Applying Stereonets and Vector Mathematics to Model Fault Intersections

https://github.com/rivak7/fault-modeling

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Introduction

Since middle school, I have been interested in geology. Outside of school, I have participated for many years in a competition called Science Olympiad, where a student can compete in multiple events of their choosing. Last year, as a part of my participation in a new event called Geologic Mapping, I was introduced to a tool that geologists use to visualize the three-dimensional orientations of planar and linear geological structures (see Figure 1). This tool is called the stereonet. For the competition, I was expected to know how to use certain rules to plot planes and find their intersections using the stereonet. While I learned how to use stereonets to do this, I became curious about the mathematical principles that make these techniques work.

In my exploration, I will examine the mathematical principles behind the stereonet to explain how the techniques I learned work to plot fault planes. I will then develop a way to find their intersections. Some of the concepts I will be focusing on include projection (using a stereonet) and vector mathematics, topics I self-studied before they were covered in the course. As such, I will address the question: How can stereonets and vector mathematics be applied to visualize fault planes and their intersections?

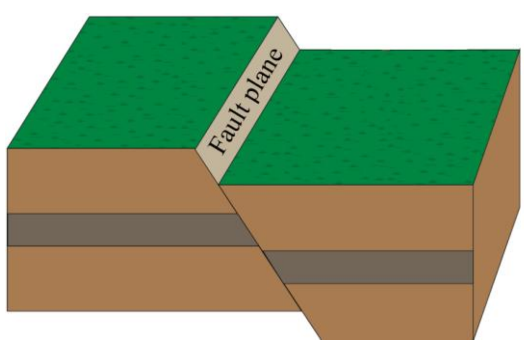

Figure 1: Faults accommodate motion between blocks of the Earth's crust. The planar surface along which this motion occurs is called a fault plane and is one of many structures that can be represented on a stereonet. I will be focusing on fault planes in this exploration because that is what Science Olympiad tests on.

Background on Stereonets

The stereonet (see Figure 2) is a type of projection that projects the bottom half of a sphere onto a circular 2D plane (imagine viewing a hemispherical bowl from above). This hemisphere is marked with a set of great circles and a set of small circles that are perpendicular to one another. When projected onto the stereonet, these circles appear just like longitude and latitude lines, respectively, on the globe [1]. Note that from now on, we will refer to the circular 2D plane (the projection itself) as the stereonet.

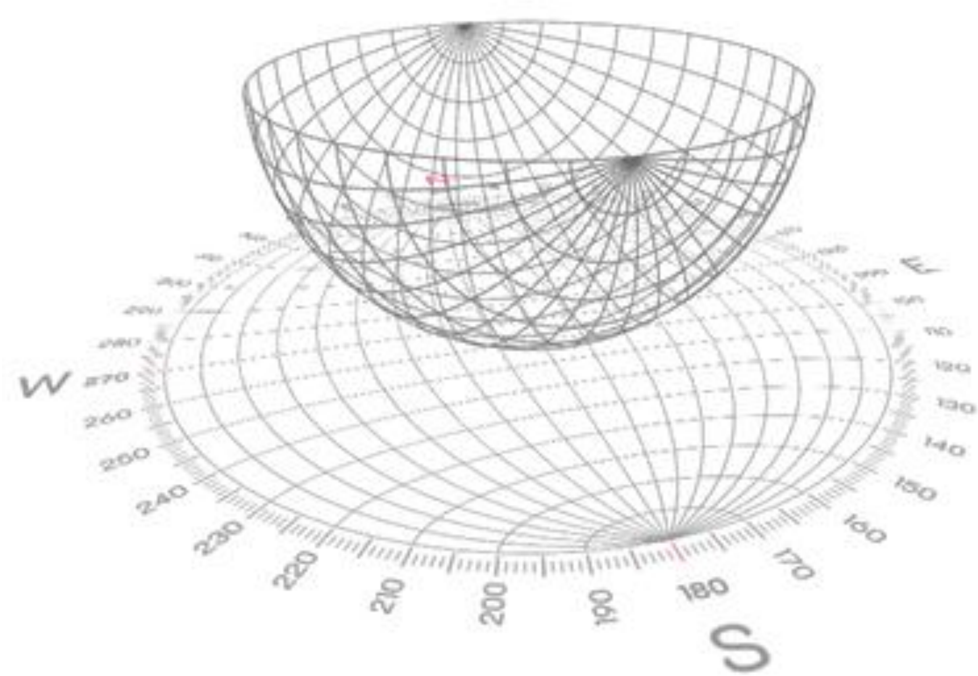

Figure 2: A stereonet is a stereographic projection of the bottom half of a sphere onto a circular 2D plane [2].

