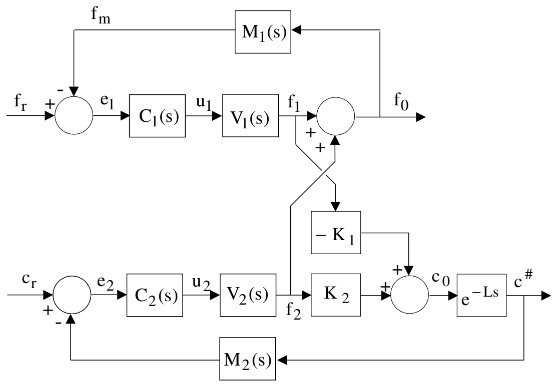

Fig. 81.3 Model of blending system with simple feedback loops

However, to minimise interaction, it is necessary to detune the controllers, *e.g.* by using low gains and large reset times. This makes for poor dynamic performance in terms of speed of response and poor disturbance rejection in terms of steady state offset. Much better control can be realised by means of a 2 × 2 multivariable controller, as depicted in block diagram form in Figure 81.4.

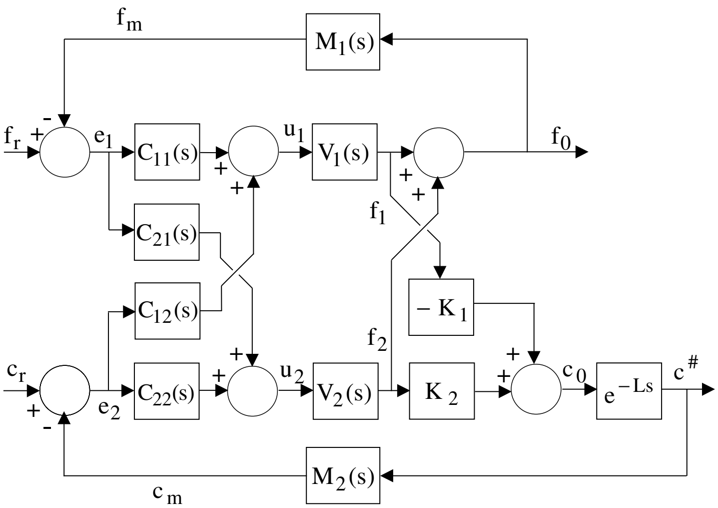

Fig. 81.4 Model of blending system with multivariable controller

81.2 Compensator Concept

The multivariable controller may be considered to consist of four compensators. In principle, compensators $C_{11}(s)$ and $C_{22}(s)$ handle the flow and concentration control loops, in a conventional way, and compensators $C_{21}(s)$ and $C_{12}(s)$ handle the interactions. Thus, for example, following an increase in flow set point f_r, compensator $C_{11}(s)$ will increase flow f_1 and hence f_0 to its desired value. However, in doing so, it will cause concentration c_0 to decrease *via* the gain K_1. This interaction is compensated for by $C_{21}(s)$ which increases f_2 to produce an equal but opposite effect on c_0 *via* K_2. If the compensators are designed correctly, other signals in the concentration control loop, such as e_2 and $c^\#$ should be unaffected by the change in f_0. Similarly, $C_{12}(s)$ protects f_0 against disturbances from the concentration control loop.

81.3 Control System Model

Two feedback loops are established such that both f_0 and $c^\#$ may be controlled against set points. The measurement functions may be articulated in matrix form:

$$\begin{bmatrix} f_m(s) \\ c_m(s) \end{bmatrix} = \begin{bmatrix} M_1(s) & 0 \\ 0 & M_2(s) \end{bmatrix} \begin{bmatrix} f_0(s) \\ c^\#(s) \end{bmatrix} \qquad (81.2)$$

which may be denoted:

$$\underline{m}(s) = M(s).\underline{x}(s)$$

The comparators are handled simply by vector addition:

