Applying Stereographic Projection and Vector Mathematics to Model Fault Intersections

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Table of Contents

Introduction

Since middle school, I have been interested in geology. Outside of school, I have participated for many years in a competition called Science Olympiad, where a student can compete in multiple events of their choosing. Last year, as a part of my participation in a new event called Geologic Mapping, I was introduced to a method geologists use to visualize the three-dimensional orientations of planar and linear geological structures. This method is the called the stereonet. As a part of the competition, I was expected to know how to use certain rules to plot fault planes and find their intersections using the stereonet. While I learned how to use stereonets to do this, I became curious about the mathematical principles that make these techniques work. Furthermore, the visualization of fault planes using stereonets has some interesting applications such as predicting seismic hazard potential in prone regions, such where I live (near Seattle), which makes this topic even more interesting to me.

In my exploration, I will first examine the mathematical principles behind the stereonet to explain how the techniques I learned work to plot fault planes and find their intersections. Some of the concepts I will be focusing on include stereographic projection and vector mathematics. As such, I will address the question: How can stereographic projection and vector mathematics be applied to analyze fault planes and their intersections?

Background on Stereonets

The stereonet (also called Wulff net) is a type of stereographic projection that projects the bottom half of a sphere onto a circular 2D plane. It helps to visualize the stereonet as the bottom half of a sphere as viewed from above. The lines on the stereonet are a set of great circles and a set of small circles that are perpendicular to one another (just like longitude and latitude lines, respectively, on the globe) (Burchfiel and Studnicki-Gizbert 1).

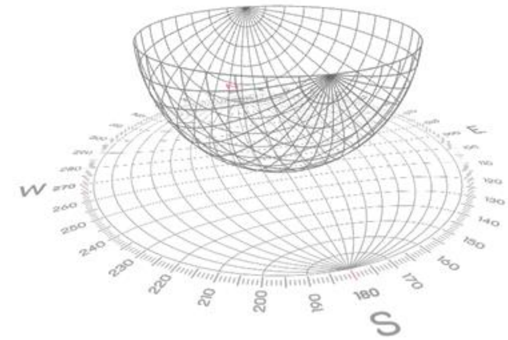

Figure 1: a stereonet is a stereographic projection of the bottom half of a sphere onto a circular 2D plane (visiblegeology.com).

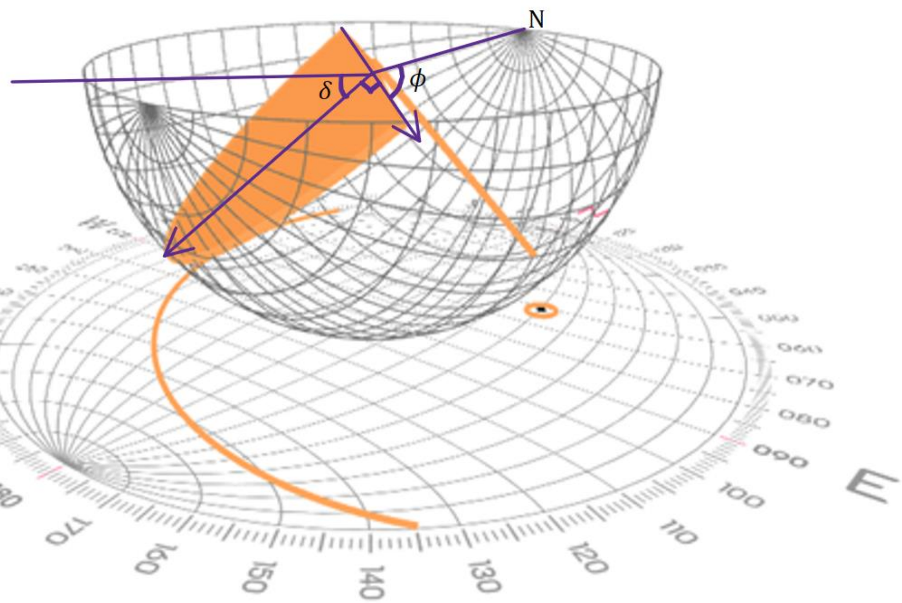

Figure 2: a plane is represented as a great circle on the stereonet.