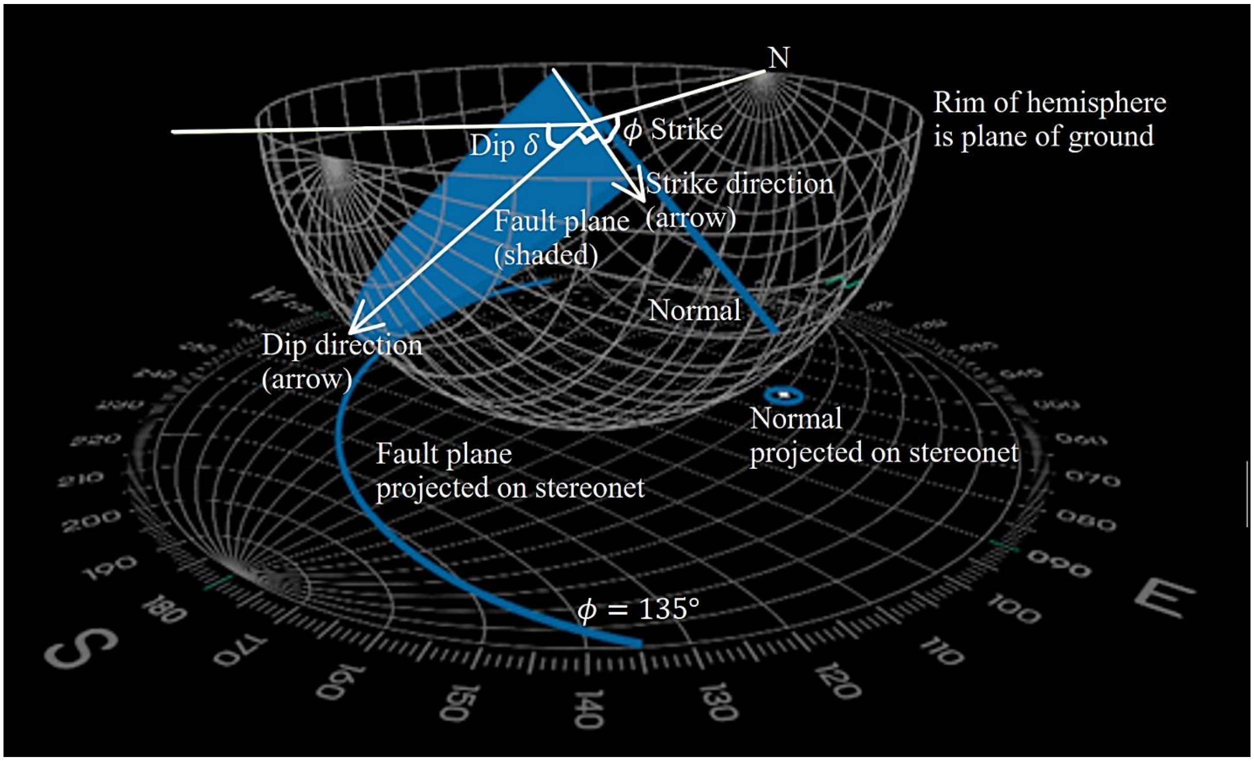

Figure 3: A plane (such as a fault plane) projected on the stereonet is an arc. A line's projection is a point. The normal vector (a line) of the fault plane is shown here and will be referred to later. Note how the strike ϕ (defined below) can be directly read off the edge of the stereonet.

On stereonets, planes (see Figure 3) are arcs plotted using *strike* (ϕ) and *dip* (δ), which are two angles that represent the orientation of the plane and the inclination of the plane with respect to the horizontal, respectively. The strike is the clockwise angle from north, and by the convention of the *right-hand rule*, is 90° counterclockwise from the dip direction. Without this convention, there would be two possible strikes for the same plane, separated by 180°. Meanwhile, the dip represents the angle between the plane and the horizontal. To represent a plane on a stereonet, we would plot something like what is shown in Figure 4 on the next page. To understand stereonets, it is extremely helpful to remember that the vantage point is a bird's eye view looking through the ground as if it were transparent. Refer back to Figure 3 as needed throughout this paper.

