# Lab 5 Submission

Q4. $$\frac{\partial q}{\partial t} = K \frac{\partial^2 q}{\partial x^2}$$

$$\frac{q_j^{(n+1)} - q_j^{(n)}}{\delta t} = \frac{K}{2} \left\{ \frac{q_{j+1}^{(n+1)} - 2q_j^{(n+1)} + q_{j-1}^{(n+1)}}{\delta x^2} + \frac{q_{j+1}^{(n)} - 2q_j^{(n)} + q_{j-1}^{(n)}}{\delta x^2} \right\}$$

$$q_j^{(n+1)} - q_j^{(n)} = \frac{K\delta t}{2\delta x^2} \left\{ q_{j+1}^{(n+1)} - 2q_j^{(n+1)} + q_{j-1}^{(n+1)} + q_{j+1}^{(n)} - 2q_j^{(n)} + q_{j-1}^{(n)} \right\}$$

let $q_j^{(n)} = A^n e^{ikj\delta x}$,

$$A^{n+1} e^{ikj\delta x} - A^n e^{ikj\delta x} = \frac{K\delta t}{2\delta x^2} \left\{ A^{n+1} e^{ik(j+1)\delta x} - 2A^{n+1} e^{ikj\delta x} + A^{n+1} e^{ik(j-1)\delta x} + A^n e^{ik(j+1)\delta x} - 2A^n e^{ikj\delta x} + A^n e^{ik(j-1)\delta x} \right\}$$

Divide both sides by $A^n e^{ikj\delta x}$,

$$(A-1) = \frac{K\delta t}{2\delta x^2} \left\{ Ae^{ik\delta x} - 2A + Ae^{-ik\delta x} + e^{ik\delta x} - 2 + e^{-ik\delta x} \right\}$$

Using the Euler Formula $\cos(k\delta x) = \frac{1}{2}(e^{ik\delta x} + e^{-ik\delta x})$,

$$(A-1) = \frac{K\delta t}{2\delta x^2} \left\{ 2A\cos(k\delta x) + 2\cos(k\delta x) - 2(A+1) \right\}$$

$$= \frac{K\delta t}{\delta x^2} \left\{ A\cos(k\delta x) + \cos(k\delta x) - (A+1) \right\}$$

$$(A-1) = \frac{K\delta t}{\delta x^2} \cos(k\delta x)\left[A+1\right] - \frac{K\delta t}{\delta x^2}\left[A+1\right]$$

$$\frac{\delta x^2}{\delta t} = K\left(\frac{A+1}{A-1}\right)\left[\cos(k\delta x) - 1\right]$$

Using the double angle formula,

$$\frac{\delta x^2}{\delta t} = K\left(\frac{A+1}{A-1}\right)\left[-2\sin^2\left(\frac{k\delta x}{2}\right)\right]$$

$$A\delta x^2 - \delta x^2 = -2AK\delta t\sin^2\left(\frac{k\delta x}{2}\right) - 2K\sin^2\left(\frac{k\delta x}{2}\right)\delta t$$

$$A\delta x^2 + 2AK\sin^2\left(\frac{k\delta x}{2}\right)\delta t = \delta x^2 - 2K\sin^2\left(\frac{k\delta x}{2}\right)\delta t$$

$$A = \frac{\delta x^2 - 2K\sin^2\left(\frac{k\delta x}{2}\right)\delta t}{\delta x^2 + 2K\sin^2\left(\frac{k\delta x}{2}\right)\delta t} \Rightarrow \frac{1 - \frac{2K\delta t}{\delta x^2}\sin^2\left(\frac{k\delta x}{2}\right)}{1 + \frac{2K\delta t}{\delta x^2}\sin^2\left(\frac{k\delta x}{2}\right)}$$

Q2. $$\frac{d^2q}{dx^2} - \sin^2 x \frac{dq}{dx} + \lambda q = 0 \quad , \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2} \quad \text{with BC: } q\left(-\frac{\pi}{2}\right) = 0 \text{ \& } q\left(\frac{\pi}{2}\right) = 0$$

