CS & IT ENGINEERING

Theory of Computation

PDA : closure properties

Lecture No. 06

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TOPICS TO BE COVERED

01 closure properties

02 CFLs Vs DCFLs

03

04

05

Closure properties

- Finite languages ✓
- Infinite languages ✓
- Regular languages ✓
- DCFLs ?
- CFLs ?

Closure properties for CFLs

1. $L_1 \cup L_2$
2. ~~2~~ $L_1 \cap L_2$
3. ~~3~~ $\overline{L}$
4. ~~4~~ $L_1 - L_2$
5. $L_1 \cdot L_2$
6. L^{Rev}
7. L^*
8. L^+

9. ~~9~~ Subset (L)
10. prefix (L)
11. suffix (L)
12. Substring (L)
13. $f(L)$
14. $h(L)$
15. ε-free $h(L)$

16. $h^{-1}(L)$
17. ~~17~~ L_1 / L_2

18 Finite $\cup$	~~24~~ Inf $\cup$
~~19~~ " $\cap$	~~25~~ " $\cap$
~~20~~ " $-$	~~26~~ " $-$
21 " $\cdot$	~~27~~ " $\cdot$
22 " $\subseteq$	~~28~~ " $\subseteq$
23 " f	~~29~~ " f

Not closed for CFLs :

$\cap, \overline{L}, /$

$L_1 - L_2$

$\subseteq$

Inf ... -

fin $\cap$

fin -

I C Q D S I_{all} F_{ID}

$\boxed{ICQDSI_{all}F_{ID}}$ X

Closure properties for DCFLs

1. $L_1 \cup L_2$
2. $L_1 \cap L_2$
3. ~~3~~ $\overline{L}$
4. $L_1 - L_2$
5. $L_1 \cdot L_2$
6. L^{Rev}
7. L^*
8. L^+
9. Subset (L)
10. ~~10~~ prefix (L)
11. suffix (L)
12. Substring(L)
13. $f(L)$
14. $h(L)$
15. ε-free $h(L)$
16. ~~16~~ $h^{-1}(L)$
17. L_1 / L_2

18. Finite $\cup$	24. Inf $\cup$
19. " $\cap$	25. " $\cap$
20. " $-$	26. " $-$
21. " $\cdot$	27. " $\cdot$
~~22~~ " $\subseteq$	28. " $\subseteq$
23. " f	29. " f

closed for DCFLs:

$\bar{L}$, pref, h^{-1}, fin $\subseteq$,

$C \quad P \quad \left\{ I_h \quad F_s \right.$

$\boxed{C P I_h F_s}$ ✓

Regular closures for CFLs

1. $CFL \cup Reg =$
2. $CFL \cap Reg =$
3. $CFL - Reg =$
4. $CFL \cdot Reg =$
5. $CFL \,/\, Reg =$

$\left.\right\}$ CFL

Regular closures for DCFLs

1. $DCFL \cup Reg =$
2. $DCFL \cap Reg =$
3. $DCFL - Reg =$
4. $DCFL \cdot Reg =$
5. $DCFL / Reg =$

DCFL

Union

- Closed for CFLs
- Not closed for DCFLs

$CFL_1 \cup CFL_2 \Rightarrow CFL$

CFG_1 (S_1) $\quad$ CFG_2 (S_2)

$S \rightarrow S_1 \mid S_2$

CFG_1

CFG_2

$DCFL_1 \cup DCFL_2 \Rightarrow$ Need not be DCFL

(Always CFL)

i) $\underbrace{a^n b^n}_{DCFL} \cup \underbrace{a^n b^n}_{DCFL} \Rightarrow \underbrace{a^n b^n}_{DCFL}$

* ii) $\underbrace{\{a^* b^n c^n\}}_{DCFL} \cup \underbrace{\{a^n b^n c^*\}}_{DCFL} \Rightarrow$ not DCFL