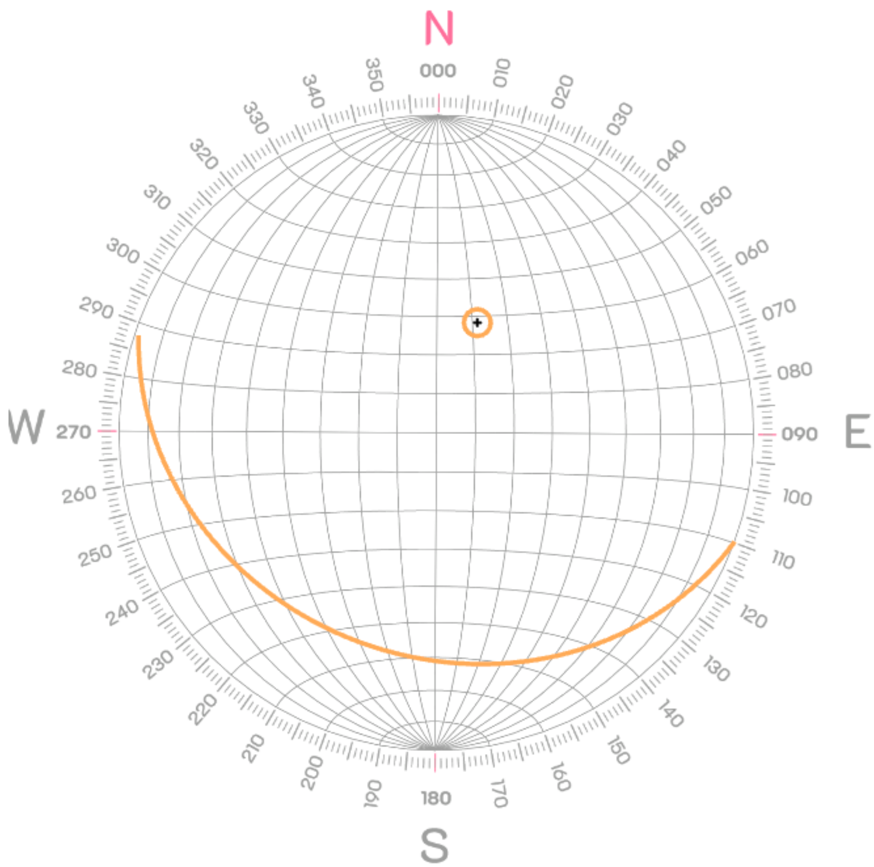

Figure 4: Stereonet plot of a plane with strike 110° and dip 30°.

Following the right-hand-rule convention previously mentioned and identifying the dip direction from the center of the arc (pointing roughly SSW), we read a strike of 110°, rather than 290°, from Figure 4. The dip is more difficult to read. An eyeball approximation measuring the angular separation of the center of the arc from the edge of the stereonet gives 30°, the dip. This is because each gridline is 10°, and a straight line connecting the center of the arc to the edge of the stereonet at 110° + 90° = 200° (the dip direction) would span across 3 of these gridlines. Note that when graphing by hand, we use a setup of tracing paper on top of the grid and plot all planes as if the strike was 0° (and the dip direction was 90°). This allows us to use the gridlines, counting inward from the edge of the grid marked “E” or “090” to plot the dip precisely (see Figure 5 on the next page). After plotting the arc, we rotate the tracing paper clockwise by ϕ so that the arc is at the correct strike. This information will be important later to understand the stereonets presented when we plot multiple fault planes and their intersections. So, how can we go from the strike and dip representation of a plane to something that is more familiar to us?

