Denotational semantics for stabiliser quantum programs

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Abstract

The stabiliser fragment of quantum theory is a foundational building block for quantum error correction and the fault-tolerant compilation of quantum programs. In this article, we develop a sound, universal and complete denotational semantics for stabiliser operations which include measurement, classically-controlled Pauli operators, and affine classical operations, in which quantum error-correcting codes are first-class objects. The operations are interpreted as certain *affine relations* over finite fields. This offers an algebraically and simpler alternative to the standard operator-algebraic semantics of quantum programs, whose time complexity grows exponentially as the state space increases in size, which moreover requires arithmetic over the complex numbers.

We demonstrate the power of the resulting semantics by describing a small, proof-of-concept assembly language for stabiliser programs with fully-abstract denotational semantics.

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1 Introduction

The problem of compiling quantum algorithms into fault-tolerant hardware-level instructions is a central challenge in the design of scalable quantum systems [11, 5, 34]. To this end, quantum error-correcting codes play a central role, where stabiliser codes are the most common and well-studied quantum error correction codes [25]. For fault-tolerant compilation to scale, we need a better understanding of the compositional structure of fault-tolerance, and therefore of the stabiliser fragment. Unlike general quantum programs, stabiliser quantum programs can be simulated efficiently on a probabilistic classical computer [2]. Despite this fact, the formal denotational semantics of stabiliser quantum programs has not been thoroughly studied.

In this article, we develop a nondeterministic denotational semantics for quantum programs built from stabiliser operations, including Clifford operators, Pauli errors, Pauli measurement, affine classical operations and classically-controlled Pauli operators. Our finely tuned denotation semantics for stabiliser quantum programs is to be contrasted with the usual, much larger denotational semantics of non-stabiliser quantum programs in terms of quantum channels. Our work draws from two lines of research: the categorical semantics of quantum programming languages and quantum computing [42, 41, 40, 30]; and the symplectic representation of pure stabiliser circuits [27, 33, 29, 16, 8]. Ultimately, these results constitue the first step towards the development of formally verified *fault-tolerant* quantum compilation frameworks, integrating current approaches to compilation [19, 18, 28, 36] and verification [39, 13, 31, 46, 44, 21, 35, 38].

The categorical semantics of quantum theory builds on the mathematical semantics of finite-dimensional quantum processes with measurement and classically-controlled Pauli operators and affine classical operations. These semantics can be formally stated in the language of operator algebras [9], and are built in three stages of increasing expressivity:

1. *Pure quantum mechanics* via finite-dimensional Hilbert spaces;
2. *Mixed quantum mechanics* via completely-positive maps between matrix algebras;
3. *Quantum measurements and classical control* via completely-positive maps between finite-dimensional C^*-algebras.

These increasing stages of expressivity can be restated by applying the following functorial constructions to the †-compact-closed category, FHilb, of finite-dimensional Hilbert spaces and linear maps:

$$\textit{pure}\text{ QM} \xrightarrow{\text{CPM construction [40]}} \textit{mixed}\text{ QM} \xrightarrow{\text{Splitting †-idempotents [41]}} \text{QM w/ } \textit{measurements}$$

Finite-dimensional quantum mechanics can therefore be understood in purely categorical terms, agnostic to the theory of operator algebras. This point of view is highly amenable to generalisation and specialisation: simply replace FHilb with any other †-compact-closed category, and apply these constructions to add abstract notions of mixing and measurement.

In this article, we work with †-compact-closed categories specifically tailored to the stabiliser fragment. The first semantics is obtained directly by restricting FHilb to the stabiliser fragment; whereas, the second semantics is given by the symplectic representation of stabiliser maps. Specifically, we work throughout with odd-prime-dimensional quantum systems, which ensures that the symplectic representation is well-behaved. Whilst the full symplectic semantics breaks down in even characteristic, we can nevertheless recover the theory of CSS codes in the qubit case [10, 43, 15, 32].

Outline. Section 2 recalls the basic theory of symmetric monoidal categories and †-compact closed examples. Section 3 begins with a review of the stabiliser formalism and its symplectic formulation. We then describe novel denotational semantics for mixing in section 4 and measurement in section 5. In each section, we develop the standard operator-theoretic semantics given by restriction, and the corresponding symplectic representation, proving their equivalence:

$$\begin{tikzcd}
\text{pure stabiliser theory} \arrow[r, leftrightarrow, "{[33, \text{ Chapter 9}],[16],(\text{Section 3})}"] \arrow[d, "\text{CPM construction}"'] & \text{affine Lagrangian relations} \arrow[d, "\text{CPM construction}"] \\
\text{mixed stabiliser theory} \arrow[r, leftrightarrow, "\text{Section 4}"] \arrow[d, "\text{Splitting †-idempotents}"'] & \text{affine coisotropic relations} \arrow[d, "\text{Splitting †-idempotents}"] \\
\begin{array}{c}\text{stabiliser theory}\\ \text{with measurements}\end{array} \arrow[r, leftrightarrow, "\text{Section 5}"] & \begin{array}{c}\text{affine relations}\\ \text{with symplectic types}\end{array}
\end{tikzcd}$$

Finally, in section 6, we define a simple imperative language for stabiliser quantum programs, including Pauli measurement, affine classical operations, and classically-controlled Pauli operators, equipped with a fully abstract denotational semantics derived from section 5. We finish this section, conjecturing that this extends to a fully abstract denotational semantics up to the possibily of measurement outcomes.

Contributions. We present several novel contributions:

- Corollary 41: a symplectic, relational semantics for completely positive stabiliser maps;
- Theorem 57: we model stabiliser quantum measurements and classical control as affine relations augmented with a modality to represent quantum data;
- Propositions 44,58: we prove that the physically-realisable stabiliser programs, i.e. *stabiliser quantum channels*, are represented by the total relations;
- Theorem 67: we interpret a toy programming language in this relational semantics, and prove full abstraction.

We construct finely-tuned, yet equivalent, categories of relations which offer an algebraically simpler and computationaly tractable alternative to the standard operator-theoretic semantics, whilst supporting concrete computational tools native to the stabiliser formalism and stabiliser quantum error correction.

Notation. Throughout, p denotes an *odd* prime, so that $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ is the field of integers modulo p. Let FHilb denote the category of finite-dimensional Hilbert spaces and linear maps. The inner product is denoted by $\langle - | = \rangle$, the outer product by $|-\rangle\langle=|$, vectors by $|\varphi\rangle$, and their Hermitian adjoints by $\langle\varphi| := |\varphi\rangle^\dagger$. Denote the internal hom of linear endomorphisms on a finite-dimensional Hilbert space $\mathcal{H}$ by $\mathcal{B}(\mathcal{H}) \cong \mathcal{H}^* \otimes \mathcal{H}$.

Given a symmetric monoidal category $\mathcal{C}$ whose monoid of scalars $\mathcal{C}(I, I)$ is a group, let $\mathrm{Proj}(\mathcal{C})$ denote the symmetric monoidal category obtained by quotienting by invertible scalars.

2 Preliminaries: symmetric monoidal and †-compact closed categories

In this section, we review the basic theory of symmetric monoidal categories and †-compact closed categories. These will serve as the mathematical structures with which we allow us to formally model theories of *processes*; allowing us to formally state our denotational semantics.

2.1 Symmetric monoidal categories

We begin by recalling the definition of a symmetric monoidal category. These describe theories of processes which can be composed in sequence and in parallel, where moreover systems can be freely exchanged with each other:

▶ **Definition 1.** *A* ***symmetric monoidal category*** *(SMC) is a tuple* $(\mathcal{C}, \otimes, I, \alpha, u^L, u^R, \mathsf{swap})$ *where:*

- *$\mathcal{C}$ is a category. In other words, this is a collection objects A, B, C, between which there are morphisms $f : A \to B, g : C \to D, \cdots$ which can be composed $g \circ f : A \to C$, where there is an identity for composition $f \circ 1_A = 1_B \circ f$;*
- *$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, called the* ***monoidal product****;*
- *$I \in \mathcal{C}$ is an object of $\mathcal{C}$, called the* ***monoidal unit****;*

moreover, the other data consists of natural isomorphisms:

- *$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ called the* ***associator****;*
- *$u^L_A : I \otimes A \to A$ and $u^R_A : A \otimes I \to A$ called the* ***unitors****;*
- *$\mathsf{swap}_{A,B} : A \otimes B \to B \otimes A$ called the* ***symmetry****.*

These natural isomorphisms satisfy the usual coherence conditions: associativity, unitality, where swap *sastisfies the Yang-Baxter equation, and is self-inverse.*

▶ **Definition 2.** *Given symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$* ***symmetric monoidal functor*** *from $\mathcal{C}$ to $\mathcal{D}$ is a tuple (F, φ, φ_I) consisting of:*

- *a functor $F : \mathcal{C} \to \mathcal{D}$;*
- *natural isomorphisms $\varphi_{A,B} : FA \otimes_{\mathcal{D}} FB \to F(A \otimes_{\mathcal{C}} B)$ natural in A, B;*
- *a morphism $\varphi_I : I_{\mathcal{D}} \to F(I_{\mathcal{C}})$ which is an isomorphism.*

These data satisfy the standard coherence conditions: compatibility with associators, unitors, and symmetry. A ***symmetric monoidal equivalence*** *is a symmetric monoidal functor whose underlying functor is an equivalence of categories.*

The following two examples of symmetric monoidal categories will be used in our paper. The first example is the first approximation of a semantics for quantum processes:

▶ **Example 3.** The symmetric monoidal category FHilb of **finite-dimensional Hilbert spaces and linear maps** has:

- **objects:** finite-dimensional Hilbert spaces $\mathcal{H}$, i.e. complex vector spaces equipped with an inner product $\langle - | = \rangle_{\mathcal{H}} : \overline{\mathcal{H}} \times \mathcal{H} \to \mathbb{C}$ where $\overline{\mathcal{H}}$ is the conjugate vector space.
- **morphisms:** linear maps $\mathcal{H} \to \mathcal{K}$;
- **identities:** $1_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$;
- **composition:** composition of linear maps;
- **monoidal product:** the bilinear tensor product $\mathcal{H} \otimes \mathcal{K}$;
- **monoidal unit:** $\mathbb{C}$;
- **symmetry:** for basis vectors $|h\rangle \in \mathcal{H}$, $|k\rangle \in \mathcal{K}$, $\mathsf{swap}_{\mathcal{H},\mathcal{K}}(|h\rangle \otimes |k\rangle) := |k\rangle \otimes |h\rangle$. Note that this not depend on the chosen basis.

Whereas, the second serves as a semantics for nondeterministic processes:

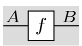
▶ **Example 4.** The symmetric monoidal category FRel of **finite sets and relations**:

- **objects:** finite sets;
- **morphisms:** relations $R \subseteq A \times B$;
- **identities:** diagonal relations $1_A := \{(a, a) \mid a \in A\}$;
- **composition:** for $R \subseteq A \times B$ and $S \subseteq B \times C$, the relational composite is given by:

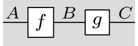
$$S \circ R := \{(a, c) \mid \exists b \in B \text{ with } (a, b) \in R \text{ and } (b, c) \in S\} \subseteq A \times C;$$

- **monoidal product:** Cartesian product $A \times B$;
- **monoidal unit:** the singleton set $\{*\}$;
- **symmetry:** $\mathsf{swap}_{A,B} := \{((a, b), (b, a)) \mid a \in A, b \in B\}$.

▶ **Definition 5.** *A symmetric monoidal category is* ***strict*** *when for all objects A, B, C: $\alpha_{A,B,C} = 1, u^L_A = 1$ and $u^R_A = 1$. In other words, in this setting we do not need to care about the bracketing of objects.*

Strict symmetric monoidal categories admit a graphical representation using **string diagrams**. Objects are drawn as wires and morphisms as boxes. So for example a morphism $f : A \to B$ is drawn as . We think of the wires as the *systems*, where the boxes are *processes* acting on the systems on the left to produce systems on the right.

For any maps $f : A \to B$, $g : B \to C$, and $h : C \to D$, the composite $g \circ f$ is drawn by horizontal horizontal pasting whereas, the monoidal product $f \otimes g$ is drawn by vertical stacking



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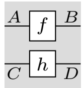

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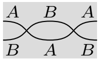
The symmetry $\mathsf{swap}_{A,B}$ is drawn by crossing wires: . The coherence conditions are reflected by the following isotopies, so that for all maps $f : A \to B$ and $g : C \to D$:



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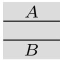

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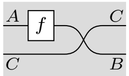

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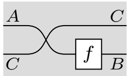

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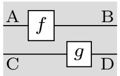

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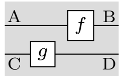

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Given any symmetric monoidal category, we can always work with string diagrams:

▶ **Theorem 6.** *Every symmetric monoidal category is symmetric monoidally equivalent to a strict symmetric monoidal category.*

Our two running examples are equivalent as symmetric monoidal categories to two well-known strict symmetric monoidal categories:

▶ **Example 7.** FHilb is symmetric monoidally equivalent to the category of complex matrices $\mathsf{Mat}_{\mathbb{C}}$, where the symmetric monoidal structure is given by the Kronecker product.

▶ **Example 8.** FRel is symmetric monoidally equivalent to the category of Boolean matrices $\mathsf{Mat}_{\mathbb{B}}$, where the symmetric monoidal structure is given by the Kronecker product.

2.2 †-symmetric monoidal categories

In the previous subsection, we reviewed the theory of symmetric monoidal categories: which model processes which can be composed in sequence, in reverse, and where sytems can be freely exchanged with each other. In this section, we add additional structure which allows for proces to be reversed:

▶ **Definition 9.** *A* ***†-symmetric monoidal category*** *(†-SMC) is a symmetric monoidal category equipped with a functor $(-)^\dagger : \mathcal{C}^{\mathsf{op}} \to \mathcal{C}$ that is identity on objects and satisfies*

$$(f^\dagger)^\dagger = f, \qquad (g \circ f)^\dagger = f^\dagger \circ g^\dagger, \quad \textit{and} \quad (f \otimes g)^\dagger = f^\dagger \otimes g^\dagger.$$

We additionally require the coherence maps are unitary, so that:

$$\alpha^\dagger = \alpha^{-1}, \qquad (u^L_A)^\dagger = (u^L_A)^{-1}, \qquad (u^R_A)^\dagger = (u^R_A)^{-1}, \quad \textit{and} \quad \mathsf{swap}^\dagger = \mathsf{swap}^{-1} = \mathsf{swap}.$$

A ***strict †-symmetric monoidal category*** *is one whose underlying symmetric monoidal category is strict.*

Both of our running examples are †-symmetric monoidal categories:

▶ **Example 10.** FHilb is a †-compact closed category with respect to the Hermitian adjoint. Concretely give a linear map $f : \mathcal{H} \to \mathcal{K}$, the Hermitian adjoint $f^\dagger : \mathcal{K} \to \mathcal{H}$ is the unique map such that for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$, $\langle fx|y\rangle_{\mathcal{K}} = \langle x|f^\dagger y\rangle_{\mathcal{H}}$.

