1500 series

CS & IT ENGINEERING

Discrete Mathematics

Lecture No.- 02

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Recap of Previous Lecture

Topic Inclusion - Exclusion

Topics to be Covered

Topic Derangement

Q

Let n books be distributed to n students. Suppose that the books are returned and distributed to the students again later on .In how many ways can the books be distributed so that no student will get the same book twice

$n! \times D_n$

$D_n = n!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots \frac{(-1)^n}{n!}\right]$

A) n! * Dn

B) Dn

C) n! ^2

D) (Dn)^2

Q Find the number of derangements of the integers from 1to 10 inclusive, satisfying the condition that the set of elements in the first 5 places is: 1,2,3,4,5, in some order

A) 14,400

B) 1936 ✓

C) 5280

D) nota

1	2	3	4	5	6	7	8	9	10

$\{ D5 \} \{ D5 \}$

$D5 \times D5$

44×44

$= 1936$

mistake: D_{10} 1 2 3 4 . . . 10

10 4 . . . 1

$$D_n \cong n! \times 0.367$$

$$(n \geq 6)$$

$$D_n = n!\left[\frac{1}{2!} - \frac{1}{3!} \cdots \frac{(-1)^n}{n!}\right]$$

$$D_6 =$$

$$D_5 = 5!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right]$$

$$D_5 = 44.$$

Q

Find the number of derangements of the integers from 1 to 10 inclusive, satisfying the condition that the set of elements in the first 5 places is: 6,7,8,9,10, in some order.

A) 14,400

B) 1936

C) 5280

D) nota

1 2 3 4 5 6 7 8 9 10

Deranged:

$5!$ × $5!$

$5! \times 5!$

120×120

$14,400$

1 2 3 4 5 6 7 8 9 10

6 7^{th} . . . . 10^{th}

_ _ _ _ _ _ _ _ _ _

6 7 8 9 10 1 2 3 4 5

10 7 8 9 6

51

Q

At a theater 10 men check their hats. In how many ways can their hats be returned so that

(i) no man receives his own hat? (D_{10})

(ii) at least 1of the men receives his own hat? $(10! - D_{10})$

atleast = Total − D_n

(iii) at least 2 of the men receive their own hats?

exactly 1 person will get right hat

exactly 0 will get same hat

$0 \quad 1 \quad \boxed{2 + 3 + 4 \quad 5 \quad \ldots\ldots \quad 10} = 10!$

Ans Ans

D_{10}

out of 10 it can be any 1.

$D_{10} + {}^{10}C_1 \times D_9$

$= 10! - \left(D_{10} + {}^{10}C_1 \cdot D_9\right)$

0 hat are correctly

1 hat is correctly placed

1 2 3 4 5

what will no. of derangements such that 1 letter is correctly placed.

$\{\_\ \_\ \_\ \_\ \_\}$

1 2 3 4 5

1 $\underline{\quad D4 \quad}$

Choose any element & fix it

this can be done in $^5C_1$ ways

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

$$D5 + {}^5C_1 \cdot D4 + \boxed{{}^5C_2 \cdot D3 + {}^5C_3 \cdot D2 \quad {}^5C_4 \cdot D1 + 1}$$

$$44 + 5 \times 9 + 10 \cdot 2 + 10 \cdot 1 + 5 \times 0 + 1 = 51$$

$$\begin{array}{r} 44 \\ 45 \\ 20 \\ 10 \\ 1 \\ \hline 120 \end{array}$$

Q

The squares of a chessboard are painted 8 different colors. The squares of each row are painted all 8 colors and no 2 consecutive squares in one column can be painted the same color. In how many ways can this be done?