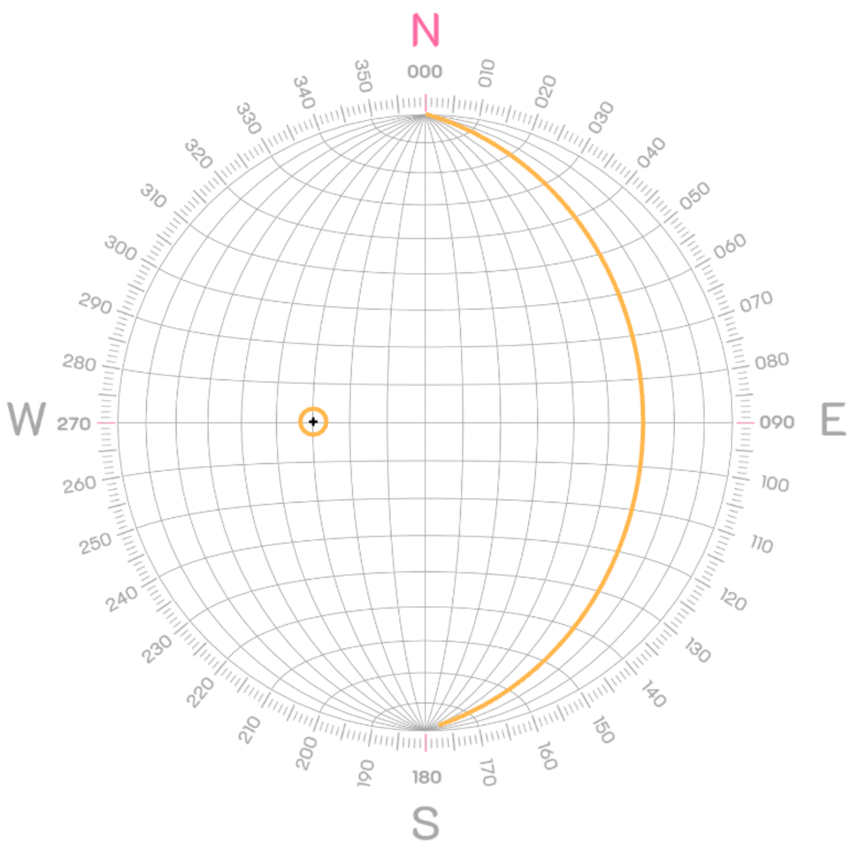

Figure 5: On tracing paper, we would first plot the arc as if the strike was 0°, making it very easy to see the dip of 30° (the center of the arc is separated by 3 gridlines measuring 10° from the eastern edge of the stereonet, so the angular separation of the center of the arc from the edge of the stereonet is 30°). Then, we would rotate the tracing paper about the center of the stereonet through a clockwise angle ϕ, so that the fault plane is in the correct orientation.

<u>Representing a Plane: Normal Vectors</u>

Consider the definition that a *normal vector* of a plane is normal to all vectors lying on the plane. Such a vector can be written in the form $\mathbf{n} = k\langle a, b, c\rangle$ where $k \in \mathbb{R}$ is an arbitrary scalar; that is, all normal vectors of a plane are scalar multiples of each other. Also note that the angular brackets used in this exploration are a compact way to represent vectors with components separated by commas.

Now, let the origin be the point where the strike and dip directions (the two arrows in Figure 3) intersect; and let the $+x$-axis point to the east, the $+y$-axis point to the north, and the $+z$-axis point straight up. Note that the origin and both arrows in Figure 3 lie on the plane. Also note that the line in Figure 3 which is normal to the plane represents the normal vector. If we can find the vectors represented by the arrows in Figure 3, then we can take their cross product to find the normal vector.

The unit vector along the strike direction is horizontal, so its z-component is zero. From the unit circle, we know that the Cartesian coordinates of an angle that is θ *counterclockwise* from the positive x-axis is $(\cos\theta\,,\sin\theta)$. Here, we want the coordinates of an angle that is ϕ *clockwise* from the positive y-axis. We can produce the relation (see Figure 6):

$$\theta = -(\phi - 90°) = 90° - \phi$$

so that the unit vector along the strike direction is

$$\langle \cos(90° - \phi)\,, \sin(90° - \phi)\,, 0\rangle = \langle \sin\phi\,, \cos\phi\,, 0\rangle. \tag{3}$$

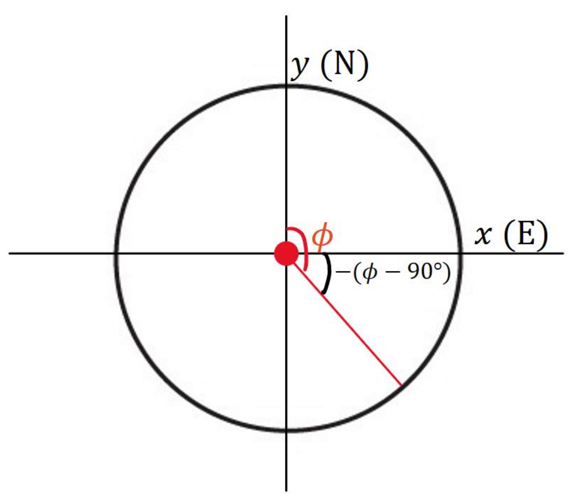

Figure 6: Relation between ϕ and θ.

Now, we will find the unit vector along the dip direction. It should be easy to see why the z-component is simply $-\sin\delta$ (see Figure 7):

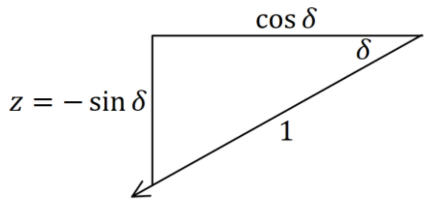

Figure 7: Components of the unit vector in the dip direction.

