Solutions

Solution 1. The model is given by

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij};$$

$$j = 1(1)n_i, i = 1(1)5$$

$$\epsilon_{ij} \sim N(0, \sigma^2), \forall i, j$$

y_{ij} : response time (in milliseconds) for the j^{th} observation corresponding to the i^{th} circuit,
μ : mean effect,
α_i : additional effect due to the i^{th} circuit,
ϵ_{ij} : error for the j^{th} observation corresponding to the i^{th} circuit.

We define

$$\overline{y_{i0}} = \frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij};$$

$$\overline{y_{00}} = \frac{1}{n}\sum_{i=1}^{5}\sum_{j=1}^{n_i} y_{ij}.$$

To test, $H_0 : \alpha_i = 0, \forall i = 1(1)5$ ag. H_1 : not H_0.
The required test statistic is given by

$$F = \frac{\sum_{i=1}^{5} n_i(\overline{y_{i0}} - \overline{y_{00}})^2}{\sum_{i=1}^{5}\sum_{j=1}^{n_i}(y_{ij} - \overline{y_{i0}})^2}(\frac{n-5}{5-1})$$

Under H_0, $F \sim F_{4,n-5}$. Here, $n = \sum_{i=1}^{5} n_i = 49$.
We reject H_0, if the observed value of F from the sample is greater than $F_{4,44;0.05} = 2.58$. From the sample, we get $F_{obs} = 0.775$.
Hence, H_0 is accepted on the basis of the given sample and we conclude that all 5 circuits have the same response time.

Solution 2. The model is given by

$$y_{ij} = \mu_i + \epsilon_{ij};$$

$$j = 1(1)n_i, i = 1(1)5;$$

$$\epsilon_{ij} \sim N(0, \sigma^2), \forall i, j$$

y_{ij} : blood pH reading from blood analysis in certain standardized units for the j^{th} observation corresponding to the i^{th} drug,
μ_i : effect due to the i^{th} drug,
ϵ_{ij} : error for the j^{th} observation corresponding to the i^{th} drug.

We define

$$\overline{y_{i0}} = \frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij};$$

$$\overline{y_{00}} = \frac{1}{n}\sum_{i=1}^{5}\sum_{j=1}^{n_i} y_{ij}.$$

i) **To test the hypothesis that all 5 drugs are equally effective:**

To test, $H_0 : \mu_i = 0, \forall i = 1(1)5$ ag. H_1 : not H_0.
The required test statistic is given by

$$F = \frac{\sum_{i=1}^{5} n_i(\overline{y_{i0}} - \overline{y_{00}})^2}{\sum_{i=1}^{5}\sum_{j=1}^{n_i}(y_{ij} - \overline{y_{i0}})^2}(\frac{n-5}{5})$$

Under H_0, $F \sim F_{5,n-5}$. Here, $n = \sum_{i=1}^{5} n_i = 17$.
We reject H_0, if the observed value of F from the sample is greater than $F_{5,12;0.05} = 3.1$. From the sample, we get $F_{obs} = 34.45$.
Hence, H_0 is rejected on the basis of the given sample and we conclude that drugs are not equally effective.

ii) To test, $H_0 : L = \mu_1 + 2\mu_2 - \mu_3 - \mu_4 - \mu_5 = 0$ ag. H_1 : not H_0.
The least square estimates, $\hat{\mu_i} = \overline{y_{i0}}$, $\forall i = 1(1)5$.
Hence, $\hat{L} = \hat{\mu_1} + 2\hat{\mu_2} - \hat{\mu_3} - \hat{\mu_4} - \hat{\mu_5} = \sum_{i=1}^{5} \alpha_i\hat{\mu_i}$ and $V(\hat{L}) = \hat{\sigma}^2 \sum_{i=1}^{5} \frac{\alpha_i^2}{n_i}$.
$\hat{\sigma}^2 = \frac{1}{17}\sum_{i=1}^{5}\sum_{j=1}^{n_i}(y_{ij} - \overline{y_{i0}})^2$.
The required test statistic is given by

$$t = \frac{\sqrt{12}\hat{L}}{\sqrt{17V(\hat{L})}}.$$

Under H_0, $t \sim t_{12}$.
We reject H_0, if the observed value of $|t|$ from the sample is greater than $t_{12;0.025} = 2.178$. From the sample, we get $t_{obs} = 1.336$.
Hence, H_0 is accepted on the basis of the given sample and we conclude that $L = 0$.

iii) **To test for differential effects between pairs of drugs:**

Critical Difference Method:

To test, $H_{0ij} : \mu_i = \mu_j$ ag. H_{1ij} : not H_{0ij}.
The required test statistic is given by

$$t_{ij} = \frac{\hat{\mu_i} - \hat{\mu_j}}{\sqrt{\frac{17}{12}\hat{\sigma}^2(\frac{1}{n_i} + \frac{1}{n_j})}}$$

.
Under H_{0ij}, $t_{ij} \sim t_{12}$.
We reject H_{0ij}, if the observed value of $|t_{ij}|$ from the sample is greater than $t_{12;0,025} = 2.178$.
From the sample, we observe that the following pairs of drugs are significantly different in their effect on pH level of blood:

$$(A, B), (A, C), (A, D), (A, E)$$

Scheffe's method for Multiple Comparison Test:

Using Scheffe's method, we obtain the same results as obtained for Critical Difference Method.

