An Analysis of Astronomical Gravity Assists

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Due: April 25, 2016

Maneuvering objects throughout outer space requires precision and utilization of all available resources. Used in the correct fashion, Newton's law of universal gravitation becomes a resource available for moving objects. In order for scientists and engineers to move spacecraft through the far reaches of the solar system and beyond, gravitational assist maneuvers must be utilized.

A gravity assist, at its core, is simply a maneuver wherein a mass uses the gravity of a much larger mass in order to change its own velocity in a way such that the smaller mass never fully "succumbs" to the gravitational field of the larger mass. That is to say the smaller mass begins outside of the gravitational field of the larger mass, and ends its maneuver outside of the gravitational field of the larger mass. As a matter of semantics, the smaller mass shall henceforth be called the spacecraft and be denoted by the letter *m*, while the larger mass shall be called the planet and be denoted by the letter *M*. When viewing this mathematically, it proves valuable to view the kinematic equation relating velocity to time and acceleration

$$v_x(t) = v_{x,i} + a_x t. \tag{1}$$

Normally, the substitution $a_x = g = 9.8$ could be performed, however when dealing with motion very far away from the surface of the Earth, this is not the case. The acceleration during a gravity assist is not constant because the force due to gravity by the planet on the spacecraft increases as the spacecraft comes closer to the planet. In order to find out exactly what the acceleration is, one can find the force of gravity by a sphere-shaped planet on the spacecraft using Newton's Law of Gravity

$$F_{Mm}^{G} = G\frac{mM}{r^2} \tag{2}$$

where G is the gravitational constant $G = 6.6738 \times 10^{-11}$ N $\cdot \frac{m^2}{kg^2}$ and r is the distance between the spacecraft and the center of the spherical planet.

Using Newton's second law of motion shows that

$$\Sigma F_x = ma_x, \tag{3}$$

and for a situation in which the spacecraft is near the planet and no other large bodies, the only force being exerted on the spacecraft is the force of gravity by the planet so that

$$a_x = \frac{F_{Mm}^{G}}{m} = \frac{\left(\frac{GmM}{r^2}\right)}{m} = \frac{GM}{r_x^2}. \tag{4}$$

This equation verifies the assumption that all object fall at the same acceleration regardless of their mass, as *m* is not present in the final equation. Having solved for the acceleration of an object due to the gravitational field of a much larger object is important, as it is essentially solving for *g*, except this solution is portable in that one can find the pull of gravity by any planet on an object near that planet.

Continuing on, simple kinematic equations cannot be relied on, as objects hurtling throughout space require deeper analysis. Luckily, Newton has already solved this problem. Logsdon describes Newton's *vis-viva* equation as follows

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right). \tag{5}$$

Here, μ is known as the standard gravitational parameter, and can be replaced as follows

$$\mu = (M + m)G. \tag{6}$$

Meanwhile in equation 5, v is the relative speed between the spacecraft and the planet, which can simply be seen as the speed of the spacecraft from the planet's reference frame. r is again the distance between the two bodies, keeping in mind it is described as being the distance between the center of mass of both objects. This is simply the distance between each object's center, as they can be approximated as spheres. Finally, a is the semi-major axis (Logsdon 1998).

The semi-major axis is normally found by taking the longest diameter of an ellipse and dividing it by two. For a hyperbola, the semi-major axis becomes the distance between the center of the two vertices and the chosen vertex multiplied by -1. For an elliptical orbit, $a > 0$ and for a parabolic orbit, a becomes infinite. For gravity assists, however, the primary concern is with hyperbolic orbits, as an object in parabolic orbit will exit with the same amount of speed it enters with. For this reason, a will be negative for most gravity assist situations and can be calculated as follows

$$a = \frac{\mu}{2\varepsilon}. \tag{7}$$

In this equation, ε represents the specific orbital energy of the two objects, which relates their potential and kinetic energy. ε remains constant according to the *vis-visa* equation (also known as the orbital energy conservation equation) and can be calculated like so