$$\begin{bmatrix} e_1(s) \\ e_2(s) \end{bmatrix} = \begin{bmatrix} f_r(s) \\ c_r(s) \end{bmatrix} - \begin{bmatrix} f_m(s) \\ c_m(s) \end{bmatrix} \qquad (81.3)$$

which may be denoted:

$$\underline{e}(s) = \underline{r}(s) - \underline{m}(s)$$

Four compensators are required. In essence, C_{11} and C_{22} handle the feedback requirements for control and C_{21} and C_{12} counteract the process interactions:

$$\begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix} = \begin{bmatrix} C_{11}(s) & C_{12}(s) \\ C_{21}(s) & C_{22}(s) \end{bmatrix} \begin{bmatrix} e_1(s) \\ e_2(s) \end{bmatrix} \qquad (81.4)$$

which may be denoted:

$$\underline{u}(s) = C(s).\underline{e}(s)$$

The compensator outputs are applied to the control valves *via* I/P converters:

$$\begin{bmatrix} f_1(s) \\ f_2(s) \end{bmatrix} = \begin{bmatrix} V_1(s) & 0 \\ 0 & V_2(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix} \qquad (81.5)$$

which may be denoted:

$$\underline{f}(s) = V(s).\underline{u}(s)$$

Equations 81.1–81.5 are a complete, but generic, description of the system.

81.4 Compensator Design

The starting point is to establish the transfer matrix of the closed loop system from Equations 81.1–81.5. Using block diagram algebra gives:

$$\begin{aligned} \underline{x}(s) &= P(s).\underline{f}(s) \\ &= P(s).V(s).C(s).\,(\underline{r}(s) - M(s).\underline{x}(s)) \end{aligned}$$

Matrix manipulation gives:

$$\underline{x}(s) = (I + P(s).V(s).C(s).M(s))^{-1} \\ .\,P(s).V(s).C(s).\underline{r}(s)$$

Let G(s) be some user defined transfer matrix that specifies the desired closed loop response:

$$\underline{x}(s) = G(s).\underline{r}(s)$$

The objective is to minimise interaction between the loops. The ideal is zero interaction which corresponds to making G(s) diagonal such that changing set point f_r only affects f_0 and changing set point c_r only affects $c^\#$:

$$\begin{bmatrix} f_0(s) \\ c^\#(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & 0 \\ 0 & G_{22}(s) \end{bmatrix} \begin{bmatrix} f_r(s) \\ c_r(s) \end{bmatrix}$$

Whence:

$$G(s) = (I + P(s).V(s).C(s).M(s))^{-1}) \\ . P(s.V(s).C(s)$$

Solving for C(s) gives the desired multivariable controller design:

$$C(s) = V(s)^{-1}.P(s)^{-1}.G(s) \\ . (I - M(s).G(s))^{-1} \tag{81.6}$$

C(s) is totally dependent upon the accuracy of the model and the sensible choice of G(s). If some G(s) is chosen that is unrealistic, perhaps looking for a faster response than the system is physically capable of achieving, then C(s) will be unrealisable.

81.5 Worked Example No 1

The dynamics have been chosen as typical of process systems. If the measurements of flow and concentration are made by means of an electromagnetic flow meter and conductivity cell respectively, time constants of 2 s are reasonable. Likewise, for the pneumatically actuated control valves, time constants of 3 s are assumed. The time delay for the blending process, which only affects the concentration measurement, is 2 s:

$$M(s) = \begin{bmatrix} \frac{1}{2s+1} & 0 \\ 0 & \frac{1}{2s+1} \end{bmatrix}$$

$$V(s) = \begin{bmatrix} \frac{1}{3s+1} & 0 \\ 0 & \frac{1}{3s+1} \end{bmatrix}$$

$$P(s) = \begin{bmatrix} 1 & 1 \\ -e^{-2s} & e^{-2s} \end{bmatrix}$$

Note that a value of unity is assumed for all the steady state gains. This implies that either all the signals have the same range, such as 4–20 mA, or that they have been scaled on a common basis, such as percentage of range.