1. $CFL_1 \cup CFL_2 \Rightarrow CFL$
2. $DCFL_1 \cup DCFL_2 \Rightarrow CFL$

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3. $CFL \cup DCFL \Rightarrow CFL$
4. $CFL \cup Reg \Rightarrow CFL$
5. $DCFL \cup Reg \Rightarrow DCFL$
6. $CFL \cup Fin \Rightarrow CFL$
7. $DCFL \cup Fin \Rightarrow DCFL$
8. $CFL \cup Inf \Rightarrow Inf$
9. $DCFL \cup Inf \Rightarrow Inf$

$\Sigma = \{a, b\}$

① $(a+b)^* \cup \{a^n b^n\} \Rightarrow (a+b)^*$ Reg

② $\{a^n b^n\} \cup \{a^n b^{2n}\} \Rightarrow$ CFL but not DCFL

③ $\phi \cup \{a^n b^n\} \Rightarrow \{a^n b^n\}$ DCFL

④ $\{\underbrace{a^n b^{2n}}_{\text{DCFL}}\} \cup \{\underbrace{a^{2n} b^n}_{\text{DCFL}}\} \Rightarrow$ CFL but not DCFL

⑤ $\{a^n b^n\} \cup \{b^n a^n\} \Rightarrow$ DCFL

② Intersection

→ Not closed for CFLs
→ " " for DCFLs

CFL$_1$ ∩ CFL$_2$ ⟹ Need not be CFL (Always CSL)

DCFL$_1$ ∩ DCFL$_2$ ⟹ Need not be DCFL (Always CSL)

Example:

$$\{a^n b^n c^*\} \cap \{a^* b^n c^n\} \Rightarrow \{a^n b^n c^n\}$$

not DCFL
not CFL

① $\{a\,b^n c^n\}_{n\geq 0} \cap \{a^n\,b\,c^n\}_{n\geq 0} \Rightarrow \{abc\}$

② $\{a^* b^n c^n\}_{n\geq 0} \cap \{a^n b^n \boxed{c}\}_{n\geq 0} \Rightarrow \{ab\boxed{c}\}$