On stereonets, planes (see Figure 2) are portions of great circles plotted using *strike* (ϕ) and *dip* (δ), which are two angles that represent the orientation and the inclination of the plane with respect to horizontal, respectively. The strike is the clockwise angle from north, and by convention, is 90° counterclockwise from the dip direction (the arrow pointing downwards in Figure 2). Note that without this convention, there would be two possible strikes for the same plane (separated by 180°). Meanwhile, the dip represents the angle between the plane and the horizontal. In practice, we would plot something like this:

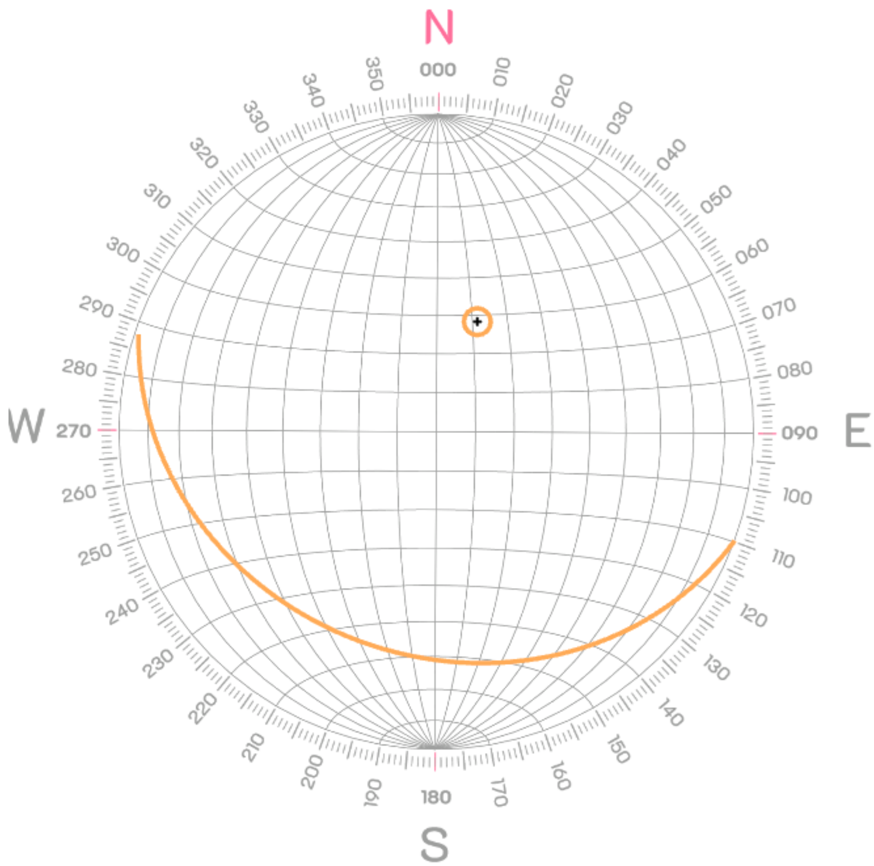

Figure 3: stereonet plot of plane with strike 110° and dip 30°.

Following convention, it is easy to read a strike of 110° from Figure 3. A quick eyeball approximation measuring the angular distance from the edge of the stereonet to the center of the arc of the fault plane gives a separation of 30°, the dip. Note that when we plot stereonets by hand, we will use a setup of tracing paper on top of a template so that we can rotate the end of the arc pointing in the strike direction to the point marked "N" to be able to use gridlines to measure dip precisely. After plotting the curve, we will rotate the tracing paper back to its original orientation so both "N" points are aligned. This information will be important later as we plot multiple fault planes and their intersections. So, how can we go from the strike and dip representation of a plane to something that is more familiar to us?

