Rivyesch Ranjan (29392004) Lab 6 – Hand Calculations

1.(a) central diff space : $$\left.\frac{\partial q}{\partial x}\right|_j^n = \frac{q_{j+1}^{(n)} - q_{j-1}^{(n)}}{2\Delta x} + O(\Delta x^2)$$

backward diff time : $$\left.\frac{\partial q}{\partial t}\right|_j^n = \frac{q_j^{(n)} - q_j^{(n-1)}}{\Delta t} + O(\Delta t)$$

$$\frac{q_j^{(n)} - q_j^{(n-1)}}{\Delta t} + u\,\frac{q_{j+1}^{(n)} - q_{j-1}^{(n)}}{2\Delta x} = 0$$

$$q_j^{(n+1)} - q_j^{(n)} = -\frac{u\Delta t}{2\Delta x}\left(q_{j+1}^{(n+1)} - q_{j-1}^{(n+1)}\right)$$ ← shifted time step by 1 ; $n-1 \to n$, $n \to n+1$

$$q_j^{n+1} + \frac{u\Delta t}{2\Delta x}\left(q_{j+1}^{(n+1)} - q_{j-1}^{(n+1)}\right) = q_j^n$$

(b) let $u_j^{(n)} = A^n e^{ikj\Delta x}$ and let $C = \frac{u\Delta t}{2\Delta x}$,

$$A^{n+1}e^{ikj\Delta x} + C\left(A^{n+1}e^{ik(j+1)\Delta x} - A^{n+1}e^{ik(j-1)\Delta x}\right) = A^n e^{ikj\Delta x}$$

Divide both sides by $A^n e^{ikj\Delta x}$,

$$A + C\left(Ae^{ik\Delta x} - Ae^{-ik\Delta x}\right) = 1$$

Since $\sin(k\Delta x) = \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2}$,

$$A + CA\,(2i\sin(k\Delta x)) = 1$$

$$A = \frac{1}{1 + 2C\sin(k\Delta x)\,i}$$

$$|A| = \frac{1}{\sqrt{1^2 + 4C^2\sin^2(k\Delta x)}}$$

$$= \frac{1}{\sqrt{1 + \left(\frac{u\Delta t}{\Delta x}\right)^2 \sin^2(k\Delta x)}}$$

Since $\sin^2(k\Delta x)$ ranges from 0 to 1 – taking the maximum, the denominator will always be greater than 1. Hence, $|A|$ will be less than 1 always

Q2. (a) For $u > 0$,

$$\frac{q_j^{(n+1)} - q_j^{(n)}}{\Delta t} + u \frac{q_j^{(n)} - q_{j-1}^{(n)}}{\Delta x} = 0$$

$$q_j^{(n+1)} = q_j^{(n)} - u\Delta t \frac{q_j^{(n)} - q_{j-1}^{(n)}}{\Delta x}$$

(b) let $u_j^{(n)} = A^n e^{ikj\Delta x}$, and let $c = \frac{u\Delta t}{\Delta x}$,

$$A^{n+1} e^{ikj\Delta x} = A^n e^{ikj\Delta x} - c\left(A^n e^{ikj\Delta x} - A^n e^{ik(j-1)\Delta x}\right)$$

Divide both sides by $A^n e^{ikj\Delta x}$,

$$A = 1 - c(1 - e^{-ik\Delta x})$$

$$A = (1-c) + ce^{-ik\Delta x}$$

$$= (1-c) + c\left[\cos(k\Delta x) - i\sin(k\Delta x)\right]$$

$$A = \left\{\left[(1-c) + c\cos(k\Delta x)\right]^2 + \left[-ci\sin(k\Delta x)\right]^2\right\}^{1/2}$$

$$A = \left[(1-c)^2 + \cancel{c^2\cos^2(k\Delta x)} + 2c(1-c)\cos(k\Delta x) + \cancel{c^2\sin^2(k\Delta x)}\right]^{1/2}$$

$$= \left[1 - 2c + 2c^2 + 2c(1-c)\cos(k\Delta x)\right]^{1/2}$$

$|A| < 1$,

$$\left[1 - 2c + 2c^2 + 2c(1-c)\cos(k\Delta x)\right]^{1/2} < 1$$

$$-c + c^2 + c(1-c)\cos(k\Delta x) < 0$$

$$(-1 + c) + (1-c)\cos(k\Delta x) < 0$$

$$(-1+c)\left[1 - \cos(k\Delta x)\right] < 0$$

$$(-1+c)\left[1 - \left(1 - 2\sin^2\left(\frac{k\Delta x}{2}\right)\right)\right] < 0$$

$$(-1+c)\left(2\sin^2\left(\frac{k\Delta x}{2}\right)\right) < 0$$

Since $\sin^2\left(\frac{k\Delta x}{2}\right)$ will always be +ve and range from 0 to 1,

$$(-1+c)2 < 0$$

$$c < 1$$

$$\frac{u\Delta t}{\Delta x} < 1 \implies \Delta t < \frac{\Delta x}{u}$$

The scheme is conditionally stable when CFL < 1

3.(a) $$\frac{T_{j,k}^{(n)} - T_{j,k}^{(n-1)}}{\Delta t} = \alpha \left[\frac{T_{j+1,k}^{n} - 2T_{j,k}^{n} + T_{j-1,k}^{n}}{\Delta x^2} + \frac{T_{j,k+1}^{n} - 2T_{j,k}^{n} + T_{j,k-1}^{n}}{\Delta y^2} \right]$$