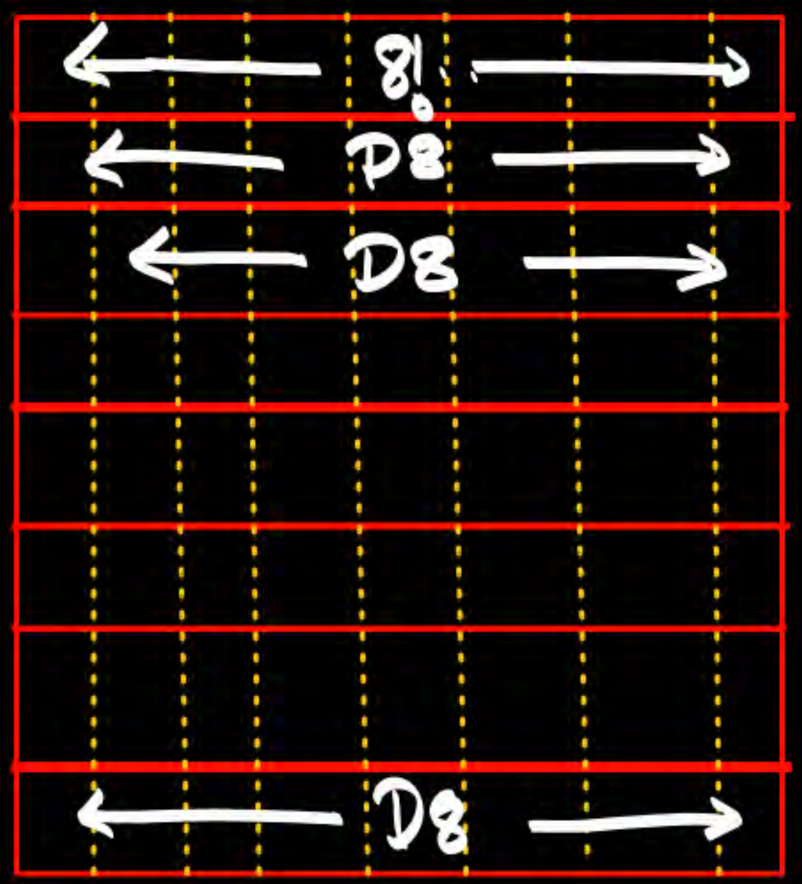
A) 8!(D8)^7

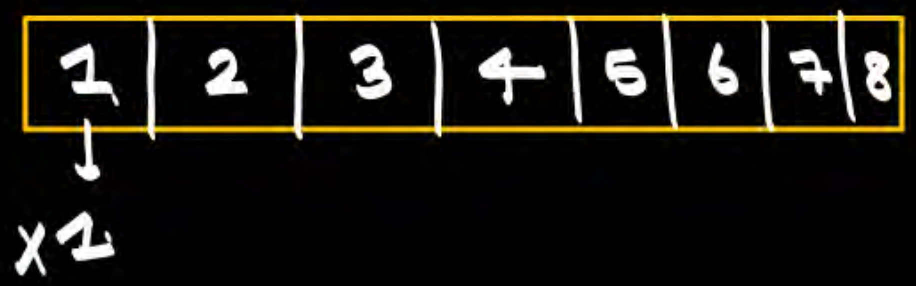
B) D8^2

C) (8!)^8

D) nota

$$8!\,(D_8)^7.$$





Find the number of permutations of the integers1to10inclusive $\{1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\}$

(a) such that exactly 4 of the integers are in their natural positions it can be any 4 → $^{10}C_4 \times D_6$.

(e) that do not begin with a 1 and do not end with 10.

Q

Find the number of $\boxed{\text{permutations}}$ of the integers1to10inclusive

$\{1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\}$

(a) such that exactly 4 of the integers are in their natural positions

$^{10}C_6 \cdot D6 + {}^{10}C_7 \cdot D7 + {}^{10}C_8 \cdot D8 \cdots {}^{10}C_{10} \cdot D10.$

(b) such that 6 or more of the integers are deranged.

(c) that do not have 1 in the first place, nor 4 in the fourth place, nor 7 in the seventh place.

Total $-(A_1 \cup A_2 \cup A_3)$

(d) such that no odd integer will be in the natural position.

(e) that do not begin with a 1 and do not end with 10.

Q

Total $-(A_1 \cup A_2 \cup A_3)$

$A_1 \to$ 1 will be @ 1st place 1 ___ $9!$

$A_2 \to$ 4 . . . @ 4 ___ 4 ___ $9!$

$A_3 \to$ 7 . . @ 7 ← ___ 7 → $9!$

$A_1 + A_2 + A_3 = 3 \cdot 9!$

$A_1 \cap A_2 =$ 1 ___ 4 ___ ← $8!$ →

A_1, A_2, A_3 → choosing 2 element: $\,^3C_2 \times 8!$

fixing 2 element ↓ $8!$

$A_1 \cap A_2 \cap A_3 \to 7!$

(c) that do not have 1 in the first place, nor 4 in the fourth place, nor 7 in the seventh place.