▶ **Example 11.** FRel is a †-compact closed category with respect to $R^\dagger := \{(b,a) \mid (a,b) \in R\}$..

▶ **Definition 12.** *A functor between †-symmetric monoidal categories is a symmetric monoidal functor satisfying $F(f^\dagger) = F(f)^\dagger$. A †-symmetric monoidal equivalence is a symmetric monoidal equivalence preserving and reflecting the dagger.*

▶ **Example 13.** There is an equivalence of †-symmetric monoidal categories $\mathsf{FHilb} \simeq \mathrm{M}_{n\times m}(\mathbb{F}_p)_{\mathbb{C}}$, where the Hermitian adjoint is transported to the complex conjugate transpose.
There is an equivalence of †-symmetric monoidal categories $\mathsf{FRel} \simeq \mathrm{M}_{n\times m}(\mathbb{F}_p)_{\mathbb{B}}$, where relational converse is transported to transpose.

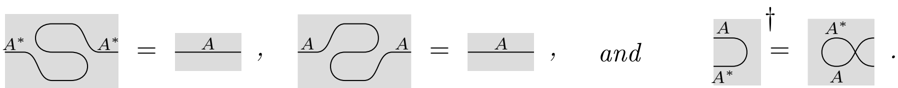
2.3 †-compact closed categories

In the previous subsection, we reviewed the theory of †-symmetric monoidal categories, modelling processes which can be composed in parallel, in sequence, where systems can be freely exchanged with each other, and where processes can be reversed. In this section, we as for additional structure which accommodates a notion of feedback, so that the outputs of proceses can be turned into inputs and vice-versa:

▶ **Definition 14.** *A †-symmetric monoidal category is **†-compact closed** (†-CCC) when every object A has a dual object A^* and morphisms $\eta_A : I \to A^* \otimes A$ and $\varepsilon_A : A \otimes A^* \to I$ called the **cup** and **cap** satisfying*

$$(1_A \otimes \varepsilon_A) \circ (\eta_A \otimes 1_A) = 1_A, \qquad (\varepsilon_A \otimes 1_{A^*}) \circ (1_{A^*} \otimes \eta_A) = 1_{A^*}, \quad \text{and} \quad \varepsilon_A^\dagger = \eta_A \circ \mathsf{swap}_{A^*,A}.$$

*A **strict †-compact closed category** is a †-compact closed category whose underlying symmetric monoidal category is strict. In string diagrams, we represent η_A by bending the wire into a cup and ε_A into cap, where the coherence conditions become the following isotopies:*



Note that a †-symmetric monoidal functor automatically preserves †-compact closed structure.

▶ **Example 15.** In FHilb, the dual is given on objects $\mathcal{H}$ by the internal hom $\mathcal{H}^* := \mathsf{FHilb}(\mathcal{H}, \mathbb{C})$ of linear functions from $\mathcal{H}$ into $\mathbb{C}$. Given an orthonormal basis $\{|h_i\rangle\}_i$ of $\mathcal{H}$, the inner product induces an orthonormal basis $\{\langle h_i|\}_i$ of $\mathcal{H}$. The cup $\eta_{\mathcal{H}}$ is given by:

$$\eta_{\mathcal{H}}(1) := \sum_i \langle h_i| \otimes |h_i\rangle$$

Note this doesn't depend on the choice of basis.

▶ **Example 16.** In FRel, the dual is trivial $A^* = A$. The cup is given by the diagonal relation:

$$\eta_A := \{(*, (a, a)) \mid a \in A\}$$

3 Preliminaries: the stabiliser theory

We review the elements of the stabiliser theory, and its representation with symplectic linear algebra. Explicitly, we define two †-CCCs for the pure stabiliser theory; one in terms of finite dimensional Hilbert spaces and the other in terms of structure-preserving relations between sets with extra structure:

1. a *concrete* †-CCC, Stab_p, given by restricting FHilb;
2. an *abstract* †-CCC, $\mathsf{AffLagRel}_{\mathbb{F}_p}$, described in terms of symplectic linear algebra.

These two †-CCCss are known to be equivalent up to nonzero scalars [33, 16]. We will take $\mathsf{AffLagRel}_{\mathbb{F}_p}$ to serve as the basis from which we build our abstract denotational semantics.

3.1 The Hilbert space picture

Consider the p-dimensional complex vector space $\mathcal{H}_p := \mathbb{C}^p$, equipped with the canonical orthonormal basis $\{|x\rangle \mid x \in \mathbb{F}_p\}$. $\mathcal{H}_p$ models the state of a quantum system called a **qupit**.

▶ **Definition 17.** *Let $\chi(x) := \exp(i2\pi x/p)$, then the **Pauli operators** on $\mathcal{H}_p$ are generated by $Z|x\rangle := \chi(x)|x\rangle$ and $X|x\rangle := |x+1\rangle$ and assemble into the qupit **Pauli group**:*

$$\mathcal{P}_p := \{\chi(y)X^xZ^z \mid x, y, z \in \mathbb{F}_p\} \subseteq \mathsf{U}(\mathcal{H}_p^{\otimes n}).$$

*The n-**qupit Pauli group** is defined to be the n-fold tensor product $\mathcal{P}_p^{\otimes n}$, so that an arbitrary Pauli operator takes the form $\chi(y)\bigotimes_{j=1}^n X_j^{x_j}Z_j^{z_j}$ for some vectors $\mathbf{x}, \mathbf{z} \in \mathbb{F}_p^n$ and $y \in \mathbb{F}_p$.*

▶ **Definition 18.** *The **Clifford group** is the unitary normaliser of $\mathcal{P}_p^{\otimes n}$:*

$$\mathcal{C}\ell_p^n := \{U \in \mathcal{U}(\mathcal{H}_p^{\otimes n}) \mid \forall P \in \mathcal{P}_p^{\otimes n}, UPU^\dagger \in \mathcal{P}_p^{\otimes n}\} \subseteq \mathsf{U}(\mathcal{H}_p^{\otimes n}).$$

▶ **Lemma 19.** *Take a maximal Abelian subgroup $S \subseteq \mathcal{P}_p^{\otimes n}$ such that $\chi(x)1_{\mathcal{H}_p}^{\otimes n} \in S$ if and only if $x = 0$. Then S determines a normalised quantum state up to a global phase $\exp(2\pi i\theta)$ where $\theta \in [0,1)$, as the unique state $|S\rangle \in \mathcal{H}_p^{\otimes n}$ such that $s|S\rangle = |S\rangle$ for all $s \in S$.*

*The equivalence class $[\exp(2\pi i\theta)|S\rangle]_{\theta\in[0,1)}$ is called the **stabiliser state** associated to the **stabiliser group** S.*

Consider a stabiliser group $S \subseteq \mathcal{P}_p^{\otimes n}$, then for any $C \in \mathcal{C}\ell_p^n$ and $s \in S$, we have that $CsC^\dagger \cdot C|S\rangle = Cs|S\rangle = C|S\rangle$. It follows that the stabiliser group $CSC^\dagger = \{CsC^\dagger \mid s \in S\}$ stabilises the state $|CSC^\dagger\rangle = C|S\rangle$. Clifford unitaries therefore map stabiliser states to stabiliser states. Therefore, it is natural to assemble these operations together into a †-CCC:

▶ **Definition 20.** *The †-CCC Stab_p of **qupit stabiliser maps** is the †-compact-closed subcategory of FHilb generated by the qupit stabiliser states and Clifford operators as well as the scalars $1/\sqrt{p}$ and $\sqrt{p}$ under tensor product, composition and the Hermitian adjoint.*

If we did not add the scalars $1/\sqrt{p}$ and $\sqrt{p}$, we would obtain a category whose morphisms are linear contractions. Therefore we add these scalars to obtain a compact closed category as the the compact closed structure increases the operator norm.

3.2 The symplectic picture

We recall how the (pure) qupit stabiliser theory can be restated in purely symplectic terms by taking the notion of a stabiliser group, and their symplectic representation, as fundamental. The following notion will serve as the objects in the "symplectic" category $\mathsf{AffLagRel}_{\mathbb{F}_p}$:

▶ **Definition 21.** *A **symplectic vector space** (V, ω_V) is a (finite-dimensional) $\mathbb{F}_p$-vector space V equipped with a non-degenerate, alternating, bilinear form $\omega_V : V \oplus V \to \mathbb{F}_p$.*

One can always choose the following concrete symplectic form:

▶ **Example 22** (Standard symplectic form)**.** Given any $n \in \mathbb{N}$, $(\mathbb{F}_p^n \oplus \mathbb{F}_p^n, \omega_n)$ is a symplectic vector space where $\omega_n\big((\mathbf{x},\mathbf{z}),(\mathbf{a},\mathbf{b})\big) := \mathbf{x}\cdot\mathbf{b} - \mathbf{z}\cdot\mathbf{a}$.

The Pauli group $\mathcal{P}_p^{\otimes n}$ is a central extension of the Abelian group $\mathbb{F}_p^n \oplus \mathbb{F}_p^n$ by $\mathbb{F}_p$ [29, § 6.2.3]. This means that there is a function which parametrises Pauli operators

$$\pi : \mathbb{F}_p \oplus \mathbb{F}_p^n \oplus \mathbb{F}_p^n \longrightarrow \mathcal{P}_p^{\otimes n} : (y, \mathbf{x}, \mathbf{z}) \longmapsto \chi(y)\chi(2^{-1}\mathbf{x}\cdot\mathbf{z}) \bigotimes_{k=1}^n X_k^{x_k} Z_k^{z_k}, \tag{1}$$

chosen such that $\pi(y, \mathbf{x}, \mathbf{z})\pi(c, \mathbf{a}, \mathbf{b}) = \pi(y + c + 2^{-1}\mathbf{x}\cdot\mathbf{a}, \mathbf{x}+\mathbf{a}, \mathbf{z}+\mathbf{b})$. This allows us to work with Pauli operators purely in terms of symplectic data; for example:

▶ **Lemma 23.** *Two Paulis $\pi(y, \mathbf{x}, \mathbf{z})$ and $\pi(c, \mathbf{a}, \mathbf{b})$ commute if and only if $\omega_n((\mathbf{x}, \mathbf{z}), (\mathbf{a}, \mathbf{b})) = 0$.*

Recall that the commutation of Paulis was needed to define stabiliser groups. Therefore, the representation π allows us to define stabiliser groups purely at the symplectic level:

▶ **Definition 24.** *Given a linear subspace S of a symplectic vector space (X, ω), its* ***symplectic complement*** *is $S^\omega := \{\mathbf{x} \in X \mid \forall \mathbf{s} \in S, \omega_n(\mathbf{x}, \mathbf{s}) = 0\}$. The subspace S is:*
- ***isotropic*** *if $S \subseteq S^\omega$;*
- ***coisotropic*** *if $S^\omega \subseteq S$;*
- ***Lagrangian*** *if it is maximally isotropic, i.e. $S = S^\omega$.*

An affine subspace is isotropic, coisotropic, or Lagrangian in case its linear part[1] is. By convention, empty subspaces are affine Lagrangian.

The isotropic condition on a subspace is equivalent to the commutation of the corresponding Pauli operators. The additional Lagrangian condition imposes that these subgroups must be maximal. The affine component accounts for the phase factor $\chi(a)$.

▶ **Proposition 25** ([27])**.** *The mapping (1) lifts to a bijection between the set of non-empty affine Lagrangian subspaces of $(\mathbb{F}_p^n \oplus \mathbb{F}_p^n, \omega_n)$ and the set of stabiliser subgroups of $\mathcal{P}_p^{\otimes n}$ (i.e., the set of stabiliser states of n qupits up to a global phase $\chi(a)$.).*

Symplectic vector spaces admit a notion of basis compatible with the symplectic form:

▶ **Proposition 26.** *Every symplectic vector space (X, ω) admits a basis of the form $\{e_j, f_j \mid 1 \geqslant j \geqslant n\}$ for some $n \in \mathbb{N}$ and for which, for any j, k,*

$$\omega(e_j, e_k) = 0, \quad \omega(f_j, f_k) = 0 \quad \textit{and} \quad \omega(e_j, f_k) = \begin{cases} 1 & \textit{if} \quad j = k; \\ 0 & \textit{otherwise.} \end{cases} \tag{2}$$

Such a basis is called a ***symplectic basis****. Furthermore, if S is an isotropic subspace of X, every basis of S extends to a symplectic basis of X via a symplectic Gram-Schmidt process.*

A symplectic basis for X is equivalent to a structure-preserving isomorphism $X \cong \mathbb{F}_p^n \oplus \mathbb{F}_p^n$:

▶ **Definition 27.** *A* ***symplectomorphism*** *$\varphi : (X, \omega_X) \to (Y, \omega_Y)$ is a linear isomorphism such that for all $\mathbf{x}, \mathbf{y} \in X$, $\omega_Y(\varphi(\mathbf{x}), \varphi(\mathbf{y})) = \omega_X(\mathbf{x}, \mathbf{y})$. An* ***affine symplectomorphism*** *is an affine map $X \to Y$ whose linear part[2] is a symplectomorphism.*

All symplectic vector spaces (X, ω_X) are even-dimensional and symplectomorphic to a standard symplectic vector space $(\mathbb{F}_p^n \oplus \mathbb{F}_p^n, \omega_n)$ for $n := \dim(X)/2$. In particular:

▶ **Proposition 28.** *There is an isomorphism between the Clifford group $\mathcal{C}\ell_p^n$, modulo global phase, and the group of affine symplectomorphisms on $(\mathbb{F}_p^n \oplus \mathbb{F}_p^n, \omega_n)$.*

Affine Lagrangian relations

Affine Lagrangian subspaces are *the* fundamental notion in the symplectic representation:

1. the *graph* of an affine symplectomorphism $\varphi : (X, \omega_X) \to (Y, \omega_Y)$ is an affine Lagrangian subspace $\mathrm{Gr}(\varphi) := \{(\mathbf{x}, \varphi(\mathbf{x})) \mid x \in X \subseteq X \oplus Y\} \subseteq (X \oplus Y, -\omega_X \oplus \omega_Y)$;

[1] The **linear part** of an affine subspace $A \subseteq \mathbb{F}_p^n$ is the subspace of $\mathbb{F}_p^n$ given by $\{\mathbf{x} - \mathbf{y} \mid \mathbf{x}, \mathbf{y} \in A\}$.
[2] φ is **affine** if there is a linear map ℓ such that $\varphi(\mathbf{x}) - \varphi(\mathbf{0}) = \ell(x)$. ℓ is called the **linear part** of φ.