As for the x- and y-components, we know that the projection of the unit vector along the dip direction onto $z = 0$ (the horizontal plane, see the line above the dip direction arrow in Figure 3) is a 90° clockwise rotation of the unit vector along the strike direction. Performing this rotation on the x- and y-components of the unit vector along the strike direction yields

$$\langle \sin\phi, \cos\phi \rangle \rightarrow \langle \cos\phi, -\sin\phi \rangle$$

(which comes from applying the rotation matrix $\mathbf{R} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$ with $t = -90°$ *counterclockwise*). Scaling both components by $\cos\delta$ (see Figure 7) yields the unit vector along the dip direction,

$$\langle \cos\phi\cos\delta, -\sin\phi\cos\delta, -\sin\delta \rangle. \tag{4}$$

Both the unit vectors along the strike and dip directions (see equations (3) and (4)) are orthogonal to the normal vector. This means that we can take the cross product of these unit vectors to obtain the normal vector:

$$\mathbf{n} = \langle \sin\phi, \cos\phi, 0 \rangle \times \langle \cos\phi\cos\delta, -\sin\phi\cos\delta, -\sin\delta \rangle = \langle -\cos\phi\sin\delta, \sin\phi\sin\delta, -\cos\delta \rangle \tag{5}$$

<u>Intersection of Planes: Representing a Line</u>

Now suppose we have two fault planes F_1 and F_2 with

$$\mathbf{n}_1 = \langle -\cos\phi_1\sin\delta_1, \sin\phi_1\sin\delta_1, -\cos\delta_1 \rangle$$
$$\mathbf{n}_2 = \langle -\cos\phi_2\sin\delta_2, \sin\phi_2\sin\delta_2, -\cos\delta_2 \rangle$$

and we want to find the line formed by the intersection of the two fault planes (see Figure 8 on the next page). To do this, we can take the cross-product of the two normal vectors which will yield a vector parallel to the line of intersection. This is because the normal vectors are, by definition, orthogonal to every line in their respective planes, and taking their cross product would yield a vector that is orthogonal to both normal vectors. Hence, this vector must be parallel to both planes, and thus their line of intersection. Proceeding, we have

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{pmatrix} \cos\delta_1 \sin\phi_2 \sin\delta_2 - \cos\delta_2 \sin\phi_1 \sin\delta_1 \\ \cos\delta_1 \cos\phi_2 \sin\delta_2 - \cos\delta_2 \cos\phi_1 \sin\delta_1 \\ \sin\phi_1 \sin\delta_1 \cos\phi_2 \sin\delta_2 - \cos\phi_1 \sin\delta_1 \sin\phi_2 \sin\delta_2 \end{pmatrix}.$$

Note that the last component can be simplified by factoring out $\sin\delta_1 \sin\delta_2$ and applying the angle addition identity for sine on what remains, such that

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{pmatrix} \cos\delta_1 \sin\phi_2 \sin\delta_2 - \cos\delta_2 \sin\phi_1 \sin\delta_1 \\ \cos\delta_1 \cos\phi_2 \sin\delta_2 - \cos\delta_2 \cos\phi_1 \sin\delta_1 \\ \sin\delta_1 \sin\delta_2 \sin(\phi_1 - \phi_2) \end{pmatrix}. \quad (6)$$

Note that the origin (0,0,0) passes through both planes as noted in the previous section and is visible in Figure 3. Hence, it also passes through the line of intersection. Then we can represent the line of intersection in vector form as

$$\mathbf{r}(\mathrm{t}) = t(\mathbf{n}_1 \times \mathbf{n}_2) \quad (7)$$

where $\mathbf{r}$ is the position vector associated with the scalar t.

