CS & IT ENGINEERING

Algorithm

Miscellaneous Topics

Lecture No.- **11**

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Recap of Previous Lecture

Topic — Longest Common Subsequence

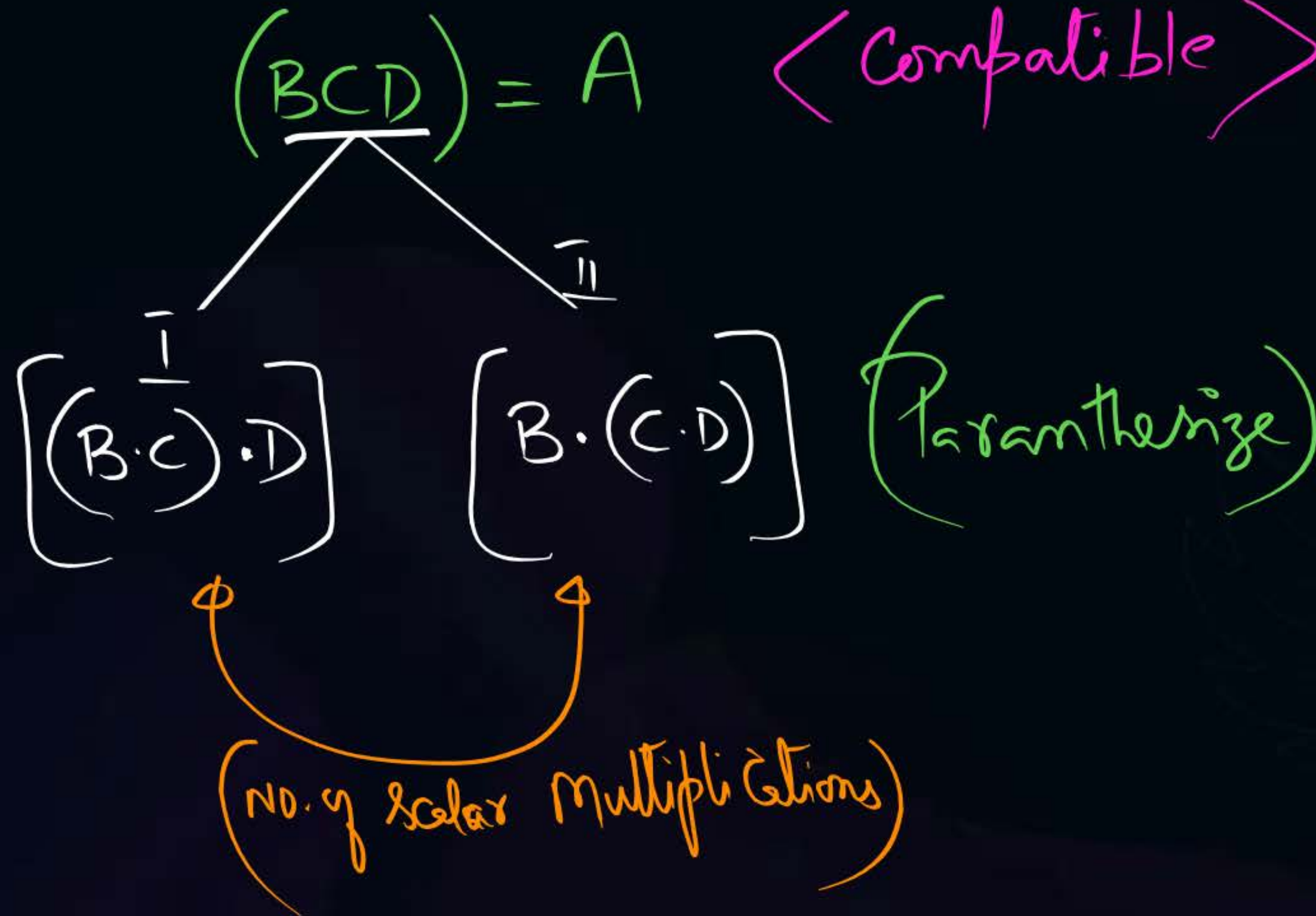**Topic** — MCP

Topics to be Covered

Topic Matrix Chain Product

Topic Sum of Subsets

Matrix chain Product (MCP):

