CS & IT ENGINEERING

Discrete Mathematics

Combinatorics

Lecture No.- 07

By- Satish Yadav sir

Recap of Previous Lecture

Topic Introduction to Combinatorics

Topics to be Covered

Topic Pigeonhole Principle

Topic : Combinatorics

Inclusion / Exclusion :

$\{1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\}$

Q:1 ⇒ what will be no. of elements which are ÷ by 2 or 5 in a set $\{1 \ldots 10\}$.

Total − $\boxed{P(2 \cup 5)}$

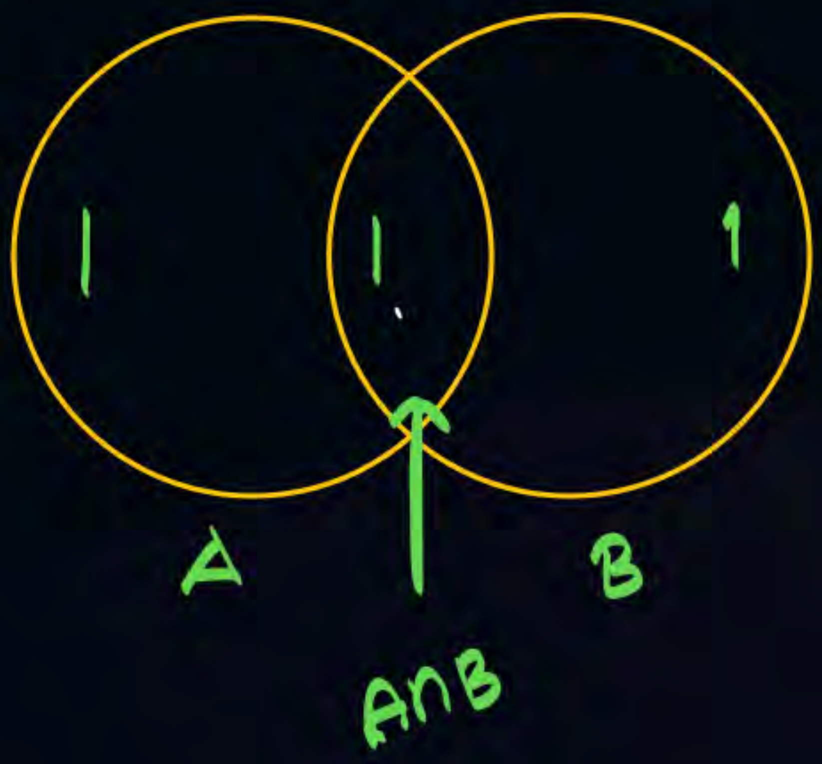
$P(2)$: no. of elements ÷ by 2 = $\frac{\text{Total elements}}{2} = \frac{10}{2} = 5$