Representing a Plane: Normal Vectors

Consider the concept that a *normal vector* of a plane is the vector that is normal to all vectors lying on the plane. It should be intuitive from geometry that such a vector can be written in the form $\mathbf{n} = k\langle a, b, c\rangle$, that is, all normal vectors of a plane are scalar multiples of each other.

Suppose that we have two points (x, y, z) and (p, q, r) lying on the plane. Then the vector $\langle x - p, y - q, z - r\rangle$ must also lie on the plane. By definition, $\langle a, b, c\rangle$ is normal to this vector, so we can write

$$\langle a, b, c\rangle \cdot \langle x - p, y - q, z - r\rangle = 0$$
$$ax - ap + by - bq + cz - cr = 0$$
$$ax + by + cz = ap + bq + cr.$$

Setting the RHS equal to a constant d, we see that in Cartesian coordinates, a plane is represented by the equation

$$ax + by + cz = d \tag{1}$$

where $\langle a, b, c\rangle$ is the normal vector. Now, it should also be easy to see why if we know a point (x_1, y_1, z_1), we can write the equation of the plane as

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0. \tag{2}$$

Now, let the origin be the point where the strike and dip directions (the two arrows in Figure 2) intersect; and the x-axis be along the east-west axis, the y-axis be along the north-south axis, and the z-axis be in the vertical direction. Note that the origin and both arrows lie on the orange plane. Also note the orange line in Figure 2 which is normal to the plane. If we can find the vectors represented by these arrows, then we can set up a system of equations to find the normal vector.

The unit vector along the strike direction is horizontal, so its z-component is zero. From the unit circle, we know that the coordinates of an angle that is θ *counterclockwise* from the positive x-axis is $(\cos\theta, \sin\theta)$. Here, we want the coordinates of an angle that is ϕ *clockwise* from the positive y-axis. We can produce the relation (see Figure 3):

$$\theta = -(\phi - 90^\circ) = 90^\circ - \phi$$

so that the unit vector along the strike direction is

$$\langle \cos(90^\circ - \phi), \sin(90^\circ - \phi), 0\rangle = \langle \sin\phi, \cos\phi, 0\rangle. \quad (3)$$

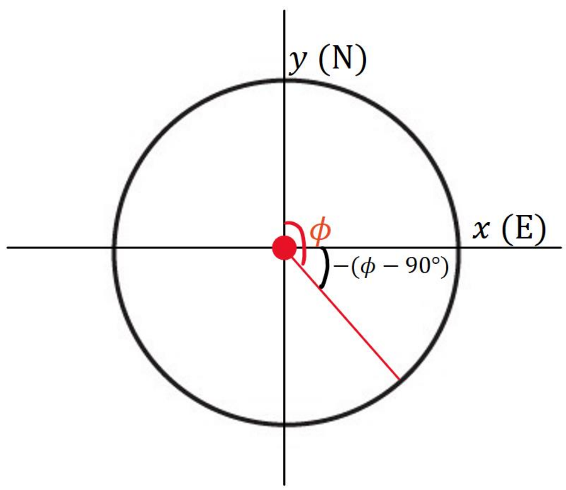

Figure 3: relation between ϕ and θ.

Now, we will find the unit vector along the dip direction. It should be easy to see why the z-component is simply $-\sin\delta$ (see Figure 4).

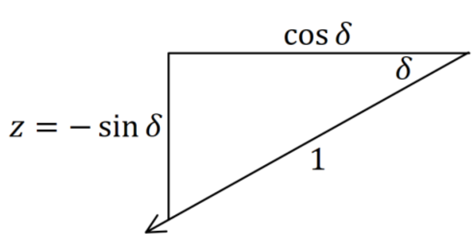

Figure 4: components of the unit vector in the dip direction.