(a) $$\frac{d^2q}{dx^2} = \frac{q_{i+1} - 2q_i + q_{i-1}}{\Delta x^2}$$

$$\frac{dq}{dx} = \frac{q_{i+1} - q_{i-1}}{2\Delta x}$$

Since $\Delta x = h$,

$2h^2$

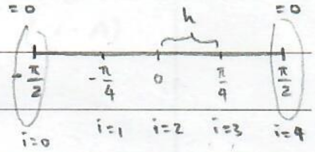
$$\frac{q_{i+1} - 2q_i + q_{i-1}}{h^2} - \sin^2 x_i \frac{q_{i+1} - q_{i-1}}{2h} + \lambda q_i = 0$$

$$2(q_{i+1} - 2q_i + q_{i-1}) - h\sin^2 x_i (q_{i+1} - q_{i-1}) + 2\lambda h^2 q_i = 0$$

$$2q_{i+1} - h\sin^2 x_i\, q_{i+1} = -2\lambda h^2 q_i - h\sin^2 x_i\, q_{i-1} + 4q_i - 2q_{i-1}$$

$$q_{i+1}(2 - h\sin^2 x_i) = (-2\lambda h^2 + 4)q_i + (-h\sin^2 x_i - 2)q_{i-1}$$

$$q_{i+1} = \frac{(-2\lambda h^2 + 4)q_i + (-h\sin^2 x_i - 2)q_{i-1}}{(2 - h\sin^2 x_i)}$$




(b) for $i = 1$, $$\frac{q_2 - 2q_1 + \overset{0}{\cancel{q_0}}}{h^2} - \sin^2 x_i \frac{q_2 - \overset{0}{\cancel{q_0}}}{2h} + \lambda q_1 = 0$$

$$\left(\frac{\sin^2 x_1}{2h} - \frac{1}{h^2}\right)q_2 + \left(\frac{2}{h^2}\right)q_1 = \lambda q_1$$

for $i = 2$, $$\frac{q_3 - 2q_2 + q_1}{h^2} - \sin^2 x_i \frac{q_3 - q_1}{2h} + \lambda q_2 = 0$$

$$\left(\frac{\sin^2 x_2}{2h} - \frac{1}{h^2}\right)q_3 + \left(\frac{2}{h^2}\right)q_2 + \left(-\frac{\sin^2 x_2}{2h} - \frac{1}{h^2}\right)q_1 = \lambda q_2$$

for $i = 3$, $$\frac{\overset{0}{\cancel{q_4}} - 2q_3 + q_2}{h^2} - \sin^2 x_3 \frac{\overset{0}{\cancel{q_4}} - q_2}{2h} + \lambda q_3 = 0$$

$$\left(\frac{2}{h^2}\right)q_3 + \left(-\frac{1}{h^2} - \frac{\sin^2 x_3}{2h}\right)q_2 = \lambda q_3$$

Forming the matrix,

$$\begin{bmatrix} \frac{2}{h^2} & \frac{\sin^2 x_1}{2h} - \frac{1}{h^2} & 0 \\ -\frac{\sin^2 x_2}{2h} - \frac{1}{h^2} & \frac{2}{h^2} & \frac{\sin^2 x_2}{2h} - \frac{1}{h^2} \\ 0 & -\frac{1}{h^2} - \frac{\sin^2 x_3}{2h} & \frac{2}{h^2} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \lambda \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

1. Since $f = At^2 + Bt + C$ * Q2C on last pg.

$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} f \, dt$$

polynomial fit :
$$\begin{cases} A t_{i-1}^2 + B t_{i-1} + C = f_{i-1} \\ A t_i^2 + B t_i^2 + C = f_i \\ A t_{i+1}^2 + B t_i^2 + C = f_{i+1} \end{cases}$$

Since $t_i = 0$, $t_{i+1} = h$ and $t_{i-1} = -h$,

$$\begin{cases} Ah^2 - Bh + C = f_{i-1} & \rightarrow (1) \\ C = f_i & \rightarrow (2) \\ Ah^2 + Bh + C = f_{i+1} & \rightarrow (3) \end{cases}$$