Total $-(A_1 \cup A_2 \cup A_3)$

(H/W) (d) such that no odd integer will be in the natural → position.

$10! - (3 \cdot 9! - \,^3C_2 \, 8! + 7!)$

(e) that do not begin with a 1 and do not end with 10.

Q

Find the number of permutations of the integers1to10inclusive $\{1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\}$

(a) such that exactly 4 of the integers are in their natural positions it can be any 4 $\rightarrow {}^{10}C_4 \times D_6$.

A: beginswith 1. $1\ \underline{\qquad\qquad}$ $9!$.

B: ends with 10 $\underline{\qquad\qquad}\ 10$ $9!$.

$A \cap B$ $1\ \underline{\qquad\qquad}\ 10$ $8!$. $\rightarrow 10! - 2 \cdot 9! + 8!$.

Total $-$ $(A \cup B)$

$10! - (9! + 9! - 8!)$

$\bar{A} \cap \bar{B} =$ ~~Total~~ $\overline{A \cup B} = \text{Total} - A \cup B$.

(e) that do not begin with a 1 and do not end with 10.

donot starts with 1 and donot ends with 10.

$\overline{A} \cap \overline{B}$

Total - (starts with 1 or ends with 10)

$\underbrace{\text{starts with 1}}_{A}$ or $\underbrace{\text{ends with 10}}_{B.}$

$$\boxed{\begin{aligned} &\overline{A} \cap \overline{B} \\ &= \overline{A \cup B} \\ &= \text{Total} - (A \cup B) \end{aligned}}$$

A: starts with 1 $\quad 1 \underbrace{\hspace{3cm}}_{9!}$

B: ends with 10 $\quad \underbrace{\hspace{3cm}}_{9!} 10$

Starts with 1 & end's with 10 $\quad 1 \underbrace{\hspace{3cm}}_{8!} 10$

$A \cup B = A + B - A \cap B.$

$= 9! + 9! - 8!$

Ans: Totatal $- (A \cup B)$

$= 10! - (2 \cdot 9! - 8!)$ ✓

Q Let $A = \{1,2,3, \dots, 7\}$. A function $f: A \to A$ is said to have a fixed point if for some $x \in A, f(x) = x$. How many one-to-one functions $f: A \to A$ have at least one fixed point?

$7! - D_7$

Q

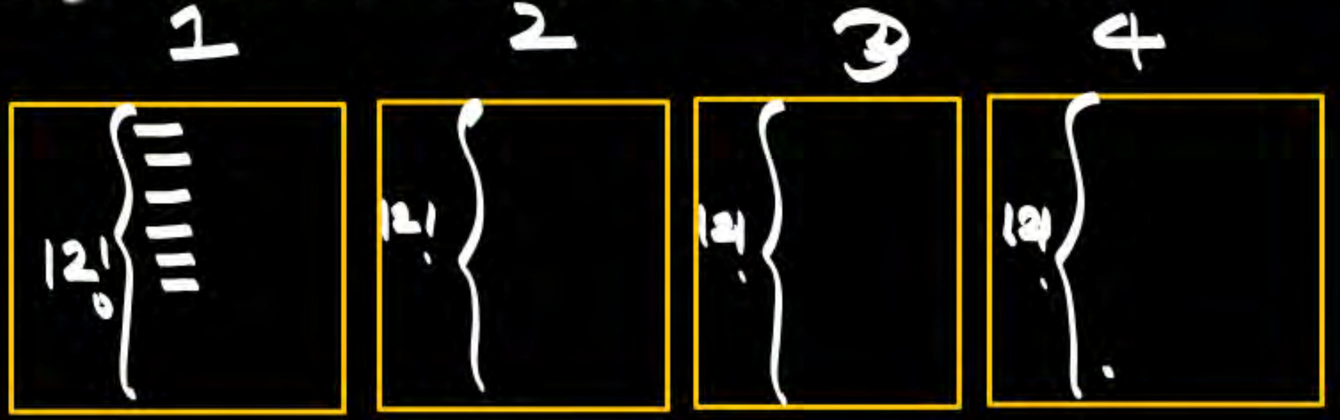
In how many ways can Mrs. Joshi distribute ten distinct books to her ten children (one book to each child) and then collect and redistribute the books so that each child has the opportunity to peruse two different books?