$$\varepsilon = -\frac{\mu^2}{2h^2}(1 - e^2) \text{ (Logsdon 1998).} \tag{8}$$

Equation 8 once again represents new quantities, the first is e, the orbital eccentricity of m. The orbital eccentricity is the measure of how much the orbit differs from that of a perfect circle. For hyperbolic orbits, which are the primary concern for gravitational assists, the eccentricity will be larger than 1. The second new quantity is the specific relative angular momentum of the bodies, which is simply the angular momentum divided by the reduced mass, so that

$$h = \frac{L}{\left(\frac{Mm}{M+m}\right)}. \tag{9}$$

Predicting and designing motion for a gravity assist maneuver comes down to using the described *vis-visa* equation. Despite the motion of a gravity not resembling a circle, one can still treat part of the problem as if it were circular motion. For example, the only force acting on the spacecraft is the gravitational force by the planet and this force is pointing from the spacecraft to the center of the planet, so that means this force is a centripetal force. With this realization, equation 2 can be slightly adjusted

$$F_{Mm}^{G} = \frac{GMm}{r^2} = \frac{mv^2}{r}. \tag{10}$$

This means the total kinetic energy of the interaction becomes

$$K = \frac{GMm}{2r}. \tag{11}$$

This leads to the total mechanical energy of the interaction

$$E_{mech} = U^g + K = -\frac{GMm}{r} + \frac{GMm}{2r} = -\frac{GMm}{2r}. \tag{12}$$

Because the specific orbital energy is related to the system's total mechanical energy and reduced mass, and easier way to calculate the specific orbital energy is revealed

$$\varepsilon = \frac{E_{mech}}{\left(\frac{Mm}{M+m}\right)} = \frac{-\frac{GMm}{2r}}{\left(\frac{Mm}{M+m}\right)} = -\frac{G(M+m)}{2r}. \tag{13}$$

Gravity assists, sometimes called gravity slingshots, are used very often in space travel as they save quite a bit of fuel. Voyager 1 and 2 used a gravity assist via Jupiter to reach Uranus and Neptune, and Cassini used four assists to reach Saturn (Dykla et al. 2004). By flying by Venus twice, then Earth, then Jupiter, Cassini managed to save nearly 75 tons of fuel by utilizing gravity assists, which can be calculated using the *vis-visa* equation (Johnson 2003).

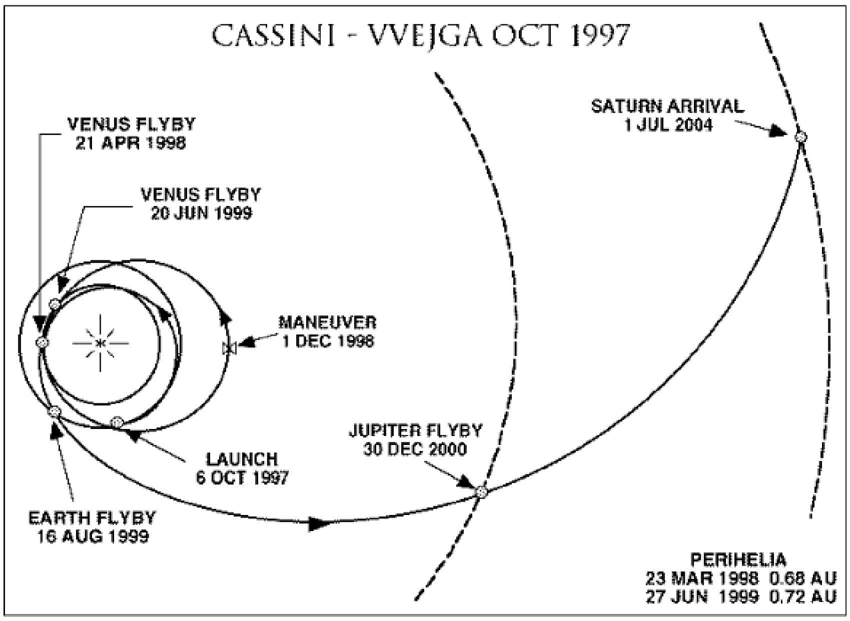

Figure 1: Trajectory for the Cassini spacecraft (Trajectory 2004).