As for the x- and y-components, we know that the projection of the unit vector along the dip direction onto $z = 0$ (the horizontal plane, see the line above the dip direction arrow in Figure 2) is a 90° clockwise rotation of the unit vector along the strike direction. Performing this rotation on the x- and y-components of the unit vector along the strike direction yields

$$\langle \sin\phi, \cos\phi \rangle \to \langle \cos\phi, -\sin\phi \rangle$$

(which comes from applying the rotation matrix $\mathbf{R} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$ with $t = -90°$ *counterclockwise*). Scaling both components by $\cos\delta$ (see Figure 4) yields the unit vector along the dip direction,

$$\langle \cos\phi\cos\delta, -\sin\phi\cos\delta, -\sin\delta \rangle. \quad (4)$$

Both the unit vectors along the strike and dip directions (see equations (3) and (4)) are orthogonal to the normal vector. This means that we can take the cross product of these unit vectors to obtain the normal vector:

$$\mathbf{n} = \langle \sin\phi, \cos\phi, 0 \rangle \times \langle \cos\phi\cos\delta, -\sin\phi\cos\delta, -\sin\delta \rangle = \langle -\cos\phi\sin\delta, \sin\phi\sin\delta, -\cos\delta \rangle \quad (5)$$

Intersection of Planes: Representing a Line

Now suppose we have two fault planes F_1 and F_2 with

$$\begin{aligned} \mathbf{n}_1 &= \langle -\cos\phi_1\sin\delta_1, \sin\phi_1\sin\delta_1, -\cos\delta_1 \rangle \\ \mathbf{n}_2 &= \langle -\cos\phi_2\sin\delta_2, \sin\phi_2\sin\delta_2, -\cos\delta_2 \rangle \end{aligned}$$

and we want to find the line formed by the intersection of the two fault planes (see Figure 5 on page 9). To do this, we can take the cross-product of the two normal vectors which will yield a vector parallel to the line of intersection. This is because the normal vectors are, by definition, orthogonal to every line in their respective planes, and taking their cross product would yield a vector that is orthogonal to both normal vectors. Hence, this vector must be parallel to both planes, and thus their line of intersection. Proceeding, we have

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{pmatrix} \cos\delta_1 \sin\phi_2 \sin\delta_2 - \cos\delta_2 \sin\phi_1 \sin\delta_1 \\ \cos\delta_1 \cos\phi_2 \sin\delta_2 - \cos\delta_2 \cos\phi_1 \sin\delta_1 \\ \sin\phi_1 \sin\delta_1 \cos\phi_2 \sin\delta_2 - \cos\phi_1 \sin\delta_1 \sin\phi_2 \sin\delta_2 \end{pmatrix}.$$

Note that the last component can be simplified by factoring out $\sin\delta_1 \sin\delta_2$ and applying the angle addition identity for sine on what remains, such that

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{pmatrix} \cos\delta_1 \sin\phi_2 \sin\delta_2 - \cos\delta_2 \sin\phi_1 \sin\delta_1 \\ \cos\delta_1 \cos\phi_2 \sin\delta_2 - \cos\delta_2 \cos\phi_1 \sin\delta_1 \\ \sin\delta_1 \sin\delta_2 \sin(\phi_1 - \phi_2) \end{pmatrix}. \tag{6}$$

Note that the origin (0,0,0) passes through both planes as noted in the previous section and is visible in Figure 2. Hence, it also passes through the line of intersection. Then we can represent the line of intersection in vector form as

$$\mathbf{r}(\mathrm{t}) = t(\mathbf{n}_1 \times \mathbf{n}_2) \tag{7}$$

where $\mathbf{r}$ is the position vector associated with the scalar t.