$$(10!) \times D_{10}$$

Q

Komal has 48 different books: 12 each in mathematics, chemistry, physics, and computer science. These books are arranged on four shelves in her office with all books on any one subject on its own shelf. When her office is cleaned, the 48 books are taken down and then replaced on the shelves - once again with all 12 books on any one subject on its own shelf. In how many ways can this be done so that

$D_4 \cdot (12!)^4$



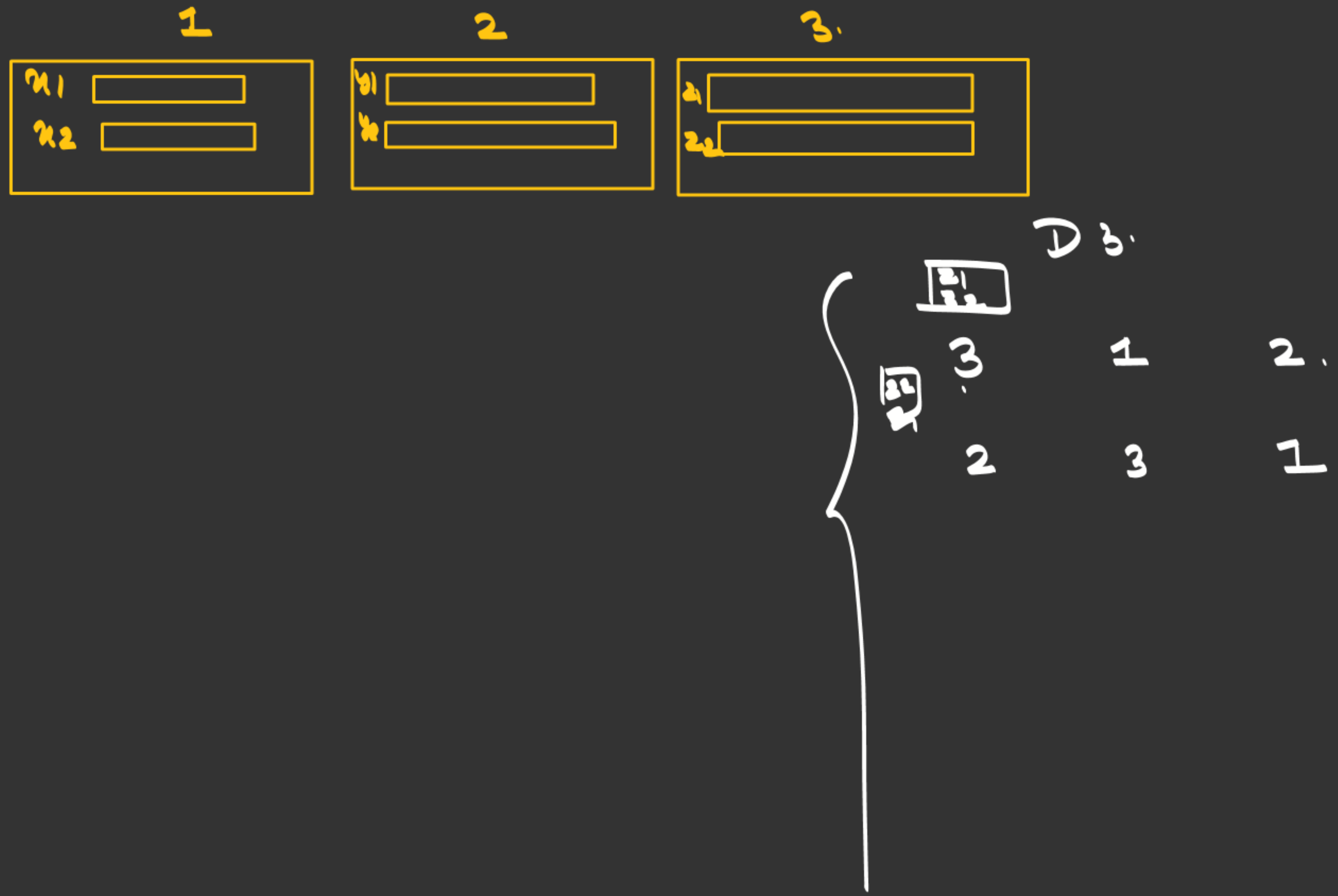
(a) no subject is on its original shelf?

