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Math 430

Part I:

Consider the vibrating string equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) \qquad x \in [0, \pi]$$

With boundary conditions

$$u(0,t) - \frac{\partial u}{\partial x}(0,t) = 0 \qquad\qquad 2u(\pi,t) + \frac{\partial u}{\partial x}(\pi,t) = 0$$

With initial conditions

$$u(x,0) = f(x) \qquad\qquad \frac{\partial u}{\partial t}(x,0) = 0 \qquad x \in [0, \pi]$$

The function $f : [0,\pi] \to \mathbb{R}$ is a continuous piecewise smooth function s.t $f(0) - f^{'}(0) = 0$ and $2\, f(\pi) + f^{'}(\pi) = 0$

Part II:

Use the separation of variables method to solve the given differential equation.

Using separation of variables:

$u(x,t) = S(x)T(t)$

Prove that the Sturm-Liouville eigenvalue problem for the space part does not have nonpositive eigenvalues. Denote by $\lambda_n = (\mu n)^2$ with $n \in N$ the positive eigenvalues

of this Sturm-Liouville eigenvalue problem. Explain graphically the locations of λ_n and $(\mu n)^2$. State clearly the formulas for a corresponding eigenfunctions. Name these eigenfunctions ϕ_n with n ∈ N, since you will need them to write the solution as an infinite series.

You get the eigenvalue problem:

$$\frac{d^2 S}{dx^2}(x) = -\lambda S(x)$$

$$\frac{d^2 T}{dt^2}(t) = -\lambda T(t)$$

Assume you have negative eigenvalues $\lambda = \mu^2$, $\mu < 0$ s.t the general solution is:

$$S(x) = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$$

$$S^{l}(x) = c_1\,\mu \sinh(\mu x) + c_2\,\mu \cosh(\mu x)$$

$$S(0) - S^{l}(0) = c_1 - [c_2\,\mu] = 0$$

$$2S(\pi) + S^{l}(\pi) = 2\,[c_1 \cosh(\mu\pi) + c_2 \sinh(\mu\pi)] + [c_1\,\mu \sinh(\mu\pi) + c_2\,\mu \cosh(\mu\pi)]$$

$$\text{“ ”} = 2\,c_1 \cosh(\mu\pi) + 2\,c_2 \sinh(\mu\pi) + c_1\,\mu \sinh(\mu\pi) + c_2\,\mu \cosh(\mu\pi)$$

$$\text{“ ”} = c_1\,\mu \sinh(\mu\pi) + 2\,c_1 \cosh(\mu\pi) + 2\,c_2 \sinh(\mu\pi) + c_2\,\mu \cosh(\mu\pi)$$

$$\text{“ ”} = c_1\,[\,\mu \sinh(\mu\pi) + 2 \cosh(\mu\pi)] + c_2[2 \sinh(\mu\pi) + \mu \cosh(\mu\pi)]$$

In[ ]:=
```
Clear[matri];
matri = {
    {1, -μ},
    {μ Sinh[μ π] + 2 Cosh[μ π], 2 Sinh[μ π] + μ Cosh[μ π]}
  };
matri // MatrixForm
Det[matri]
```

Which yields the formula for the eigenfunction that can be simplified:

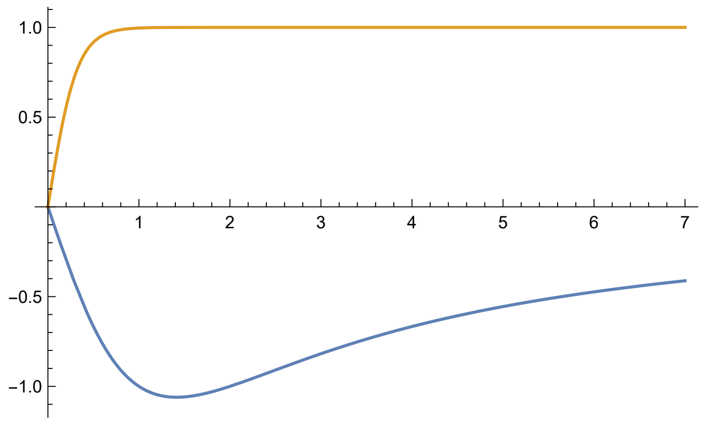
$$3\,\mu \cosh(\mu\pi) + 2 \sinh(\mu\pi) + \mu^2 \sinh(\mu\pi) = 0$$

$$3\,\mu \cosh(\mu\pi) + \sinh(\mu\pi)\left(2 + \mu^2\right) = 0$$

$$3\,\mu \cosh(\mu\pi) = -\sinh(\mu\pi)\left(2 + \mu^2\right)$$

$$-\frac{3\,\mu}{2+\mu^2} = \tanh(\mu\pi)$$

```
Plot[{-3 μ/(2 + μ^2), Tanh[μ π]}, {μ, 0, 7}]
```



This shows that there are no intersections for this equation. Therefore, only the trivial solution exists and has no negative eigenvalues.