$P(5)$: no. of elements ÷ by 5 = $\frac{10}{5} = 2$

$P(2 \cap 5) = \frac{10}{2 \cdot 5} = 1$

Ans: $P(2) + P(5) = 5 + 2 = 7 - 1 = 6$

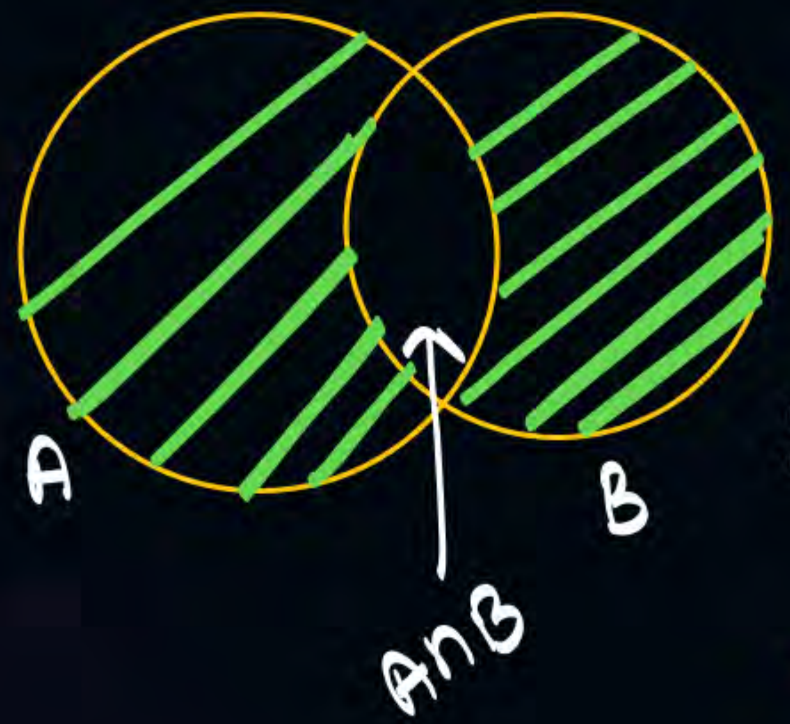
$I = \{e_1, e_2, \ldots, e_x\}$



$A \cup B = A + B - A \cap B$

$A_1 \cup A_2 = A_1 + A_2 - A_1 \cap A_2$

$n = 2$



$A - B = A - (A \cap B)$

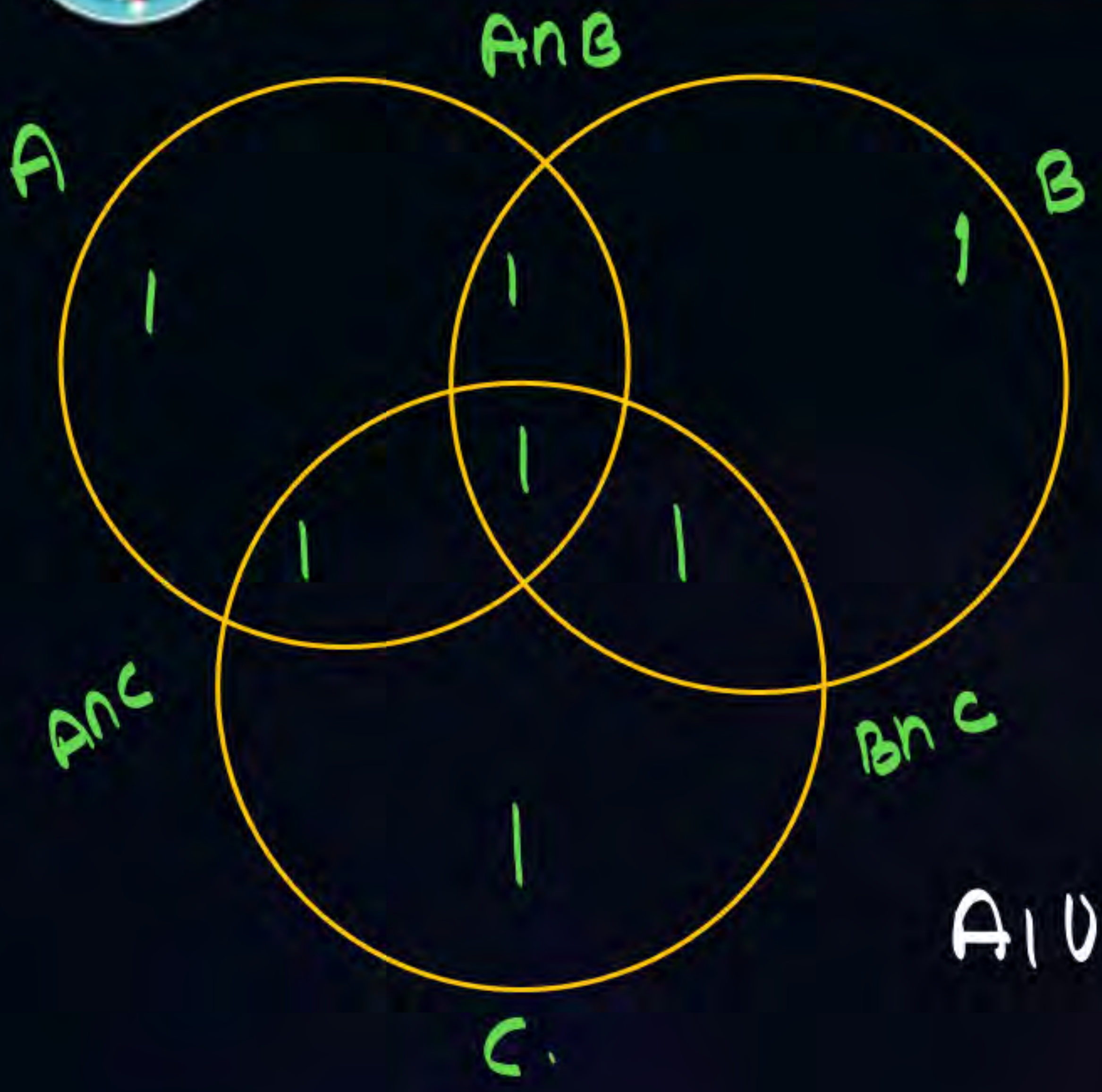
$B - A = B - (A \cap B)$

Symmetric Diff (Δ / $\oplus$) — present in A or in B but not in both

$= (A - B) \cup (B - A)$

$= (A \cup B) - (A \cap B)$

Topic : Combinatorics



$$A \cup B \cup C = A + B + C - A \cap B - B \cap C - A \cap C + A \cap B \cap C$$

$$A_1 \cup A_2 \cup A_3 = A_1 + A_2 + A_3 - A_1 \cap A_2 - A_2 \cap A_3 - A_1 \cap A_3 + A_1 \cap A_2 \cap A_3.$$

$$A_1 \cup A_2 \cup A_3 \cup A_4 = \sum A_i - \sum A_i \cap A_j + \sum A_i \cap A_j \cap A_k - A_1 \cap A_2 \cap A_3 \cap A_4.$$

Topic : Combinatorics

$$A_1 \cup A_2 \cup \dots A_n = \sum A_i - \sum A_i \cap A_j + \sum A_i \cap A_j \cap A_K$$

$$\dots\dots (-1)^{n+1} (A_1 \cap A_2 \cap A_3 \dots A_n)$$

Topic : Combinatorics