2. the composition of affine symplectomorphisms, and thus Clifford operators, is compatible with the relational composition of their graphs $\mathrm{Gr}(\varphi); \mathrm{Gr}(\psi) = \mathrm{Gr}(\psi \circ \varphi)$;
3. the action of Clifford operators on stabiliser states is compatible with the relational composition of their corresponding affine Lagrangian subspaces.

In other words, following Weinstein [45], we adopt the motto:

Everything is an affine Lagrangian relation!

▶ **Definition 29.** *The* ***category*** $\mathsf{AffLagRel}_{\mathbb{F}_p}$ ***of affine Lagrangian relations*** *has:*

- ***objects****: symplectic vector spaces* (V, ω_V)*;*
- ***morphisms*** $(V, \omega_V) \to (W, \omega_W)$*: affine Lagrangian subspaces of* $(V \oplus W, -\omega_V \oplus \omega_W)$*;*
- ***Identity and composition*** *given by the diagonal relation and relational composition;*
- ***monoidal product****: given by the direct sum* $(V \oplus W, \omega_V \oplus \omega_W)$*;*
- ***monoidal unit****: given by the trivial symplectic vector space* $I := (\mathbb{F}_p^0, 1) \cong (\mathbb{F}_p^0 \oplus \mathbb{F}_p^0, \omega_0)$*;*
- ***dagger****: given by the relational converse.*
- ***†-compact structure****: the dual is given by* $(V, \omega_V)^* := (V, -\omega_V)$ *and the cup is given by the diagonal relation* $\eta_{(V,\omega_V)} := \{(0, (\mathbf{x}, \mathbf{x}))\}$*.*

▶ **Theorem 30** ([16, 33])**.** *There is an essentially surjective and full †-compact-closed functor* $\mathrm{Rel} : \mathsf{Stab}_p \to \mathsf{AffLagRel}_{\mathbb{F}_p}$ *This restricts to a †-compact-closed equivalence when quotienting by invertible scalars:* $\mathrm{Proj}(\mathsf{Stab}_p) \simeq \mathsf{AffLagRel}_{\mathbb{F}_p}$*.*

For convenience, denote $\mathsf{AffLagRel}_{\mathbb{F}_p}(n, m) := \mathsf{AffLagRel}_{\mathbb{F}_p}((\mathbb{F}_p^n \oplus \mathbb{F}_p^n, \omega_n), (\mathbb{F}_p^m \oplus \mathbb{F}_p^m, \omega_m))$, which via theorem 30 and equation (1) are the concrete affine Lagrangian relations representing stabiliser maps $\mathcal{H}_p^{\otimes n} \to \mathcal{H}_p^{\otimes m}$. These concrete relations represent the basic operations of the stabiliser theory. However, the full stabiliser quantum theory is *much richer* than what we have described. For example, *stabiliser codes* are fundamental in quantum error correction, but they are not described by pure states. Similarly, the measurement and classical control of stabiliser circuits, which we have not yet discussed, are essential for error correction.

4 Stabiliser codes and mixed states

The stabiliser theory presented in Section 3 is, from the perspective of quantum computation, fundamentally limited: it is efficiently simulatable on a classical computer [23, 2]. Nevertheless, the algebraic structure of the theory—specifically, the concept of stabiliser subgroups—provides the foundation for quantum error correction (QEC). QEC leverages these elements to encode and manipulate information in a way that supports universal quantum computation, while retaining structural features which permit fault tolerance.

In the stabiliser formalism, one represents potential errors which can occur during computation using Pauli operators.

A **stabiliser code** is a *non-maximal* stabiliser group, i.e. an Abelian subgroup G of $\mathcal{P}_p^{\otimes n}$ such that $\chi(x) \cdot 1_{\mathcal{H}_p^{\otimes n}} \in S$ if and only if $x = 0$. Whereas a maximal stabiliser group G uniquely determines a pure stabiliser state $|G\rangle$—a one-dimensional subspace of the Hilbert space $\mathcal{H}_p^{\otimes n}$—a stabiliser code determines a higher-dimensional subspace, the *codespace*:

$$\mathcal{H}_G := \{|\varphi\rangle \in \mathcal{H}_p^{\otimes n} \mid s\,|\varphi\rangle = |\varphi\rangle \text{ for all } s \in G\} \subseteq \mathcal{H}_p^{\otimes n} \tag{3}$$

Elements of the stabiliser group impose linear constraints on the codespace. Relaxing the number of constraints therefore yields a larger subspace, while still enforcing sufficient symmetry to make it possible to detect Pauli errors.

Semantically, it is natural to view a stabiliser code not just as a subspace but as the completely mixed state on that subspace. Concretely, let $\Pi_G : \mathcal{H}_p^{\otimes n} \to \mathcal{H}_p^{\otimes n}$ denote the projection of $\mathcal{H}_p^{\otimes n}$ onto $\mathcal{H}_S$. The normalised map $\rho_G := \Pi_G / \operatorname{Tr}(\Pi_G)$ is the *mixed state* for the uniform mixed state on $\mathcal{H}_S$. This interpretation allows stabiliser codes to be treated within the framework of mixed-state quantum mechanics, and more importantly, to obtain denotational semantics for stabiliser codes via categorical constructions of mixed state quantum theory.

In this section, we introduce two semantics for mixed stabiliser quantum mechanics by:

1. *restricting* the mixed processes to those built out stabiliser maps;
2. *generalising* the symplectic representation to affine coisotropic relations.

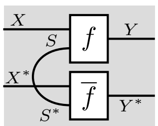
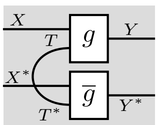
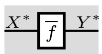
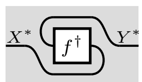
Because the first semantics is given by restriction, it is semantically hard to grapple with. On the other hand, the second semantics is novel, and much more apt to reason about stabiliser codes.

4.1 Completely-positive maps between matrix algebras

Selinger's CPM construction builds a category of mixed processes $\mathrm{CPM}(\mathsf{C})$ out of a †-CCC C by adding a notion of discarding that respects the dagger structure [40]. This plays a similar role to how the Kleisli categories Set_P and Meas_G, respectively over the power-set and Giry monads, are categorical semantics for nondeterministic and probabilistic computations:

▶ **Definition 31** ([40]). *Given a †-CCC* C*, the †-CCC* $\mathrm{CPM}(\mathsf{C})$ *has:*

- ***objects:*** *same as* C.
- ***morphisms*** $[f, S] : X \to Y$***:*** *are equivalence classes of pairs* (f, S)*, where* S *is an object of* C *and* $f : X \otimes S \to Y$ *in* C*, modulo the equivalence relation*

$(f, S) \sim (g, T) \iff$  $=$  *where*  $:=$ .

- ***all other †-compact-closed structure:*** *inherited from* C.

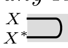
There is a canonical "doubling" functor $\iota : \mathsf{C} \to \mathrm{CPM}(\mathsf{C})$ which sends morphisms $f : X \to Y$ to $(u_X^R; f, I) : X \to Y$. This means that the CPM construction is adding a new morphism that connects both halves of this doubling. For the example of $\mathsf{C} := \mathsf{FHilb}$, this is interpreted as adding the (unnormalized) maximally mixed state:

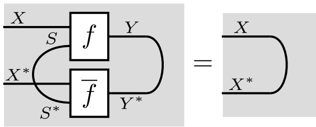
▶ **Theorem 32** ([40, Ex. 4.21]). $\mathrm{CPM}(\mathsf{FHilb})$ *is equivalent to the CCC of* completely-positive *(CP) maps between matrix algebras* $\mathcal{B}(\mathcal{H})$*, for all* $\mathcal{H} \in \mathsf{FHilb}$.

▶ **Corollary 33.** *There is a faithful †-compact functor* $\mathrm{CPM}(\mathsf{Stab}_p) \rightarrowtail \mathrm{CPM}(\mathsf{FHilb})$.

The projection $\Pi_G : \mathcal{H}_p^{\otimes n} \to \mathcal{H}_p^{\otimes n}$ onto the codespace of a stabiliser code G is a state in $\mathrm{CPM}(\mathsf{Stab}_p)$. Moreover, Clifford operators C in $\mathcal{C}\ell_p^n \to \mathsf{Stab}_p \to \mathrm{CPM}(\mathsf{Stab}_p)$ act on these projectors by conjugation $C^\dagger \Pi_G C = \Pi_{C^\dagger GC}$ as expected. In particular, given a stabiliser state $|G\rangle$, we have $C^\dagger |G\rangle\langle G| C = |C^\dagger GC\rangle\langle C^\dagger GC|$.

To restrict physical processes, we impose the additional normalisation constraint.

▶ **Definition 34.** *Given any* $X \in \mathsf{C}$*, let* Tr_X *denote the morphism* $X \to I \in \mathrm{CPM}(\mathsf{C})$ *given by the equivalence class* . *A morphism* $[f, S] : X \to Y$ *in* $\mathrm{CPM}(\mathsf{C})$ *is* ***causal*** *if and only if* $\operatorname{Tr}_Y[f, S] = \operatorname{Tr}_X$*, diagrammatically:*



Denote the symmetric monoidal subcategory of causal morphisms in $\mathrm{CPM}(\mathsf{C})$ *by* $\mathrm{Caus}(\mathrm{CPM}(\mathsf{C}))$.

Concretely, the morphisms $\mathrm{Tr}_{\mathcal{H}}$ in $\mathrm{CPM}(\mathsf{FHilb})$, are given by the linear-algebraic trace in FHilb. In the language of operator algebras:

▶ **Corollary 35.** $\mathrm{Caus}(\mathrm{CPM}(\mathsf{FHilb}))$ *is equivalent to the SMC of completely-positive* trace-preserving *(CPTP) maps between matrix algebras* $\mathcal{B}(\mathcal{H}) \cong \mathcal{H}^* \otimes \mathcal{H}$ *for all* $\mathcal{H} \in \mathsf{FHilb}$.

In other words, these are CP maps which preserve the trace norm, in analogy to how Markov processes preserve the L^1 norm.

▶ **Example 36.** The normalisation $\rho_G = \Pi_G / \mathrm{Tr}(\Pi_G)$ of the projector Π_G onto the code space of a stabiliser code G is completely positive and trace preserving; whereas without the normalisation factor, the projector Π_G is only trace-preserving when $\mathrm{Tr}(\Pi_G) = 1$.

4.2 Stabiliser codes as affine coisotropic relations

In this subsection, we apply the CPM construction to the †-CCC of affine Lagrangian relations, obtaining a relational semantics for $\mathrm{CPM}(\mathsf{Stab}_p)$. In particular, we show that this produces the poset-enriched †-CCC of affine *coisotropic* relations, relaxing the dimensionality requirement for Lagrangian relations.

▶ **Definition 37.** *The †-CCC* $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$ *of* ***affine coisotropic relations*** *has the same structure as* $\mathsf{AffLagRel}_{\mathbb{F}_p}$*, where now the morphisms* $(V, \omega_V) \to (W, \omega_W)$ *are affine coisotropic subspaces of* $(V \oplus W, -\omega_V \oplus \omega_W)$.

It is well-understood that "phaseless" stabiliser codes are in bijection with isotropic subspaces [27], and hence also with coisotropic subspaces via the symplectic complement. However, once Pauli phases are introduced, this second bijection breaks down: phased stabiliser codes correspond exactly to affine coisotropic subspaces but *not* to affine isotropic subspaces. Thus, affine coisotropic relations are the correct algebraic setting for the stabiliser theory with non-maximal stabiliser groups.

▶ **Example 38.** The total subspace can be regarded as an affine coisotropic relation:

$$\mathrm{Im}_{(V,\omega_V)} := \{(0, \mathbf{v}) \mid \forall \mathbf{v} \in V\} : I \to (V, \omega_V) \tag{4}$$

We use the name $\mathrm{Im}_{(V,\omega_V)}$ because postcomposition with an affine Lagrangian, or affine coisotropic relation $R : (V, \omega_v) \to (W, \omega_W)$ is identified with the set-theoretic image:

$$\mathrm{Im}_V ; R = \{(0, \mathbf{w}) \mid \exists \mathbf{v} : (\mathbf{v}, \mathbf{w}) \in R\} = \mathbb{F}_p^0 \oplus \mathrm{Im}(R) \cong \mathrm{Im}(R) \tag{5}$$

Adding the image as a generator to $\mathsf{AffLagRel}_{\mathbb{F}_p}$ yields $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$:

▶ **Proposition 39.** *Every non-empty affine coisotropic subspace of* $\mathbb{F}_p^n \oplus \mathbb{F}_p^n$ *of dimension* $n + m$ *is the image of an affine Lagrangian coisometry* $\mathbb{F}_p^m \oplus \mathbb{F}_p^m \to \mathbb{F}_p^n \oplus \mathbb{F}_p^n$.

Proof. We prove the claim for linear Lagrangian coisometries and coisotropic linear subspaces, after which the affine generalisation follows immediately.