Assume you have positive eigenvalues $\lambda = \mu^2$, $\mu > 0$ such that the general solution is:

$$S(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$$
$$S^l(x) = -c_1\, \mu \sin(\mu x) + c_2\, \mu \cos(\mu x)$$

```
Clear[tS];
tS[x_] = c₁ Cos[μ x] + c₂ Sin[μ x]
Cos[x μ] c₁ + Sin[x μ] c₂
FullSimplify[tS[0] - tS'[0], And[μ ∈ Constant]]
c₁ - μ c₂
FullSimplify[2 * tS[π] + tS'[π], And[μ ∈ Constant]]
Sin[π μ] (-μ c₁ + 2 c₂) + Cos[π μ] (2 c₁ + μ c₂)
```

Substituting in these new equations for the boundary conditions:

$$S(0) - S^l(0) = c_1 - c_2\, \mu = 0$$

$$2S(\pi) + S^l(\pi) = 2[c_1 \cos(\mu\pi) + c_2 \sin(\mu\pi)] +$$
$$[-c_1\, \mu \sin(\mu\pi) + c_2\, \mu \cos(\mu\pi)]$$

$$\text{“ ”} = 2$$
$$c_1 \cos(\mu\pi) + 2\, c_2 \sin(\mu\pi) - c_1\, \mu \sin(\mu\pi) + c_2\, \mu \cos(\mu\pi)$$

$$\text{“ ”} = c_1[2 \cos(\mu\pi) - \mu \sin(\mu\pi)] + c_2[2 \sin(\mu\pi) + \mu \cos(\mu\pi)]$$

Now what’s presented is a linear algebra problem where it’s needed to find a nontrivial solution for c_1 and c_2. To do this, you will take the determinant of the matrix and denote it our eigenfunction ϕ_n.

```
Clear[matri];
matri = {
    {1, -μ},
    {2 Cos[μ π] - μ Sin[μ π], μ Cos[μ π] + 2 Sin[μ π]}
   };
matri // MatrixForm
FullSimplify[Det[matri]]
```

$$\begin{pmatrix} 1 & -\mu \\ 2\,\text{Cos}[\pi\,\mu] - \mu\,\text{Sin}[\pi\,\mu] & \mu\,\text{Cos}[\pi\,\mu] + 2\,\text{Sin}[\pi\,\mu] \end{pmatrix}$$

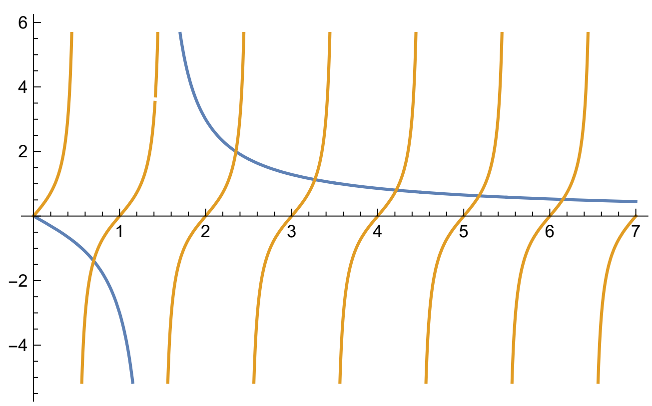
Setting the determinant to zero will yield an equation for intersections being nontrivial solutions:

$$3\,\mu\cos(\mu\pi) + \sin(\mu\pi)\left(2 - \mu^2\right) = 0$$

$$\sin(\mu\pi)\left(2 - \mu^2\right) = -3\,\mu\cos(\mu\pi)$$

$$\tan(\mu\pi) = -\frac{3\,\mu}{2-\mu^2}$$

```
Plot[{-3 μ/(2 - μ^2), Tan[μ π]}, {μ, 0, 7}]
```



Where there are infinitely many intersection points for infinitely many eigenvalues. Now it can be seen that in the first row, it must be that $c_1 = \mu,\ c_2 = 1$ where the eigenfunction is:

$$\phi_n(x) = \mu\cos(\mu x) + \sin(\mu x)$$

Now, I have to investigate the points where, $3\,\mu\cos(\mu\pi) + \sin(\mu\pi)\left(2 - \mu^2\right) = 0$.

