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Lab 04 Hand Calculations

Q1 $\dfrac{dy}{dt} = -\dfrac{\tan^2\theta}{y^{1.5}} \dfrac{d^2}{4}\sqrt{2g}$

(a) for solution up to order error term of h^3

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2!}y''(t) + O(h^3)$$

finding the derivatives,

$$y' = \frac{dy}{dt} = \frac{-\tan^2\theta}{y^{1.5}} \frac{d^2}{4}\sqrt{2g}$$

$$y'' = \frac{d}{dt}\left(\frac{dy}{dt}\right) = -\tan^2\theta \cdot \frac{d^2}{4}\sqrt{2g} \cdot -1.5y^{-2.5}$$

$$= \frac{1.5\tan^2\theta}{y^{2.5}} \frac{d^2}{4}\sqrt{2g}$$

Hence,

~~y(t+h) = y~~

$$y_{i+1} = y_i + h\left(\frac{-\tan^2\theta}{y^{1.5}} \frac{d^2}{4}\sqrt{2g}\right) + \frac{h^2}{2!}\left(\frac{1.5\tan^2\theta}{y^{2.5}} \frac{d^2}{4}\sqrt{2g}\right) + O(h^3)$$

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Q3. $m\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + kx = 0$

(a) Reducing to system of 1^{st} order ODEs

$$\frac{dx}{dt} = v \rightarrow (1)$$

$$\frac{dv}{dt} = -\frac{kx}{m} - \frac{c}{m}v \rightarrow (2)$$

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(b) Sub $k = 0.5$, $c = 1.5$, $m = 1$ ~~and~~

$$\frac{d^2x}{dt^2} + 1.5\frac{dx}{dt} + 0.5x = 0$$

$x(0) = x_0 = 1.5$

$v(0) = 0$

$\Delta t = 0.1\,s$

$t = 0\,s \rightarrow t = 0.1\,s$

$$x_{i+1} = x_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = f_1(t, z)$$

$$* \; v_{i+1} = v_i + \frac{h}{6}(g_1 + 2g_2 + 2g_3 + g_4) = f_2(t, x, v)$$

$$f_2(t, x, v) = -\frac{k}{m}x - \frac{c}{m}v$$

$$= -0.5x - 1.5v$$

for $i = 0$

$k_1 = f_1(t_0, v_0)$

$= 0$

$g_1 = f_2(t_0, x_0, v_0)$

$= -\frac{k}{m}x_0 - \frac{c}{m}(0) = -0.75$

$k_2 = f_1\left(t_0 + \frac{h}{2}, v_0 + \frac{g_1 h}{2}\right)$

$= 0 + (-0.75)\frac{0.1}{2}$

$= -0.0375$

$g_2 = f_2\left(t_0 + \frac{h}{2}, x_0 + \frac{hk_1}{2}, v_0 + \frac{g_1 h}{2}\right)$

$= -0.5(1.5 + 0) - 1.5\left(0 - \frac{0.75 \times 0.1}{2}\right)$

$= -0.693750$

$k_3 = f_1\left(t_0 + \frac{h}{2}, v_0 + \frac{g_2 h}{2}\right)$

$= 0 - \frac{0.69375 \times 0.1}{2}$

$= -0.034688$

$g_3 = f_2\left(t_0 + \frac{h}{2}, x_0 + \frac{k_2 h}{2}, v_0 + \frac{g_2 h}{2}\right)$

$= -0.5\left(1.5 + \frac{(-0.0375)0.1}{2}\right) - 1.5\left(\frac{-0.69375 \times 0.1}{2}\right)$

$= -0.697031$

$k_4 = f_1(t_0 + h, v_0 + g_3 h)$

$= 0 + (-0.697031)0.1$

$= -0.0697031$

$g_4 = f_2\left(t_0 + \cancel{\frac{h}{2}}, x_0 + \cancel{\frac{k_3 h}{2}}, v_0 + \cancel{\frac{g_3 h}{2}}\right)$

$= -0.5(1.5 + (-0.034688)0.1)$

$- 1.5(-0.697031)0.1$

$= -0.643711$

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Substituting $\rightarrow$ v_{i+1} and x_{i+1}

$$x_1 = 1.5 + \frac{0.1}{6}\left(0 + 2(-0.0375 - 0.034688) - 0.0697031\right)$$

$$= 1.496432$$

$$z_1 = 0 + \frac{0.1}{6}\left(-0.75 + 2(-0.69375 - 0.697031) - 0.643711\right)$$

$$= -0.0695\cancel{8788}\,8$$

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Q5 $\frac{dy}{dt} = -3y + 6e^{-t}$ $\qquad y(0) = 1.0$

(a) $y_{i+1} = y_i + h(-3y) + h(6e^{-t})$

$y_i^c = y_i + \varepsilon$

$$y_{i+1}^c = y_i^{(c)}(1-3h)^i \longrightarrow (1-3h)^i(y_i^{(c)} + \varepsilon_0)$$

$$(1-3h)^i y_i + \underbrace{(1-3h)^i \varepsilon_0}_{E_t}$$

for $E_t \to 0$, $i \to \infty$

$|1-3h| < 1$

~~-1 < 1-3h < 1~~ $\qquad -1 < 1-3h < 1$

$-2 < -3h < 0$

$\frac{2}{3} > h > 0 \implies 0 < h < \frac{2}{3}$

Alternative Method: (from lecture slides)

$\frac{dy}{dt} = f(t,y) \longleftrightarrow \frac{dy}{dt} = Ky$

approximated as above,

$f(t,y) = Ky$

Chain Rule,

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(Ky) = \frac{\partial}{\partial y}\cdot K + y \cdot \cancel{\frac{\partial K}{\partial y}}^{\,0}$$ since K is constant

$$\frac{\partial f}{\partial y} = K$$

from equation given , $f(t,y) = \dfrac{dy}{dt} = -3y + 6e^{-t}$

$$\frac{\partial f}{\partial y} = -3$$

$\therefore\ K = -3$

$$-2 < h\frac{\partial f}{\partial y} < 0$$

obtained similar to shown in previous pg (previous solution)

$$-2 < hK < 0$$

$$-2 < -3h < 0$$

$$\frac{2}{3} > h > 0 \implies 0 < h < \frac{2}{3}$$

(Same as previous method)

(b) $y_{i+1} = y_i + hf(t_{i+1}, y_{i+1})$

$$= y_i + h(Ky_{i+1}) + h(6e^{-t_{i+1}})$$

$$y_{i+1} = \left(\frac{1}{1-hK}\right)^i y_i + E_t$$

$$\left(\frac{1}{1-hK}\right)^i \rightarrow 0$$

$$-1 < \frac{1}{1-hK} < 1 \longrightarrow \frac{1}{1-hK} > -1$$

$$-1 + hK < 1$$

$$hK < 2$$

$$\frac{1}{1-hK} < 1$$

$$1 - hK > 1$$

$$-hK > 0$$

$$hK < 0$$

Sub $K = -3$,

$\cancel{h<}$ $h > \frac{2}{3}$, $h > 0$

$\therefore$ always stable