Let S be such a coisotropic subspace so that S^ω is isotropic. Consider a basis of S^ω which by proposition 26 extends to a symplectic basis of (V,ω). This yields a symplectomorphism $\varphi : (V,\omega) \to (\mathbb{F}_p^n \oplus \mathbb{F}_p^n, \omega_n)$ that takes this symplectic basis of V to the standard basis, and such that $\varphi(S^\omega) = \{(\mathbf{x}, \mathbf{0}_{2n-k}) \mid \mathbf{x} \in \mathbb{F}_p^k\}$. Note that the subspace $D := \{(\mathbf{x}, \mathbf{0}_k) \mid x \in \mathbb{F}_p^k\}$ is Lagrangian in $(\mathbb{F}_p^k \oplus \mathbb{F}_p^k, \omega_k)$ so that $\varphi(S^\omega) = D \oplus \{\mathbf{0}_{2(n-k)}\}$. It follows that $C := \mathrm{Gr}(\varphi); (D \oplus 1_{2(n-k)})$ is precisely a Lagrangian relation for which $\ker C = C^{\mathsf{R}}(\{\mathbf{0}_{2(n-k)}\}) = S^\omega$. Then, C is a composition of isometries, and thus an isometry. Then $S = (S^\omega)^\omega = (\ker C)^\omega = \mathrm{im}(C^{\mathsf{R}})$, i.e. S is the image of the Lagrangian coisometry C^{R}. ◀

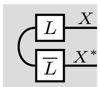
▶ **Theorem 40.** *There is a †-compact isomorphism* $\mathrm{CPM}(\mathsf{LagRel}_{\mathbb{F}_p}) \cong \mathsf{CoisotRel}_{\mathbb{F}_p}$ *sending:*

$$[f,(S,\omega_S)] : (X,\omega_S) \to (Y,\omega_Y) \quad \longmapsto \quad (1_{(X,\omega_X)} \oplus \mathrm{Im}_{(S,\omega_S)}); f : (X,\omega_S) \to (Y,\omega_Y)$$

Proof. We prove the proposition for Lagrangian and coisotropic linear relations, after which the affine extension follows immediately.

This assignment is clearly functorial and identity-on-objects, and preserves the †-compact-closed structure. Moreover, since both $\mathrm{CPM}(\mathsf{LagRel}_{\mathbb{F}_p})$ and $\mathsf{CoisotRel}_{\mathbb{F}_p}$ are compact-closed, it suffices to prove that the states in both categories are in canonical bijection. We already have surjectivity by proposition 39, so that all we need to prove is injectivity.

Given Lagrangian relations $L : S \to X$ and $M : T \to X$ such that $\mathrm{Im}(L) \neq \mathrm{Im}(M)$, then

$$\mathbf{x} \in \mathrm{Im}(L) \quad \text{if and only if} \quad \begin{bmatrix}\mathbf{x}\\ \mathbf{x}\end{bmatrix} \in$$  $$= \left\{ \begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} \;\middle|\; \exists \mathbf{z} : \begin{array}{c}(\mathbf{z},\mathbf{x}) \in L\\ (\mathbf{z},\mathbf{y}) \in \overline{L}\end{array} \right\}. \tag{6}$$

But by assumption there is some $\mathbf{x}$ such that $\mathbf{x} \in \mathrm{Im}(L)$ and $\mathbf{x} \notin \mathrm{Im}(M)$, and it is therefore immediate that $[L,(S,\omega_S)] \neq [M,(T,\omega_T)]$. ◀

▶ **Corollary 41.** *There is a †-compact functor* $\mathrm{Rel} : \mathrm{CPM}(\mathsf{Stab}_p) \to \mathsf{AffCoisotRel}_{\mathbb{F}_p}$*, which restricts to an equivalence when quotienting by scalars* $\mathsf{AffCoisotRel}_{\mathbb{F}_p} \simeq \mathrm{Proj}(\mathrm{CPM}(\mathsf{Stab}_p))$.

Proof. This follows immediately from the equivalence $\mathsf{AffCoisotRel}_{\mathbb{F}_p} \cong \mathrm{CPM}(\mathsf{AffLagRel}_{\mathbb{F}_p}) \simeq \mathrm{CPM}(\mathrm{Proj}(\mathsf{Stab}_p))$ and observing that, in the case of Stab_p, we obtain the same category if we quotient by scalars before or after applying the CPM construction. ◀

To include mixed states and stabiliser codes in our semantics, we update our motto:

Everything is an affine coisotropic relation!

▶ **Example 42.** The following quantum channels are represented in $\mathrm{Rel}(\mathrm{CPM}(\mathsf{Stab}_p))$:
- the *maximally mixed state* by $\mathrm{Im}_{(V,\omega_V)}$;
- the *quantum trace* by its relational converse $\mathrm{Im}^\dagger_{(V,\omega_V)}$;
- the *completely depolarising channel* by $\mathrm{Im}^\dagger_{(V,\omega_V)}; \mathrm{Im}_{(V,\omega_V)}$;
- the *Z-flip channel* by $\mathscr{E}_X := \{((z,x),(z',x)) \mid x,z,z' \in \mathbb{F}_p\}$.

▶ **Definition 43.** *A relation* $R : X \to Y$ *with converse* $R^\dagger$ *is* ***total*** *when* $\mathrm{Im}(R^\dagger) = X$*. Given a category* C *of relations, let* $\mathrm{Total}(\mathsf{C})$ *denote the subcategory of total maps.*

By restricting the mixed stabiliser theory to the trace-preserving maps (the physical processes), the functor $\mathrm{CPM}(\mathsf{Stab}_p) \to \mathsf{AffCoisotRel}_{\mathbb{F}_p}$ restricts to an equivalence on the nose, without quotienting by scalars:

▶ **Proposition 44.** *The functor* $\mathrm{Rel} : \mathrm{CPM}(\mathsf{Stab}_p) \to \mathsf{AffCoisotRel}_{\mathbb{F}_p}$ *restricts to a symmetric monoidal equivalence* $\mathrm{Caus}(\mathrm{CPM}(\mathsf{Stab}_p)) \simeq \mathrm{Total}(\mathsf{AffCoisotRel}_{\mathbb{F}_p})$ *making the following diagram commute:*

$$\begin{tikzcd}
\mathrm{Total}(\mathsf{AffCoisotRel}_{\mathbb{F}_p}) \arrow[rr, tail] \arrow[d, phantom, sloped, "\simeq"] & & \mathsf{AffCoisotRel}_{\mathbb{F}_p} \arrow[d, phantom, sloped, "\simeq"] \\
\mathrm{Caus}(\mathrm{CPM}(\mathsf{Stab}_p)) \arrow[r, tail] \arrow[d, tail] & \mathrm{CPM}(\mathsf{Stab}_p) \arrow[r, two heads] \arrow[d, tail] & \mathrm{Proj}(\mathrm{CPM}(\mathsf{Stab}_p)) \arrow[d, tail] \\
\mathrm{Caus}(\mathrm{CPM}(\mathsf{FHilb})) \arrow[r, tail] \arrow[d, phantom, sloped, "\simeq"] & \mathrm{CPM}(\mathsf{FHilb}) \arrow[r, two heads] \arrow[d, phantom, sloped, "\simeq"] & \mathrm{Proj}(\mathrm{CPM}(\mathsf{FHilb})) \arrow[d, phantom, sloped, "\simeq"] \\
\begin{array}{c}\textit{CPTP maps between}\\ \textit{matrix algebras}\end{array} \arrow[r, tail] & \begin{array}{c}\textit{CP maps between}\\ \textit{matrix algebras}\end{array} \arrow[r, two heads] & \mathrm{Proj}\left(\begin{array}{c}\textit{CP maps between}\\ \textit{matrix algebras}\end{array}\right)
\end{tikzcd}$$

Proof. It is immediate that $\mathrm{Rel} : \mathrm{Caus}(\mathrm{CPM}(\mathsf{Stab}_p)) \to \mathrm{Total}(\mathsf{AffCoisotRel}_{\mathbb{F}_p})$ is an essentially surjective, full, monoidal functor, making the diagram commute. It remains to prove faithfulness. Take two maps $[f, S], [g, T] : X \to Y$ in $\mathrm{CPM}(\mathsf{Stab})$ such that $[g, T] = \lambda \cdot [f, S]$ some $\lambda \neq 0$. Then $\mathrm{Tr}_Y[g, T] = \lambda \, \mathrm{Tr}_Y[f, S] = \lambda \, \mathrm{Tr}_X$. Therefore $[g, T]$ is causal iff $\lambda = 1$ i.e. $[g, T] = [f, S]$, thus, each projective equivalence class of morphisms $\mathrm{Caus}(\mathrm{CPM}(\mathsf{Stab}_p))$ contains at most one representative. Therefore, the equivalence $\mathsf{AffCoisotRel}_{\mathbb{F}_p} \simeq \mathrm{Proj}(\mathrm{CPM}(\mathsf{Stab}_p))$ uniquely lifts along Proj on causal maps. ◀

4.3 Stabiliser quantum error correction and the order on projectors

The symplectic representation of stabiliser quantum mechanics is the cornerstone of finite-dimensional quantum error correction [24]; where the odd prime dimensional description is developed by Ashikhmin and Knill [4]. In this subsection will recall this correspondence in more detail. Then we give a compositional account for the refinement of stabiliser quantum error correction codes.

To understand the connection between the symplectic geometry and quantum error correction, we revisit the notion of a stabiliser code. Given a nonempty affine coisotropic subspace $S = L + \mathbf{a} \subseteq (\mathbb{F}_p^{2n}, \omega_n)$, this induces a Abelian subgroup of the Pauli group

$$G_S := \{\pi\big(\omega_n(\mathbf{a}, \mathbf{b}), \mathbf{b}\big) \ \mid \ \mathbf{b} \in L^{\omega_n}\} \subseteq \mathcal{P}_p^{\otimes n}$$

such that $\chi(x) \cdot 1_{\mathcal{H}_p^{\otimes n}} \in G_S$ if and only if $x = 0$. In other words, G_S is a stabiliser code. The projector onto the codespace $\mathcal{H}_{G_S} \subseteq \mathcal{H}_p^{\otimes n}$ is given by:

$$\Pi_{G_S} := \frac{1}{|L^{\omega_n}|} \sum_{\mathbf{b} \in L^{\omega_n}} \pi\big(\omega_n(\mathbf{a}, \mathbf{b}), \mathbf{b}\big) : \mathcal{H}_p^{\otimes n} \to \mathcal{H}_p^{\otimes n}.$$

By convention the empty affine coisotropic subspace induces the abelian subgroup $G_\varnothing := \{1_{\mathcal{H}_p^{\otimes n}}\} \subseteq \mathcal{P}_p^{\otimes n}$, so that $\Pi_{G_\varnothing} := 0$. This convention is chosen so that for any affine coisotropic subspace $S \subseteq (\mathbb{F}_p^{2n}, \omega_n)$, we have that $\mathrm{Rel}(\Pi_{G_S}) = S$.

Because global phases are quotiented together in mixed quantum theory, we can represent arbitrary Pauli operators by elements $\mathbf{e} \in \mathbb{F}_p^{2n}$ as $\pi(\mathbf{0}, \mathbf{e})$. Interpreting Pauli operators as errors, we capture the error correction properties of the stabiliser code G_S either by reference to the isotropic subspace L^ω or by reference to the projector Π_{G_S}:

An error $\pi(0, \mathbf{e}) \in \mathcal{P}_p^{\otimes n}$ *is*	*Symplectic condition*	*Projector condition*
Trivial	$\mathbf{e} \in L^{\omega_n}$	$\Pi_{G_S}\pi(\mathbf{0}, \mathbf{e})\Pi_{G_S} = \Pi_{G_S}$
Detectable	$\mathbf{e} \notin L$	$\Pi_{G_S}\pi(\mathbf{0}, \mathbf{e})\Pi_{G_S} = 0$
Undetectable and nontrivial	$\mathbf{e} \in L \setminus L^{\omega_n}$	$\Pi_{G_S}\pi(\mathbf{0}, \mathbf{e})\Pi_{G_S} \neq 0$, $\Pi_{G_S}\pi(\mathbf{0}, \mathbf{e})\Pi_{G_S} \neq \Pi_{G_S}$

Moreover, a finite set $\mathcal{E} \subseteq \mathbb{F}_p^{2n}$ of errors is *correctable* if and only if:

$$\forall \mathbf{e} \neq \mathbf{f} \in \mathcal{E} : \mathbf{f} - \mathbf{e} \notin L \quad \iff \quad \forall \mathbf{e} \neq \mathbf{f} \in \mathcal{E} : \Pi_{G_S} \pi(\mathbf{0}, \mathbf{f} - \mathbf{e}) \Pi_{G_S} = 0.$$

The *code distance* $d(S) \in \mathbb{N}$ of a nonempty affine coisotropic subspace $S = L + \mathbf{a}$ is the minimal number of tensor factors on which a nontrivial undetectable Pauli acts. Stabiliser codes are designed to maximize the code distance. This is most easily understood in the symplectic picture where:

$$d(S) \coloneqq \min \left\{ \ \left|\{i \in \{0, \ldots, n-1\} : (e_{x,i}, e_{z,i}) \neq (0,0)\}\right| \ : \ \forall \mathbf{e} = (\mathbf{e}_x, \mathbf{e}_z) \in L \setminus L^{\omega_n} \ \right\}$$

The order on nonempty affine Lagrangian subspaces, and their corresponding projectors is trivial. However, for affine coisotropic subspaces, it tells us when the corresponding projectors are more or less pure than each other: because $\mathrm{Rel}(\Pi_{G_S}) = S$ there is an inclusion of affine coisotropic subspaces $R \subseteq S$ if and only if there is an inclusion of the images of their corresponding projectors $\mathrm{Im}(\Pi_{G_R}) \subseteq \mathrm{Im}(\Pi_{G_S})$. In other words, when the constraints imposed by R can be relaxed to the contraints imposed by S.

This order also allows us to quantify the error correction properties of codes. Given affine coisotropic subspaces $R, S \subseteq (\mathbb{F}_p^{2n}, \omega_n)$, if $R \subseteq S$ then:

- A trivial for S is also trivial for R.
- A detectable error detectable for S is also detectable for R.
- A set of correctable errors for S is also correctable by R.

Of course, this comes with a trade-off: the code distance is anti monotone with respect to this order so that $R \subseteq S$ implies $d(S) \leqslant d(R)$.