A find root function will find an accurate root given an appropriate approximation:

```
(μ /. FindRoot[3 μ Cos[μ π] + (2 - μ^2) Sin[μ π] == 0, {μ, #}]) &[0.7]
```

```
0.6988439926220367`
```

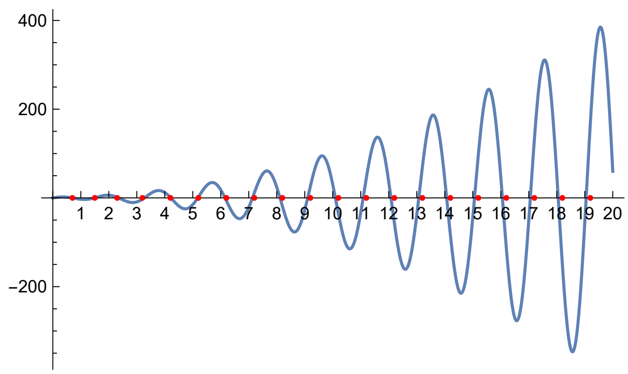
This is a list of 20 initial guesses as a basis.

```
Join[{.7, 1.5, 2.3, 3.2}, Range[4.2, 20, 1]]
```

```
{0.7`, 1.5`, 2.3`, 3.2`, 4.2`, 5.2`, 6.2`, 7.2`, 8.2`, 9.2`,
 10.2`, 11.2`, 12.2`, 13.2`, 14.2`, 15.2`, 16.2`, 17.2`, 18.2`, 19.2`}
```

To verify the accuracy of this:

```
Plot[3 μ Cos[μ π] + (2 - μ^2) Sin[μ π], {μ, 0, 20},
 Epilog → {{Red, Point[{#, 0}] & /@ Join[{.7, 1.5, 2.3, 3.2}, Range[4.2, 20, 1]]}},
 Ticks → {Range[0, 20, 1], Automatic}]
```



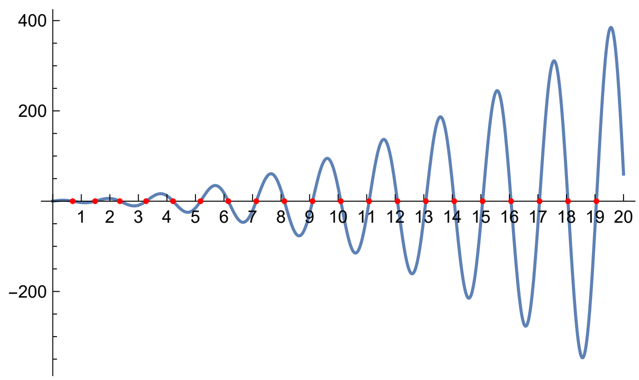
Now, I see that this is very accurate and will be a good starting point. I am able to map a more accurate root-finder to the set of guesses.

```
Clear[Teves];
Teves =
 Map[(μ /. FindRoot[3 μ Cos[μ π] + (2 - μ^2) Sin[μ π] == 0, {μ, #}]) &,
  Join[{.7, 1.5, 2.3, 3.2}, Range[4.2, 20, 1]]]
```

```
{0.6988439926220367`, 1.485288735668841`, 2.352246989756362`, 3.2692475403061545`,
 4.215162976142927`, 5.178052184283803`, 6.151358483910703`, 7.13138031152095`,
 8.115932059614435`, 9.103661943450714`, 10.093697462073768`, 11.085453868349026`,
 12.078526219688026`, 13.07262600312969 5`, 14.067542510634956`, 15.063118492518084`,
 16.05923434201292`, 17.05579754190259`, 18.05273545128162`, 19.049990268014447`}
```

I can now compare these new values to the initial approximations from above.

```
Plot[3 μ Cos[μ π] + (2 - μ^2) Sin[μ π], {μ, 0, 20},
 Epilog → {{Red, Point[{#, 0}] & /@ Teves}}, Ticks → {Range[0, 20, 1], Automatic}]
```