$^4C_1 \times D_3 \times (12!)^4$

(b) one subject is on its original shelf?

$D_4 \times (D_{12})^4$

(c) no subject is on its original shelf and no book is in its original position?



1
2
3.
x1
x2
y1
z1
z2
D3.
3 1 2.
2 3 1

Q

In how many ways can we put 5 letters L1, L2, L3, L4, L5, in 5 envelopes e1, e2, e3, e4, and e5 (at 1 letter per envelope) so that

L1 L2 L3 L4 L5

i. no letter is correctly placed. D_5

ii. at least 1 letter is correctly placed $5! - D_5$

120 119

iii. exactly 2 letters are correctly placed $^5C_2 \cdot D_3$

iv. at most 1 letter is correctly placed $\quad 0 \quad 1$

$D_5 + {}^5C_1 \cdot D_4$

$120 - 1 = \underline{119}$

v. at least 1 letter is wrongly placed vi. exactly 1 letter is wrongly placed ?

0

Q

A high school decides to host a gift exchange for their students. If 150 students participate in the exchange, how many different ways can gift-givers be assigned such that,

(a) no student is assigned to themselves?

(b) there are 50 grade 9's, 30 grade 10's, 30 grade 11's, and 40 grade 12's participating, and the students only draw names from students in their own grade?

Q

Kalil is interviewing for jobs at 5 different companies where each job has a two part interview. He has 5 interview-appropriate outfits. Kalil wants to wear each outfit once for each round of interviews, but does not want to wear the same outfit to the second-part of an interview as he wore to the first. How many ways can he do this?

For the set of positive integers $\{1, 2, 3, 4, \ldots, n-1, n\}$, we know that the first 6 digits appear in the first 6 positions. If there are 2 385 derangement of this set, what is the value of n?

A waiter (who is not particularly good at their job) has 8 customers at lunch. Every person order a different meal. How many different ways can the waiter bring people their food such that:

(a) no one get the meal they ordered?

(b) at least one person gets the food they ordered?

(c) exactly two people get the food they ordered?

(d) exactly one person gets someone else's food?

Twelve friends host a potluck (a *party* where everyone brings a dish), six of them are vegetarians while the other six are not. Every individual bring both a drink and a main dish (suppose that the vegetarians only bring vegetarian main dishes). How many ways can these friends bring home leftovers such that each friend brings home one drink and one main dish and

(a) no friend brings home either of the things they brought to the potluck?

(b) the vegetarians all bring home the meal they brought but not their drink, while the non-vegetarians bring home the drink they brought but not a different main dish?

(c) No one brings home *both* of the items they brought.
Hint: Use PIE instead of trying to adapt the derangement formula.

Find the number of derangements of the integers from 1to 10inclusive, satisfying the condition that the set of elements in the first 5 places is

(a) 1,2,3,4,5, in some order, ANS : D5 . D5 = 1936.

(b) 6,7,8,9,10, in some order. (5!)^2 = 14,400

4. An advertising agency has 1,000 clients. Suppose that T is the set of clients that use television advertising, R is the set of clients that use radio advertising, and N is the set of clients who use newspaper advertising. Suppose that $|T| = 415$, $|R| = 350$, $|N| = 280$, 100 clients use all 3 types of advertising, 175 use television and radio, 180 use radio and newspapers, and $|T \cap N| = 165$.
(a) Find $|T \cap R \cap \overline{N}|$.
(b) How many clients use radio and newspaper advertising but not television?
(c) How many use television but do not use newspaper advertising and do not use radio advertising?
(d) Find $|\overline{T} \cap \overline{R} \cap \overline{N}|$.

