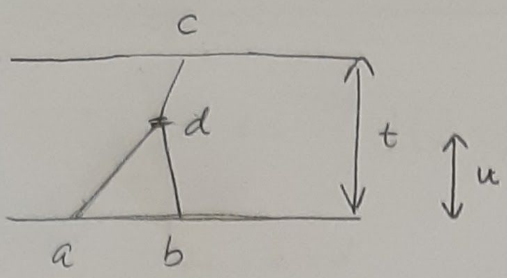

Rate matrix Q

$$P(b,c,d,u \mid a) = \left(e^{Qu}\right)_{ad} \left(e^{Qu}\right)_{db} \left(e^{Q(t-u)}\right)_{dc} \frac{e^{-u}}{\int_0^t e^{-v}\,dv}$$

$$= \frac{1}{1-e^{-t}}\, e^{-u} \left(e^{Qu}\right)_{ad} \left(e^{Qu}\right)_{db} \left(e^{Q(t-u)}\right)_{dc}$$

$$P(b,c,d \mid a) = \int_0^t P(b,c,d,u \mid a)\,du$$

$$P(b,c \mid a) = \sum_d P(b,c,d \mid a)$$

Jukes-Cantor rate matrix. Rate μ for mutation.

$$e^{Qu} = \frac{1}{4}\cancel{I} + e^{-u\mu}\,\frac{1}{4}\begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

$\frac{\mu}{4} = \frac{\theta}{3}$, i.e. $\mu = \frac{4}{3}\theta$

$$\begin{pmatrix} -\theta & \frac{\theta}{3} & \frac{\theta}{3} & \frac{\theta}{3} \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

Terms in the integral

$$e^{-u}\left(\tfrac{1}{4}+\alpha e^{-u\mu}\right)\left(\tfrac{1}{4}+\beta e^{-u\mu}\right)\left(\tfrac{1}{4}+\gamma e^{-(t-u)\mu}\right)$$

where

$$\alpha=\begin{cases}3/4 & a=d\\ -1/4 & a\neq d\end{cases}\qquad \beta=\begin{cases}3/4 & b=d\\ -1/4 & b\neq d\end{cases}\qquad \gamma=\begin{cases}3/4 & c=d\\ -1/4 & c\neq d\end{cases}$$

$$=e^{-u}\left(\tfrac{1}{16}+\tfrac{1}{4}(\alpha+\beta)e^{-u\mu}+\alpha\beta e^{-2u\mu}\right)\left(\tfrac{1}{4}+\gamma e^{-(t-u)\mu}\right)=$$

$$\tfrac{1}{64}e^{-u}\ +$$

$$\tfrac{1}{16}\gamma e^{-u}e^{-(t-u)\mu}\ +$$

$$\tfrac{1}{16}(\alpha+\beta)e^{-u}e^{-u\mu}\ +$$

$$\tfrac{1}{4}(\alpha+\beta)\gamma e^{-u}e^{-u\mu}e^{-(t-u)\mu}\ +$$

$$\tfrac{1}{4}\alpha\beta e^{-u}e^{-2u\mu}\ +$$

$$\alpha\beta\gamma e^{-u}e^{-2u\mu}e^{-(t-u)\mu}\ =$$

$$\tfrac{1}{64}e^{-u}+\tfrac{1}{16}\gamma e^{-t\mu}e^{(\mu-1)u}+\tfrac{1}{16}(\alpha+\beta)e^{-(1+\mu)u}\ +$$

$$\tfrac{1}{4}(\alpha+\beta)\gamma e^{-t\mu}e^{-u}+\tfrac{1}{4}\alpha\beta e^{-(1+2\mu)u}+\alpha\beta\gamma e^{-t\mu}e^{-(1+\mu)u}$$

All integrals are of the form

$$I(\varepsilon) = \int_0^t e^{-\varepsilon \mu}\, du = \int_0^t e^{-\varepsilon u}\, du$$
$_{,t}$

$$\underset{\varepsilon \neq 0}{=} \left[-\frac{1}{\varepsilon} e^{-\varepsilon u} \right]_0^t = \frac{1}{\varepsilon}\left(1 - e^{-\varepsilon t}\right)$$

$\varepsilon = 0$: $I(t, \varepsilon) = t$

$\mu \neq 1$!!

$$P(b,c,d \mid a) = \frac{1}{64} I(1) + \frac{1}{16} \gamma e^{-t\mu} I(1-\mu) +$$

$$\frac{1}{16}(\alpha+\beta) I(1+\mu) + \frac{1}{4}(\alpha+\beta)\gamma e^{-t\mu} I(1) +$$

$$\frac{1}{4}\alpha\beta\, I(1+2\mu) + \alpha\beta\gamma e^{-t\mu} I(1+\mu)$$

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computation : Given t, μ. Fill array 4^4

for a in 1:4
  b
  c
  d

  α = ifelse(a=d, 3/4, - 1/4)
  β b=d
  γ c=d