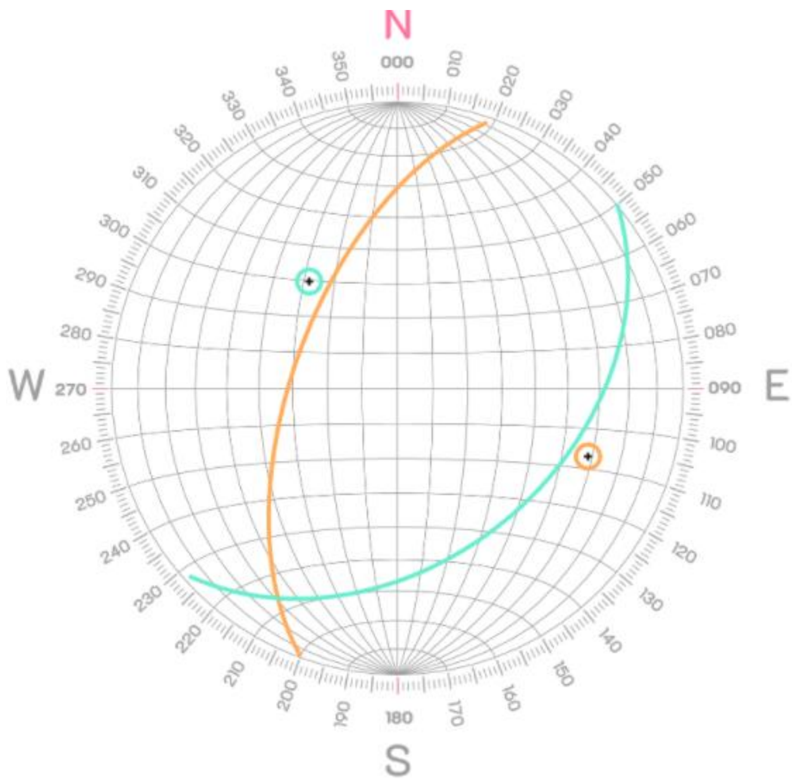

Figure 8: F_1 and F_2 with ϕ_1 = 50°, δ_1 = 40°, ϕ_2 = 200°, δ_2 = 60°.

On the stereonet, note that the line of intersection is represented by the point at the intersection of the two arcs representing the planes. This is because the dot is the projection of the downward plunging line onto the horizontal surface below it (see Figure 9). In fact, the circled crosses are also representations of lines—in particular, they are the projections of the normal vectors of each plane onto the stereonet. When plotted on the stereonet, the normal vectors are called poles, and unsurprisingly, they always have an angular separation of 90° from their corresponding planes.

Like with planes, we can also describe lines using two angles, except this time they are called *trend* (θ) and *plunge* (λ) which are analogous to strike and dip, respectively. The trend describes the orientation of the line as measured clockwise from north, while the plunge measures the angle of inclination from the horizontal.

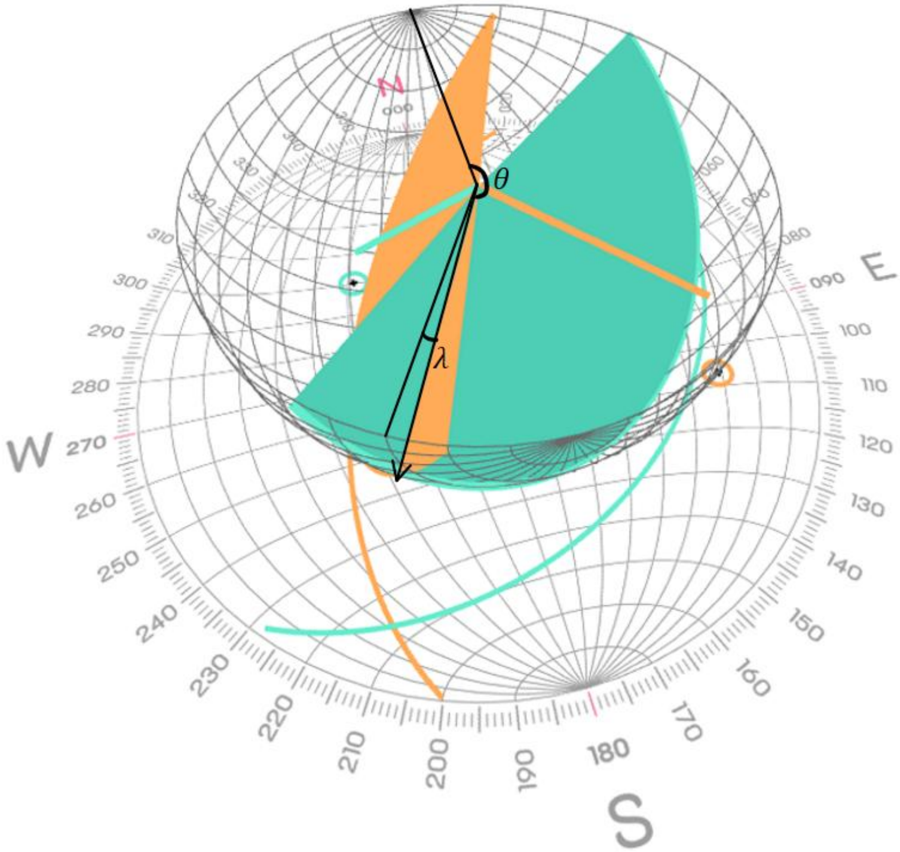

Figure 9: Same set of planes visualized from above. Arrow represents the line of intersection.