(1) + (3) : $2Ah^2 + 2C = f_{i-1} + f_{i+1}$

$$2Ah^2 = f_{i-1} + f_{i+1} - 2f_i$$

$$A = \frac{f_{i-1} + f_{i+1} - 2f_i}{2h^2}$$

(1) - (3) : $-2Bh = f_{i-1} - f_{i+1}$

$$B = \frac{f_{i-1} - f_{i+1}}{-2h} = \frac{f_{i+1} - f_{i-1}}{2h}$$

$$C = f_i$$

Substituting into main equ. above,

$$y_{i+1} - y_i = \int_{t_i}^{t_{i+1}} \left[ \left( \frac{f_{i-1} - 2f_i + f_{i+1}}{2h^2} \right) t^2 + \left( \frac{f_{i+1} - f_{i-1}}{2h} \right) t + f_i \right] dt$$

$$= \frac{f_{i-1} - 2f_i + f_{i+1}}{2h^2} \cdot \left[\frac{t^3}{3}\right]_{t_i}^{t_{i+1}} + \frac{f_{i+1} - f_{i-1}}{2h} \cdot \left[\frac{t^2}{2}\right]_{t_i}^{t_{i+1}} + f_i \left[t\right]_{t_i}^{t_{i+1}}$$

Since $t_{i+1} = +h$ and $t_i = 0$,

$$= \frac{f_{i-1} - 2f_i + f_{i+1}}{2h^2} \cdot \frac{h^3}{3} + \frac{f_{i+1} - f_{i-1}}{2h} \cdot \frac{h^2}{2} + f_i \cdot h$$

$$= \frac{f_{i-1} - 2f_i + f_{i+1}}{6} h + \frac{f_{i+1} - f_{i-1}}{4} h + f_i h$$

$$= \left(\frac{5f_{i+1} + 8f_i - f_{i-1}}{12}\right) h$$

Q5. $$q_j^{n+1} = q_j^n + \frac{K}{2}\frac{\delta t}{\delta x^2}\left[(1-\theta)(q_{j+1}^n - 2q_j^n + q_{j-1}^n) + (1+\theta)(q_{j+1}^{n+1} - 2q_j^{n+1} + q_{j-1}^{n+1})\right]$$

Rearranging,

$$q_j^{n+1} - \frac{K}{2}\frac{\delta t}{\delta x^2}(q_{j+1}^{n+1} - 2q_j^{n+1} + q_{j-1}^{n+1})(1+\theta) = q_j^n + \frac{K}{2}\frac{\delta t}{\delta x^2}(q_{j+1}^n - 2q_j^n + q_{j-1}^n)(1-\theta)$$

let $A = \frac{K}{2}\frac{\delta t}{\delta x^2}(1+\theta)$ and $B = \frac{K}{2}\frac{\delta t}{\delta x^2}(1-\theta)$

$$q_j^{n+1} - Aq_{j+1}^{n+1} + 2Aq_j^{n+1} - Aq_{j-1}^{n+1} = q_j^n + Bq_{j+1}^n - 2Bq_j^n + Bq_{j-1}^n$$

$$-Aq_{j-1}^{n+1} + (1+2A)q_j^{n+1} - Aq_{j+1}^{n+1} = Bq_{j-1}^n + (1-2B)q_j^n + Bq_{j+1}^n$$

for node $j=1$,

$$-Aq_0^{n+1} + (1+2A)q_1^{n+1} - Aq_2^{n+1} = Bq_0^n + (1-2B)q_1^n + Bq_2^n$$

due to boundary condition $q(0,t) = q_0 = 1$,

$$(1+2A)q_1^{n+1} - Aq_2^{n+1} = (1-2B)q_1^n + Bq_2^n + (A+B)q_0$$

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for node $j=4$,

$$-Aq_3^{n+1} + (1+2A)q_4^{n+1} - Aq_5^{n+1} = Bq_3^n + (1-2B)q_4^n + Bq_5^n$$

Hence all intermediate nodes from $j=2$ to $j=M-1$ have the following form,

$$-Aq_{j-1}^{n+1} + (1+2A)q_j^{n+1} - Aq_{j+1}^{n+1} = Bq_{j-1}^n + (1-2B)q_j^n + Bq_{j+1}^n$$

for node $j=M$,

$$-Aq_{M-1}^{n+1} + (1+2A)q_M^{n+1} - Aq_{M+1}^{n+1} = Bq_{M-1}^n + (1-2B)q_M^n + Bq_{M+1}^n$$