4. (a) 75.
(b) $|R \cap N \cap \overline{T}| = 80$
(c) $|T \cap \overline{N} \cap \overline{R}| = 175$.
(d) $|\overline{T} \cap \overline{R} \cap \overline{N}| = |\overline{T \cup R \cup N}| = 1{,}000 - 625 = 375$.

10. Find the number of permutations of the integers 1 to 10 inclusive ·
    (a) such that exactly 4 of the integers are in their natural positions (that is, exactly 6 of the integers are deranged).
    (b) such that 6 or more of the integers are deranged.
    (c) that do not have 1 in the first place, nor 4 in the fourth place, nor 7 in the seventh place.
    (d) such that no odd integer will be in the natural position.
    (e) that do not begin with a 1 and do not end with 10.

10. (a) $C(10,6)D_6$.
    (b) $\binom{10}{6}D_6 + \binom{10}{7}D_7 + \binom{10}{8}D_8 + \binom{10}{9}D_9 + \binom{10}{10}D_{10}$
    (c) $10! - (3)9! + (3)8! - 7!$.
    (d) $10! - \binom{5}{1}9! + \binom{5}{2}8! - \binom{5}{3}7! + \binom{5}{4}6! - \binom{5}{5}5!$.
    (e) $10! - (2)9! + 8!$.

17. At a theater 10 men check their hats. In how many ways can their hats be returned so that
    (a) no man receives his own hat?
    (b) at least 1 of the men receives his own hat?
    (c) at least 2 of the men receive their own hats?

17. (a) D_{10}.
    (b) $10! - D_{10}$.
    (c) $10! - D_{10} - 10D_9$.

25. The squares of a chessboard are painted 8 different colors. The squares of each row are painted all 8 colors and no 2 consecutive squares in one column can be painted the same color. In how many ways can this be done?

25. The first row can be painted 8! ways. Each row after the first can be painted D_8 ways. Hence the number of ways is $8!(D_8)^7$.

5. Determine the number of positive integers n, $1 \le n \le 2000$, that are

a) not divisible by 2, 3, or 5

b) not divisible by 2, 3, 5, or 7

c) not divisible by 2, 3, or 5, but are divisible by 7

(a) c_1: number n is divisible by 2
c_2: number n is divisible by 3
c_3: number n is divisible by 5
$N(c_1) = \lfloor 2000/2 \rfloor = 1000,\ N(c_2) = \lfloor 2000/3 \rfloor = 666,$
$N(c_3) = \lfloor 2000/5 \rfloor = 400,\ N(c_1c_2) = \lfloor 2000/(2)(3) \rfloor = 333,$
$N(c_2c_3) = \lfloor 2000/(3)(5) \rfloor = 133,\ N(c_1c_3) = \lfloor 2000/(2)(5) \rfloor = 200,$
$N(c_1c_2c_3) = \lfloor 2000/(2)(3)(5) \rfloor = 66.$
$N(\bar{c}_1\bar{c}_2\bar{c}_3) = 2000 - (1000 + 666 + 400) + (333 + 200 + 133) - 66 = 534$

(b) Let c_1, c_2, c_3 be as in part (a). Let c_4 denote the number n is divisible by 7. Then $N(c_4) = 285,\ N(c_1c_4) = 142,\ N(c_2c_4) = 95,\ N(c_3c_4) = 57,\ N(c_1c_2c_4) = 47,\ N(c_1c_3c_4) = 28,\ N(c_2c_3c_4) = 19,\ N(c_1c_2c_3c_4) = 9.$ $N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = 2000 - (1000 + 666 + 400 + 285) + (333 + 200 + 133 + 142 + 95 + 57) - (66 + 47 + 28 + 19) + 9 = 458$
(c) $534 - 458 = 76.$

2. a) List all the derangements of 1, 2, 3, 4, 5 where the first three numbers are 1, 2, and 3, in some order.

b) List all the derangements of 1, 2, 3, 4, 5, 6 where the first three numbers are 1, 2, and 3, in some order.