The process of finding the trend and plunge from a stereonet is surprisingly simple: θ can be read directly off the edge of the stereonet, while λ can be found by rotating the tracing paper so that the grid markings can be used to read off the angular distance from the edge to the point representing the line of intersection on the stereonet. For the example presented in Figures 8-9, we have $\theta \approx 210°$ and $\lambda \approx 16°$ just by using this method. But how does equation (7) relate to this simple representation, and is there a way to relate the strike and dip of each plane to the trend and plunge of their line of intersection?

Relating Trend and Plunge with Strike and Dip

Suppose we have a line with trend θ and plunge λ formed by the intersection of two planes F_1 and F_2 with strikes ϕ_1, ϕ_2 and dips δ_1, δ_2. We wish to relate θ and λ to the quantities $\phi_1, \phi_2, \delta_1, \delta_2$. Suppose we have the equation for a line that passes through the origin,

$$\mathbf{r}(t) = t\mathbf{v}$$

for which we will find θ and λ. Note that we already know that $\mathbf{v} = \mathbf{n_1} \times \mathbf{n_2}$ from equation (7), but to simplify our notation we will consider $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ where each component matches that in equation (6).

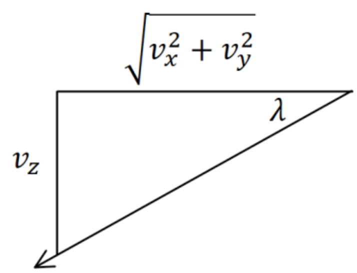

Figure 10: Right triangle relation to find λ.

We find it convenient to solve for λ first using the visualization in Figure 10, representing the line of intersection as viewed edge-on with the horizontal. The top segment is in the plane of the horizontal, the tail of the arrow lies at the origin, and the plane is rotated through an angle θ clockwise from N, as defined by the trend. Hence, we can relate:

$$\lambda = \arctan\left(\frac{-v_z}{\sqrt{v_x^2 + v_y^2}}\right) \tag{9}$$

Solving for θ is a little more complicated. We consider the horizontal plane and components v_x, v_y to describe the trend:

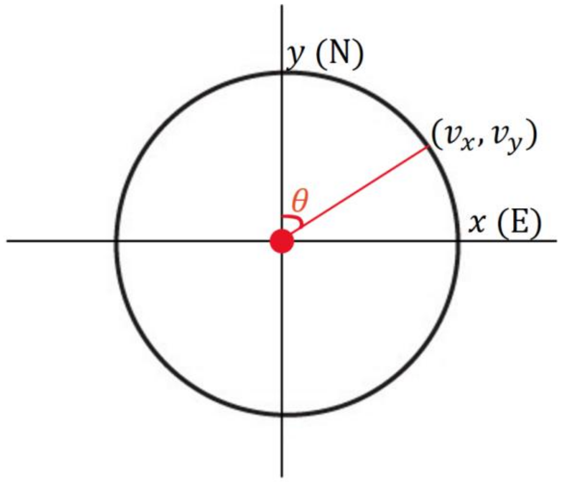

Figure 11: Consider a bird's eye view (from the $+z$ axis) of the horizontal plane to find θ.

At first glance, we might notice in Figure 11 that $\theta = \arctan(v_x/v_y)$ and be tempted to generalize this to all $\theta \in [0°, 360°)$, but after careful consideration we realize that we must account for the range of the $\arctan()$ function. Still, it is simple to determine expressions for θ in each of the following cases:

v_x	v_y	θ	
Positive	Positive	$\arctan(v_x/v_y)$	(10)
Positive	Negative	$180° - \arctan(-v_x/v_y)$	(11)
Negative	Negative	$180° + \arctan(v_x/v_y)$	(12)
Negative	Positive	$360° - \arctan(-v_x/v_y)$	(13)

Table 1: Expressions for θ corresponding to each quadrant, determined by signs of v_x, v_y.

Now that we have expressions relating θ and λ to the components of $\mathbf{v}$, we are essentially done. Note that we could substitute the components from equation (6) into equations (9)-(13) but doing so would be unnecessarily complicated. Rather, it is easy to first substitute values for $\phi_1, \delta_1, \phi_2, \delta_2$ into equation (6) and then apply equations (9)-(13) as needed. In the case of fault planes F_1 and F_2 with $\phi_1 = 50°$, $\delta_1 = 40°$, $\phi_2 = 200°$, $\delta_2 = 60°$, these substitutions yield

$$\mathbf{v} = \langle -0.473, -0.829, -0.278 \rangle,$$

and applying equations (9) and (12) (noting that both v_x and v_y are negative) yield the exact values (to one decimal place) of $\lambda = 16.2°$ and $\theta = 209.7°$ which are consistent with the graphical stereonet estimations of the plunge (16°) and trend (210°) made on page 10.