Shifting time step,

$$\frac{T_{j,k}^{(n+1)} - T_{j,k}^{(n)}}{\Delta t} = \alpha \left[\frac{T_{j+1,k}^{(n+1)} - 2T_{j,k}^{(n+1)} + T_{j-1,k}^{(n+1)}}{2} + \frac{T_{j,k+1}^{n+1} - 2T_{j,k}^{n+1} + T_{j,k-1}^{n+1}}{\Delta y^2} \right]$$

$$T_{j,k}^{(n+1)} - \alpha \Delta t \left(\frac{T_{j+1,k}^{(n+1)} - 2T_{j,k}^{(n+1)} + T_{j-1,k}^{(n+1)}}{\Delta x^2} + \frac{T_{j,k+1}^{(n+1)} - 2_{j,k}^{(n+1)} + T_{j,k-1}^{(n+1)}}{\Delta y^2} \right) = T_{j,k}^{n}$$

(b) $$A^{n+1} e^{iRj\Delta x} e^{isk\Delta y} - \alpha \Delta t \left[\frac{1}{\Delta x^2} \left(A^{n+1} e^{i(R+1)j\Delta x} e^{isk\Delta y} - 2A^{n+1} e^{iRj\Delta x} e^{isk\Delta y} + A^{n+1} e^{i(R-1)j\Delta x} e^{isk\Delta y} \right) \right.$$

$$\left. + \frac{1}{\Delta y^2} \left(A^{n+1} e^{iRj\Delta x} e^{i(s+1)k\Delta y} - 2A^{n+1} e^{iRj\Delta x} e^{isk\Delta y} + A^{n+1} e^{iRj\Delta x} e^{i(s-1)k\Delta y} \right) \right] = A^{n} e^{iRj\Delta x} e^{isk\Delta y}$$

Divide both sides by $A^{n} e^{iRj\Delta x} e^{isk\Delta y}$,

$$A - \alpha t \left[\frac{1}{\Delta x^2} \left(Ae^{ij\Delta x} - 2A + Ae^{-ij\Delta x} \right) + \frac{1}{\Delta y^2} \left(Ae^{ik\Delta y} - 2A + Ae^{-ik\Delta y} \right) \right] = 1$$

$$A - \alpha t \left[\frac{1}{\Delta x^2} \left(2A \cos(j\Delta x) - 2A \right) + \frac{1}{\Delta y^2} \left(2A \cos(k\Delta y) - 2A \right) \right] = 1$$

$$A - \alpha t \left[\frac{2A}{\Delta x^2} \left(1 - 2\sin^2\left(\frac{j\Delta x}{2}\right) - 1 \right) + \frac{2A}{\Delta y^2} \left(1 - 2\sin^2\left(\frac{k\Delta y}{2}\right) - 1 \right) \right] = 1$$

$$A - \frac{4\alpha\Delta t}{\Delta x^2} A \sin^2\left(\frac{j\Delta x}{2}\right) + \frac{4\alpha\Delta t}{\Delta y^2} A \sin^2\left(\frac{k\Delta y}{2}\right) = 1$$

$$A = \frac{1}{1 + \frac{4\alpha\Delta t}{\Delta x^2} \sin^2\left(\frac{j\Delta x}{2}\right) + \frac{4\alpha\Delta t}{\Delta y^2} \sin^2\left(\frac{k\Delta y}{2}\right)}$$

Since $\sin^2\left(\frac{j\Delta x}{2}\right)$ and $\sin^2\left(\frac{k\Delta y}{2}\right)$ range from 0 to 1, taking the maximum to account for all scenario, the denominator will always be greater than 1. Hence, A will be less than 1 always

(c) From Equation derived in 3(a),

$$[I]\,T^{n+1} - \alpha\Delta t\,[L]\,T^{n+1} = [I]\,T^{n} + BC$$

$$([I] - \alpha\Delta t[L])\,T^{n+1} = [I]\,T^{n} + BC$$

$$T^{n+1} = \underbrace{([I] - \alpha\Delta t[L])^{-1}}_{[A]}([I]T^{n} + BC)$$

and $[B] = [I]$

4. (a) Determine the BTCS scheme of Eqn 3

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Set up parameters of the problem
Define the anonymous function for BCf
 For each time step
        Discretise time domain for current time step
        For each different x & y number of points
                discretise the x & y domain (xg and yg)     start time
                call MatSetup function to find L and BC
                Find matrix A
                Start Time                                  stop time
                For each time level
                        Calculate Temperature
                end
                Stop Time
         Reshape solution to include boundary grid points
         Plot figures / graphs

         end

   end
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