(GATE)

How many int are not ÷ by 2, 3, or 5 in a set $\{1, \ldots 123\}$?

$$P(2 \text{ OR } 3 \text{ OR } 5) = P(2) + P(3) + P(5) - P(2\cdot3) - P(3\cdot5) - P(2\cdot5) + P(2\cdot3\cdot5)$$

$$= \left\lfloor\frac{123}{2}\right\rfloor + \left\lfloor\frac{123}{3}\right\rfloor + \left\lfloor\frac{123}{5}\right\rfloor - \left\lfloor\frac{123}{\text{lcm}(2\cdot3)}\right\rfloor - \left\lfloor\frac{123}{\text{lcm}(3\cdot5)}\right\rfloor - \left\lfloor\frac{123}{\text{lcm}(2\cdot5)}\right\rfloor + \left\lfloor\frac{123}{2\cdot3\cdot5}\right\rfloor$$

$$\text{Ans} = 123 - P(2 \cup 3 \cup 5) = 33.$$

no. of elements which are ÷ by 4, 6 or from a set $\{1 \ldots \ldots 1000\}$?

$$P(4 \text{ OR } 6 \text{ OR } 8) = \left\lfloor \frac{1000}{4} \right\rfloor + \left\lfloor \frac{1000}{6} \right\rfloor + \ ' \ - \ - \left\lfloor \frac{1000}{lcm(4,6)} \right\rfloor$$

$\{1 \quad 4 \quad 6 \quad 8 \quad 12 \ \ldots \ 18 \quad 20 \quad 24 \quad 28 \ldots \quad 1000\}$

Topic : Combinatorics

no of elements which are $\div$ by 4 or 6 from a set $\{1 \ldots 1000\}$.