③ $\{\boxed{a}\,b^*\,\boxed{c}\} \cap \{a^* b^n c^n\} \Rightarrow \{abc\}$

1. $CFL_1 \cap CFL_2 \Rightarrow$ CSL
2. $DCFL_1 \cap DCFL_2 \Rightarrow$ CSL

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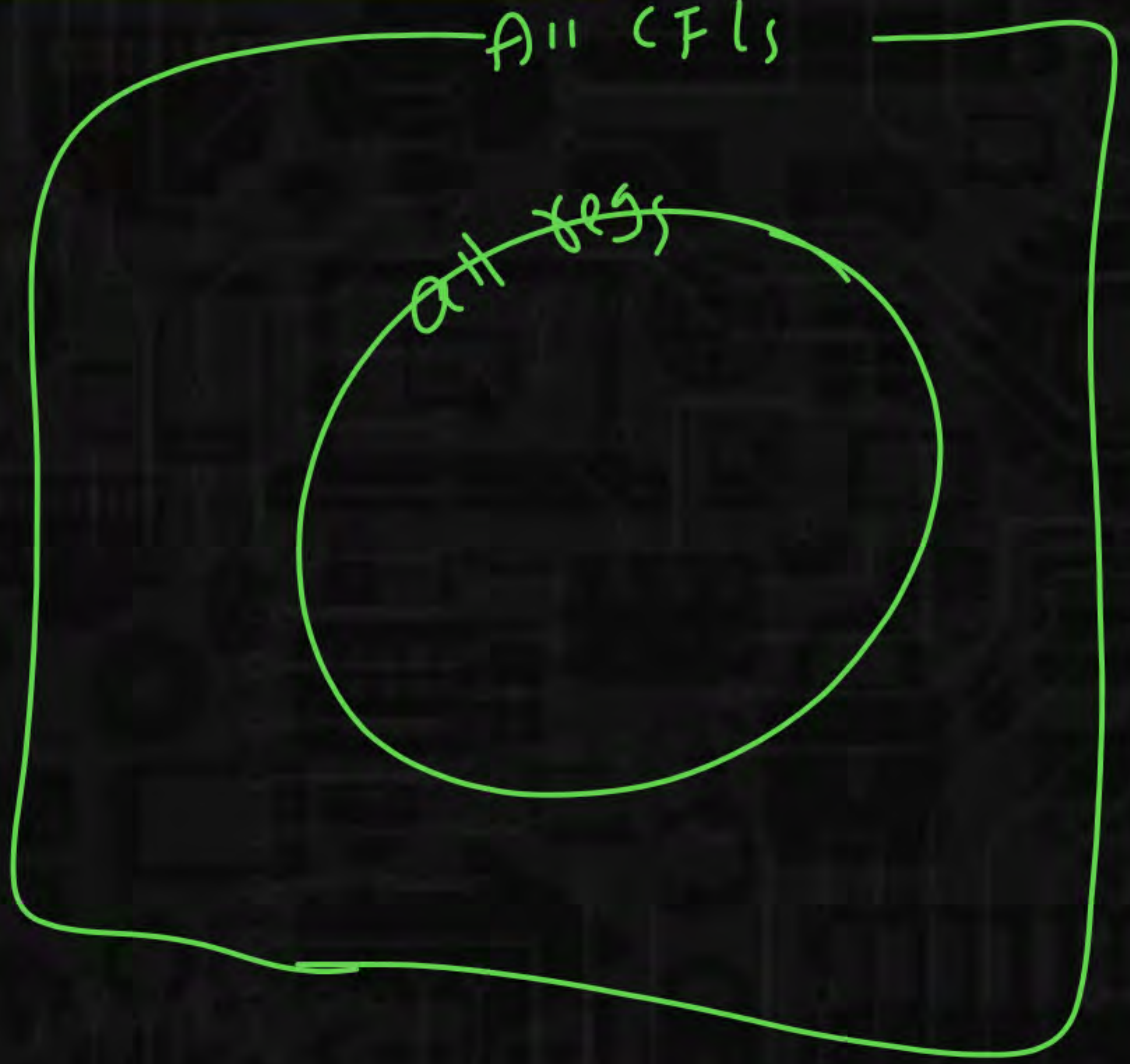
3. $CFL \cap DCFL \Rightarrow$ CSL
4. $CFL \cap Reg \Rightarrow$ CFL
5. $CFL \cap Fin \Rightarrow$ Fin
6. $CFL \cap Inf \Rightarrow$?
7. $DCFL \cap Reg \Rightarrow$ DCFL
8. $DCFL \cap Fin \Rightarrow$ Fin
9. $DCFL \cap Inf \Rightarrow$?

CFL $\boxed{\begin{array}{c} \cup \\ \cap \\ \_ \\ \_ \\ \_ \end{array}}$ Reg $\Rightarrow$ CFL

$\{a^n b^n\} \cap a^* b^* \Rightarrow a^n b^n$

CFL $\cap$ Reg $\Rightarrow$ Need not be CFL

Set of CFLs $\cap$ Set of Regs $\Rightarrow$ Set of Regs



always CFL

CFL $\cap$ Reg $\Rightarrow$ $\cancel{\text{Reg}}$

Set of all CFLs $\cap$ Set of all Regs

$\Rightarrow$ Set of all Regs

③ Complement

- → Not closed for CFLs
- → closed for DCFLs

$\overline{CFL} \Rightarrow$ Need not be CFL
(Always CSL)

Proof :

$L = \overline{a^n b^n c^n}$ is CFL

$\overline{L} = a^n b^n c^n$ is not CFL

$\overline{DCFL} \Rightarrow DCFL$

Proof :