$(BCD) = A$ $\langle$ Compatible $\rangle$

$A_{n \times m} \times B_{m \times k}$

$$\begin{bmatrix} (B \cdot C) \cdot D \end{bmatrix} \quad \begin{bmatrix} B \cdot (C \cdot D) \end{bmatrix}$$

I   II

(Paranthesize)

(No. of scalar Multiplications)

Defn: Given a chain q Matrices $\langle A_1, A_2, A_3 \ldots A_n \rangle$ where Matrix A_i is q dimension $\boxed{P_{i-1} \times P_i}$

The Problem q M.C.P is to fully Parenthesize the chain, Such that the total no. q <u>Scalar Multiplications</u> is Minimum ;

Ex1: $\langle BCD \rangle$

I   II

Ex2: $\langle A_1 A_2 A_3 A_4 \rangle$

$K=1$ 2   3

$(A_1)(A_2 A_3 A_4)$ $(A_1 A_2)(A_3 A_4)$ $(A_1 A_2 A_3)(A_4)$

III

$(A_2)(A_3 A_4)$ $(A_2 A_3)(A_4)$ $(A_1)(A_2 A_3)$ $(A_1 A_2)(A_3)$

I   II   IV   V
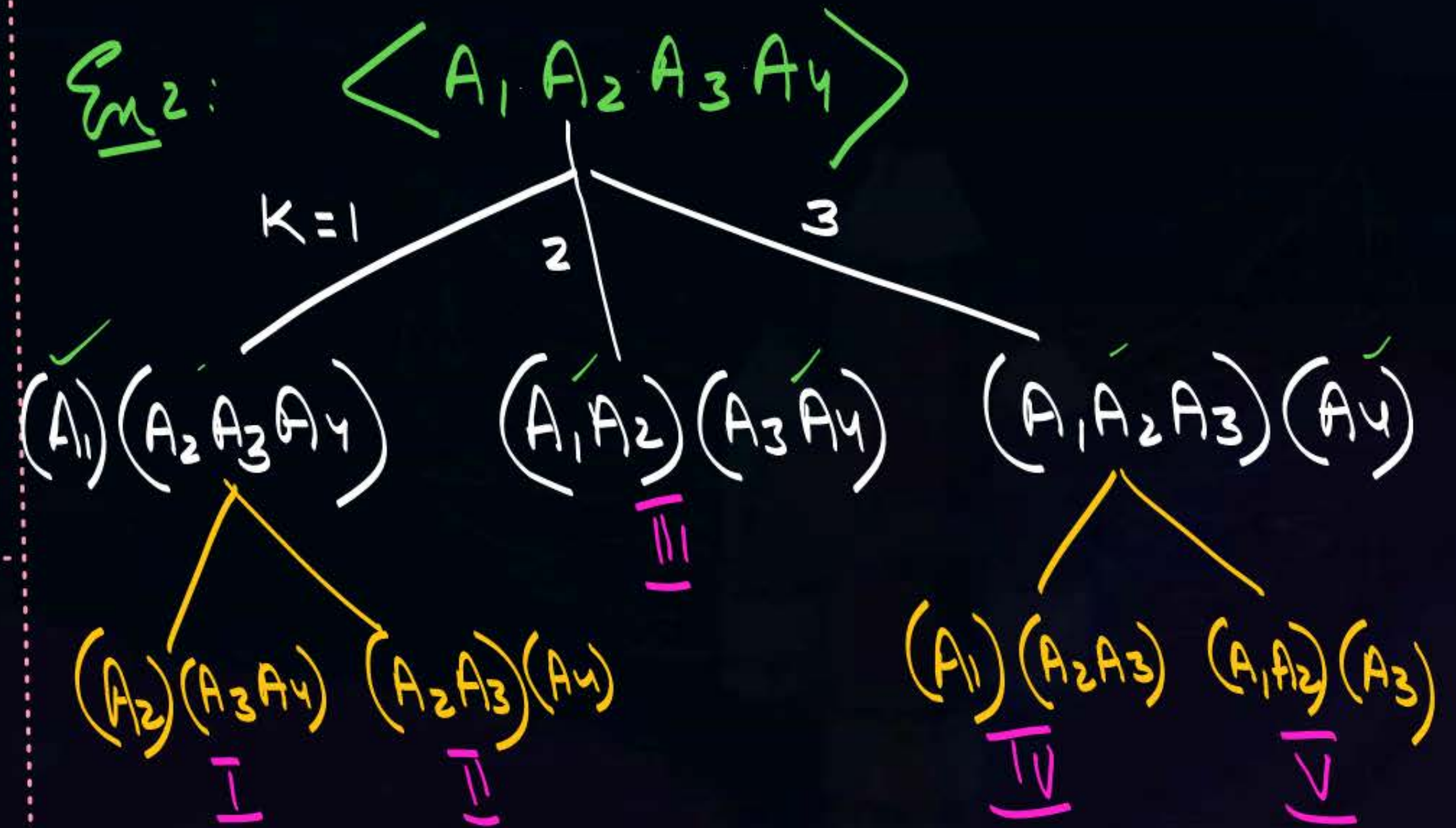
<u>Catalan NO:</u> $\dfrac{1}{(n+1)} 2n c_n$