$$\left\lfloor \frac{1000}{4} \right\rfloor + \left\lfloor \frac{1000}{6} \right\rfloor - \left\lfloor \frac{1000}{\text{lcm}(4 \cdot 6)} \right\rfloor \left(\frac{1000}{24} \right.$$

4 6 8 c 12 16 18 20 24 28 30 32 36

overc

Topic : Combinatorics

$x_1 + x_2 + x_3 = 20$

$2 \le x_1 \le 6$

$3 \le x_2 \le 7$

$1 \le x_3 \le 5$

Required

x_1	x_2	x_3
2	3	1
3	4	2
4	5	3
5	6	4
6	7	5
7	8	6

0

Total

$A_1 \cup A_2 \cup A_3$

A1 A2 A3

$x_1 \ge 7 \quad x_2 \ge 8 \quad x_3 \ge 6$

$A_1 : \{ x_1 + x_2 + x_3 = 20 \mid x_1 \ge 7 \}$

$x_2 \ge 3 \quad x_3 \ge 1$

$A_2 : \{ x_1 + x_2 + x_3 = 20 \mid x_2 \ge 8 \}$

$x_1 \ge 2 \quad x_3 \ge 1$

$A_3 : \{ x_1 + x_2 + x_3 = 20 \mid x_3 \ge 6 \}$

$x_1 \ge 2 \quad x_2 \ge 3$

Required = Total − $A_1 \cup A_2 \cup A_3$

$= \{ x_1 + x_2 + x_3$, $x_1 \ge 2$, $x_2 \ge 3$, $x_3 \ge 1 \} - \{ A_1 \cup A_2 \cup A_3 \}$

Topic : Combinatorics

$$^{14+2}C_2 = {}^{16}C_2.$$

$$\text{Total} = U = \{x_1 + x_2 + x_3 = 20 \mid x_1 \geq 2 \quad x_2 \geq 3 \quad x_3 \geq 1\}.$$

$A_1 \rightarrow 7.$

$A_2 \rightarrow 8$

$A_3 \rightarrow 6.$

Q: $x_1 + x_2 + x_3 = 20$

$2 \leq x_1 \leq 5$ no. of non negative soltⁿ ?

$4 \leq x_2 \leq 7$

$-2 \leq x_3 \leq 9$.

Topic : Combinatorics

Derangement : (D_n) nothing is @ right place.

places: 1 2 3

1	2	3	×
1	3	2	×
2	1	3	×
2	3	1	✓
3	1	2	✓
3	2	1	×

D_3

$D_3 = 2$.

$D_3 =$ Total − atleast 1 element @ right position.

$D_3 = 3! - ($atleast 1 —— " $)$

Topic : Combinatorics

$D_4 = 4! - $(atleast 1 element @ right position)

$= 4! - (A_1 \cup A_2 \cup A_3 \cup A_4)$

$\underline{A_1}$ $\underline{A_2}$ $\underline{A_3}$ $\underline{A_4}$.

$A_1 \rightarrow$ Total no. of ways.
1st element @ right position.

$A_2 \rightarrow$ —— " ——— 2nd ——— " ———.

$A_3 \rightarrow$ ——— " ——— 3rd ——— " ———.

A_4: ——— " — 4 ———.

$A_1 \cap A_2 =$ Total ways.
1 & 2 elements.
@ right position

$|A_1| = 3!$. $|A_2| = 3!$.
$|A_3| = 3!$ $|A_4| = 3!$.

$\_\ \_\ \_\ \_$.

$A_1 \Rightarrow$ $1\ \underline{2}\ \underline{3}\ \underline{4}$ A_2 $\underline{1}\ 2\ \underline{3}\ \underline{4}$. $A_1 \cap A_2 = 1\ 2\ \_\ \_$.

$1\ 2\ 4\ 3$ $1\ 2\ 4\ 3$

$|A_1| = 3!$. $2!$

$3!$. $3!$.