Solution 3. The model is given by

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij};$$

$$j = 1(1)4, i = 1(1)4$$

$$\epsilon_{ij} \sim N(0, \sigma^2), \forall i, j$$

y_{ij} : tensile strength (lb/in^2) of cement for the j^{th} observation corresponding to the i^{th} mixing technique,
μ : mean effect,
α_i : additional effect due to the i^{th} mixing technique,
ϵ_{ij} : error for the j^{th} observation corresponding to the i^{th} mixing technique.

We define

$$\overline{y_{i0}} = \frac{1}{4}\sum_{j=1}^{4} y_{ij};$$

$$\overline{y_{00}} = \frac{1}{16}\sum_{i=1}^{4}\sum_{j=1}^{4} y_{ij}.$$

a) **<u>To test the hypothesis of similar effect of different mixing techniques:</u>**

To test, $H_0 : \alpha_i = 0, \forall i = 1(1)4$ ag. H_1 : not H_0.
The required test statistic is given by

$$F = \frac{\sum_{i=1}^{4} 4(\overline{y_{i0}} - \overline{y_{00}})^2}{\sum_{i=1}^{4}\sum_{j=1}^{4}(y_{ij} - \overline{y_{i0}})^2}\left(\frac{16-4}{4-1}\right)$$

Under H_0, $F \sim F_{3,12}$.
We reject H_0, if the observed value of F from the sample is greater than $F_{3,12;0.05} = 3.49$. From the sample, we get $F_{obs} = 12.73$.
Hence, H_0 is rejected on the basis of the given sample and we conclude that the different mixing techniques give rise to different tensile strengths.

b) **<u>Comparisons between pairs of means:</u>**

<u>Student Newman Keul's Test:</u>
By this method, the difference between any two means under test is significant when the range of the observed means of each and every subgroup containing the two means under test is significant according to the studentized critical range. From the sample, we observe that only, the mixing methods $(1, 3)$ are not signifcantly different in their effect on tensile strength. All other pairs of mixing methods are significantly different.

<u>Duncan's Test:</u>
The test procedure in Duncan's multiple comparison test is the same as in the Student-NewmanKeuls test except the observed ranges are compared with Duncan's $\alpha\%$ critical range. Using this method, we obtain the same results as above.

<u>Tukey's Method:</u>
Now, we use Tukey's method to make comparisons b/w pairs of means. From the sample, we observe that the pairs $(1, 2), (1, 3), (2, 3)$ of mixing techniques are not significantly different. All other pairs are signifcantly different.

Solution 4. The model is given by

$$y_{ij} = \mu_i + e_{ij};$$

$$j = 1(1)5, i = 1(1)5;$$

$$e_{ij} \sim N(0, \sigma^2), \forall i, j$$

y_{ij} : tensile strength (lb/in^2) of synthetic fibre for the j^{th} observation corresponding to the i^{th} cotton weight percentage,

μ_i : effect due to the i^{th} cotton weight percentage,
e_{ij} : error for the j^{th} observation corresponding to the i^{th} cotton weight percentage.

We define

$$\overline{y_{i0}} = \frac{1}{5}\sum_{j=1}^{5} y_{ij};$$

$$\overline{y_{00}} = \frac{1}{25}\sum_{i=1}^{5}\sum_{j=1}^{5} y_{ij}.$$

i) **To test the hypothesis that all 5 cotton weight percentages are equally effective:**

To test, $H_0 : \mu_i = 0, \forall i = 1(1)5$ ag. H_1 : not H_0.
The required test statistic is given by

$$F = \frac{\sum_{i=1}^{5} n_i(\overline{y_{i0}} - \overline{y_{00}})^2}{\sum_{i=1}^{5}\sum_{j=1}^{n_i}(y_{ij} - \overline{y_{i0}})^2}(\frac{25-5}{5})$$

Under H_0, $F \sim F_{5,20}$.
We reject H_0, if the observed value of F from the sample is greater than $F_{5,20;0.05} = 2.71$. The ANOVA table that was obtained is given below:

Table 1: ANOVA Table

Source	df	Sum of Squares	Mean Square	F value
Drug	5	6131	1226.2	152.1
Error	20	161	8.1	
Total	25	6292		

From the sample, we get $F_{obs} = 152.1$.
Hence, H_0 is rejected on the basis of the given sample and we conclude that the different percentages of cotton weights give rise to different tensile strengths in the synthetic fibres.

ii) **Test to find differential effects of pairs of cotton weight percentages:**

Duncan's Test:
From the sample, we observe that the following pairs of cotton weight percentages are not significantly different in their effect on tensile strength of the fibre: ("20","25"), ("15","35"). All other pairs are significantly different.