$\Omega(2^n)$

Derivation of D.P based Recurrence for M.C.P

→ Let the resultant Matrix $A_{i..j}$ be of the Product

$$\langle A_i \cdot A_{i+1} \cdot A_{i+2} \cdots A_j \rangle$$

$$\underline{Ex.}$$

→ Any optimal Parenthesization, must Split the chain about the Matrix A_K & A_{K+1} S.T the total No. of Scalar Multiplications is Min.

$$A_{1..n} : \langle A_1, A_2, A_3 \cdots A_n \rangle$$

$$K = 1, 2, 3, \cdots n-1$$

→ Let $m[i,j]$ denote the no. of Scalar Multiplications to get the Matrix $A_{i..j}$

$$\langle A_i \; A_{i+1} \cdots \cdots \cdots A_j \rangle = A_{i..j} = m[i,j]$$

$$\langle \underbrace{(A_i \; A_{i+1} \cdots A_K)}_{(A_{i..K})} \underbrace{(A_{K+1} \cdots A_j)}_{(A_{K+1..j})} \rangle$$

$$[i \leq K \leq j-1]$$

$$\boxed{m[i,j] = Min\left\{ m[i,K] + m[K+1,j] + P_{i-1} * P_k * P_j \right\}}$$
$$i \leq K < j$$

$$m[i,j] = \min_{i \leq k < j} \{ m[i,k] + m[k+1,j] + P_{i-1} \times P_k \times P_j \}$$