Conclusion

Through this exploration we have connected the graphical representations of planes and lines using a stereonet to their more familiar vector representations. Doing so has allowed me to appreciate the link between pen-and-paper tools that geologists use and their mathematical basis, revealing the insights behind the techniques I originally learned for Science Olympiad. This application has practical uses as well, and I have synthesized my findings into a Microsoft Excel spreadsheet on GitHub that can calculate the normal vector of a plane given the strike and dip, and the trend and plunge of the line of intersection formed by any two planes.[1] However, I understand that there are a number of other useful concepts in geologic mapping that are tested in Science Olympiad, such as apparent and true dip, the geometry of folds, topographic maps, and many more which I have not treated in my exploration. Perhaps my future pursuits could lead me to explore these concepts more deeply.

A major practical limitation of my exploration is that fault surfaces often have nonconstant dip; that is, they might be curved. Take the following fault system that can be commonly found in places where the crust is undergoing extension, as exemplified by the Basin and Range province in the Southwestern United States (see Figure 12 on next page):

[1] Available at https://github.com/rivak7/fault-modeling.

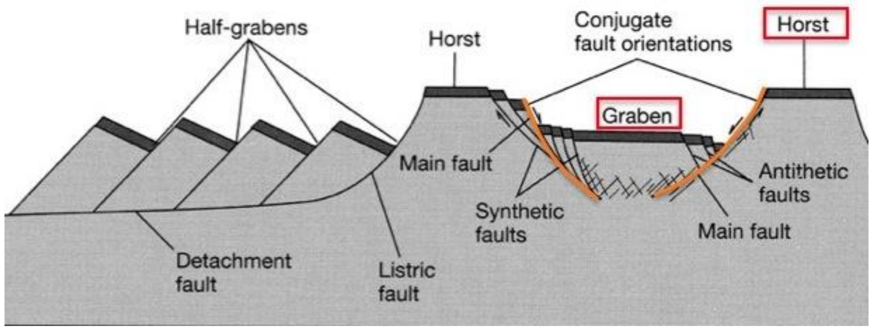

Figure 12: The curved surface of the listric/detachment fault indicates a changing dip. Ignore the terminology in the labels; defining them is not necessary for our purpose. [3]

The fault surface is no longer planar, so it would be difficult to represent using the model we have previously developed. However, if we can construct some function $\delta(d)$ that relates the dip δ with the depth d, then we can use it to find the tangent plane to the fault surface at a particular depth by differentiating, $\delta_{\text{tangent}} = \delta'(d)$. Then if we wanted to, say, find the line of intersection of a half-graben with the detachment fault (both labeled in Figure 12), we could use the same previously established framework to do so.

Beyond Science Olympiad, visualizing fault planes and their intersections can have important implications in seismology. Specifically, the geometric arrangement of fault planes is often a small part of complex numerical simulations that incorporate mathematics beyond the scope of this course to model extensional, compressional, and shear forces in Earth's crust and the resulting strain in the rock. They can thus be used to predict earthquake events wherein these forces cause sudden rupture in a fault, releasing elastically stored energy. A subsequent exploration might consider how a simulation might use numerical integration methods to estimate the energy released in an earthquake, and it would be interesting to investigate how such methods could be applied to a hypothetical earthquake on the Cascadia Subduction Zone, located only a few hundred kilometers away from where I live.

References

[1] B. C. Burchfiel and C. Studnicki-Gizbert. *12.113 Structural Geology*. Fall 2005. Massachusetts Institute of Technology: MIT OpenCourseWare. License: Creative Commons BY-NC-SA. Accessed Nov. 28, 2024. [Online]. Available: https://ocw.mit.edu/courses/12-113-structural-geology-fall-2005/51dd383015d21972508194ae0c8fe491_lab3_stereonets.pdf

[2] *Visible Geology*.[2] (2024). Seequent, The Bentley Subsurface Company. Accessed: Nov. 28, 2024. [Online]. Available: https://www.visiblegeology.com/

[3] F. DeCourten, *The Broken Land: Adventures in Great Basin Geology.* Salt Lake City, UT: University of Utah Press, 2003.

[2] All figures with stereonets were produced using visiblegeology.com and photographically edited for clarity.