In 1966, it was discovered that an optimum flight path, allowing a spacecraft to travel from Earth to Jupiter, Saturn, Uranus, and Neptune using gravity slingshots occurred once every 175 years due to the necessary positions for Uranus and Neptune (Flandro). This planetary alignment was to be later dubbed the Grand Tour. While the original Grand Tour plan of four spacecraft did not come to fruition, the project was salvaged through the Voyager program. Voyager 1 carried out its program of visiting Jupiter and Saturn’s moon Titan before proceeding into interstellar space. Voyager 2, however, completed the Grand Tour in full and is now traveling through interstellar space (Lieberman). The Grand Tour represents an ultimate showing of the power of gravitational assists to propel spacecraft throughout the solar system and beyond.

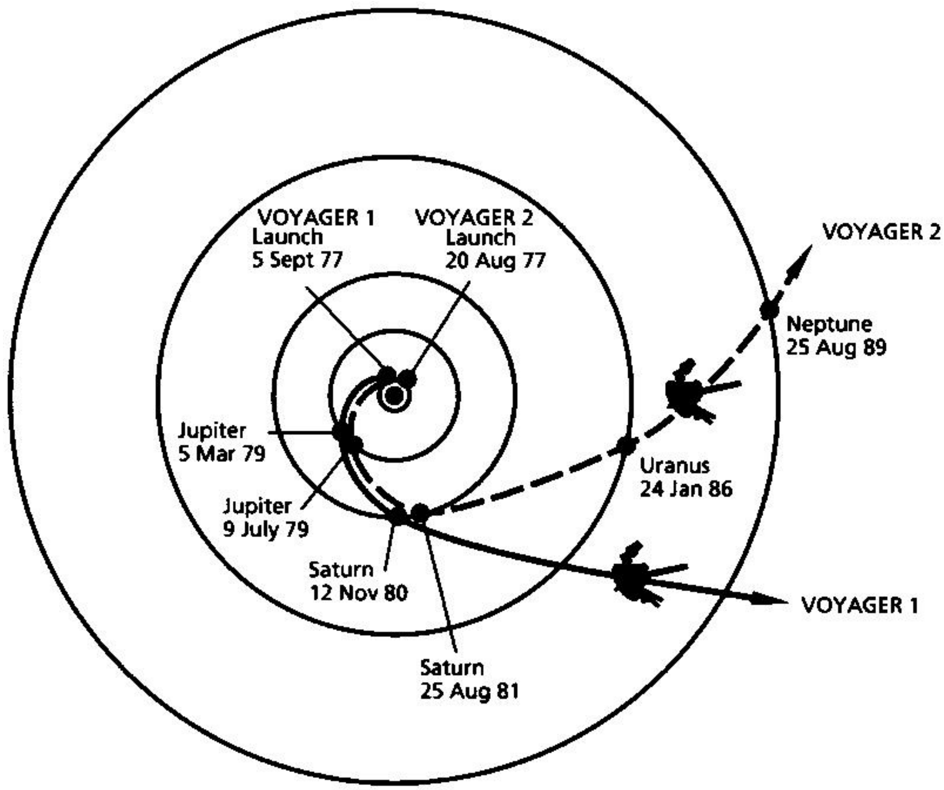

Figure 2: Flight Paths for Voyager 1 and Voyager 2 (Voyager).

An interesting thing to note for gravity assists, is that there is a trend for when a spacecraft will gain speed and when it will lose speed from a maneuver. When a spacecraft approaches the planet from behind, as can be seen in Fig. 2, the final speed of the spacecraft will increase. When a spacecraft approaches in front of the planet, as can be seen in Fig. 3, the final speed of the spacecraft will be lower than the initial speed. When analyzing the velocity triangles in the figures, it can be seen that the greatest possible speed that a spacecraft can gain is twice the speed of the planet. The required trajectory to gain the maximum possible speed will require the spacecraft to plunge directly into the planet from the opposite direction of the planet's motion (Johnson 2003).