$$m[i,i] = 0$$

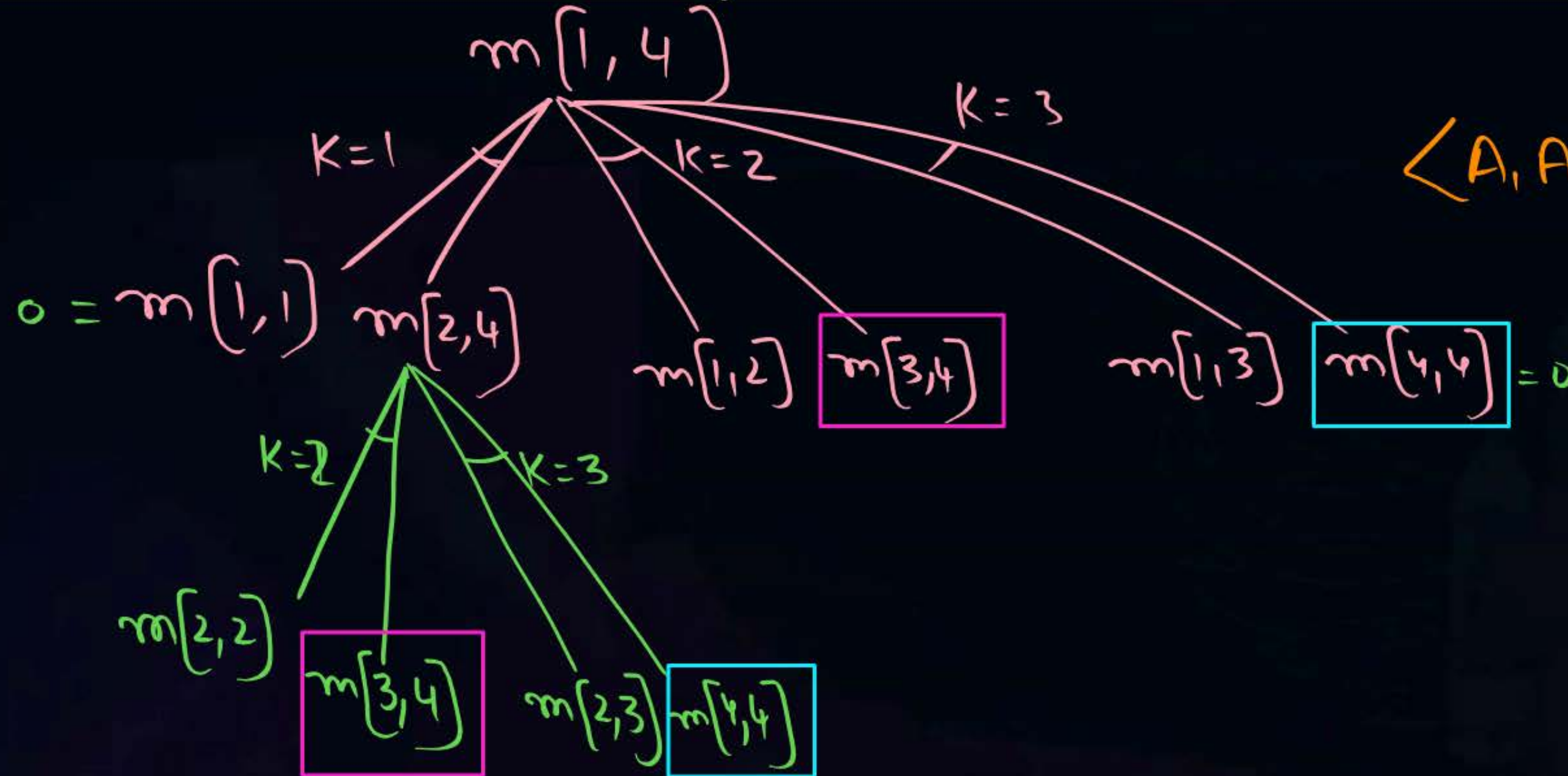$$s[i,j] = K \quad \langle \text{Point of Split} \rangle$$

$j - i = 0$

$j - i = 1$

$j - i = 2$

$\langle A_1 A_2 A_3 A_4 \rangle = A_{1 \cdots 4}$

$$A_{1..6} = \left\langle \underbrace{\left(\overset{A_i}{A_1} A_2 \overset{A_k}{A_3} \right)}_{I} \underbrace{\left(A_4 A_5 \overset{A_j}{A_6} \right)}_{II} \right\rangle$$

$$2 \times 6$$

$$\left(A_{1..3} \right) \longleftrightarrow \left(A_{4..6} \right)$$

$$\underset{P_0 \times P_3}{2 \times 8} \qquad \underset{P_3 \times P_6}{8 \times 6}$$

$$m[1,3] + m[4,6] + \underline{P_0 \cdot P_3 \cdot P_6}$$

$$P_{i-1} * P_k * P_j$$

$$A_1 \rightarrow P_0 \times P_1 \rightarrow 2 \times 3$$
$$A_2 \rightarrow P_1 \times P_2 \rightarrow 3 \times 5$$
$$A_3 \rightarrow P_2 \times P_3 \rightarrow 5 \times 8$$
$$A_4 \rightarrow P_3 \times P_4 \rightarrow 8 \times 4$$
$$A_5 \rightarrow P_4 \times P_5 \rightarrow 4 \times 7$$
$$A_6 \rightarrow P_5 \times P_6 \rightarrow 7 \times 6$$

$$\langle A_1 A_2 A_3 A_4 \rangle = \langle 2 \times 3 ; \ 3 \times 5 ; \ 5 \times 8 ; \ 8 \times 4 \rangle = (174) \checkmark$$

$$\langle A_1 A_2 A_3 A_4 \rangle \qquad (174)$$

$2 \cdot 3 \cdot 4$

$0 + 216 +$ $K=1$ (240) $\overset{2}{(230)}$ $K=3$

$\dfrac{\overset{1}{2}16}{\underset{\cdot 0}{24}}$

216

$\langle (A_1) \cdot (A_2 A_3 A_4) \rangle$ $\langle (A_1 A_2)(A_3 A_4) \rangle$ $\langle (A_1 A_2 A_3)(A_4) \rangle$