3. How many derangements are there for 1, 2, 3, 4, 5?

4. How many permutations of 1, 2, 3, 4, 5, 6, 7 are not derangements?

5. a) Let $A = \{1, 2, 3, \ldots, 7\}$. A function $f: A \to A$ is said to have a *fixed point* if for some $x \in A$, $f(x) = x$. How many one-to-one functions $f: A \to A$ have at least one fixed point?

b) In how many ways can we devise a secret code by assigning to each letter of the alphabet a different letter to represent it?

2. (a) There are only two derangements with this property: 23154 and 31254.
(b) Here there are four such derangements:
(i) 231546 (ii) 231645 (iii) 312546 (iv) 312645

3. The number of derangements for 1,2,3,4,5 is $5![1 - 1 + (1/2!) - (1/3!) + (1/4!) - (1/5!)] = 5![(1/2!) - (1/3!) + (1/4!) - (1/5!)] = (5)(4)(3) - (5)(4) + 5 - 1 = 60 - 20 + 5 - 1 = 44.$

4. There are $7! = 5040$ permutations of 1,2,3,4,5,6,7. Among these there are $7![1 - 1 + (1/2!) - (1/3!) + (1/4!) - (1/5!) + (1/6!) - (1/7!)] = 1854$ derangements. Consequently, we have $5040 - 1854 = 3186$ permutations of 1,2,3,4,5,6,7 that are not derangements.

5. (a) $7! - d_7\ (d_7 \doteq (7!)e^{-1})$; (b) $d_{26} \doteq (26!)e^{-1}$

6. How many derangements of 1, 2, 3, 4, 5, 6, 7, 8 start with (a) 1, 2, 3, and 4, in some order? (b) 5, 6, 7, and 8, in some order?

7. For the positive integers $1, 2, 3, \ldots, n-1, n$, there are 11,660 derangements where 1, 2, 3, 4, and 5 appear in the first five positions. What is the value of n?

6. (a) There are $(d_4)^2 = 9^2 = 81$ such derangements.
(b) In this case we get $(4!)^2 = 24^2 = 576$ derangements.

7. Let $n = 5 + m$. Then $11,660 = d_5 \cdot d_m = 44(d_m)$, and so $d_m = 265 = d_6$. Consequently, $n = 11$.

9. In how many ways can Mrs. Ford distribute ten distinct books to her ten children (one book to each child) and then collect and redistribute the books so that each child has the opportunity to peruse two different books?

9. $(10!)d_{10} \doteq (10!)^2(e^{-1})$

12. Ms. Pezzulo teaches geometry and then biology to a class of 12 advanced students in a classroom that has only 12 desks. In how many ways can she assign the students to these desks so that (a) no student is seated at the same desk for both classes? (b) there are exactly six students each of whom occupies the same desk for both classes?

12. (a) $(12!)d_{12}$ (b) $(12!)\binom{12}{6}d_6$

1. Determine how many $n \in \mathbf{Z}^+$ satisfy $n \leq 500$ and are not divisible by 2, 3, 5, 6, 8, or 10.

1. We need only consider the divisors 2,3, and 5. Let c_1 denote divisibility by 2, c_2 divisibility by 3, and c_3 divisibility by 5.

$N = 500$; $N(c_1) = \lfloor 500/2 \rfloor = 250$; $N(c_2) = \lfloor 500/3 \rfloor = 166$; $N(c_3) = \lfloor 500/5 \rfloor = 100$; $N(c_1c_2) = \lfloor 500/6 \rfloor = 83$; $N(c_1c_3) = \lfloor 500/10 \rfloor = 50$; $N(c_2c_3) = \lfloor 500/15 \rfloor = 33$; $N(c_1c_2c_3) = \lfloor 500/30 \rfloor = 16$.

$N(\bar{c}_1\bar{c}_2\bar{c}_3) = 500 - (250 + 166 + 100) + (83 + 50 + 33) - 16 = 134$.

19. Caitlyn has 48 different books: 12 each in mathematics, chemistry, physics, and computer science. These books are arranged on four shelves in her office with all books on any one subject on its own shelf. When her office is cleaned, the 48 books are taken down and then replaced on the shelves—once again with all 12 books on any one subject on its own shelf. In how many ways can this be done so that (a) no subject is on its original shelf? (b) one subject is on its original shelf? (c) no subject is on its original shelf and no book is in its original position? [For example, the book originally in the third (from the left) position on the first shelf must not be replaced on the first shelf and must not be in the third (from the left) position on the shelf where it is placed.]