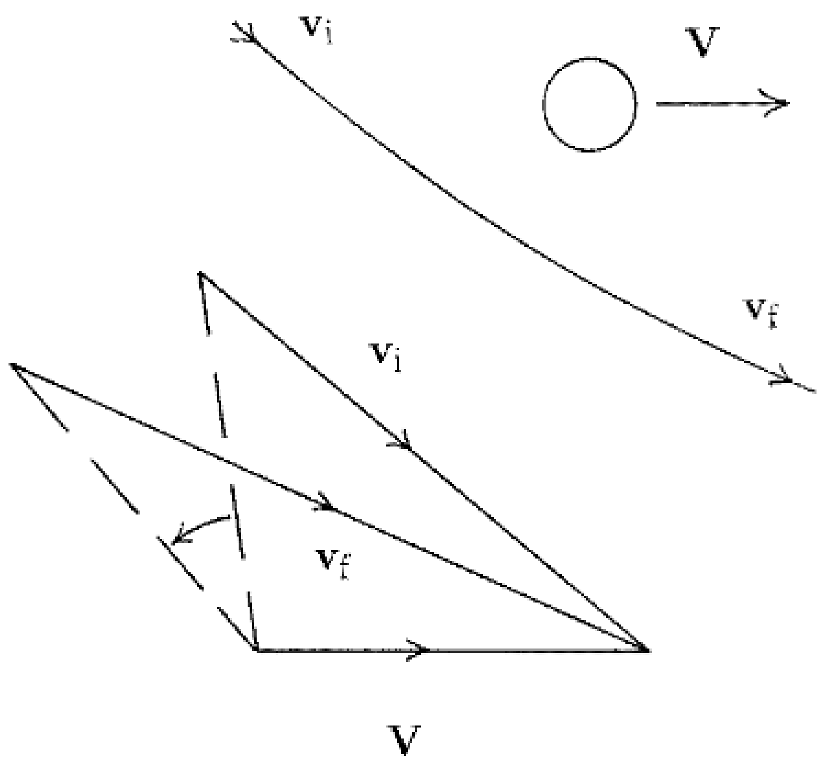

Figure 2: Velocity diagram for a gravity assist behind the planet. Notice that for both this figure, and figure 3 that V_i -V (dashed line) does not change in magnitude as orbital energy (earlier defined in equation 8) is conserved via the *vis-visa* equation (Johnson 2003).

With the realization that spacecraft can actually lose speed at a fast rate without burning any fuel, trajectory designers have much more freedom to launch a spacecraft into the vast outer space with confidence that the spacecraft will not run out of fuel before reaching its destination. When a spacecraft is intended to be launched into orbit for a planet far away from the Earth, it will need to lose kinetic energy in order to come to a bound orbit with the planet. In order to do this, one can burn quite a bit of fuel, or they can perform a gravity assist along that planet's moon in order to lose kinetic energy so that the total mechanical energy of the spacecraft and planet system is negative. This will allow the spacecraft to be placed in a bound orbit around the target planet while reserving its fuel supply for later use (Bartlett and Hord 1985).

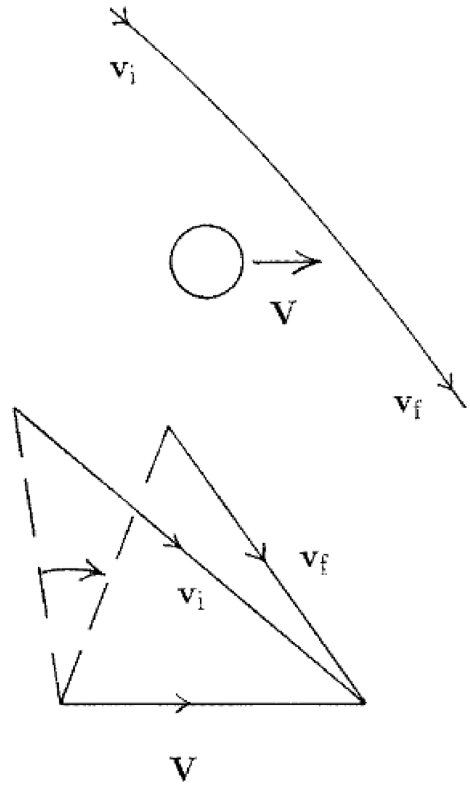

Figure 3: Velocity diagram for a gravity assist in front of planet. Notice a lower final speed (Johnson 2003).