$2 \cdot 3 \cdot 5 + 5 \cdot 8 \cdot 4 +$

$2 \cdot 5 \cdot 4$

$K=2$ $3 (216)$ $K=1$ $\overset{2}{} \checkmark$

(220)

$3 \times 8 \quad 8 \times 4$

$\langle (A_2)(A_3 A_4) \rangle$ $\langle (A_2 A_3)(A_4) \rangle$ $\langle (A_1)(A_2 A_3) \rangle$ $\langle (A_1 A_2)(A_3) \rangle$

$3 \times 5 \qquad 5 \times 4$

$3 \cdot 5 \cdot 8 + 0 +$ $K=2 \qquad K=3$

$(0 + 5 \cdot 8 \cdot 4 +$ $3 \cdot 8 \cdot 4$

$\quad 3 \cdot 5 \cdot 4)$

$\dfrac{\begin{matrix}160\\60\end{matrix}}{220}$ $\dfrac{\begin{matrix}120\\96\end{matrix}}{216}$

Final Parenthes $= \langle ((A_1 A_2) \cdot (A_3)) \cdot (A_4) \rangle$

$30 + 80 + 2 \cdot 8 \cdot 4 = \dfrac{\begin{matrix}110\\64\end{matrix}}{} \Big] 174$

$$\underset{\substack{l=4 \\ j-i=3}}{m(1,4)} = \underset{K=1,2,3}{min} \left\{ \underset{l=1}{m[1,1]} + \underset{l=3}{m[2,4]} + P_0 \cdot P_1 \cdot P_4 \; , \right.$$

$$\underset{l=2}{m[1,2]} + \underset{l=2}{m[3,4]} + P_0 \cdot P_2 \cdot P_4 \; ,$$

$$\left. \underset{l=3}{m[1,3]} + \underset{l=1}{m[4,4]} + P_0 \cdot P_3 \cdot P_4 \right\rangle$$

$l=1 \Rightarrow$ Bound $\left(m[i,i] \right)$

$l=2 \Rightarrow j-i=1$

$l=3 \Rightarrow j-i=2$

$l=4 \Rightarrow j-i=3$

Topic : Matrix Chain Product

(PW)

Algorithm Matrix-Chain-Product (p)

1 $n \leftarrow length[p] - 1$

2 for $i \leftarrow 1$ to n

3   do $m[i, i] \leftarrow 0$

4 for $\ell \leftarrow 2$ to n ℓ is the chain length.

5   for $i \leftarrow 1$ to $n - \ell + 1$

6          $(j \leftarrow i + \ell - 1)$

7              $m[i, j] \leftarrow \infty$

8   for $k \leftarrow i$ to $j - 1$ $\langle$ Pts. of Split $\rangle$

9                      $q \leftarrow m[i, k] + m[k + 1, j] + P_{i-1}P_kP_j$

10   if $(q < m[i, j])$

11   then $m[i, j] \leftarrow q$

12                                 $s[i, j] \leftarrow k$

13   return m and s

$P = \langle P_0 \, P_1 \, P_2 \cdots P_n \rangle$ order

order of Matrices

$\ell = 1 \quad j - i = 0$

$\Big\}$ Base cond. $\rightarrow O(n^3)$

$\langle A_1, A_2 \cdots A_n \rangle$

$n = \begin{pmatrix} No. \text{ of Matrices in} \\ \text{the chain} \end{pmatrix}$

$\ell = 2$

$i = 1 \text{ to } 3$

$j = \cancel{1+2} = 12$

$\ell = 2$

$m[1, 2] \quad m[2, 3] \quad m[3, 4]$

$\ell = 3$

$m[1, 3] \quad m[2, 4]$

$\langle A_1 A_2 A_3 A_4 \rangle$

$i = 2$

$j = 2 + 2 - 1$

$= 3$

$n = 4$

Space: $O(n^2)$

Slide 5

7) Sum of Subsets (SoS):

Defn: Given a Set of n-elements (integers) & also another Element (number) 'M'. The problem of S.O.S is to determine, if there exists a Subset of the given elements whose Sum Equals to 'M';