L is DCFL

$\Downarrow$

construct DPDA

$\Downarrow$ f $\longleftrightarrow$ nf

DPDA

$\Downarrow$

$\overline{L}$ is DCFL

$$L = \underbrace{\overline{a^n b^n c^n}}_{?} = (a+b+c)^* - \underbrace{\{a^n b^n c^n\}}_{\text{not CFL}}$$

$$= (a+b+c)^* - \{a^m b^n c^k \mid m=n=k\}$$

$\overline{a^n b^n c^n}$

is CFL

$$= \boxed{\{a^m b^n c^k \mid (m \neq n) \vee (m \neq k) \vee (n \neq k)\} \cup \Sigma^* ba \Sigma^* \cup \Sigma^* ca \Sigma^* \cup \Sigma^* cb \Sigma^*}$$

$\Sigma = \{a, b, c\}$

$\overline{a^n b^n c^n}$ is CFL

$\not\Downarrow$

$a^n b^n c^n$ is not CFL

$(a+b+c)^*$ is CFL

$\not\Downarrow$

ϕ is CFL

$L_1 = \overline{\{ww \mid w \in \{a,b\}^*\}}$ is CFL but not DCFL

$\overline{L_1} = \{ww \mid w \in \{a,b\}^*\}$ is not CFL

$L_2 = \{ww^R \mid w \in \{a,b\}^*\}$ is CFL

$\overline{L_2} = \overline{\{ww^R \mid \quad " \quad\}}$ is also CFL

(4) Difference

→ Not closed for CFLs

→ " " for DCFLs

$$\boxed{L_1 - L_2 = L_1 \cap \overline{L_2}}$$

$\cap$

→ not closed for CFLs & DCFLs

(5) Concatenation

→ closed for CFLs

→ Not closed for DCFLs

$CFL_1 \cdot CFL_2 \Rightarrow$ Always CFL

$CFL_1 \Downarrow CFG_1 (S_1)$, $CFL_2 \Downarrow CFG_2 (S_2)$

$\Rightarrow$ $S \to S_1 S_2$, CFG_1, CFG_2

$DCFL_1 \cdot DCFL_2 \Rightarrow$ Need not be DCFL (Always CFL)

$\underbrace{\{a^n b^n \mid n \geq 0\}}_{DCFL} \cdot \underbrace{\{a^m b^{2m} \mid m \geq 0\}}_{DCFL} \Rightarrow \{a^n b^n a^m b^{2m}\}$ not DCFL (It is CFL)

$\underbrace{aaa}_{push}\ \boxed{b}$?

(6) Reversal

- → closed for CFLs
- → Not closed for DCFLs

$(CFL)^{Rev} \Rightarrow CFL$

$\underset{L}{CFL} \Rightarrow CFG \xRightarrow{\text{every production reverse it}} \underset{\text{modified}}{CFG} \Rightarrow \underset{(L^{Rev})}{CFL}$

$(DCFL)^{Rev} \Rightarrow$ Need not be DCFL (Always CFL)

$L = \boxed{\{c a^n b^n\} \cup \{d a^m b^{2m}\}}$ is DCFL

$L^{Rev} = \boxed{\{b^n a^n c\} \cup \{b^{2m} a^m d\}}$ is not DCFL

⑦ Kleene star
⑧ Kleene plus

→ closed for CFLs

$(CFL)^* \Rightarrow CFL$

$(CFL)^+ \Rightarrow CFL$

CFL (L) $\Rightarrow$ CFG (S) $\xRightarrow{\text{Add } X \to XS \mid \varepsilon}$ $\boxed{\begin{array}{l} X \to XS \mid \varepsilon \\ \text{CFG} \end{array}}$ $\Rightarrow L^*$

Add $X \to XS \mid S$

→ Not closed for DCFLs

$(DCFL)^* \Rightarrow$ Need not be DCFL

$(DCFL)^+ \Rightarrow$ Need not be DCFL

(Always CFL)

$$L = \{c\,a^n b^n\} \cup \{a^m b^{2m}\}$$

cab ✓

cabb ✗

is DCFL

c a . . . . .