Choose a sensible closed loop transfer matrix:

$$G(s) = \begin{bmatrix} \frac{1}{4s+1} & 0 \\ 0 & \frac{e^{-2s}}{4s+1} \end{bmatrix}$$

In the case of the flow loop, which has first order lags of 2 s and 3 s, a closed loop response equivalent to a first order lag of 4 s is reasonable. Likewise for the concentration loop except that a 2 s delay in the closed loop response has been allowed for the measurement delay.

Substitution of these values into Equation 81.6 and manipulation yields

$$C(s) = \begin{bmatrix} \frac{(3s+1)(2s+1)}{4s(4s+3)} & \frac{-(3s+1)(2s+1)}{2\left((4s+1)(2s+1)-e^{-2s}\right)} \\ \frac{(3s+1)(2s+1)}{4s(4s+3)} & \frac{(3s+1)(2s+1)}{2\left((4s+1)(2s+1)-e^{-2s}\right)} \end{bmatrix}$$

All four compensators consist of combinations of lead, lag, integrator and delay terms. This is quite normal for multivariable control, the compensators having been designed to give a specified response for a particular plant.

81.6 Decouplers

The principal disadvantage of multivariable controllers is that the compensators $C_{11}(s)$ to $C_{22}(s)$ do not have the structure of a PID controller. Their form is not intuitive which is undesirable for operational reasons. An alternative approach is to use decouplers, as depicted in Figure 81.5.

In this case the two compensators, $C_1(s)$ and $C_2(s)$, may be realised by conventional PID controllers:

$$C(s) = K_C\left(1 + \frac{1}{T_R s} + T_D s\right)$$

The decouplers, $D_1(s)$ and $D_2(s)$, and compensators are related according to:

$$\begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix} = \begin{bmatrix} C_1(s) & C_2(s).D_2(s) \\ C_1(s).D_1(s) & C_2(s) \end{bmatrix} \begin{bmatrix} e_1(s) \\ e_2(s) \end{bmatrix} \tag{81.7}$$

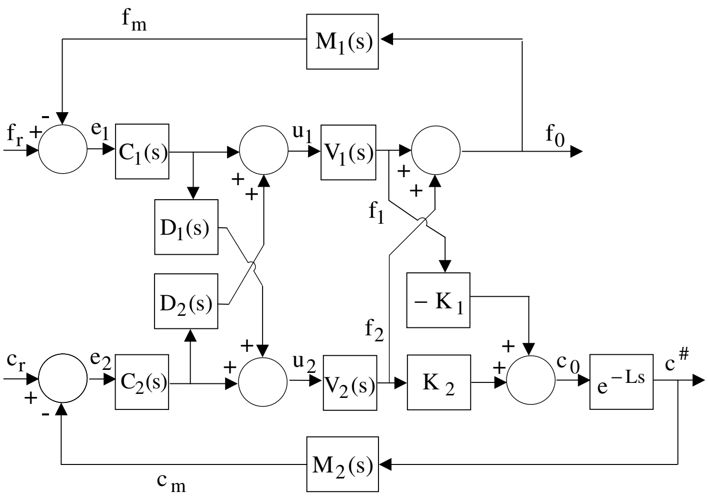

Fig. 81.5 Multivariable control of blending system using decouplers

The design technique essentially consists of choosing appropriate $C_1(s)$ and $C_2(s)$, empirically or otherwise, and then finding $D_1(s)$ and $D_2(s)$ for the known $P(s)$, $M(s)$ and $V(s)$ to give a specified $G(s)$.