Ex: $n = 5$; A: $\langle \underset{1}{10}, \underset{2}{20}, \underset{3}{30}, \underset{4}{40}, \underset{5}{50} \rangle$, M = 50

T/F

Y/N

Brute Force: $O(2^n)$

1) $\{1, 4\}$
2) $\{2, 3\}$
3) $\{5\}$

$\langle$ Decision Problem $\rangle$

Derivation of D.P based recurrence fn S.o.S

$$n \; ; \quad A \langle A_1 A_2 \cdots \boxed{A_n} \rangle \; ; \quad M \; ; \quad X : \langle 1 \cdots n \rangle$$

Let $SoS(n, M)$ repr. the soln to S.o.S, with n, numbers & Element M

$$SoS(n, M) = T/F \quad \left[\begin{array}{l} \text{Whethere there exists a Subset of given } 'n' \\ \text{Elements that Sum to M} \end{array} \right]$$

$$SoS(n, M) = SoS(n-1, M) \quad , A_n > M$$

$$= \left(SoS(n-1, M) \text{ or } \right. \qquad , A_n \leq M$$
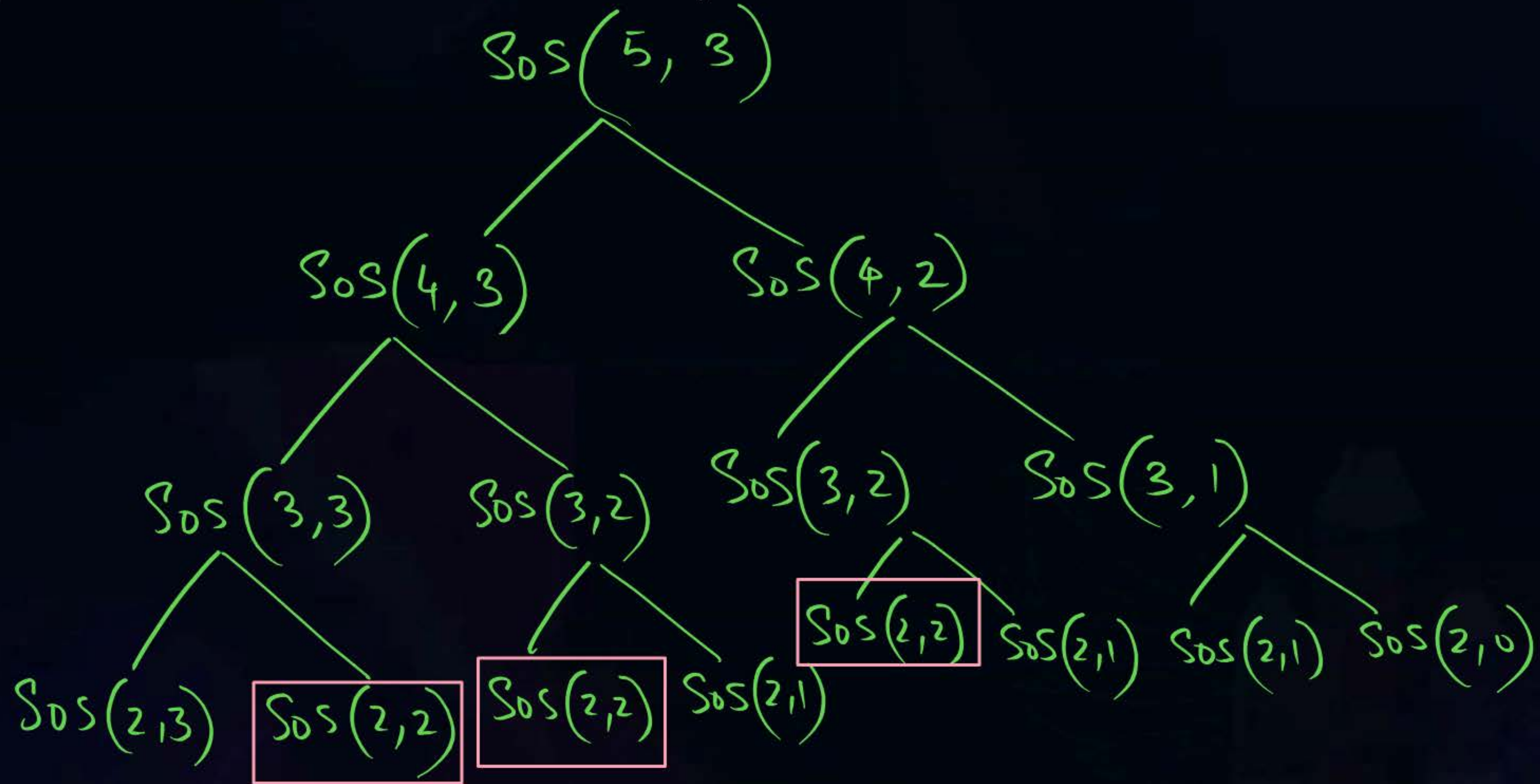
$$\left. SoS(n-1, M-A_n) \right]$$