Since the boundary condition on the R.H.S is Neumann's, we do not have a fixed value to use. We approximate to find $M+1$ using central finite difference method :

$$\frac{dq_M}{dx} = \frac{q_{M+1} - q_{M-1}}{2dx}$$

Since given that $\left.\frac{dq}{dx}\right|_{x=10} = 0$,

$$\frac{q_{M+1} - q_{M-1}}{2dx} = 0$$

$$q_{M+1} = q_{M-1}$$

$$-Aq_{M-1}^{n+1} + (1+2A)q_M^{n+1} - Aq_{M-1}^{n+1} = Bq_{M-1}^{n} + (1-2B)q_M^{n} + Bq_{M-1}^{n}$$

$$-2Aq_{M-1}^{n+1} + (1+2A)q_M^{n+1} = 2Bq_{M-1}^{n} + (1-2B)q_{M+1}^{n}$$

Forming the matrix,

$$[L] = \begin{bmatrix} 1+2A & -A & & & \\ -A & 1+2A & -A & & \\ & \ddots & \ddots & \ddots & \\ & & -A & 1+2A & -A \\ & & & -2A & 1+2A \end{bmatrix}$$

$$= \begin{bmatrix} 1+K\frac{dt}{\delta x^2}(1+\theta) & -\frac{K}{2}\frac{dt}{\delta x^2}(1+\theta) & & & \\ -\frac{K}{2}\frac{dt}{\delta x^2}(1+\theta) & 1+K\frac{dt}{\delta x^2}(1+\theta) & & & \\ & & \ddots & & \\ & & -\frac{K}{2}\frac{dt}{\delta x^2}(1+\theta) & 1+K\frac{dt}{\delta x^2}(1+\theta) & -\frac{K}{2}\frac{dt}{\delta x^2}(1+\theta) \\ & & & -K\frac{dt}{\delta x^2}(1+\theta) & 1+K\frac{dt}{\delta x^2}(1+\theta) \end{bmatrix}$$

$$[R] = \begin{bmatrix} 1-2B & B & & & \\ B & 1-2B & B & & \\ & \ddots & \ddots & \ddots & \\ & & +B & 1-2B & B \\ & & & 2B & 1-2B \end{bmatrix}$$

$$[R] = \begin{bmatrix} 1-K\frac{\delta t}{\delta x^2}(1-\theta) & \frac{K}{2}\frac{\delta t}{\delta x^2}(1-\theta) & & & \\ \frac{K}{2}\frac{\delta t}{\delta x^2}(1-\theta) & 1-K\frac{\delta t}{\delta x^2}(1-\theta) & \frac{K}{2}\frac{\delta t}{\delta x^2}(1-\theta) & & \\ & & \ddots & & \\ & & \frac{K}{2}\frac{\delta t}{\delta x^2}(1-\theta) & 1-K\frac{\delta t}{\delta x^2}(1-\theta) & \frac{K}{2}\frac{\delta t}{\delta x^2}(1-\theta) \\ & & & K\frac{\delta t}{\delta x^2}(1-\theta) & 1-K\frac{\delta t}{\delta x^2}(1-\theta) \end{bmatrix}$$

$$[BC] = \begin{bmatrix} (A+B)q_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} K\frac{\delta t}{\delta x^2} q_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Q2C. $$\frac{dq}{dx} = k$$

$$\frac{dk}{dx} = (\sin^2 x)k - \lambda q$$

boundary conditions are $q\left(-\frac{\pi}{2}\right) = 0$ and $q\left(\frac{\pi}{2}\right) = 0$

The other condition has to be guessed and this can be of any value as the solution of eigenvalues itself will not be any different. The only difference will be the amplitude of the curve which does not concern us for this problem

$$k\left(-\frac{\pi}{2}\right) = 1$$