The partial order enrichment of $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$ and $\mathrm{Total}(\mathsf{AffCoisotRel}_{\mathbb{F}_p})$ gives us a *compositional* account for this quantitative property for mixed stabiliser quantum *processes*:

▶ **Theorem 45.** *with respect to this order:*

- $\mathrm{CPM}(\mathsf{Stab}_p)$ *is enriched in preordered sets;*
- $\mathrm{Proj}(\mathrm{CPM}(\mathsf{Stab}_p))$ *is enriched in partially ordered sets;*
- $\mathrm{Caus}(\mathrm{CPM}(\mathsf{Stab}_p))$ *is enriched in partially ordered sets.*

Proof. Since $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$ is compact closed, any morphism $R : (V, \omega_V) \to (W, \omega_W)$ canonically induces a state $\lfloor R \rfloor : I \to (V \oplus W, \omega_V \oplus \omega_W)$, called its *name.* For example in $\mathrm{CPM}(\mathsf{Stab}_p)$ the name is proportional to the Choi matrix. Therefore, there is an inclusion of affine coisotropic relations $R \subseteq S$ if and only if there is an inclusion of the images of their corresponding projectors $\mathrm{Im}(\Pi_{G_{\lfloor R \rfloor}}) \subseteq \mathrm{Im}(\Pi_{G_{\lfloor S \rfloor}})$.

Just as dividing the projector onto the code space by the trace norm $\Pi_{G_S} / \mathrm{Tr}(\Pi_{G_S})$ is a mixed stabiliser state; dividing a mixed stabiliser state by its *operator norm* is a projector onto the code space of a stabiliser code. Because the operator normalized Choi-matrices of morphisms between any two Hilbert spaces in $\mathrm{Caus}(\mathrm{CPM}(\mathsf{Stab}_p))$ are distinct from each other, the isomorphism $\mathrm{Rel} : \mathrm{Caus}(\mathrm{CPM}(\mathsf{Stab}_p)) \cong \mathrm{Total}(\mathsf{AffCoisotRel}_{\mathbb{F}_p})$ induces a partial order on the trace preserving mixed stabiliser quantum processes.

On the other hand, when working in $\mathrm{CPM}(\mathsf{Stab}_p)$, two proportional, yet distinct morphisms of the same type induce the same projector, so the order fails to be anti-symmetric. ◀

Note that for projectors, this order corresponds to the usual Löwner order [3, Section 6].

5 Measurement and classical types

In the previous section, we saw how the CPM construction provides an abstract setting for:

1. general mixed state quantum mechanics when applied to FHilb;
2. more specifically, stabiliser codes when applied to $\mathsf{AffLagRel}_{\mathbb{F}_p} \simeq \mathsf{Stab}_p$.

As previously mentioned, stabiliser codes are used to detect and correct errors on noisy quantum channels: encoding quantum information redundantly as a mixed state. However, from, an operational point of view, to detect errors, one has to measure part of the code space; and to correct errors, one must apply operations to the code space conditional on the measurement outcomes.

Given an indexed orthonormal basis $B = \{|\lambda_1\rangle, \cdots, |\lambda_n\rangle\}$ for a finite-dimensional Hilbert space $\mathcal{H}$, the measurement in this basis is represented by the projector $\mathcal{E}_B : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ which sends pure states to probabilistic mixtures of pure states according to the *Born rule*:

$$\mathcal{E}_B(|\varphi\rangle\!\langle\varphi|) \coloneqq \sum_{j=1}^{n} |\lambda_j\rangle\!\langle\lambda_j|\, |\varphi\rangle\!\langle\varphi|\, |\lambda_j\rangle\!\langle\lambda_j| = \sum_{j=1}^{n} |\langle\lambda_j|\varphi\rangle|^2\, |\lambda_j\rangle\!\langle\lambda_j| \tag{7}$$

The indices of the basis are interpreted as the measurement outcomes occurring with probability $|\langle\lambda_j|\varphi\rangle|^2$. This projector $\mathcal{E}_B$ is an endomorphism on $\mathcal{H}$ in CPM(FHilb). In particular, a Pauli-X basis measurement is an endomorphism on $\mathcal{H}_p$ in CPM(Stab_p). Therefore, in some sense, $\mathcal{E}_B$ is the "classical" subobject of the "quantum" object $\mathcal{H}$ which has been measured in the basis B. By promoting these subobjects to objects, in the following subsection we obtain a categorical semantics for quantum theory with classical and quantum types; reproducing the usual setting for finite-dimensional quantum mechanics. Later, we perform an analogous construction to stabiliser circuits to obtain a fully relational semantics.

5.1 Adding classical types by splitting dagger-idempotents

We review the †-idempotent completion of a †-CCC, recalling:

▶ **Definition 46.** *A* ***†-idempotent*** *in a †-SMC is a map f such that $f^\dagger = f$ and $f;f = f$.*

In the setting of finite-dimensional, mixed quantum theory:

▶ **Example 47** ([30, Thm. 2.5], [14, Prop. 3.5])**.** The †-idempotents on $\mathcal{H}$ in CPM(FHilb) are in bijection with C^*-subalgebras of the matrix algebra $\mathcal{B}(H) \cong \mathcal{H}^* \otimes \mathcal{H}$.

The identity on $\mathcal{H}$ is an idempotent and corresponds to the trivial C^*-subalgebra $\mathcal{B}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$. On the other hand, projectors onto subspaces induced by measurement, such as measurement onto a basis $\mathcal{E}_B : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ correspond to commutative C^*-subalgebras of $\mathcal{B}(\mathcal{H})$. We promote these subobjects to objects:

▶ **Definition 48** ([41, Def. 3.13])**.** *Given a †-CCC* C*, the †-idempotent completion,* $\mathrm{Split}^\dagger(\mathsf{C})$*, is the †-CCC with:*

- ***objects:*** *pairs (A, a) where A is an object of* C *and $a : A \to A$ is a †-idempotent;*
- ***morphisms:*** *$f : (A, a) \to (B, b)$ are morphisms $f : A \to B$ in* C *such that $a; f; b = f$;*
- ***identities:*** *$1_{(A,a)} \coloneqq a$;*
- ***rest of †-compact structure*** *given pointwise in* C.

There is a canonical embedding $\mathcal{C} \to \mathrm{Split}^\dagger(\mathcal{C})$ sending objects $A \mapsto (A, 1_A)$ and acting as the identity on morphisms. When applied to CPM(FHilb), the †-idempotent completion reproduces the standard setting for finite-dimensional quantum mechanics:

▶ **Theorem 49** ([30, Thm. 2.5], [14, Prop. 3.5]). $\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{FHilb}))$ *is equivalent to the CCC of completely-positive maps between finite-dimensional C^*-algebras.*

The objects of the form $(\mathcal{H}, 1_\mathcal{H})$ represent the matrix algebras, interpreted as the purely quantum systems in $\mathrm{CPM}(\mathsf{FHilb})$. On the other hand, the new objects added by †-idempotent completion correspond to non-matrix C^*-algebras, interpreted as being more classical. For the example of a quantum measurement induced by an orthonormal basis B, the object $(\mathcal{H}, \mathcal{E}_B)$ is interpreted as a classical system measured according to the basis B. The canonical map $\mathcal{E}_B : (\mathcal{H}, 1_\mathcal{H}) \to (\mathcal{H}, \mathcal{E}_B)$ is interpreted as the measurement induced by B; whereas the map $\mathcal{E}_B : (\mathcal{H}, \mathcal{E}_B) \to (\mathcal{H}, 1_\mathcal{H})$ is interpreted as the state preparation induced by B. Measurement followed by state preparation yields the quantum system projected onto the measurement basis; whereas, state preparation followed by measurement yields the identity on the classical system:

Measuring then preparing:
$$\begin{tikzcd} (\mathcal{H}, 1_\mathcal{H}) \arrow[d, two heads, "\mathcal{E}_B"'] \arrow[dr, "\mathcal{E}_B"] & \\ (\mathcal{H}, \mathcal{E}_B) \arrow[r, two heads, "\mathcal{E}_B"'] & (\mathcal{H}, 1_\mathcal{H}) \end{tikzcd}$$

Preparing then measuring:
$$\begin{tikzcd} (\mathcal{H}, \mathcal{E}_B) \arrow[r, two heads, "\mathcal{E}_B"] \arrow[dr, equal, "\mathcal{E}_B"'] & (\mathcal{H}, 1_\mathcal{H}) \arrow[d, two heads, "\mathcal{E}_B"] \\ & (\mathcal{H}, \mathcal{E}_B) \end{tikzcd}$$

Arbitrary completely-positive maps between C^*-algebras cannot be physically implemented. Just as in the previous section, we must impose an additional constraint:

▶ **Definition 50.** *A morphism $[f, S] : (X, x) \to (Y, y)$ in $\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{C}))$ is* ***causal*** *if and only if $\mathrm{Tr}_Y[f, S] = \mathrm{Tr}_X : (X, x) \to (I, 1_I)$. We denote $\mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{C})))$ the symmetric monoidal subcategory of causal morphisms in $\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{C}))$.*

In the setting of finite-dimensional quantum theory; this reproduces the usual operator algebraic setting for finite-dimensional quantum mechanics:

▶ **Corollary 51.** $\mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{FHilb})))$ *is equivalent to the symmetric monoidal category of completely-positive trace-preserving (CPTP) maps between finite-dimensional C^*-algebras.*

In other words, the morphisms in $\mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{FHilb})))$ correspond to finite-dimensional **quantum channels**, and the states correspond to **density matrices**. Importantly, the state preparation and measurement maps are quantum channels. For notational convenience, denote the category of quantum channels by $\mathsf{QuantChan} := \mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{FHilb})))$.

5.2 The stabiliser theory with affine nondeterministic classical control

In the previous subsection, we recalled how the symmetric monoidal category $\mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{FHilb})))$ is equivalent to the standard setting for quantum channels. That is to say, the finite-dimensional quantum circuits with measurement and classical control. In this subsection, by applying the same constructions to $\mathrm{CPM}(\mathsf{Stab}_p) \rightarrowtail \mathrm{CPM}(\mathsf{FHilb})$; we show that the canonical setting for stabilizer quantum mechanics with Pauli measurement and Pauli state preparation admits a concise, entirely relational description. To this end:

▶ **Definition 52.** *The †-compact-closed category $\mathsf{AffRel}_{\mathbb{F}_p}$ of* ***affine relations*** *has finite-dimensional $\mathbb{F}_p$-vector spaces as objects and affine subspaces as morphisms. Composition is given by relational composition, whilst the identity and compact-closed structure are given by the diagonal relation. The dagger is given by the relational converse.*

▶ **Lemma 53.** *There is a faithful †-compact-closed functor $Q : \mathsf{AffLagRel}_{\mathbb{F}_p} \rightarrowtail \mathsf{AffRel}_{\mathbb{F}_p}$ which forgets symplectic structure.*

Instead of forming $\mathrm{Split}^\dagger(\mathsf{AffCoisotRel}_{\mathbb{F}_p})$ on the nose, we can add additional affine relations to the image of $Q : \mathsf{AffCoisotRel}_{\mathbb{F}_p} \rightarrowtail \mathsf{AffRel}_{\mathbb{F}_p}$ which †-split †-idempotents:

▶ **Proposition 54.** *The †-idempotents in* $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$ *†-split through the forgetful functor* $\mathsf{AffCoisotRel}_{\mathbb{F}_p} \rightarrowtail \mathsf{AffRel}_{\mathbb{F}_p}$*. In particular, Pauli-$X$ measurement splits through the relations:*

$$\mu_X := \left\{ \left(\begin{bmatrix} x \\ z \end{bmatrix}, x \right) \in \mathbb{F}_p^2 \oplus \mathbb{F}_p \right\} : Q(\mathbb{F}_p^2, \omega_2) \to \mathbb{F}_p, \ \eta_X := \left\{ \left(x, \begin{bmatrix} x \\ z \end{bmatrix} \right) \in \mathbb{Z}_p \oplus \mathbb{F}_p^2 \right\} : \mathbb{F}_p \to Q(\mathbb{F}_p^2, \omega_2)$$

Proof. Consider the relation in $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$ corresponding to the Z-flip channel:

$$\mathscr{E}_X := \mathrm{Rel}(\mathcal{E}_X) = \left\{ \left(\begin{bmatrix} x \\ z \end{bmatrix}, \begin{bmatrix} x \\ z' \end{bmatrix} \right) \in \mathbb{F}_p^2 \oplus \mathbb{F}_p^2 \right\} : (\mathbb{F}_p^2, \omega_2) \to (\mathbb{F}_p^2, \omega_2)$$

Any †-idempotent in $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$ is affine symplectomorphic to $\mathscr{E}_X^{\oplus n} \oplus 1_{(\mathbb{F}_p^m \otimes \mathbb{F}_p^m, \omega_m)}$ for some $n, m \in \mathbb{N}$. Moreover, $Q(\mathscr{E}_X) = \mu_X ; \eta_X$ splits as $\eta_X ; \mu_X = 1_{\mathbb{F}_p}$. ◀

The process of †-splitting †-idempotents through $Q : \mathsf{AffCoisotRel}_{\mathbb{F}_p} \rightarrowtail \mathsf{AffRel}_{\mathbb{F}_p}$ adds a *quantum* modality Q to $\mathsf{AffRel}_{\mathbb{F}_p}$; imposing compatibility with the symplectic structure:

▶ **Definition 55.** *Let* $\mathsf{AffRel}_{\mathbb{F}_p}^Q$ *denote the †-CCC with:*

- ***Objects:*** *Generated by finite direct sums of finite dimensional symplectic vector spaces* $Q(V, \omega_V) \in Q(\mathsf{AffCoisotRel}_{\mathbb{F}_p})$*, and finite dimensional vector spaces* $W \in \mathsf{AffRel}_{\mathbb{F}_p}$*;*
- ***Morphisms:*** *Generated by* $Q(\mathsf{AffCoisotRel}_{\mathbb{F}_p})$ *in addition to* $\mu_X : Q(\mathbb{F}_p^2, \omega_2) \to \mathbb{F}_p$ *and* $\nu_X : \mathbb{F}_p \to Q(\mathbb{F}_p^2, \omega_2)$ *under the direct sum and relational composition;*
- ***†-compact structure:*** *Given pointwise in* $Q(\mathsf{AffCoisotRel}_{\mathbb{F}_p})$*,* $\mathsf{AffRel}_{\mathbb{F}_p}$*, where* $\mu_X^\dagger := \nu_X$*.*

By restricting to either class of objects, it is immediate that:

▶ **Lemma 56.** $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$ *and* $\mathsf{AffRel}_{\mathbb{F}_p}$ *are full †-compact closed subcategories of* $\mathsf{AffRel}_{\mathbb{F}_p}^Q$*.*

Moreover, because $Q : \mathsf{AffCoisotRel}_{\mathbb{F}_p} \rightarrowtail \mathsf{AffRel}_{\mathbb{F}_p}$ is faithful, it is immediate that:

▶ **Theorem 57.** *There is a †-compact closed equivalence* $\mathrm{Split}^\dagger(\mathsf{AffCoisotRel}_{\mathbb{F}_p}) \simeq \mathsf{AffRel}_{\mathbb{F}_p}^Q$*.*

In other words, this category is obtained by glueing together the †-CCCs $\mathsf{AffLagRel}_{\mathbb{F}_p}$ and $\mathsf{AffRel}_{\mathbb{F}_p}$ along the map μ_X which projects onto the X subspace and its transpose. We interpret the symplectic objects $Q(V, \omega_V)$ as the quantum types; the objects W with no symplectic structure as the classical types; μ_X as the measurement in the Pauli-X basis; and ν_X as state preparation in the Pauli-X basis.