19. a) $d_4(12!)^4$
b) $\binom{4}{1}d_3(12!)^4$
c) $d_4(d_{12})^4$

4. Annually, the 65 members of the maintenance staff sponsor a "Christmas in July" picnic for the 400 summer employees at their company. For these 65 people, 21 bring hot dogs, 35 bring fried chicken, 28 bring salads, 32 bring desserts, 13 bring hot dogs and fried chicken, 10 bring hot dogs and salads, 9 bring hot dogs and desserts, 12 bring fried chicken and salads, 17 bring fried chicken and desserts, 14 bring salads and desserts, 4 bring hot dogs, fried chicken, and salads, 6 bring hot dogs, fried chicken, and desserts, 5 bring hot dogs, salads, and desserts, 7 bring fried chicken, salads, and desserts, and 2 bring all four food items. Those (of the 65) who do not bring any of these four food items are responsible for setting up and cleaning up for the picnic. How many of the 65 maintenance staff will (a) help to set up and clean up for the picnic? (b) bring only hot dogs? (c) bring exactly one food item?

c_1: **Staff member brings hot dogs**
c_2: **Staff member brings fried chicken**
c_3: **Staff member brings salads**
c_4: **Staff member brings desserts**
$N = 65$
$N(c_1) = 21$; $N(c_2) = 35$; $N(c_3) = 28$; $N(c_4) = 32$
$N(c_1c_2) = 13$; $N(c_1c_3) = 10$; $N(c_1c_4) = 9$; $N(c_2c_3) = 12$; $N(c_2c_4) = 17$; $N(c_3c_4) = 14$
$N(c_1c_2c_3) = 4$; $N(c_1c_2c_4) = 6$; $N(c_1c_3c_4) = 5$; $N(c_2c_3c_4) = 7$
$N(c_1c_2c_3c_4) = 2$.

(a) $N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = 65-[21+35+28+32]+[13+10+9+12+17+14]-[4+6+5+7]+2 = 65-116+75-22+2 = 4$.
(b) $N(\bar{c}_2\bar{c}_3\bar{c}_4) = N-[N(c_2)+N(c_3)+N(c_4)]+[N(c_2c_3)+N(c_2c_4)+N(c_3c_4)]-N(c_2c_3c_4)$, so $N(c_1\bar{c}_2\bar{c}_3\bar{c}_4) = N(c_1)-[N(c_1c_2)+N(c_1c_3)+N(c_1c_4)]+[N(c_1c_2c_3)+N(c_1c_2c_4)+N(c_1c_3c_4)]-N(c_1c_2c_3c_4) = 21-[13+10+9]+[4+6+5]-2 = 21-32+15-2 = 2$.

(c) $N(\bar{c}_1c_2\bar{c}_3\bar{c}_4) = N(c_2)-[N(c_1c_2)+N(c_2c_3)+N(c_2c_4)]+[N(c_1c_2c_3)+N(c_1c_2c_4)+N(c_2c_3c_4)]-N(c_1c_2c_3c_4) = 35-[13+12+17]+[4+6+7]-2 = 35-42+17-2 = 8$

$N(\bar{c}_1\bar{c}_2c_3\bar{c}_4) = N(c_3)-[N(c_1c_3)+N(c_2c_3)+N(c_3c_4)]+[N(c_1c_2c_3)+N(c_1c_3c_4)+N(c_2c_3c_4)]-N(c_1c_2c_3c_4) = 28-[10+12+14]+[4+5+7]-2 = 28-36+16-2 = 6$.
$N(\bar{c}_1\bar{c}_2\bar{c}_3c_4) = N(c_4)-[N(c_1c_4)+N(c_2c_4)+N(c_3c_4)]+[N(c_1c_2c_4)+N(c_1c_3c_4)+N(c_2c_3c_4)]-N(c_1c_2c_3c_4) = 32-[9+17+14]+[6+5+7]-2 = 32-40+18-2 = 8$.
So the answer is $2+8+6+8 = 24$.


THANK YOU