a . . . . .

$$L^* = \left(\{c\,a^n b^n\} \cup \{a^m b^{2m}\}\right)^*$$

is not DCFL

cab ✓

cabb ✓

$n>0$

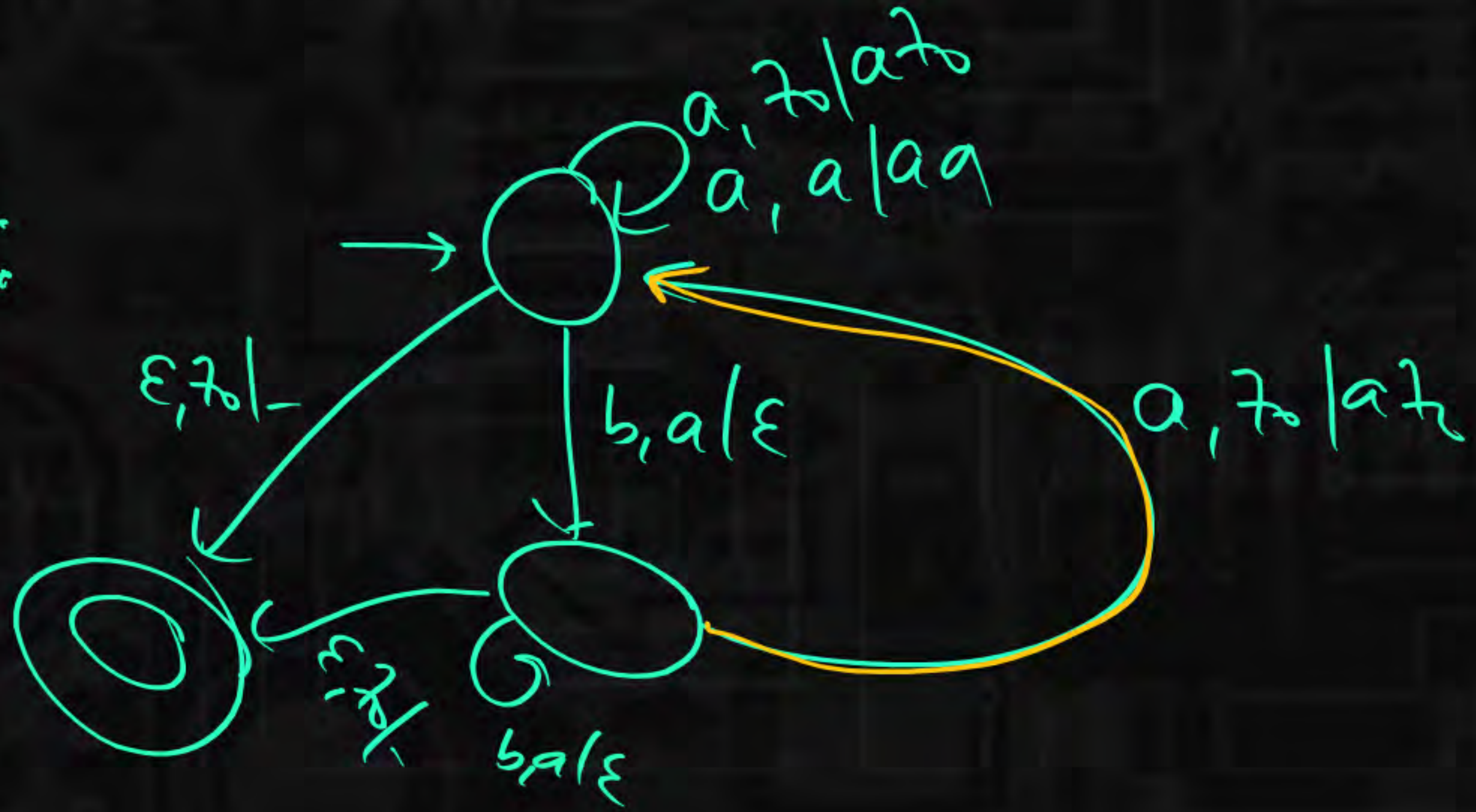
$n=0$

$$(\quad)^2 = (c\quad)^1 (abb)^1 \; c\,\boxed{a}$$

$L = \{a^n b^n\}$ is DCFL

$L^* = \{a^n b^n\}^*$ is DCFL

L^*:



9 Subset

→ not closed for regs

DCFLs

CFLs

CSLs

Recs

RELs

$(a+b)^*$ is reg, DCFL, CFL

ϕ

reg

DCFL

$a^n b^n$

not reg

DCFL

CFL

$\{a^n b^n a^m b^{2m}\}$

not DCFL

$\underbrace{a^n b^n a^n}_{\text{not CFL}}$

Subset of regular language is need not be reg

Subset of DCFL is need not be DCFL

Subset of CFL is need not be CFL

(10) Prefix(L)

(11) Suffix(L)

(12) Substring(L)

→ closed for CFLs

→ Not closed for DCFLs

→ closed for DCFLs

Prefix of DCFL

Complement of DCFL

⇒ DCFL

Proof for Prefix of CFL :

$L = a^n b^n \implies S \rightarrow \overbrace{aSb}^{\text{add all prefixes of } aSb} \mid \overbrace{\varepsilon}^{\text{add all prefixes of } \varepsilon} \implies \underbrace{S \rightarrow \varepsilon \mid a \mid aS \mid aSb}_{\text{Prefix of CFL}}$

CFL CFG

$\text{Prefix}(L) = \{ a^m b^n \mid m \geq n \}$

Homomorphism & Substitution

- → closed for CFLs
- → Not closed for DCFLs

$L=\{ca^nb^n\}\cup\{a^mb^{2m}\}$

DCFL

$L \xrightarrow{h} h(L)=\{0^n1^n\}\cup\{0^m1^{2m}\}$

not DCFL

(It is CFL)