Student Newman Keul's Test:

We obtain the same results as above.

Scheffe's Test:

Using this method, we observe that the following pairs of cotton weight percentages are not significantly different in their effect on tensile strength of the fibre: ("15","20"), ("15","35"), ("20","25"), ("20"
All other pairs are significantly different.

Tukey's Test:

Using this method, we observe that the following pairs of cotton weight percentages are not significantly different in their effect on tensile strength of the fibre: ("15","35"), ("20","25"), ("20","35"), ("25"
All other pairs are significantly different.

iii) To test:

$$H_{01}: \mu_4 = \mu_5$$

$$H_{02}: \mu_1 + \mu_3 = \mu_4 + \mu_5$$

$$H_{03}: \mu_1 = \mu_3$$

$$H_{04}: 4\mu_2 = \mu_1 + \mu_3 + \mu_4 + \mu_5$$

We can write the hypotheses given above in the format $H_0 : L^T\beta = 0$, where L is vector of length 5 and $\beta = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)^T$

The required test statistic is given by

$$t = \frac{\sqrt{n-5}L^T\hat{\beta}}{\sqrt{nMSEL^T(X^TX)^{-1}L}} \overset{H_0}{\sim} t_{20}$$

where, X is the deisgn matrix and $\hat{\beta}$ is the vector of estimated coefficients.

We reject H_{0i} at level $\alpha = 0.05$ if $|t_{obs}| > t_{0.025,20}$..

On the basis of our observations from the given sample, H_{01} and H_{03} are rejected.

Solution 5. The model is given by

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij};$$

$$j = 1(1)6, i = 1(1)4$$

$$\epsilon_{ij} \sim N(0, \sigma^2), \forall i, j$$

y_{ij} : response for the j^{th} patient corresponding to the i^{th} sleeping medicine,
μ : mean effect,
α_i : additional effect due to the i^{th} sleeping medicine,
ϵ_{ij} : error for the j^{th} patient corresponding to the i^{th} sleeping medicine.

We define

$$\overline{y_{i0}} = \frac{1}{6}\sum_{j=1}^{6} y_{ij};$$

$$\overline{y_{00}} = \frac{1}{24}\sum_{i=1}^{4}\sum_{j=1}^{6} y_{ij}.$$

i) To test, $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$ ag. H_1 : not H_0.
The required test statistic is given by

$$F = \frac{\sum_{i=1}^{4} 6(\overline{y_{i0}} - \overline{y_{00}})^2}{\sum_{i=1}^{4}\sum_{j=1}^{6}(y_{ij} - \overline{y_{i0}})^2}(\frac{24-4}{4-1})$$

Under H_0, $F \sim F_{3,20}$.
We reject H_0, if the observed value of F from the sample is greater than $F_{3,20;0.05} = 3.1$. From the sample, we get $F_{obs} = 5.99$.
Hence, H_0 is rejected on the basis of the given sample and conclude that the sleeping medicines affect the patients differently.

ii) **Test for differential effects of pairs of sleeping medicines:**

t-test:
From the sample, we observe that the following pairs of sleeping medicines are not significantly

different in their effect on the patient: $(B, D), (A, D), (A, C)$. All other pairs are significantly different.

Student Newman Keul's Test:

Using this method, we conclude that the following pairs of sleeping medicines are not significantly different in their effect on the patient: $(B, D), (A, D), (A, C), (C, D)$. All other pairs are significantly different.

Duncan's Test:

Here, the results are similar to those obtained from t-test.

Tukey's Test:

Here, the results are similar to those obtained from Student Newman Keul's Test.

Scheffe's Test:

Here, the results are similar to those obtained from Student Newman Keul's Test.

iii) To test:

$$H_0 : L = \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 = 0 \quad \text{versus} \quad H_1 : L \neq 0$$

We test this hypothesis by constructing the confidence interval for L using a) Tukey's procedure and b) Scheffe's procedure.

Our findings are given on the tables below:

Table 2: Tukey's procedure)

Lower bound	Upper bound
-37.25	73.25

Table 3: Scheffe's procedure

Lower bound	Upper bound
-68.62	104.62

Since the interval between the lower and upper bounds contains 0 in both the cases, hence we accept our null hypothesis at level $\alpha = 0.05$ as a conclusion of both the tests.