Now that this is accomplished I can return to the determinant equation and check these roots. The result is 20 eigenvalues the correspond to eigenvalues found

```
Clear[Tefs];
Tefs[x_] = (3 # Cos[# x] + Sin[2 # x]) & /@ Teves
{2.09653197786611` Cos[0.6988439926220367` x] + Sin[1.3976879852440733` x],
 4.4558662070065225` Cos[1.485288735668841` x] + Sin[2.970577471337682` x],
 7.056740969269086` Cos[2.352246989756362` x] + Sin[4.704493979512724` x],
 9.807742620918464` Cos[3.2692475403061545` x] + Sin[6.538495080612309` x],
 12.64548892842878` Cos[4.215162976142927` x] + Sin[8.430325952285854` x],
 15.53415655285141` Cos[5.178052184283803` x] + Sin[10.356104368567607` x],
 18.45407545173211` Cos[6.151358483910703` x] + Sin[12.30271696782140 5` x],
 21.394140934562852` Cos[7.13138031152095` x] + Sin[14.2627606230419` x],
 24.347796178843303` Cos[8.115932059614435` x] + Sin[16.23186411922887` x],
 27.31098583035214` Cos[9.103661943450714` x] + Sin[18.207323886901428` x],
 30.281092386221303` Cos[10.093697462073768` x] + Sin[20.187394924147537` x],
 33.256361605047076` Cos[11.085453868349026` x] + Sin[22.17090773669805` x],
 36.235578659064075` Cos[12.078526219688026` x] + Sin[24.157052439376052` x],
 39.21787800938908` Cos[13.072626003129695` x] + Sin[26.1452520062593 9` x],
 42.20262753190487` Cos[14.067542510634956` x] + Sin[28.135085021269912` x],
 45.18935547755425` Cos[15.063118492518084` x] + Sin[30.12623698503617` x],
 48.17770302603876` Cos[16.05923434201292` x] + Sin[32.11846868402584` x],
 51.16739262570776` Cos[17.05579754190259` x] + Sin[34.11159508380518` x],
 54.15820635384486` Cos[18.05273545128162` x] + Sin[36.10547090256324` x],
 57.14997080404334` Cos[19.049990268014447` x] + Sin[38.099980536028895` x]}
```

Verify the eigenvalue equation:

```
FullSimplify[Table[D[Tefs[x][[k]], {x, 2}] + (Teves[[k]])^2 Tefs[x][[k]], {k, 1, Length[Teves]}]]
{0.`, 0.`, 0.`, 0.`, 0.`, 0.`, 0.`, 0.`, 0.`,
 0.`, 0.`, 0.`, 0.`, 0.`, 0.`, 0.`, 0.`, 0.`, 0.`, 0.`}
```

Verify the boundary conditions (b.c):

```
Table[Chop[{(((Tefs2[x][[k]]) /. {x → 0}) + (D[Tefs2[x][[k]], {x, 1}]) /. {x → 0}),
   ((Tefs2[x][[k]]) /. {x → Pi})}], {k, 1, Length[Teves2]}]
{{1.3976879852440733`, 0.4024321094447939`}, {2.970577471337682`, -1.0675530511219595`},
 {4.704493979512724`, 1.9472653828420148`}, {6.538495080612309`, -2.9163288712439304`},
 {8.430325952285854`, 3.913933697596791`}, {10.356104368567607`, -4.919520326107797`},
 {12.302716967821405`, 5.9267211414211864`}, {14.2627606230419`, -6.93363810502905`},
 {16.23186411922887`, 7.939779894651503`}, {18.20732388690142`, -8.945098525811696`},
 {20.187394924147537`, 9.949676446360925`}, {22.17090773669805`, -10.953623126119197`},
 {24.157052439376052`, 11.957042022358607`}, {26.1452520062593`, -12.96002175084747`},
 {28.135085021269912`, 13.962635456926884`}, {30.12623698503617`, -14.964942693630617`},
 {32.11846868402584`, 15.966991790571862`}, {34.11159508380518`, -16.968822042751864`},
 {36.10547090256324`, 17.970465541947483`}, {38.099980536028895`, -18.97194864744179`}}
```