$h(c)=\{\epsilon\}$ $h(a)=\{0\}$ $h(b)=\{1\}$

Homomorphism & Substitution

- closed for CFLs
- Not closed for DCFLs

$L = \{c\,a^n b^n\} \cup \{a^m b^{2m}\}$

CFL

h

$S \to A \mid B$

$A \to cX$

$X \to aXb \mid \varepsilon$

$B \to aBbb \mid \varepsilon$

$h(c) = \{\varepsilon\}$, $P \to \varepsilon$

$h(a) = \{0\}$, $Q \to 0$

$h(b) = \{1\}$, $R \to 1$

CFL

$h(L) = \{0^n 1^n\} \cup \{0^m 1^{2m}\}$

$S \to A \mid B$

$A \to PX$

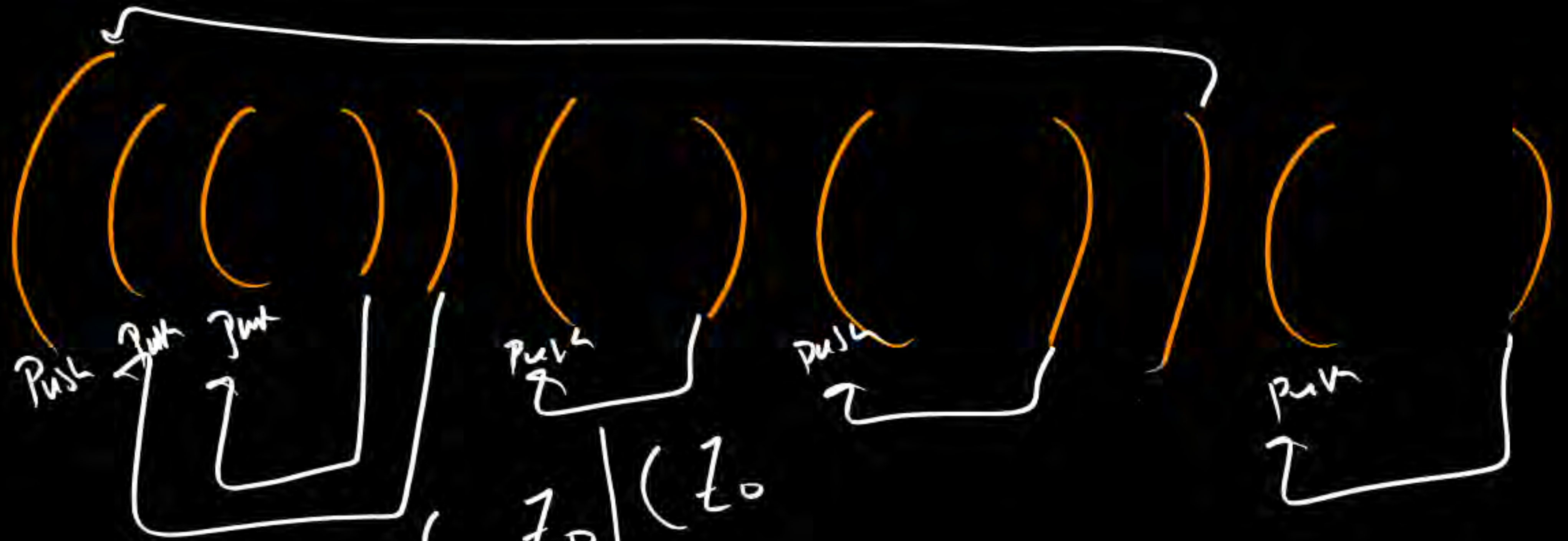
$X \to QXR \mid \varepsilon$

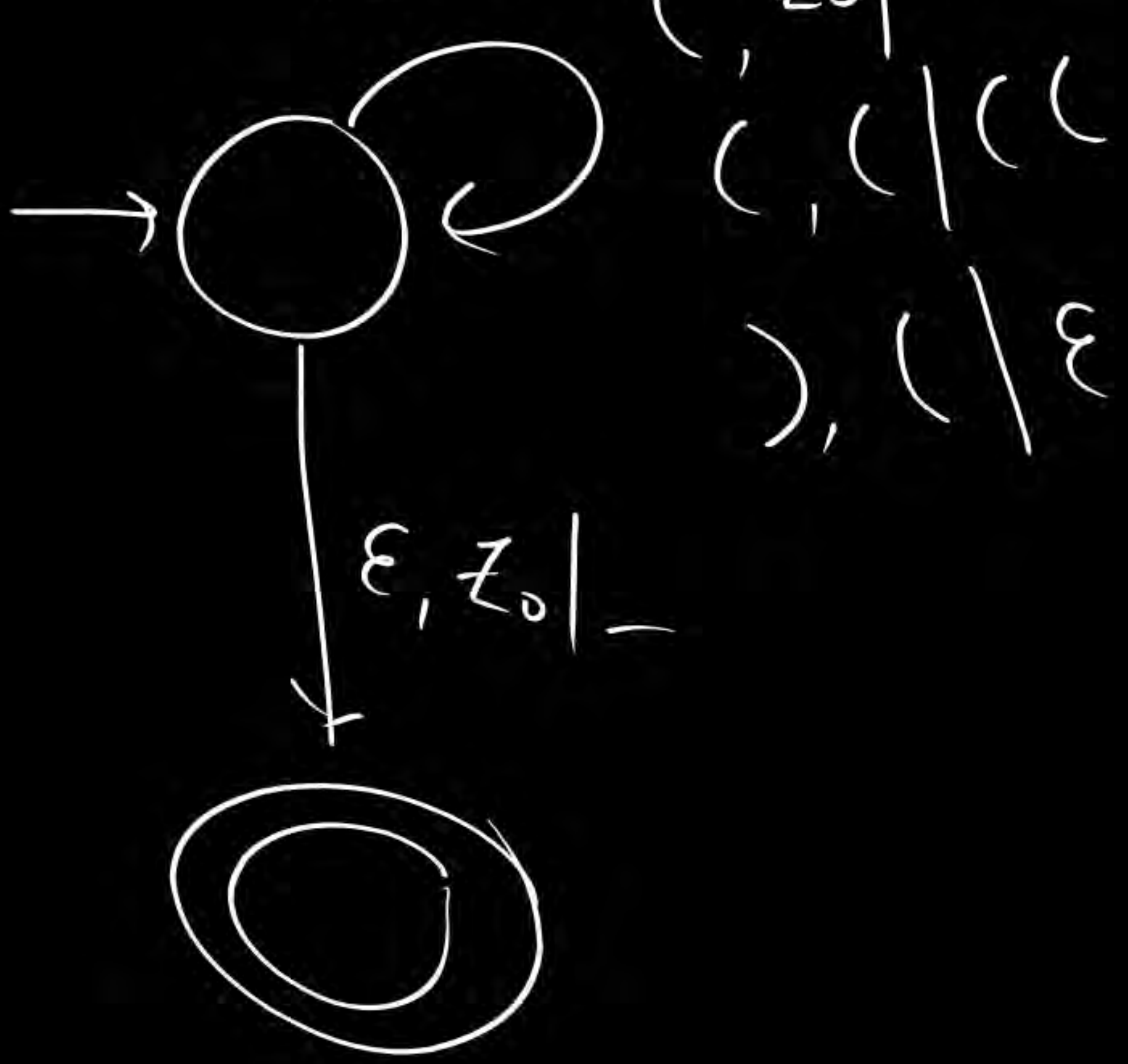
$B \to QBRR \mid \varepsilon$

$P \to \varepsilon$

$Q \to 0$

$R \to 1$





L = set of all balanced parenthesis

$\boxed{\text{is DCFL}}$

CFLs

Remember → Not closed

$\cap$, $\overline{L}$, Quotient } Diff | $\subseteq$ | fin $\cap$, fin Diff | Inf ....

DCFL

Remember → Closed

$\overline{L}$, Pref } Fin $\subseteq$, h^{-1}