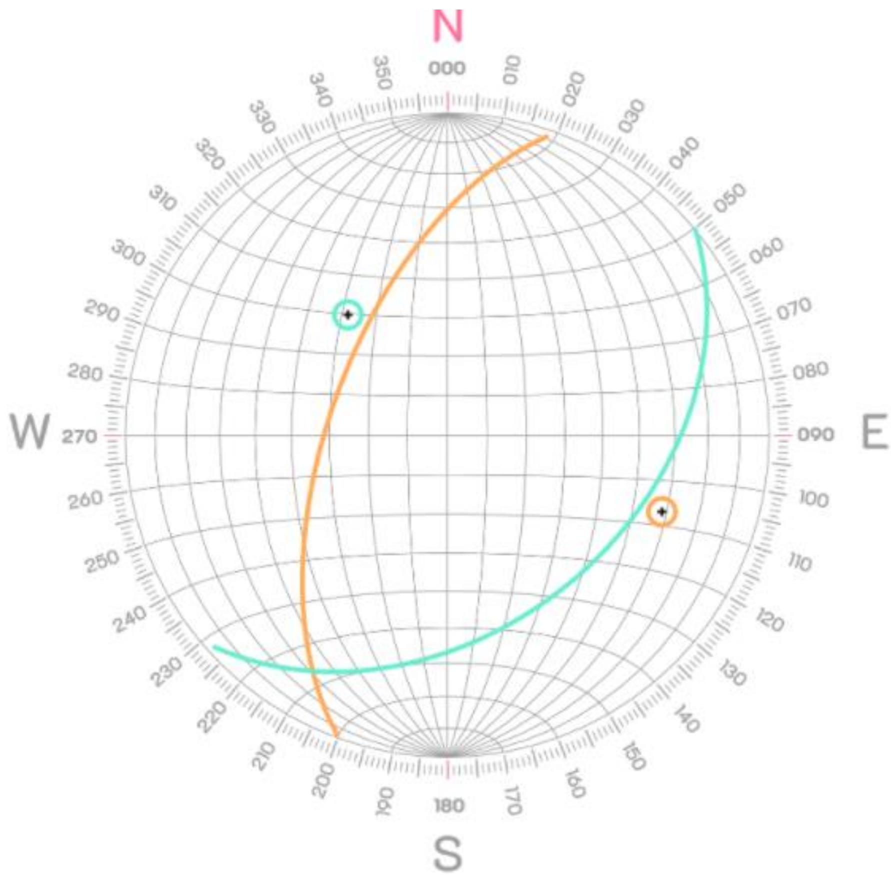

Figure 5: F_1 and F_2 with ϕ_1 = 50°, δ_1 = 40°, ϕ_2 = 200°, δ_2 = 60°.

On the stereonet, note that the line of intersection is represented by the point at the intersection of the two arcs representing the planes. This is because the dot is the projection of the downward plunging line onto the horizontal surface below it (see Figure 6). In fact, the circled crosses are also representations of lines—in particular, they are the projections of the normal vectors of each plane onto the stereonet. When plotted on the stereonet, the normal vectors are called poles, and unsurprisingly, they always have an angular separation of 90° from their corresponding planes.

Like with planes, we can also describe lines using two angles, except this time they are called *trend* (θ) and *plunge* (λ) which are analogous to strike and dip, respectively. Trend describes the direction in which the line plunges, while plunge describes how much it plunges.