Verify if the eigenfunctions are mutually orthogonal:

```
Chop[Table[NIntegrate[Tefs[x][[j]] × Tefs[x][[k]], {x, 0, Pi}, MaxRecursion → 200,
   AccuracyGoal → 10, PrecisionGoal → 10], {j, 1, Length[Teves]}, {k, 1, Length[Teves]}]]

{{3.1695542638905314`, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 6.05270609377826`, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 11.447421777114457`, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 19.655186583157327`, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 30.84225592070824`, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 45.08997600994953`, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 62.43678721212305`, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 82.90144541390568`, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 106.49362568472772`, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 133.21861894407832`, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 163.0794787162756`, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 196.07805305976999`, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 232.2155076596170`, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 271.492604618836`, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 313.90985795629`, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 359.46762393040694`, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 408.1661554329887`, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 460.0056358132477`, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 514.9862005204802`, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 573.1079513107504`}}
```

The next thing needed to investigate is the given initial conditions:

```
Clear[Tf4];
Tf4[x_] = 1/5 (-1 - x - (π + 1/2 - 3/(2 π)) x^2 + x^3)
```

$$\frac{1}{5}\left(-1-x-\left(\frac{1}{2}-\frac{3}{2\pi}+\pi\right)x^2+x^3\right)$$

The output is true s.t it satisfies the conditions:

$f(0) - f'(0) = 0$ and $2f(\pi) + f'(\pi) = 0$

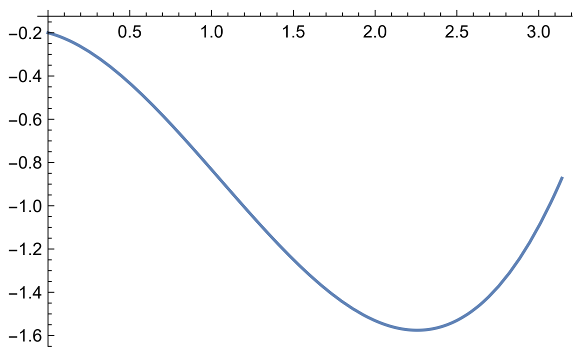
This function was chosen because it satisfies the B.C's

```
{Simplify[Tf4[0] - Jf4'[0] == 0], Simplify[2 * Tf4[π] + Tf4'[π] == 0]}
```

```
{1 + 5 Jf4′[0] == 0, True}
```

Now to collect the coefficients for the Fourier expansion from the given function:

```
Plot[Tf4[x], {x, 0, π}]
```

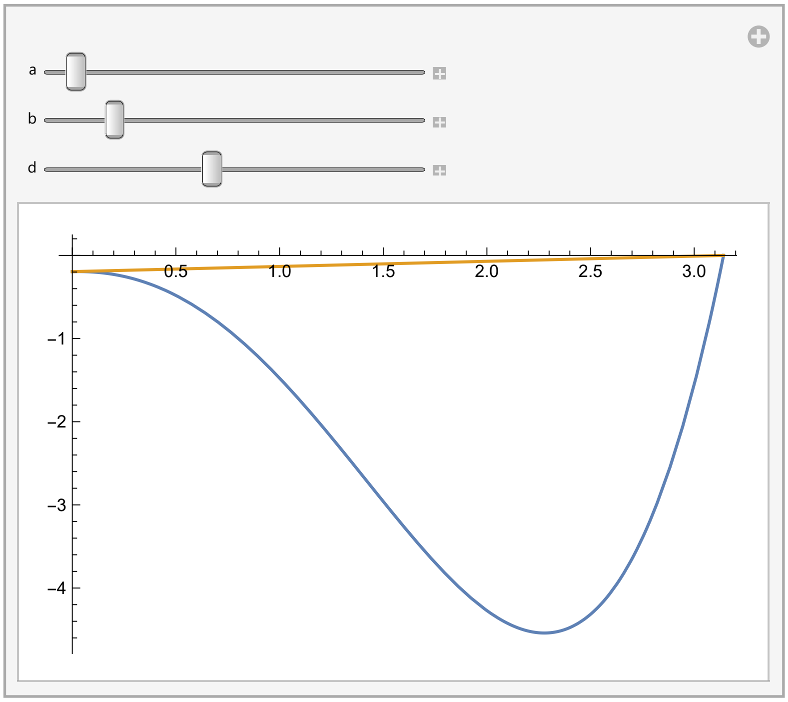



```
Clear[Tcoefs];
Tcoefs = Chop[Table[NIntegrate[(Tf4[x]) Tefs[x][[j]], {x, 0, π},
    MaxRecursion → 200, AccuracyGoal → 10, PrecisionGoal → 10], {j, 1, Length[Teves]}]]
```

```
{-2.125166478344659`, 7.0180660628580025`, -2.245677335834305`, 1.775519037559901`,
 -0.755105465535405`, 0.8788918989863862`, -0.3976748129087717`, 0.5748334065845673`,
 -0.2631056170640187`, 0.4277853692214557`, -0.1961340607133586`, 0.3415104121188204`,
 -0.15661970709431294`, 0.28469726980380905`, -0.1306161842134328`, 0.24436750419739628`,
 -0.11219097591941529`, 0.21420260427905313`, -0.09843131058455024`, 0.19075885561606132`}
```