I) CFL $\begin{array}{|c|}\hline \cup \\ \hline \cap \\ \hline - \\ \hline \cdot \\ \hline / \\ \hline \end{array}$ Reg $\Rightarrow$ CFL

(may be reg or not reg)

II) Reg $-$ CFL $\Rightarrow$ need not be CFL

$$\boxed{\text{Reg} - \text{CFL} \Rightarrow \text{Reg} \cap \overline{\text{CFL}} \Rightarrow \text{Reg} \cap \text{CSL} \Rightarrow \text{CSL}}$$

III) DCFL $\begin{array}{|c|}\hline \cup \\ \hline \cap \\ \hline - \\ \hline \cdot \\ \hline / \\ \hline \end{array}$ Reg $\Rightarrow$ DCFL

IV) Reg $-$ DCFL $\Rightarrow$ DCFL

$\text{Reg} - \text{DCFL} \Rightarrow \text{Reg} \cap \overline{\text{DCFL}}$

$\Rightarrow \text{Reg} \cap \text{DCFL}$

$\Rightarrow \text{DCFL}$

1. $\{a^n b^{n^2}\}$ $\Rightarrow$ not CFL

$\Rightarrow$ CSL

2. $\{a^n b^n a^n\}$ $\Rightarrow$ CSL

3. $\{a^n b^{n!}\}$ $\Rightarrow$ CSL

4. $\{a^n b^{\text{prime}}\}$ $\Rightarrow$ CSL

5. $\{a^n b^{2^n}\}$ $\Rightarrow$ CSL


THANK YOU GW
SOLDIERS !