$\underline{A_1 \cap A_2}$ $\underline{A_2 \cap A_3}$. there are 4 places

$A_1 \cap A_3$ we can choose 2

in $^4C_2$ ways.

Topic : Combinatorics

$$D_4 = 4! - (A_1 \cup A_2 \cup A_3 \cup A_4)$$

$$= 4! - \left(\sum A_i - \sum A_i \cap A_j + \sum A_i \cap A_j \cap A_k - A_1 \cap A_2 \cap A_3 \cap A_4 \right)$$

$$= 4! - \left({}^4C_1 \cdot 3! - {}^4C_2 \cdot 2! + {}^4C_3 \cdot 1! - 1 \right)$$

$$= 4! - \underbrace{4 \times 3!}_{} + {}^4C_2 \cdot 2! - {}^4C_3 - \underbrace{1}_{}$$

$$= \cancel{4!} - \cancel{4!} + \frac{4!}{2! \cdot \cancel{2!}} \cdot \cancel{2!} - \frac{4!}{3!} - \frac{4!}{4!} = \frac{4!}{2!} - \frac{4!}{3!} + \frac{4!}{4!}$$

Topic : Combinatorics

$$D_4 = \frac{4!}{2!} - \frac{4!}{3!} + \frac{4!}{4!} = 4!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right] = 9$$

$$D_n = n!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots\cdots \frac{(-1)^n}{n!}\right]$$

Topic : Combinatorics

$D_1 = 0$

$D_2 = 1$

$D_3 = 2$

$D_4 = 9$

$D_5 = 44$

$D_6 = 265$

$$D_n = n!\left[\underbrace{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots \frac{(-1)^n}{n!}}\right] \quad (n \geq 7)$$

$$D_n \cong n! \times 0.367$$

$$D_6 \cong 6! \times 0.367$$

$$D_n = n! \times 0.367$$

$$\left\{ \frac{D_n}{n!} \cong \underline{\underline{0.36}} \right.$$

$n = 10$

$$D_{10} \cong 10! \times 0.36$$

$$\frac{D_{10}}{10!} \cong 0.36$$

$n = 10$ element

$n = 11$

$$\frac{D_{11}}{11!} \cong 0.36$$

10!

11!

100

Topic : Combinatorics

$e = 2.718$

$e^{-1} = 0.36$

$$e^{x} = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$$

put $x = -1$

$$e^{-1} = \cancel{1 + \frac{(-1)}{1!}} + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \frac{(-1)^4}{4!}$$

$$0.3678 = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}$$

Find the number of derangements of the integers from 1to 10inclusive, satisfying the condition that the set of elements in the first 5 places is

(a) 1,2,3,4,5, in some order, ANS : D5 . D5 = 1936.

(b) 6,7,8,9,10, in some order. (5!)^2 = 14,400

4. An advertising agency has 1,000 clients. Suppose that T is the set of clients that use television advertising, R is the set of clients that use radio advertising, and N is the set of clients who use newspaper advertising. Suppose that $|T| = 415$, $|R| = 350$, $|N| = 280$, 100 clients use all 3 types of advertising, 175 use television and radio, 180 use radio and newspapers, and $|T \cap N| = 165$.

(a) Find $|T \cap R \cap \overline{N}|$.

(b) How many clients use radio and newspaper advertising but not television?

(c) How many use television but do not use newspaper advertising and do not use radio advertising?

(d) Find $|\overline{T} \cap \overline{R} \cap \overline{N}|$.

4. (a) 75.

(b) $|R \cap N \cap \overline{T}| = 80$

(c) $|T \cap \overline{N} \cap \overline{R}| = 175$.

(d) $|\overline{T} \cap \overline{R} \cap \overline{N}| = |\overline{T \cup R \cup N}| = 1{,}000 - 625 = 375$.

10. Find the number of permutations of the integers 1 to 10 inclusive ·
   (a) such that exactly 4 of the integers are in their natural positions (that is, exactly 6 of the integers are deranged).
   (b) such that 6 or more of the integers are deranged.
   (c) that do not have 1 in the first place, nor 4 in the fourth place, nor 7 in the seventh place.
   (d) such that no odd integer will be in the natural position.
   (e) that do not begin with a 1 and do not end with 10.