For the system of Figure 81.5, the decouplers can be determined by inspection. Firstly consider the decoupler $D_1(s)$. Following a change in error e_1, the affect on C_0 due to the interaction inherent in the process is given by:

$$c_0(s) = -K_1V_1(s)C_1(s).e_1(s)$$

The corresponding affect on C_0 due to the decoupler is given by:

$$c_0(s) = K_2V_2(s)D_1(s)C_1(s).e_1(s)$$

For the decoupler to be effective, these two affects must be equal and opposite: that is they cancel each other. Thus:

$$-K_1V_1(s)C_1(s)\,e_1(s) = -K_2V_2(s)D_1(s)C_1(s)\,e_1(s)$$

Hence:

$$D_1(s) = \frac{K_1V_1(s)}{K_2V_2(s)} \approx \frac{K_1}{K_2}$$

By a similar argument, for a change in error e_2 to have no net affect on f_0, the decoupler $D_2(s)$ required is simply:

$$D_2(s) = -1$$

81.7 Sampled Data Model

Multivariable controllers are realised by means of software. The dynamics of the sampling process can be ignored if they are fast compared with the process. However, if they are of the same order of magnitude, as with this blending system, they must be taken into account. Either way, it is convenient to do the design in the Z domain because the subsequent compensators are easily realisable.

The block diagram of the sampled data multivariable control system for the blending plant is as depicted in Figure 81.6.

Note that both the error and output signals are sampled, but only the outputs are held by zero order hold devices. Whereas previous values of the error signals may be stored in memory, the output signals must be held between sampling instants to create pseudo continuous signals. It is a common mistake to assume hold devices for the error signals.

Equations 81.1–81.3 still apply. Allowing for the samplers, the equation which describes the behaviour of the compensators becomes:

$$\begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix} = \begin{bmatrix} C_{11}(s) & C_{12}(s) \\ C_{21}(s) & C_{22}(s) \end{bmatrix} \begin{bmatrix} e_1^*(s) \\ e_2^*(s) \end{bmatrix} \tag{81.8}$$

which may be denoted:

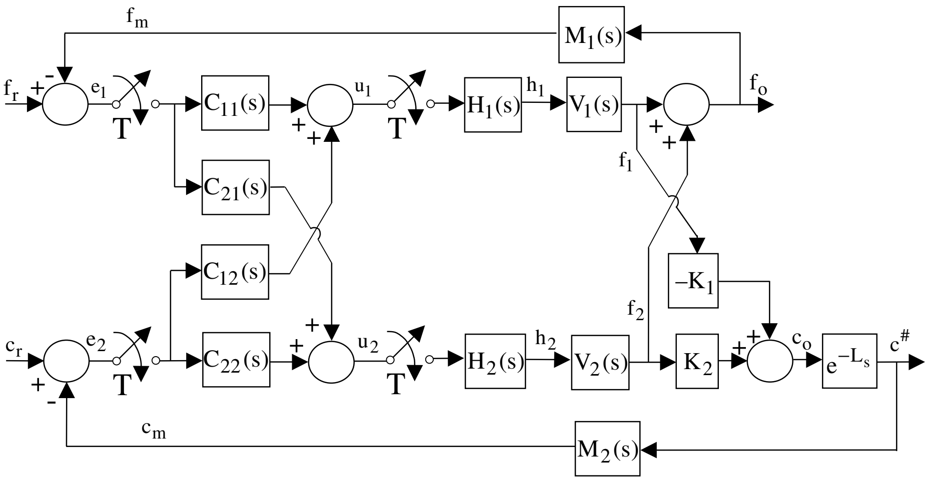

Fig. 81.6 Control of blending system using impulse compensators

$$\underline{u}(s) = C(s).\underline{e}^*(s)$$

An additional equation must be introduced to accommodate the hold devices:

$$\begin{bmatrix} h_1(s) \\ h_2(s) \end{bmatrix} = \begin{bmatrix} H_1(s) & 0 \\ 0 & H_2(s) \end{bmatrix} \begin{bmatrix} u_1^*(s) \\ u_2^*(s) \end{bmatrix} \qquad (81.9)$$

which may be denoted:

$$h(s) = H(s).\underline{u}^*(s)$$

The outputs of the holds are then applied *via* I/P converters to the two valves which manipulate the process inputs:

$$\begin{bmatrix} f_1(s) \\ f_2(s) \end{bmatrix} = \begin{bmatrix} V_1(s) & 0 \\ 0 & V_2(s) \end{bmatrix} \begin{bmatrix} h_1(s) \\ h_2(s) \end{bmatrix} \qquad (81.10)$$

which may be denoted:

$$\underline{f}(s) = V(s).\underline{h}(s)$$

81.8 Impulse Compensator Design

Block diagram algebra yields:

$$\begin{aligned} \underline{e}(z) &= \underline{r}(z) - \underline{m}(z) \\ &= \underline{r}(z) - MPVH(z).\underline{u}(z) \\ &= \underline{r}(z) - MPVH(z).C(z).\underline{e}(z) \end{aligned}$$

Solving for $\underline{e}(z)$ gives:

$$\begin{aligned} (I + MPVH(z).C(z))\,.\underline{e}(z) &= \underline{r}(z) \\ \underline{e}(z) &= (I + MPVH(z).C(z))^{-1}\,.\underline{r}(z) \end{aligned}$$

Assuming there are imaginary samplers on the controlled variables:

$$\begin{aligned} \underline{x}(z) &= PVH(z).C(z).\underline{e}(z) \\ &= PVH(z).C(z).\,(I + MPVH(z).C(z))^{-1}\,.\underline{r}(z) \\ &= G(z).\underline{r}(z) \end{aligned}$$

G(z) is the closed loop transfer matrix which determines the overall system performance:

$$\begin{aligned} G(z) = {} & PVH(z).C(z) \\ & .\,(I + MPVH(z).C(z))^{-1} \end{aligned} \qquad (81.11)$$

If G(s) is diagonal then any change in either set point will only affect one of the controlled variables, *i.e.* the system is decoupled. For a specified G(s), and for given M(s), P(s), V(s) and H(s), there must be a unique C(s):

$$\begin{aligned} G(z).\,(I + MPVH(z).C(z)) &= PVH(z).C(z) \\ G(z) &= (PVH(z) - G(z).MPVH(z))\,.C(z) \qquad (81.12) \\ C(z) &= (PVH(z) - G(z).MPVH(z))^{-1}\,.G(z) \end{aligned}$$

81.9 Worked Example No 2

Assume that M(s), V(s) and P(s) are as before, and that the zero order holds have a one second delay:

$$H(s) = \begin{bmatrix} \dfrac{1-e^{-s}}{s} & 0 \\ 0 & \dfrac{1-e^{-s}}{s} \end{bmatrix}$$

The 1 sec delay associated with the zero order holds implies a 1 sec sampling period.

Choose a closed loop transfer matrix, taking into account the dynamics of the sampling process:

$$G(s) = \begin{bmatrix} \dfrac{e^{-s}}{4s+1} & 0 \\ 0 & \dfrac{e^{-3s}}{4s+1} \end{bmatrix}$$

The rationale for choosing G(s) is essentially the same as in the continuous case, except that a 1 sec delay has been included in the flow loop to allow for the hold. Likewise for the concentration loop where a 3 sec delay has been allowed for both the measurement delay and the hold.

Assume a sampling period of T = 1 s.

Extensive manipulation yields

$$C(z) = \begin{bmatrix} \dfrac{0.441\left(1-1.323z^{-1}+0.4346z^{-2}\right)}{1-1.441z^{-1}+0.430z^{-2}} & \dfrac{-0.441\left(1-1.323z^{-1}+0.4346z^{-2}\right)}{1-1.385z^{-1}+0.4724z^{-2}-0.05597z^{-3}-0.04239z^{-4}} \\ \dfrac{0.441\left(1-1.323z^{-1}+0.4346z^{-2}\right)}{1-1.441z^{-1}+0.430z^{-2}} & \dfrac{0.441\left(1-1.323z^{-1}+0.4346z^{-2}\right)}{1-1.385z^{-1}+0.4724z^{-2}-0.05597z^{-3}-0.04239z^{-4}} \end{bmatrix} \quad (81.13)$$

which is of the form:

$$C(z) = \begin{bmatrix} \alpha(z) & -\beta(z) \\ \alpha(z) & \beta(z) \end{bmatrix} \quad (81.14)$$