$$= F \qquad , n = 0, M > 0 \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{Base Cond}$$

$$= T \qquad , n \geq 0 \; ; M = 0$$

1) $n = 5$; $A: \langle 1, 1, 1, 1, 1 \rangle$; $M = 3$

$SOS(5, 3)$

$SOS(4, 3)$ $SOS(4, 2)$

$SOS(3, 3)$ $SOS(3, 2)$ $SOS(3, 2)$ $SOS(3, 1)$

$SOS(2, 3)$ $SOS(2, 2)$ $SOS(2, 2)$ $SOS(2, 1)$ $SOS(2, 2)$ $SOS(2, 1)$ $SOS(2, 1)$ $SOS(2, 0)$

$n=5; \quad M=6; \quad A: \langle 2,8,4,11,9 \rangle$

$X \langle 0..5, \ 0..6 \rangle$

$X^{[0..n, \ 0..m]}_{(n, m)}$

$X[i,j] = T/F$

⟨ whether there exists a subset from the given 'i' elements, whose sum is 'j' ⟩

i	A_i	X \\ j	0	1	2	3	4	5	6
		0	T	F	F	F	F	F	F
2		1	T	F	T	F	F	F	F
8		2	T	F	T	F	F	F	F
4		3	T	F	T	F	T	F	T
11		4	T	F	T		T		
9		5	T	F	T		T		

if M is very large like 2^n

SOS can be implemented using bottom-up DP with Tabulation:

Algorithm SOS (n , M, A)

A [1.....n]

{

integer X[0..n , 0..M];

1. for i ← 0 to n
        for j ← 0 to M
            if (i ≥ 0 and j = 0)
                X [i , j]= T
            else
                if (i = 0 and j > 0)
                    X [i, j] = F;
                else

Base

if (A [i] > j)
    X [i,j] = X [i −1, j]
else
    X [i, j] = X [i−1, j] V X [i−1, j −A [i]]

}

Time: $O(n * M)$

Space: $O(n * M)$

∴ Time: $O(n \cdot 2^n)$

8) Bellman-Ford Algorithm [Single Source Shortest Paths] (P)(W)

(i) Dijkstra's Algo [Greedy Method] always works, provided the graph has +ve wt. edges;

(ii) If the Graph has -ve wt. edges but no -ve wt. Cycle then Dijkstra's Algo. MAY | MAY NOT work;

(iii) If the Graph has -ve wt. edges & no -ve wt. cycle then Bellman-Ford Algo always works
⟨D.P⟩ correctly

-8

10

-20   2

(Temperature)

(iv) If the Graph has -ve wt. cycle reachable from Source then NO-ALGO works;

Bellman Ford Algorithm

1) $S \to a = 3$ ✓
2) $S \to b = -1$ ✓
3) $S \to g = 3$ (?) ✗

4) $S \to c = \boxed{5}$ | $(S \to c \to D) = \boxed{11}$

$S \to c \to d \to c = 8$ | $(S \to c \to d \to c \to d) = 14$

$= 11$

$= 14$ | $= 17$

$= 20$

5) $S \to e = 2$
$= -1$
$= -4$

6) $S \to e \to f = 5$
$= 2$
$= -1$

Slide 7

Bellman Ford Algo

Apply Dijkstra's S.S.S.P Algo.

$v_0 = S$,

	S	t	x	y	z
$\{S\}$	—	✓	✓	✓	✓

(d-values)

1) $(t \to x)$

2) $(t \to z - x)$

THANK - YOU