10. (a) $C(10,6)D_6$.
   (b) $\binom{10}{6}D_6 + \binom{10}{7}D_7 + \binom{10}{8}D_8 + \binom{10}{9}D_9 + \binom{10}{10}D_{10}$
   (c) $10! - (3)9! + (3)8! - 7!$.
   (d) $10! - \binom{5}{1}9! + \binom{5}{2}8! - \binom{5}{3}7! + \binom{5}{4}6! - \binom{5}{5}5!$.
   (e) $10! - (2)9! + 8!$.

17. At a theater 10 men check their hats. In how many ways can their hats be returned so that
   (a) no man receives his own hat?
   (b) at least 1 of the men receives his own hat?
   (c) at least 2 of the men receive their own hats?

17. (a) D_{10}.
   (b) $10! - D_{10}$.
   (c) $10! - D_{10} - 10D_9$.

25. The squares of a chessboard are painted 8 different colors. The squares of each row are painted all 8 colors and no 2 consecutive squares in one column can be painted the same color. In how many ways can this be done?

25. The first row can be painted 8! ways. Each row after the first can be painted D_8 ways. Hence the number of ways is $8!(D_8)^7$.

5. Determine the number of positive integers n, $1 \leq n \leq 2000$, that are

a) not divisible by 2, 3, or 5

b) not divisible by 2, 3, 5, or 7

c) not divisible by 2, 3, or 5, but are divisible by 7

(a) c_1: number n is divisible by 2
c_2: number n is divisible by 3
c_3: number n is divisible by 5
$N(c_1) = \lfloor 2000/2 \rfloor = 1000,\ N(c_2) = \lfloor 2000/3 \rfloor = 666,$
$N(c_3) = \lfloor 2000/5 \rfloor = 400,\ N(c_1c_2) = \lfloor 2000/(2)(3) \rfloor = 333,$
$N(c_2c_3) = \lfloor 2000/(3)(5) \rfloor = 133,\ N(c_1c_3) = \lfloor 2000/(2)(5) \rfloor = 200,$
$N(c_1c_2c_3) = \lfloor 2000/(2)(3)(5) \rfloor = 66.$
$N(\bar{c}_1\bar{c}_2\bar{c}_3) = 2000 - (1000 + 666 + 400) + (333 + 200 + 133) - 66 = 534$

(b) Let c_1, c_2, c_3 be as in part (a). Let c_4 denote the number n is divisible by 7. Then $N(c_4) = 285,\ N(c_1c_4) = 142,\ N(c_2c_4) = 95,\ N(c_3c_4) = 57,\ N(c_1c_2c_4) = 47,\ N(c_1c_3c_4) = 28,\ N(c_2c_3c_4) = 19,\ N(c_1c_2c_3c_4) = 9.\ N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = 2000 - (1000 + 666 + 400 + 285) + (333 + 200 + 133 + 142 + 95 + 57) - (66 + 47 + 28 + 19) + 9 = 458$
(c) $534 - 458 = 76.$

2. **a)** List all the derangements of 1, 2, 3, 4, 5 where the first three numbers are 1, 2, and 3, in some order.

b) List all the derangements of 1, 2, 3, 4, 5, 6 where the first three numbers are 1, 2, and 3, in some order.

3. How many derangements are there for 1, 2, 3, 4, 5?

4. How many permutations of 1, 2, 3, 4, 5, 6, 7 are not derangements?

5. **a)** Let $A = \{1, 2, 3, \ldots, 7\}$. A function $f: A \to A$ is said to have a *fixed point* if for some $x \in A$, $f(x) = x$. How many one-to-one functions $f: A \to A$ have at least one fixed point?

b) In how many ways can we devise a secret code by assigning to each letter of the alphabet a different letter to represent it?