Finally, our moto becomes:

Everything is an affine relation, with quantum data captured by a symplectic modality!

which, admittedly, is not *quite* as catchy as the previous motos.

However, just as for $\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{FHilb}))$; the category $\mathrm{Proj}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{Stab}_p)) \cong \mathsf{AffRel}_{\mathbb{F}_p}^Q$ has morphisms which do not correspond to operations which can be physically implemented. We restrict ourselves to the completely-positive maps:

▶ **Proposition 58.** *The induced functor* $\mathrm{Rel} : \mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{Stab}_p)) \to \mathsf{AffRel}_{\mathbb{F}_p}^Q$ *restricts to a symmetric monoidal equivalence* $\mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{Stab}_p))) \simeq \mathrm{Total}(\mathsf{AffRel}_{\mathbb{F}_p}^Q)$ *making the following diagram commute:*

$$\begin{tikzcd}
\mathrm{Total}(\mathsf{AffRel}_{\mathbb{F}_p}^Q) \arrow[rr, tail] \arrow[d, phantom, "\wr\mid"] & & \mathsf{AffRel}_{\mathbb{F}_p}^Q \arrow[d, phantom, "\wr\mid"] \\
\mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{Stab}_p))) \arrow[r, tail] \arrow[d, tail] & \mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{Stab}_p)) \arrow[r, two heads] \arrow[d, tail] & \mathrm{Proj}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{Stab}_p))) \arrow[d, tail] \\
\mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{FHilb}))) \arrow[r, tail] \arrow[d, phantom, "\wr\mid"] & \mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{FHilb})) \arrow[r, two heads] \arrow[d, phantom, "\wr\mid"] & \mathrm{Proj}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{FHilb}))) \arrow[d, phantom, "\wr\mid"] \\
\begin{array}{c}\textit{CPTP maps between} \\ \textit{f.d. } C^*\textit{-algebras}\end{array} \arrow[r, tail] & \begin{array}{c}\textit{CP maps between} \\ \textit{f.d. } C^*\textit{-algebras}\end{array} \arrow[r, two heads] & \mathrm{Proj}\left(\begin{array}{c}\textit{CP maps between} \\ \textit{f.d. } C^*\textit{-algebras}\end{array}\right)
\end{tikzcd}$$

Proof. This follows from essentially the same argument as for proposition 44. ◀

Note that classically-controlled Pauli operators can be represented in $\mathrm{Total}(\mathsf{AffRel}^{Q}_{\mathbb{F}_p})$ because they can be constructed with Clifford operators as well as Pauli state preparation and measurements. For notational convenience, from here onwards, denote the subcategory of $\mathsf{QuantChan}$ whose morphisms are qupit **stabiliser quantum channels** by $\mathsf{StabChan}_p := \mathrm{Caus}(\mathrm{Split}^{\dagger}(\mathrm{CPM}(\mathsf{Stab}_p)))$.

Because all of the morphisms in $\mathrm{Total}(\mathsf{AffRel}^{Q}_{\mathbb{F}_p})$ are affine subspaces over $\mathbb{F}_p$, this means that exact equality of stabiliser quantum channels is computable in deterministic polynomial time. This is a deterministic analogue of the celebrated Gottesman-Knill theorem [2].

Arbitrary classical operations

By relaxing the requirement that the relations between classical objects are affine relations we obtain the following category:

▶ **Definition 59.** *Let $\mathsf{Rel}^{Q}_{\mathbb{F}_p}$ denote the †-compact closed category given by the objects and morphisms of $\mathsf{AffRel}^{Q}_{\mathbb{F}_p}$ in addition to the* non-affine *relation between classical types:*

$$\left\{\left(\begin{bmatrix} a \\ b \end{bmatrix}, a \cdot b\right) \in \mathbb{F}_p^2 \oplus \mathbb{F}_p\right\} : \mathbb{F}_p^2 \to \mathbb{F}_p$$

Indeed, by adding the relation which multiplies classical dits, it follows immediately that:

▶ **Lemma 60.** *The morphisms from $\mathbb{F}_p^n$ to $\mathbb{F}_p^m$ in $\mathsf{Rel}^{Q}_{\mathbb{F}_p}$ are precisely set-relations between from the set $\mathbb{F}_p^n$ to the set $\mathbb{F}_p^m$, ie. subsets of $\mathbb{F}_p^n \oplus \mathbb{F}_p^m$.*

Therefore, the total relations between classical objects are precisely functions between the underlying sets. These are exactly the classical corrections which can be performed deterministically. However, by relaxing the affine constraints between classical objects, the morphisms between quantum objects in $\mathsf{Rel}^{Q}_{\mathbb{F}_p}$ also fail in general to be affine coisotropic subspaces, or even just affine subspaces at all. In particular, this means that the symplectic representation of stabiliser circuits fails:

▶ **Proposition 61.** *There is no functor $\mathrm{Total}(\mathsf{Rel}^{Q}_{\mathbb{F}_p}) \to \mathsf{StabChan}_p$ which extends $\mathrm{Total}(\mathsf{AffRel}^{Q}_{\mathbb{F}_p}) \simeq \mathsf{StabChan}_p$ along $\mathrm{Total}(\mathsf{AffRel}^{Q}_{\mathbb{F}_p}) \rightarrowtail \mathrm{Total}(\mathsf{Rel}^{Q}_{\mathbb{F}_p})$.*

This is because the non-affine classical control of stabiliser codes can produce mixed states which are no longer proportional to uniform mixtures of pure states; non-affine corrections between basis elements can create mixtures of pure states with different weights. In other words, $\mathrm{Total}(\mathsf{Rel}^{Q}_{\mathbb{F}_p})$ can not tell us the probability of measurement outcomes.

6 Case study: a small imperative language for stabiliser QEC

In this section, we introduce a minimal imperative language SPL (Stabiliser Programming Language) for stabiliser quantum channels. In other words, this is a language for quantum error correction, including measurements and classical control. This language is strongly inspired by the language QPL [42], but restricted to stabiliser operations and total, non-deterministic, *affine* classical operations. We have implemented SPL and its denotational semantics in python [17].

We give SPL small-step operational semantics on pairs $[C|\rho]$ of terms acting on density operators as CPTP maps (similar to that of Ying [47, Section 3.2]), and a fully abstract

$$\frac{\Gamma \vdash c \,\triangleright\, \Delta \quad \Delta \vdash d \,\triangleright\, \Sigma}{\Gamma \vdash c \,\fatsemi\, d \,\triangleright\, \Sigma} \qquad \frac{}{\Gamma \vdash \textbf{init}\ \underline{x} \,\triangleright\, \underline{x} : \mathtt{pit}, \Gamma} \qquad \frac{}{\Gamma \vdash \textbf{qinit}\ \underline{x} \,\triangleright\, \underline{x} : \mathtt{qpit}, \Gamma}$$

$$\frac{}{\underline{\mathbf{x}} : \mathtt{pit}^n, \Gamma \vdash \underline{\mathbf{y}} = A * \underline{\mathbf{x}} \,\triangleright\, \underline{\mathbf{x}} : \mathtt{pit}^n, \underline{\mathbf{y}} : \mathtt{pit}^m, \Gamma} \qquad \frac{}{\underline{x} : \mathtt{qpit}, \Gamma \vdash \textbf{meas}\ \underline{x} \,\triangleright\, \underline{x} : \mathtt{pit}, \Gamma}$$

$$\frac{}{\Gamma \vdash \textbf{skip} \,\triangleright\, \Gamma} \qquad \frac{}{\underline{x} : \mathtt{qpit}, \Gamma \vdash \textbf{disc}\ \underline{x} \,\triangleright\, \Gamma} \qquad \frac{}{\underline{\mathbf{x}} : \mathtt{qpit}^n, \Gamma \vdash \underline{\mathbf{x}} \mathrel{*}= U \,\triangleright\, \underline{\mathbf{x}} : \mathtt{qpit}^n, \Gamma}$$

$$\frac{}{\underline{x} : \mathtt{pit}, \underline{y} : \mathtt{qpit}, \Gamma \vdash \textbf{ctrl}_P\ \underline{x}\ \underline{y} \,\triangleright\, \underline{x} : \mathtt{pit}, \underline{y} : \mathtt{qpit}, \Gamma}$$

Figure 1 Formation rules for SPL. $n \in \mathbb{N}^{>0}$ and $\tau \in \mathsf{Ty}$, $\underline{\mathbf{x}} : \tau^n$ is shorthand for $\{\underline{x_1} : \tau, \cdots, \underline{x_n} : \tau\}$ such that $\underline{\mathbf{x}} = (\underline{x_1}, \cdots, \underline{x_n}) \in \mathsf{Reg}^n$. New variables are always assumed to be fresh.

denotational semantics in the SMC $\mathrm{Total}(\mathsf{AffRel}^{Q}_{\mathbb{F}_p})$. This case study serves as a proof-of-concept to demonstrate that our symplectic semantics can be used as the foundation of a quantum compilation stack whose target code is fault-tolerant by construction and with a denotational semantics amenable to formal verification. The purpose of SPL is to show that $\mathrm{Total}(\mathsf{AffRel}^{Q}_{\mathbb{F}_p})$ serves as a denotational semantics for stabiliser quantum programs, and is to be contrasted with more powerful, and computationally expressive languages such as Quipper [26] and Proto-Quipper [22] which are not specifically tailored to the stabiliser fragment.

6.1 Syntax

SPL has quantum and classical types $\mathsf{Ty} ::= \mathtt{pit} \mid \mathtt{qpit}$. The terms are generated from the following grammar with respect to some fixed, linearly ordered set Reg indexing registers:

$$c, d \ ::= \ c \fatsemi d \mid \textbf{init}\ \underline{x} \mid \underline{\mathbf{y}} = A * \underline{\mathbf{x}} \mid \textbf{disc}\ \underline{x} \mid \textbf{qinit}\ \underline{x} \mid \underline{\mathbf{x}} \mathrel{*}= U \mid \textbf{meas}\ \underline{x} \mid \textbf{ctrl}_P\ \underline{x}\ \underline{y} \ \mid \textbf{skip}.$$

for all $n, m \in \mathbb{N}$, $U \in \mathcal{C}\ell^n_p$, $P \in \mathcal{P}^{\otimes n}_p$, $\mathbb{F}_p$-affine transformations $A : \mathbb{F}^n_p \to \mathbb{F}^m_p$, and $\underline{x}, \underline{y} \in \mathsf{Reg}$, $\underline{\mathbf{x}} \in \mathsf{Reg}^n$, $\underline{\mathbf{y}} \in \mathsf{Reg}^m$.

The term $c \fatsemi d$ represents the sequential composition of subterms; **init** $\underline{x}$ represents the initialisation of $\underline{x}$ as the p-ary digit 0; $\underline{\mathbf{y}} = A * \underline{\mathbf{x}}$ applies the affine transformation A to $\underline{\mathbf{x}}$ and stores the result on $\underline{\mathbf{y}}$; **disc** $\underline{x}$ takes the trace of $\underline{x}$; **qinit** $\underline{x}$ represents initialisation of $\underline{x}$ as the qu$p$it $|0\rangle$; $\underline{\mathbf{x}} \mathrel{*}= U$ applies the Clifford operator U on $\underline{\mathbf{x}}$; **meas** $\underline{x}$ represents the Pauli-X measurement on $\underline{x}$; $\textbf{ctrl}_P\ \underline{x}\ \underline{y}$ applies the Pauli operator P on $\underline{y}$, classically controlled by $\underline{x}$ in the Pauli-X basis; and **skip** represents the identity.

SPL is equipped with an *environment-transforming* type system, which enforces linear usage of quantum data. Typed environments are partial functions $\Gamma : \mathsf{Reg} \to \mathsf{Ty}$ which bind registers to be either qupits or pits, and which we sometimes represent as $\{\underline{x} : \tau,\ \underline{y} : \sigma,\ \underline{z} : \mu,\ \cdots\}$ for $\underline{x}, \underline{y}, \underline{z} \cdots \in \mathsf{Reg}$ and $\tau, \sigma, \mu, \cdots \in \mathsf{Ty}$. We impose that the domain $\mathrm{dom}(\Gamma)$ of Γ, i.e. the set of bound registers $\{\underline{x}, \underline{y}, \underline{z}, \cdots\}$, is *finite*. Judgments are triples $\Gamma \vdash t \,\triangleright\, \Delta$ consisting of a term t and typed environments Γ, Δ. A judgment $\Gamma \vdash t \,\triangleright\, \Delta$ is **well-formed** if it is derivable from the formation rules given in figure 1.

6.2 Operational semantics

In this subsection we define a structured operational semantics for SPL, which is strongly inspired by Ying's operational semantics for quantum programs [47, Section 3.2].

We interpret typed environments as objects in $\mathsf{StabChan}_p \rightarrowtail \mathsf{QuantChan}$:

$\{\underline{in} : \mathtt{qpit}\} \vdash$

$\mathbf{qinit}\ \underline{x}\ ⨾\ \mathbf{qinit}\ \underline{out}\ ⨾$ `% initialize registers`

$\underline{x}\ {*}{=}\ F\ ⨾\ (\underline{x}, \underline{out})\ {*}{=}\ C_X\ ⨾$ `% prepare Bell pair`

$(\underline{in}, \underline{x})\ {*}{=}\ C_X\ ⨾\ \underline{in}\ {*}{=}\ F\ ⨾\ \mathbf{meas}\ \underline{in}\ ⨾\ \mathbf{meas}\ \underline{x}\ ⨾$ `% Bell measurement`

$\mathbf{ctrl}_Z\ \underline{in}\ \underline{out}\ ⨾\ \mathbf{ctrl}_X\ \underline{x}\ \underline{out}\ ⨾$ `% phase correction`

$\mathbf{qinit}\ \underline{in}\ ⨾\ \mathbf{qinit}\ \underline{x}\ ⨾\ \mathbf{disc}\ \underline{in}\ ⨾\ \mathbf{disc}\ \underline{x}$ `% discard ancillae`

$\triangleright\ \{\underline{out} : \mathtt{qpit}\}$

Figure 2 Qupit teleportation in SPL. Where F is the Fourier transform; C_X is the (coherently) controlled X gate; and Z and X are the Pauli Z and X gates. Input is given on register $\underline{in} : \mathtt{qpit}$ and output is returned on register $\underline{out} : \mathtt{qpit}$.