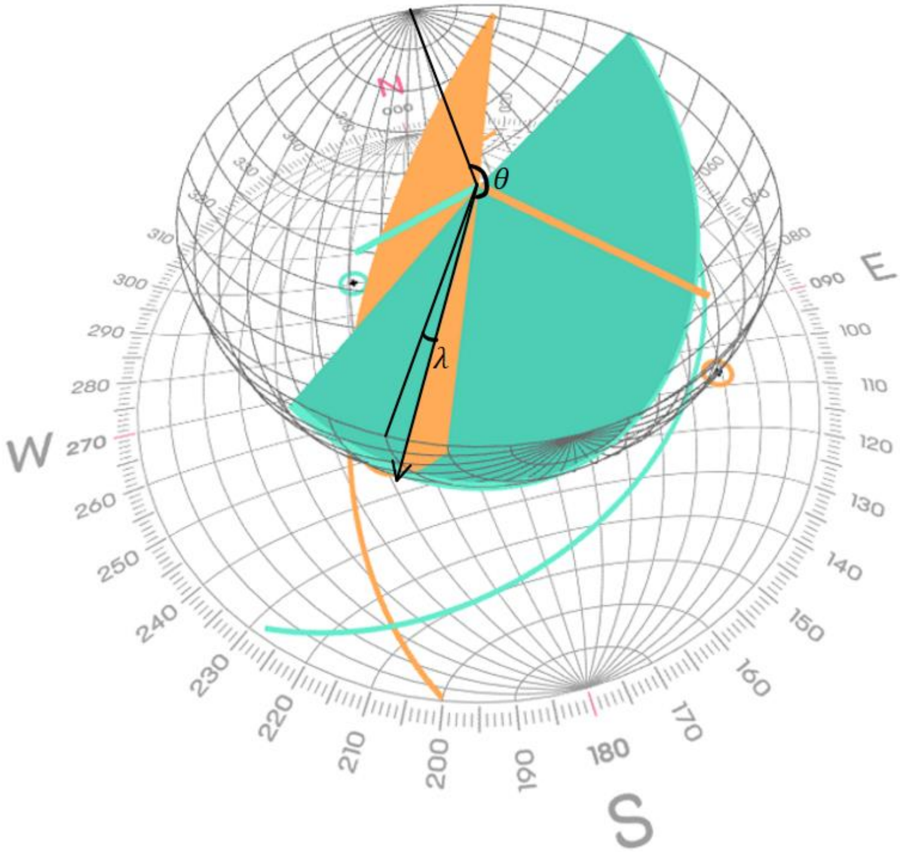

Figure 6: same set of planes visualized from above. Arrow represents the line of intersection.

The process of finding the trend and plunge from a stereonet is surprisingly simple: θ can be read directly off the edge of the stereonet, while λ can be found by rotating the tracing paper so that the grid markings can be used to read off the angular distance from the edge to the point representing the line of intersection on the stereonet. For the example presented in Figures 5 and 6, we have $\theta \approx 210°$ and $\lambda \approx 16°$ just by using this method. But how does equation (7) relate to this simple representation, and is there a way to relate the strike and dip of each plane to the trend and plunge of their line of intersection?

Relating Trend and Plunge with Strike and Dip

Suppose we have a line with trend θ and plunge λ formed by the intersection of two planes F_1 and F_2 with strikes ϕ_1, ϕ_2 and dips δ_1, δ_2. We wish to relate θ and λ to the quantities $\phi_1, \phi_2, \delta_1, \delta_2$. Suppose we have the equation for a line that passes through the origin,

$$\mathbf{r}(t) = t\mathbf{v}$$

for which we will find θ and λ. Note that we already know that $\mathbf{v} = \mathbf{n_1} \times \mathbf{n_2}$ from equation (7), but to simplify our notation we will consider $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ where each component matches that in equation (6).

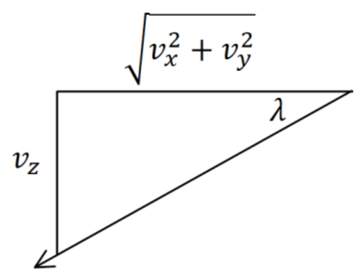

Figure 7: right triangle relation to find λ.