It can be seen by inspection that all four compensators are realisable, *i.e.* there are no terms in z^{+1}. The coefficients of z^{-1} in successive terms of both the numerator and denominator series are of reducing magnitude and, with the exception of the least significant term of the second denominator, are of alternate sign. This is necessary for stability. The compensators may be realised by means of software using any of the techniques described in Chapter 78. Remember that the variables in the

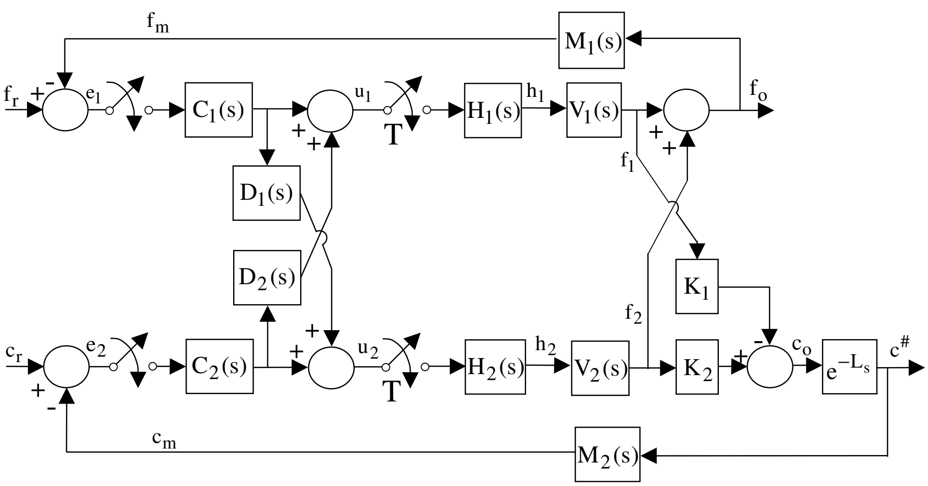

Fig. 81.7 Control of blending system using sampled data decouplers

model are all in deviation form so a steady state bias must be added to the controller outputs for implementation purposes.

81.10 Sampled Data Decoupler

The block diagram of a sampled data decoupler type of control system for the blending plant is as depicted in Figure 81.7.

Apart from the compensators and decouplers being of an impulse nature, the approach to design is essentially the same as for the continuous system of Figure 81.5. Control action is of the form

$$\begin{bmatrix} u_1(z) \\ u_2(z) \end{bmatrix} = \begin{bmatrix} C_1(z) & C_2D_2(z) \\ C_1D_1(z) & C_2(z) \end{bmatrix} \begin{bmatrix} e_1(z) \\ e_2(z) \end{bmatrix} \tag{81.15}$$

Comparison of Equations 81.14 and 81.15 reveals that, to meet the same closed loop performance criterion G(s) for the physical example of the blending system, the decouplers must be

$$D_1(s) = -D_2(s) = 1$$

Given the assumption that $K_1 = K_2 = 1$, this result is entirely consistent with that obtained in Section 81.6.

Note that it is often sufficient to realise the two compensators $C_1(z)$ and $C_2(z)$ by means of the discrete form of PID controllers developed in Chapter 78:

$$C(z) = \frac{a_0 + a_1z^{-1} + a_2z^{-2}}{1 - z^{-1}} \tag{78.3}$$

The approximation may simply be based on the first terms of $\alpha(z)$ and $\beta(z)$ which are the dominant terms. Otherwise, the PID compensator is taken as a factor out of $C_1(z)$ and $C_2(z)$. The parameters of the compensator would be presented to the operators in terms of the familiar gain, reset and rate times, whose values may be tuned empirically.

81.11 Comments

Although all these compensators have been designed for servo operation as opposed to regulo control, *i.e.* the effects of disturbances haven't been taken into account explicitly, they can nevertheless be just as effective at disturbance rejection.