In describing gravitational assists, it can be puzzling to see how one object, the spacecraft, seems to gain energy and momentum freely. This, however, is untrue. The total momenta of the system is the same after the interaction as it is before the interaction. Being a perfectly elastic collision in a closed system, the sum of the mechanical energy of the system consisting of the spacecraft and the planet is also the same after the interaction as it is beforehand (Van Allen 2003). This means that the spacecraft actually imparts an extremely small change in speed of the planet. This can be seen mathematically by using previously defined equations, such as equation 2, from the viewpoint of the spacecraft acting upon the planet. Because the mass of the planet is so much greater than the mass of the spacecraft, the impartation of lost velocity of the planet is extremely small and can be ignored for most cases.

Gravitational assists are examples of three-body problems. In the previous examples, the sun doesn't simply disappear to lend way to a simple two-body problem physically. However, in these approximations the impact of the sun on the system can be ignored. In this sense, gravity assists are examples of restricted three-body problems (Van Allen 2003). Just as the mass of the spacecraft in these examples is considered infinitesimal compared to the planets, the mass of the planets can be considered infinitesimal when compared to the sun. For this reason, when viewing the three-body problem as a whole, the sun has an extremely small force acting upon it and can be ignored.

While the effects of the sun are ignored for many gravitational assist problems, there are just as many times when the sun or other such large body can be useful for propelling spacecraft throughout solar systems. One such example presents itself as attempting to launch a spacecraft from Earth to Mars using a small amount of propellant, this is done through a Hohmann Transfer orbit. A Hohmann Transfer orbit is one in which the current orbit of an object about another is adjusted so that the furthest point of the orbit, the aphelion, is located at the spacecraft's target destination. This is done through an initial orbital adjustment of the spacecraft on Earth via escaping the Earth's orbit onto a heliocentric orbit with the correct velocity to reach Mars. While this is certainly not a hyperbolic orbit, it is still clear that the gravitational pull of the sun in this maneuver is a tool being utilized to propel a spacecraft onto a target destination, and as such is compatible with the previously defined *vis-viva* equation (Interplanetary).

Fuel use can be lowered even more so through the use of low-energy transfers. Low-energy transfers are transfers, or translations, in which the variation in speed of a spacecraft is minimized in order to transport the spacecraft across long distances using as little fuel as possible. Scientists and Engineers calculate the exact trajectory using computer simulations in backward-time. They pick a destination for the spacecraft and work backward to find exactly what needs to be done in order for the spacecraft to reach that point, usually resulting in initial small adjustments to velocity. The resulting trajectory for an experimental mission to place a spacecraft into lunar orbit from the Earth using a low-energy transfer required a total of 3,000 spirals away from the Earth and a 1.5 year travel time. This solution later inspired an actual mission, the ESA's SMART 1 in 2003 (Belbruno 2007, pp 49-53).

When performing low-energy transfers, it is important to note that the trajectories are not optimized for the typical chemical burning rocket engines that allow for Hohhman transfers. Instead, low-energy transfers utilize an engine that emits ionized atoms, typically xenon, in order to provide a small amount of acceleration for the spacecraft. While these engines are much weaker than chemical engines, they can be left on so that the velocity accumulates to an amount much greater than can be achieved of a typical chemical engine by utilizing electricity gained from solar panels (Belbruno 2007, pp. 26-28). These engines typically lead to much longer travel times because it takes quite a while for them to get up to speed, this makes them ideal for long distance travel and for unmanned voyages. Because these engines can only provide an extremely small amount of acceleration, they lend themselves well to gravity assist maneuvers in order to quickly change the velocity of the spacecraft.

Gravity assistance maneuvers are a testament to humankind's resourcefulness, proving that the same laws that limit one's view of the Earth are also able to be utilized to expand their view of the universe. With the aid of the *vis-viva* equation, charting a path through the solar system has become a possibility, one that leads to the actualization of missions. The usefulness of gravity slingshots has led to the production of equipment better suited to utilize Newton’s law of universal gravitation in the form of ion rocket engines. Gravity assists prove, through their continued use, to be an extremely effective maneuver and will continue to be the defining method for long distance space travel.

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