2. (a) There are only two derangements with this property: 23154 and 31254.
(b) Here there are four such derangements:
(i) 231546 (ii) 231645 (iii) 312546 (iv) 312645

3. The number of derangements for 1,2,3,4,5 is $5![1 - 1 + (1/2!) - (1/3!) + (1/4!) - (1/5!)] = 5![(1/2!) - (1/3!) + (1/4!) - (1/5!)] = (5)(4)(3) - (5)(4) + 5 - 1 = 60 - 20 + 5 - 1 = 44.$

4. There are $7! = 5040$ permutations of 1,2,3,4,5,6,7. Among these there are $7![1 - 1 + (1/2!) - (1/3!) + (1/4!) - (1/5!) + (1/6!) - (1/7!)] = 1854$ derangements. Consequently, we have $5040 - 1854 = 3186$ permutations of 1,2,3,4,5,6,7 that are not derangements.

5. (a) $7! - d_7\ (d_7 \doteq (7!)e^{-1})$; (b) $d_{26} \doteq (26!)e^{-1}$

6. How many derangements of 1, 2, 3, 4, 5, 6, 7, 8 start with (a) 1, 2, 3, and 4, in some order? (b) 5, 6, 7, and 8, in some order?

7. For the positive integers $1, 2, 3, \ldots, n-1, n$, there are 11,660 derangements where 1, 2, 3, 4, and 5 appear in the first five positions. What is the value of n?

6. (a) There are $(d_4)^2 = 9^2 = 81$ such derangements.
(b) In this case we get $(4!)^2 = 24^2 = 576$ derangements.

7. Let $n = 5 + m$. Then $11,660 = d_5 \cdot d_m = 44(d_m)$, and so $d_m = 265 = d_6$. Consequently, $n = 11$.

9. In how many ways can Mrs. Ford distribute ten distinct books to her ten children (one book to each child) and then collect and redistribute the books so that each child has the opportunity to peruse two different books?

9. $(10!)d_{10} \doteq (10!)^2(e^{-1})$

12. Ms. Pezzulo teaches geometry and then biology to a class of 12 advanced students in a classroom that has only 12 desks. In how many ways can she assign the students to these desks so that (a) no student is seated at the same desk for both classes? (b) there are exactly six students each of whom occupies the same desk for both classes?

12. (a) $(12!)d_{12}$ (b) $(12!)\binom{12}{6}d_6$

1. Determine how many $n \in \mathbf{Z}^+$ satisfy $n \leq 500$ and are not divisible by 2, 3, 5, 6, 8, or 10.

1. We need only consider the divisors 2,3, and 5. Let c_1 denote divisibility by 2, c_2 divisibility by 3, and c_3 divisibility by 5.

$N = 500$; $N(c_1) = \lfloor 500/2 \rfloor = 250$; $N(c_2) = \lfloor 500/3 \rfloor = 166$; $N(c_3) = \lfloor 500/5 \rfloor = 100$; $N(c_1c_2) = \lfloor 500/6 \rfloor = 83$; $N(c_1c_3) = \lfloor 500/10 \rfloor = 50$; $N(c_2c_3) = \lfloor 500/15 \rfloor = 33$; $N(c_1c_2c_3) = \lfloor 500/30 \rfloor = 16$.

$N(\bar{c}_1\bar{c}_2\bar{c}_3) = 500 - (250 + 166 + 100) + (83 + 50 + 33) - 16 = 134.$

19. Caitlyn has 48 different books: 12 each in mathematics, chemistry, physics, and computer science. These books are arranged on four shelves in her office with all books on any one subject on its own shelf. When her office is cleaned, the 48 books are taken down and then replaced on the shelves — once again with all 12 books on any one subject on its own shelf. In how many ways can this be done so that (a) no subject is on its original shelf? (b) one subject is on its original shelf? (c) no subject is on its original shelf and no book is in its original position? [For example, the book originally in the third (from the left) position on the first shelf must not be replaced on the first shelf and must not be in the third (from the left) position on the shelf where it is placed.]