▶ **Definition 62.** *Given a typed environment* Γ*, let* $\langle\Gamma\rangle$ *be the dependent tensor product:*

$$\langle\Gamma\rangle := \bigotimes_{\underline{x}\in\mathrm{dom}(\Gamma)} \begin{cases}(\mathcal{H}_p, 1_{\mathcal{H}_p}) & \textit{if} \quad \Gamma(\underline{x}) = \mathtt{qpit}; \quad \textit{else} \quad (\mathcal{H}_p, \mathcal{E}_X) \quad \textit{if} \quad \Gamma(\underline{x}) = \mathtt{pit}\end{cases}$$

and $\mathcal{D}(\Gamma) := \mathsf{QuantChan}(\mathbb{C}, \langle\Gamma\rangle)$ *be the set of density operators on* $\langle\Gamma\rangle$*.*

To give our operational semantics, we establish notation to represent stabiliser quantum channels acting on subspaces of a larger ambient space. Take typed environments Γ, Δ and ordered subsets (ie. lists) $\underline{\mathbf{x}} \subseteq \mathrm{dom}(\Gamma)$ and $\underline{\mathbf{y}} \subseteq \mathrm{dom}(\Delta)$, where moreover, $\mathrm{dom}(\Gamma) \setminus \underline{\mathbf{x}} = \mathrm{dom}(\Delta) \setminus \underline{\mathbf{y}}$. Given a stabiliser quantum channel $\mathcal{C} : \langle\Gamma|_{\underline{\mathbf{x}}}\rangle \to \langle\Delta|_{\underline{\mathbf{y}}}\rangle$ let $\mathcal{C}_{\underline{\mathbf{x}},\underline{\mathbf{y}}} : \langle\Gamma\rangle \to \langle\Gamma'\rangle$ be the stabiliser quantum channel acting as $\mathcal{C}$ on the subspace $\langle\Delta\rangle \subseteq \langle\Gamma\rangle$ and trivially on its orthogonal complement $\langle\Gamma \setminus \Delta\rangle \subseteq \langle\Gamma\rangle$.

▶ **Definition 63.** *A* ***configuration*** *is a pair consisting of a well-formed judgement* $\Gamma \vdash t \triangleright \Sigma$ *and a density operator* $\rho \in \mathcal{D}(\Gamma)$*, denoted* $[\Gamma \vdash t \triangleright \Sigma \mid \rho \in \mathcal{D}(\Gamma)]$*, or* $[t|\rho]$ *for short.*

The **small-step operational semantics** of SPL is defined by the following reduction rules, where the typed environments are omitted for notational convenience:

$$[\mathbf{skip}\,⨾\,t|\rho] \rightsquigarrow [t|\rho] \quad [(\mathbf{init}\,\underline{x})\,⨾\,t|\rho] \rightsquigarrow [t|\iota(|0\rangle)_{\underline{x},\underline{x}} \circ \rho] \quad [(\underline{\mathbf{y}} = A{*}\underline{\mathbf{x}})\,⨾\,t|\rho] \rightsquigarrow [t|\iota(\mathcal{M}^A)_{\underline{\mathbf{x}},\underline{\mathbf{y}}} \circ \rho]$$
$$[(\mathbf{disc}\,\underline{x})\,⨾\,t|\rho] \rightsquigarrow [t|(\mathrm{Tr}_{\mathcal{H}_p})_{\underline{x},\underline{x}} \circ \rho] \quad [(\mathbf{qinit}\,\underline{x})\,⨾\,t|\rho] \rightsquigarrow [t|\iota(|0\rangle)_{\underline{x},\underline{x}} \circ \rho] \quad [(\underline{\mathbf{x}} {*}{=} U)\,⨾\,t|\rho] \rightsquigarrow [t|\iota(U)_{\underline{\mathbf{x}},\underline{\mathbf{x}}} \circ \rho]$$
$$[(\mathbf{meas}\,\underline{x})\,⨾\,t|\rho] \rightsquigarrow [t|(\mathcal{E}_X)_{\underline{x},\underline{x}} \circ \rho] \quad [(\mathbf{ctrl}_P\,\underline{x}\,\underline{y})\,⨾\,t|\rho] \rightsquigarrow [t|CP_{(\underline{x},\underline{y}),(\underline{x},\underline{y})} \circ \rho],$$

where

- $\iota : \mathsf{Stab}_p \to \mathrm{CPM}(\mathsf{Stab}_p)$ takes pure stabilizer maps to stabiliser quantum channels;
- $\mathcal{E}_X : (\mathcal{H}_p, 1_{\mathcal{H}_p}) \to (\mathcal{H}_p, \mathcal{E}_X)$ denotes the Pauli-X measurement;
- $\mathrm{Tr}_{\mathcal{H}_p} : (\mathcal{H}_p, \mathcal{E}_X) \to I$ denotes the trace;
- $\mathcal{M}^A := \sum_{\mathbf{x}\in\mathbb{F}_p^m} |A\mathbf{x}\rangle\langle\mathbf{x}| : (\mathcal{H}_p, \mathcal{E}_x)^{\otimes m} \to (\mathcal{H}_p, \mathcal{E}_x)^{\otimes n}$;
- $CP := (\mathcal{H}_p, \mathcal{E}_X) \otimes (\mathcal{H}_p, 1_{\mathcal{H}_p}) \to (\mathcal{H}_p, \mathcal{E}_X) \otimes (\mathcal{H}_p, 1_{\mathcal{H}_p}) : \rho \mapsto (\bigoplus_{k=0}^{p-1} P^k)\rho(\bigoplus_{k=0}^{p-1} P^k)$ is the classically controlled $P \in \mathcal{P}_p$.

The types of density operators can be inferred from the typed environments. For example, reduction rules for pit vs qupit initialisation produce different density operators:

$$[\Gamma \vdash \mathbf{init}\,\underline{x}\ ⨾\ t \triangleright \Delta \mid \rho \in \mathcal{D}(\Gamma)] \rightsquigarrow [\underline{x} : \mathtt{pit}, \Gamma \vdash t \triangleright \Delta \mid \iota(|0\rangle)_{\underline{x},\underline{x}} \circ \rho \in \mathcal{D}(\underline{x} : \mathtt{pit}, \Gamma)]$$
$$[\Gamma \vdash \mathbf{qinit}\,\underline{x}\ ⨾\ t \triangleright \Delta \mid \rho \in \mathcal{D}(\Gamma)] \rightsquigarrow [\underline{x} : \mathtt{qpit}, \Gamma \vdash t \triangleright \Delta \mid \iota(|0\rangle)_{\underline{x},\underline{x}} \circ \rho \in \mathcal{D}(\underline{x} : \mathtt{qpit}, \Gamma)]$$

▶ **Definition 64.** *Say that two quantum channels are* ***observationally equivalent*** *in case they produce the same measurement statistics according to the Born rule when acting on arbitrary density matrices.*

▶ **Theorem 65.** *The operational semantics* $\rightsquigarrow^*$ *for* SPL *is sound, complete, and universal for* $\mathsf{StabChan}_p$*.*

Proof. Because the operational semantics are given by a deterministic transition system, given any configuration $[t|\rho]$, there is a unique ρ' such that $[c|\rho] \rightsquigarrow^* [\mathbf{skip}|\rho']$. Therefore, any well-formed term t yields a unique stabiliser quantum channel $[t|-]$. The observational equivalence of well-formed judgements c and d under $\rightsquigarrow^*$ therefore amounts to equality as stabiliser quantum channels, $[c|-] = [d|-]$, and thus, equality as quantum channels. ◀

In other words, terms in SPL represent the same channel if and only if they are observationally equivalent:

▶ **Definition 66.** *SPL terms c, d are* ***observationally equivalent*** *if for any closed context $C(-)$, and well-formed configurations $[C(c)|\rho]$ and $[C(d)|\rho]$, $[C(c)|\rho] \rightsquigarrow^* [\mathbf{skip}|\rho']$ implies that $[C(d)|\rho] \rightsquigarrow^* [\mathbf{skip}|\rho']$, and vice-versa.*

6.3 Denotational semantics

We give SPL a denotational semantics in $\mathrm{Total}(\mathsf{AffRel}^{Q}_{\mathbb{F}_p})$. On types, let $[\![\mathtt{pit}]\!] \coloneqq \mathbb{F}_p$ and $[\![\mathtt{qpit}]\!] \coloneqq Q(\mathbb{F}_p^2, \omega_2)$. Define the denotation of a typed environment to be the dependent direct sum $[\![\Gamma]\!] \coloneqq \bigoplus_{\underline{x} \in \mathrm{dom}(\Gamma)} [\![\Gamma(\underline{x})]\!]$.

The denotation of well-formed judgments $\Gamma \vdash t \triangleright \Delta$ is given by the maps $\mathsf{AffRel}^{Q}_{\mathbb{F}_p}([\![\Gamma]\!], [\![\Delta]\!])$ defined inductively from the denotation of generating terms. As before, we need to establish notation to represent affine relations acting on a subset of the registers of the context. Take ordered subsets $\underline{\mathbf{x}} \subseteq \mathrm{dom}(\Gamma)$ and $\underline{\mathbf{y}} \subseteq \mathrm{dom}(\Delta)$, where moreover, $\mathrm{dom}(\Gamma) \setminus \underline{\mathbf{x}} = \mathrm{dom}(\Delta) \setminus \underline{\mathbf{y}}$. Given a relation $S : [\![\Gamma|_{\underline{\mathbf{x}}}]\!] \to \left[\!\!\left[\Delta|_{\underline{\mathbf{y}}}\right]\!\!\right]$ let $S_{\underline{\mathbf{x}},\underline{\mathbf{y}}} : [\![\Gamma]\!] \to [\![\Gamma']\!]$ denote the relation acting as S on the subspace $[\![\Gamma|_{\underline{\mathbf{x}}}]\!] \subseteq [\![\Gamma]\!]$ and trivially everywhere else $[\![\Gamma|_{\mathrm{dom}(\Gamma)\setminus\underline{\mathbf{x}}}]\!] \subseteq [\![\Gamma]\!]$. The denotation of terms is defined inductively:

$$[\![c \,⨾\, d]\!] \coloneqq [\![c]\!] \,;\, [\![d]\!] \qquad [\![\mathbf{skip}]\!] \coloneqq 1_{[\![\Gamma]\!]} \qquad [\![\mathbf{init}\ \underline{x}]\!] \coloneqq (\{(0,0)\} : \mathbb{F}_p^0 \to \mathbb{F}_p\})_{\varnothing,\underline{x}}$$

$$[\![\underline{\mathbf{y}} = A{*}\underline{\mathbf{x}}]\!] \coloneqq (\mathrm{Gr}(A))_{\underline{\mathbf{x}},\underline{\mathbf{y}}} \qquad [\![\underline{\mathbf{x}} \mathrel{*}= U]\!] \coloneqq (\mathrm{Rel}(U))_{\underline{\mathbf{x}},\underline{\mathbf{x}}} \qquad [\![\mathbf{disc}\ \underline{x}]\!] \coloneqq \{(x,0) \mid x \in \mathbb{F}_p\}_{\underline{x},\varnothing}$$

$$[\![\mathbf{meas}\ \underline{x}]\!] \coloneqq (\mathrm{Gr}(\pi_1))_{\underline{x},\underline{x}} \qquad [\![\mathbf{qinit}\ \underline{x}]\!] \coloneqq (\{(0,(0,0)) : \mathbb{F}_p^0 \to Q(\mathbb{F}_p^2, \omega_2)\}; \eta_X)_{\varnothing,\underline{x}}$$

$$[\![\mathbf{ctrl}_P\ \underline{x}\ \underline{y}\ ]\!] \coloneqq \left\{ \left(\left(s, \begin{bmatrix} t \\ u \end{bmatrix}\right), \left(s, \begin{bmatrix} t + s \cdot a \\ u + s \cdot b \end{bmatrix}\right) \right) \,\middle|\, s, t, u \in \mathbb{F}_p \right\}_{\underline{x} \sqcup \underline{y}, \underline{x} \sqcup \underline{y}}$$

where π_k is the direct-sum projection onto the k-th component, and $P = \pi(0, a, b) \in \mathcal{P}_p$. We have omitted the typed contexts, which are given in figure 1.

▶ **Theorem 67** (Full abstraction)**.** *Well-formed judgements c and d are observationally equivalent if and only if $[\![c]\!] = [\![d]\!]$.*

Proof. Since all generating judgements are stabiliser, it follows from a straightforward induction that $[c|-]$ lies in the embedding of $\mathsf{StabChan}_p$ into the category of CPTP maps between finite-dimensional C^*-algebras. By construction $[\![c]\!] = \mathrm{Rel}[c|-]$, therefore the claim follows from proposition 58. ◀

▶ Remark 68. It is conceptually straightforward, albeit notationally cumbersome, to move to a sampling operational semantics by decomposing measurements into projections along each outcome. The same denotational semantics is fully abstract for the observational equivalence of programs defined as statistical indistinguishability of configurations.