Now the boundaries are completed, what's needed is the equation for the initial shape of the entire string.The initial shape of the flexible part of the string is given by:

$$f(x) = \frac{1}{5}\left(-1 - x - \left(\pi + \frac{1}{2} - \frac{3}{2\pi}\right)x^2 + x^3\right),\ x \in [0,\pi]$$



Part III:

The natural mode of vibration :

$$\frac{d^2 T}{dt^2}(t) = -\lambda T(t)$$

Which yields the fundamental set of solutions:

$$\cos(\mu_n ct) \quad \text{and} \quad \sin(\mu_n ct)$$

We are given that, with initial condition being t = 0, we will use the cosine time part from the eigenvalue problem.The natural mode of vibration is:

$$\cos(\mu_n ct)[\mu\cos(\mu x) + \sin(\mu x)]$$

Infinite series

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos(\mu_n ct)[c_1 \cos(\mu x) + c_2 \sin(\mu x)]$$

Where the a_n is from the initial condition s.t,

$$f(x) = \sum_{n=1}^{\infty} a_n[\cos(\mu x) + \sin(\mu x)] \quad \text{because } u(x,0) = f(x)$$

It can now be retrieved from a_n

$$a_n = \frac{\int_0^\pi f(x)\,\phi_n(x)\,dx}{\int_0^\pi \phi_n{}^2(x)\,dx}$$

(Dialog) In[82]:=

```
ff1[x_] = 1/5 (-1 - x - (π + 1/2 - 3/(2 π)) x^2 + x^3)
```

(Dialog) Out[82]=

$$\frac{1}{5}\left(-1 - x - \left(\frac{1}{2} - \frac{3}{2\pi} + \pi\right) x^2 + x^3\right)$$

(Dialog) In[82]:=

```
eigf[x_] = Cos[μ x] + Sin[μ x]
```

(Dialog) Out[82]=

Cos[x μ] + Sin[x μ]

(Dialog) In[82]:=

```
FullSimplify[Integrate[Expand[ff1[x] * eigf[x]], {x, 0, π}], And[μ ∈ Constant]]
```

(Dialog) Out[82]=

$$\frac{1}{10\,\pi\,\mu^4}\left(-6\,\mu + 2\,\pi\,\left(6 + \mu + 2\,\pi\,\mu + \mu^2 - \mu^3\right) + \left(6\,\mu + \pi\,\left(-12 + \mu\,\left(-2 + 2\,\mu\,(2 + \mu) + \pi^2\,\mu\,(2 + \mu) - \pi\,(-2 + \mu)\,(4 + \mu)\right)\right)\right) \mathrm{Cos}[\pi\,\mu] - \left(6\,\mu + \pi\,\left(12 + \mu\,\left(-2 + 8\,\pi - 2\,(2 + (-1 + \pi)\,\pi)\,\mu + (2 + (-1 + \pi)\,\pi)\,\mu^2\right)\right)\right) \mathrm{Sin}[\pi\,\mu]\right)$$

Illustration of vibrating string equation:

```
Manipulate[Plot[Evaluate[nmv4[x, t][[k]]], {x, 0, Pi},
  PlotStyle → {{Thickness[0.01], Blue}}, Epilog → {{PointSize[0.012], Point[{Pi, 0}]},
    {Red, Thickness[0.005], Line[{{4, 0}, {0, Evaluate[nmv4[0, t][[k]]]}}]},
    {PointSize[0.012], Point[{4, 0}]}
   }, PlotRange → {{-0.1, 4 + 0.1}, {-1.5, 1.5}}, AspectRatio → 1 / 5,
  Frame → True, FrameTicks → {{{}, {}}, {Join[{4}, Range[0, Pi, Pi / 4]], {}}},
  Axes → False, ImageSize → 600], {t, 0, N[2 Pi / eves4[[1]]], 0.05},
 {{k, 1}, Range[10], ControlType → Setter}]
```