19. a) $d_4(12!)^4$
b) $\binom{4}{1}d_3(12!)^4$
c) $d_4(d_{12})^4$

4. Annually, the 65 members of the maintenance staff sponsor a "Christmas in July" picnic for the 400 summer employees at their company. For these 65 people, 21 bring hot dogs, 35 bring fried chicken, 28 bring salads, 32 bring desserts, 13 bring hot dogs and fried chicken, 10 bring hot dogs and salads, 9 bring hot dogs and desserts, 12 bring fried chicken and salads, 17 bring fried chicken and desserts, 14 bring salads and desserts, 4 bring hot dogs, fried chicken, and salads, 6 bring hot dogs, fried chicken, and desserts, 5 bring hot dogs, salads, and desserts, 7 bring fried chicken, salads, and desserts, and 2 bring all four food items. Those (of the 65) who do not bring any of these four food items are responsible for setting up and cleaning up for the picnic. How many of the 65 maintenance staff will (a) help to set up and clean up for the picnic? (b) bring only hot dogs? (c) bring exactly one food item?

c_1: **Staff member brings hot dogs**
c_2: **Staff member brings fried chicken**
c_3: **Staff member brings salads**
c_4: **Staff member brings desserts**
$N = 65$
$N(c_1) = 21;\ N(c_2) = 35;\ N(c_3) = 28;\ N(c_4) = 32$
$N(c_1c_2) = 13;\ N(c_1c_3) = 10;\ N(c_1c_4) = 9;\ N(c_2c_3) = 12;\ N(c_2c_4) = 17;\ N(c_3c_4) = 14$
$N(c_1c_2c_3) = 4;\ N(c_1c_2c_4) = 6;\ N(c_1c_3c_4) = 5;\ N(c_2c_3c_4) = 7$
$N(c_1c_2c_3c_4) = 2.$

(a) $N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = 65 - [21+35+28+32] + [13+10+9+12+17+14] - [4+6+5+7] + 2 = 65 - 116 + 75 - 22 + 2 = 4.$
(b) $N(\bar{c}_2\bar{c}_3\bar{c}_4) = N - [N(c_2)+N(c_3)+N(c_4)] + [N(c_2c_3)+N(c_2c_4)+N(c_3c_4)] - N(c_2c_3c_4)$, so $N(c_1\bar{c}_2\bar{c}_3\bar{c}_4) = N(c_1) - [N(c_1c_2)+N(c_1c_3)+N(c_1c_4)] + [N(c_1c_2c_3)+N(c_1c_2c_4)+N(c_1c_3c_4)] - N(c_1c_2c_3c_4) = 21 - [13+10+9] + [4+6+5] - 2 = 21 - 32 + 15 - 2 = 2.$

(c) $N(\bar{c}_1c_2\bar{c}_3\bar{c}_4) = N(c_2) - [N(c_1c_2) + N(c_2c_3) + N(c_2c_4)] + [N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_2c_3c_4)] - N(c_1c_2c_3c_4) = 35 - [13+12+17] + [4+6+7] - 2 = 35 - 42 + 17 - 2 = 8$

$N(\bar{c}_1\bar{c}_2c_3\bar{c}_4) = N(c_3) - [N(c_1c_3)+N(c_2c_3)+N(c_3c_4)] + [N(c_1c_2c_3)+N(c_1c_3c_4)+N(c_2c_3c_4)] - N(c_1c_2c_3c_4) = 28 - [10+12+14] + [4+5+7] - 2 = 28 - 36 + 16 - 2 = 6.$
$N(\bar{c}_1\bar{c}_2\bar{c}_3c_4) = N(c_4) - [N(c_1c_4)+N(c_2c_4)+N(c_3c_4)] + [N(c_1c_2c_4)+N(c_1c_3c_4)+N(c_2c_3c_4)] - N(c_1c_2c_3c_4) = 32 - [9+17+14] + [6+5+7] - 2 = 32 - 40 + 18 - 2 = 8.$
So the answer is $2+8+6+8 = 24.$

THANK - YOU