Arbitrary classical operations

By enlarging the permissible classical operations of SPL with arbitrary classical operations, we can express a larger class of quantum channels:

▶ **Definition 69.** *Let* NLSPL *denote the language given by adding an additional operation:* $\underline{z} = \mathbf{mul}*(\underline{x}, \underline{y})$*, alongside the formation rule:*

$$\frac{}{\underline{x} : \texttt{pit}, \underline{y} : \texttt{pit}, \Gamma \vdash \underline{z} = \mathbf{mul}\star(\underline{x}, \underline{y}) \ \triangleright\ \underline{x} : \texttt{pit}, \underline{y} : \texttt{pit}, \underline{z} : \texttt{pit}, \Gamma}$$

▶ **Definition 70.** *Omitting contexts we have an additional small-step reduction rule:*

$$[\underline{z} = \mathbf{mul}\star(\underline{x}, \underline{y}) \mathbin{;} t|\rho] \rightsquigarrow \left[t \middle| \iota \left(\textstyle\sum_{j,k=0}^{p-1} |j \cdot k\rangle\langle j, k|\right)_{(\underline{x},\underline{y}),\underline{z}} \circ \rho\right]$$

By construction it is immediate that:

▶ **Lemma 71.** *The operational semantics* $\rightsquigarrow^*$ *for* NLSPL *is sound, complete, and universal for the observational equivalence of stabiliser quantum channels with arbitary classical operations.*

On the other hand, NLSPL also admits a denotational semantics:

▶ **Lemma 72.** *The denotational semantics of* SPL *into* $\mathrm{Total}(\mathsf{AffRel}^{Q}_{\mathbb{F}_p})$ *extends to a denotational semantics of* NLSPL *into* $\mathsf{Rel}^{Q}_{\mathbb{F}_p}$ *given by:*

$$[\![\underline{z} = \mathbf{mul}\star(\underline{x}, \underline{y})]\!] := \left\{ \left(\begin{bmatrix} a \\ b \end{bmatrix}, a \cdot b \right) \in \mathbb{F}_p^2 \oplus \mathbb{F}_p \right\}_{(\underline{x},\underline{y}),\underline{z}} : \mathbb{F}_p \oplus \mathbb{F}_p \to \mathbb{F}_p$$

We weaken observational equivalence to forget about the probability distribution of measurements, recording only the *support* of measurements:

▶ **Definition 73.** *Say that two quantum channels are* ***nondeterministically observationally*** *equivalent in case they produce the same* ***possible*** *measurement outcomes according to the Born rule when acting on arbitrary density matrices.*

This motivates the following conjecture:

▶ **Conjecture 74.** *Well-formed judgements* c *and* d *in* NLSPL *are nondeterministically observationally equivalent if and only if* $[\![c]\!] = [\![d]\!]$.

This seems nontrivial to prove; however, it is well-motivated. Morphisms in $\mathsf{Rel}^{Q}_{\mathbb{F}_p}$ are relations, therefore they record the possible ways in which stabilisers are related to each other. The question is if the nonlinearity of classical operations interacts well with the composition of quantum channels. Despite the breadth of research on stabiliser quantum mechanics, adding classically controlled Pauli operations dependent non-affinely on Pauli measurements is more complex than one might naïvely assume.

7 Conclusion

We have developed a denotational semantics for stabiliser quantum programs which allows for the manipulation of stabiliser codes, Pauli measurements, and affine classical affine operations and classically controlled Pauli operators. We demonstrated the power of this semantics by giving a fully abstract denotational semantics to a toy imperative stabiliser language.

We extended our language with arbitrary classical control, alongside a denotational semantics. We gave an operational semantics in terms of quantum channels generated by stabiliser quantum channels and arbitrary clasical operations, acting on density operators. We conjecture that this is fully abstract with respect to the equivalence relation induced by *possible* measurement outcomes.

In the case of qubits, the affine, symplectic representation of stabiliser maps breaks down so that $\mathrm{Proj}(\mathsf{Stab}_2) \not\simeq \mathsf{AffLagRel}_{\mathbb{F}_2}$ [16]. By restricting the unitary operations to be generated by the controlled-not gate, the Pauli group and the swap gate, we obtain the maximal subcategory of qubit stabiliser maps on which the symplectic representation still holds [15, p. 156]. This is the natural setting for CSS codes [10, 43], which are widely used in QEC [1].

The language we have described in this paper is extremely primitive and low-level; despite the abstract geometric structure of its denotational semantics. In future work, we intend to develop a higher level programming language with primitives reflecting the elegant structure of the semantics. For example, the ability to natively represent graph states and graph-like operations, correctable and detectable errors, and the ability to make use of the enrichment in partially ordered sets would be very useful.

It is also future work to explore denotational semantics for stabiliser quantum programs using their graphical calculus. There is a complete ZX-calculus for affine Lagrangian relations [8], which is equivalent to the qupit ZX-calculus [7, 37] modulo scalars. This is an interesting direction for future work because the ZX-calculus has already been successful for constructing fault-tolerant quantum circuits [6], and the design and verification of QEC codes [12, 20].

References

1 Qubit CSS code. In Victor V. Albert and Philippe Faist, editors, *The Error Correction Zoo*. 2024. URL: `https://errorcorrectionzoo.org/c/qubit_css`.

2 Scott Aaronson and Daniel Gottesman. Improved simulation of stabilizer circuits. *Physical Review A*, 70(5), November 2004. URL: `https://arxiv.org/abs/quant-ph/0406196`, `doi:10.1103/physreva.70.052328`.

3 M. Laura Arias and Alejandra Maestripieri. On partial orders of operators. *Annals of Functional Analysis*, 14(1), December 2022. URL: `http://dx.doi.org/10.1007/s43034-022-00244-y`, `doi:10.1007/s43034-022-00244-y`.

4 A. Ashikhmin and E. Knill. Nonbinary quantum stabilizer codes. *IEEE Transactions on Information Theory*, 47(7):3065–3072, 2001. URL: `https://arxiv.org/abs/quant-ph/0005008`, `doi:10.1109/18.959288`.

5 Michael E. Beverland, Prakash Murali, Matthias Troyer, Krysta M. Svore, Torsten Hoefler, Vadym Kliuchnikov, Guang Hao Low, Mathias Soeken, Aarthi Sundaram, and Alexander Vaschillo. Assessing requirements to scale to practical quantum advantage. `arXiv:2211.07629`.

6 Hector Bombin, Daniel Litinski, Naomi Nickerson, Fernando Pastawski, and Sam Roberts. Unifying flavors of fault tolerance with the ZX calculus. *Quantum*, 8:1379, June 2024. `doi:10.22331/q-2024-06-18-1379`.

7 Robert I. Booth and Titouan Carette. Complete ZX-calculi for the stabiliser fragment in odd prime dimensions. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022. `doi:10.4230/LIPICS.MFCS.2022.24`.

8 Robert I. Booth, Titouan Carette, and Cole Comfort. Graphical symplectic algebra. arXiv:2401.07914.

9 Paul Busch, Pekka Lahti, Juha-Pekka Pellonpää, and Kari Ylinen. *Quantum Measurement*. Springer International Publishing, 2016. doi:10.1007/978-3-319-43389-9.

10 A. R. Calderbank and Peter W. Shor. Good quantum error-correcting codes exist. *Physical Review A*, 54(2):1098–1105, August 1996. doi:10.1103/physreva.54.1098.

11 Earl T. Campbell, Barbara M. Terhal, and Christophe Vuillot. Roads towards fault-tolerant universal quantum computation. 549(7671):172–179. Publisher: Nature Publishing Group. doi:10.1038/nature23460.

12 Nicholas Chancellor, Aleks Kissinger, Stefan Zohren, Joschka Roffe, and Dominic Horsman. Graphical structures for design and verification of quantum error correction. *Quantum Science and Technology*, 8(4):045028, September 2023. doi:10.1088/2058-9565/acf157.

13 Kean Chen, Yuhao Liu, Wang Fang, Jennifer Paykin, Xin-Chuan Wu, Albert Schmitz, Steve Zdancewic, and Gushu Li. Verifying fault-tolerance of quantum error correction codes. arXiv:2501.14380.

14 Bob Coecke, Chris Heunen, and Aleks Kissinger. Categories of quantum and classical channels. *Quantum Information Processing*, 15(12):5179–5209, October 2014. doi:10.1007/s11128-014-0837-4.

15 Cole Comfort. *A diagrammatic approach to networks of spans and relations*. PhD thesis, 2023. doi:10.5287/ORA-AVNKVQO9N.

16 Cole Comfort and Aleks Kissinger. A graphical calculus for Lagrangian relations. In *arXiv.org*, volume 372, pages 338–351. EPTCS. doi:10.4204/EPTCS.372.24.

17 Cole R. Comfort. Stabiliser programming language (SPL). https://github.com/ColeComfort/SPL, 2025. GitHub repository.

18 Alexander Cowtan and Simon Burton. CSS code surgery as a universal construction. 8:1344. doi:10.22331/q-2024-05-14-1344.

19 Andrew Cross, Zhiyang He, Patrick Rall, and Theodore Yoder. Improved QLDPC surgery: Logical measurements and bridging codes. arXiv:2407.18393.

20 Ross Duncan and Maxime Lucas. Verifying the Steane code with Quantomatic. *Electronic Proceedings in Theoretical Computer Science*, 171:33–49, December 2014. doi:10.4204/eptcs.171.4.

21 Wang Fang and Mingsheng Ying. Symbolic execution for quantum error correction programs. 8:189:1040–189:1065. doi:10.1145/3656419.

22 Peng Fu, Kohei Kishida, Neil J. Ross, and Peter Selinger. Proto-Quipper with dynamic lifting. *Proceedings of the ACM on Programming Languages*, 7(POPL):309–334, January 2023. doi:10.1145/3571204.

23 Daniel Gottesman. The Heisenberg representation of quantum computers. arXiv:quant-ph/9807006.

24 Daniel Gottesman. *Stabilizer Codes and Quantum Error Correction*. PhD thesis, California Institute of Technology. arXiv:quant-ph/9705052.

25 Daniel Gottesman. An introduction to quantum error correction and fault-tolerant quantum computation, 2009. doi:10.48550/ARXIV.0904.2557.

26 Alexander S. Green, Peter LeFanu Lumsdaine, Neil J. Ross, Peter Selinger, and Benoît Valiron. Quipper: a scalable quantum programming language. *ACM SIGPLAN Notices*, 48(6):333–342, June 2013. doi:10.1145/2499370.2462177.

27 David Gross. Finite phase space methods in quantum information. URL: https://www.thp.uni-koeln.de/gross/files/diplom.pdf.

28 Zhiyang He, Alexander Cowtan, Dominic J. Williamson, and Theodore J. Yoder. Extractors: QLDPC architectures for efficient pauli-based computation. arXiv:2503.10390.

29 Markus Heinrich. *On stabiliser techniques and their application to simulation and certification of quantum devices*. PhD thesis, Universität zu Köln, 2021. URL: https://kups.ub.uni-koeln.de/50465/.

30 Chris Heunen, Aleks Kissinger, and Peter Selinger. Completely positive projections and biproducts. 171:71–83. doi:10.4204/EPTCS.171.7.

31 Qifan Huang, Li Zhou, Wang Fang, Mengyu Zhao, and Mingsheng Ying. Efficient formal verification of quantum error correcting programs. 9:1068–1093. doi:10.1145/3729293.

32 Aleks Kissinger. Phase-free ZX diagrams are CSS codes (...or how to graphically grok the surface code). arXiv, 2022. doi:10.48550/ARXIV.2204.14038.

33 Yurii A. Neretin. *Lectures on Gaussian Integral Operators and Classical Groups*. Number 91 in EMS Series of Lectures in Mathematics. European Mathematical Society.

34 Alexandru Paler, Ilia Polian, Kae Nemoto, and Simon J Devitt. Fault-tolerant, high-level quantum circuits: form, compilation and description. *Quantum Science and Technology*, 2(2):025003, April 2017. doi:10.1088/2058-9565/aa66eb.

35 Jennifer Paykin, Robert Rand, and Steve Zdancewic. QWIRE: a core language for quantum circuits. In *Proceedings of the 44th ACM SIGPLAN Symposium on Principles of Programming Languages*, POPL '17, page 846–858. ACM, January 2017. doi:10.1145/3009837.3009894.

36 Clément Poirson, Joschka Roffe, and Robert I. Booth. Engineering CSS surgery: compiling any CNOT in any code. arXiv:2505.01370.

37 Boldizsár Poór, Robert I. Booth, Titouan Carette, John van de Wetering, and Lia Yeh. The qupit stabiliser ZX-travaganza: Simplified axioms, normal forms and graph-theoretic simplification. *Electronic Proceedings in Theoretical Computer Science*, 384:220–264, August 2023. doi:10.4204/eptcs.384.13.

38 Robert Rand, Jennifer Paykin, Dong-Ho Lee, and Steve Zdancewic. ReQWIRE: Reasoning about reversible quantum circuits. *Electronic Proceedings in Theoretical Computer Science*, 287:299–312, January 2019. doi:10.4204/eptcs.287.17.

39 Robert Rand, Aarthi Sundaram, Kartik Singhal, and Brad Lackey. Gottesman types for quantum programs. volume 340, page 279–290. Open Publishing Association, September 2021. doi:10.4204/eptcs.340.14.

40 Peter Selinger. Dagger compact closed categories and completely positive maps: (extended abstract). In *Electronic Notes in Theoretical Computer Science*, volume 170 of *Proceedings of the 3rd International Workshop on Quantum Programming Languages (QPL 2005)*, pages 139–163. doi:10.1016/j.entcs.2006.12.018.

41 Peter Selinger. Idempotents in dagger categories. volume 210, pages 107–122. doi:10.1016/j.entcs.2008.04.021.

42 Peter Selinger. Towards a quantum programming language. In Peter Selinger, editor, *Proceedings of the 2nd International Workshop on Quantum Programming Languages (QPL 2004)*, volume 33, pages 1–16, Turku, Finland, 2004. TUCS General Publication. URL: https://www.mathstat.dal.ca/~selinger/papers.html.

43 A. M. Steane. Error correcting codes in quantum theory. *Physical Review Letters*, 77(5):793–797, July 1996. doi:10.1103/physrevlett.77.793.

44 Aarthi Sundaram, Robert Rand, Kartik Singhal, and Brad Lackey. Hoare meets Heisenberg: A lightweight logic for quantum programs. arXiv:2101.08939.

45 Alan Weinstein. *The symplectic "category"*, page 45–51. Springer Berlin Heidelberg, 1982. doi:10.1007/bfb0092426.

46 Anbang Wu, Gushu Li, Hezi Zhang, Gian Giacomo Guerreschi, Yuan Xie, and Yufei Ding. QECV: Quantum error correction verification. arXiv:2111.13728.

47 Mingsheng Ying. *Foundations of Quantum Programming*. Elsevier, 2016. doi:10.1016/c2014-0-02660-3.