We find it convenient to solve for λ first using the visualization in Figure 7, representing the line of intersection as viewed face-on. The top segment is in the plane of the horizontal, the head of the arrow lies at the origin, and the plane is rotated through an angle θ clockwise from N, as defined by the trend. Hence, we can relate:

$$\lambda = \arctan\left(\frac{-v_z}{\sqrt{v_x^2 + v_y^2}}\right) \tag{9}$$

Solving for θ is a little more complicated. We consider the horizontal plane and components v_x, v_y to describe the trend:

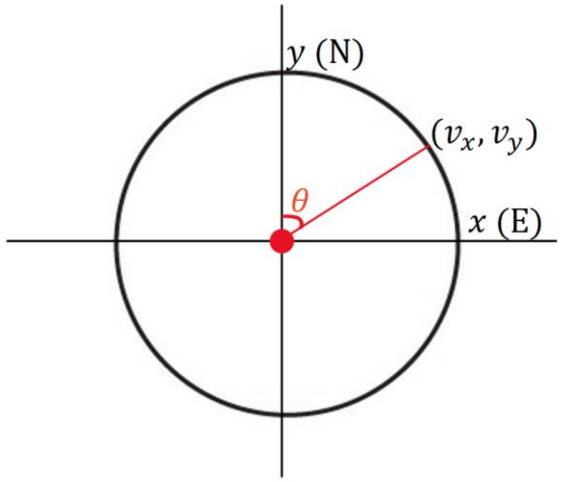

Figure 8: consider a bird's eye view (from the $+z$ axis) of the horizontal plane to find θ.

At first glance, we might notice in Figure 8 that $\theta = \arctan\,(v_x/v_y)$ and be tempted to generalize this to all $\theta \in [0°, 360°)$, but after careful consideration we realize that we must be more careful with our signs. Still, it is simple to determine expressions for θ in each of the following cases:

v_x	v_y	θ
Positive	Positive	$\arctan\left(v_x/v_y\right)$ (10)
Positive	Negative	$180° - \arctan\left(-v_x/v_y\right)$ (11)
Negative	Negative	$180° + \arctan\left(v_x/v_y\right)$ (12)
Negative	Positive	$360° - \arctan\left(-v_x/v_y\right)$ (13)

Table 1: expressions for θ corresponding to each quadrant, determined by signs of v_x, v_y.

Now that we have expressions relating θ and λ to the components of $\mathbf{v}$, we are essentially done. Note that we could substitute the components from equation (6) into equations (9)-(13) but doing so would be unnecessarily complicated. Rather, it is easy to first substitute values for $\phi_1, \delta_1, \phi_2, \delta_2$ into equation (6) and then apply equations (9)-(13) as needed. In the case of fault planes F_1 and F_2 with $\phi_1 = 50°$, $\delta_1 = 40°$, $\phi_2 = 200°$, $\delta_2 = 60°$, these substitutions yield

$$\mathbf{v} = \langle -0.473, -0.829, -0.278 \rangle,$$

and applying equations (9) and (12) (noting that both v_x and v_y are negative) yield the exact values (to one decimal place) of $\lambda = 16.2°$ and $\theta = 209.7°$ which are consistent with the graphical stereonet estimations of the plunge (16°) and trend (210°) made on page 10.

Conclusion

Through this exploration we have connected the graphical representations of planes and lines using a stereonet to their more familiar vector representations. Doing so has allowed me to appreciate the link between pen-and-paper tools that geologists use in the field and their complex mathematical basis, revealing the insights behind the techniques I originally learned. This application has practical uses as well, and I have synthesized my findings into a Microsoft Excel spreadsheet on GitHub that can calculate the normal vector of a plane given the strike and dip, and the trend and plunge of the line of intersection formed by any two planes.[1] However, I understand that there are a number of concepts in structural geology, outside of what I have been exposed to in the Geologic Mapping event, such as rake and pitch which I have not treated in my exploration. Perhaps my future pursuits could lead me to explore these concepts more deeply.

[1] Available at https://github.com/rivak7/fault-modeling.

Works Cited

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Visible Geology. Seequent, https://www.visiblegeology.com/. Accessed 28 Nov. 2024.[2]

[2] All figures with stereonets were produced using visiblegeology.com.