02/06/20

Set Theory

Set: collection of things/objects.

Eg: $A = \{1,2,3\}$ —— roaster notation

1 present in A

$1 \in A$

$4 \notin A$

Note:

* When writing 'belongs to' the left side must be object or element and the right side must be a set.
* Same object cannot appear twice in a set. ($\{1,2,2\} = \{1,2\}$)
* Order of elements in set is not important.

It is not considered wrong even if same objects repeat. But the repetition doesn't matter

Denoting a set:

$A = \{1,4,9,16,25,36,49,64,81\}$: Roaster form

$A = \{x^2 \mid x^2 < 100 \ \& \ x > 0\}$: Set builder form

Cardinality:

Cardinality of a set is no of elements in the set.

Eg: $A = \{1,2,3\}$

Cardinality of set 'A' $= |A| = 3$

Equal sets: (Axiom of extension)

Two set are said to be equal if and only if they have same elements. i.e., $A = B \Leftrightarrow \forall x (x \in A \leftrightarrow x \in B)$

i.e., $A \subseteq B$ and $B \subseteq A$

Emptyset / Nullset:

A set is said to empty set if cardinality is zero. It is denoted by - ϕ.

Eg: $\{ \}$

Subset ($\subseteq$):

'A' is called subset of 'B' iff every element of 'A' is also an element of 'B'.

It is denoted as $A \subseteq B$

$$\forall x \, (x \in A \rightarrow x \in B)$$

Note:

A set with one element is known as a singleton set

* Also 'B' is called superset of A.

Proper subset ($\subset$)

'A' is called proper subset of 'B' iff every element of 'A' is an element of 'B' and A is not equal to B.

$$A \subset B \Leftrightarrow A \subseteq B \text{ and } |A| \neq |B|$$

$$\boxed{\forall x \, (x \in A \rightarrow x \in B) \wedge \exists x \, (x \in B \wedge x \notin A)}$$

Note:

* while writing $\subset$ (or) $\subseteq$ left side & Right side must be sets.

Note:

$A \subseteq B$ and $B \subseteq C \rightarrow A \subseteq C$

$A \subset B$ and $B \subset C \rightarrow A \subset C$

$A \subseteq B$ and $B \subset C \rightarrow A \subset C$

$A \subset B$ and $B \subseteq C \rightarrow A \subset C$

Note:

For every set 'S'

$\phi \subseteq S$

$S \subseteq S$

* $\phi \subset S$ is not true for every set because for $S = \phi$, the relation doesn't hold.

~~Powse~~

Powerset : ($P(A)$ | 2^A)

Powerset is a set of ~~of~~ all ~~set~~ subsets of a given ~~to~~ set.

In powerset each element is a set.

Eg: $A = \{1, 2\}$

$P(A) = 2^A = \{\phi, \{1\}, \{2\}, \{1,2\}\}$

Cardinality of a power set is

$$|P(A)| = 2^n$$

where $|A| = n$

Eg: Let $A = \{\phi\}$

$P(A) = \{\phi, \{\phi\}\}$

Eg: $\phi = \{\ \}$

$P(\phi) = \{\phi\}$

Q1 / G-15 For a set A, the power set of A is denoted by 2^A. If $A = \{5, \{6\}, \{7\}\}$ which of the following options are TRUE?

I. $\phi \in 2^A$ II. $\phi \subseteq 2^A$

III. $\{5, \{6\}\} \in 2^A$ IV. $\{5, \{6\}\} \subseteq 2^A$

a) I & III b) II & III c) I, II, III d) I, II, IV

I. $\phi \in 2^A$,

Here ϕ is a set

~~$\therefore \phi \in 2^A$ is false~~ $\therefore$ True

II. power set contains ϕ as an element

$\phi \in 2^A$

But $\phi \subseteq$ to every set (Empty set is subset to all the sets)

$\therefore$ II true

III. $\{5, \{6\}\}$ is subset of A

$\therefore$ It is element of power set

$\{5, \{6\}\} \in 2^A$

IV $\{5, \{6\}\}$ is an element of 2^A

it is not subset

However

$\{\{5, \{6\}\}\} \subseteq 2^A$ is true.

$\therefore$ I, II & III

(Q2) Let $A = \{1, \{1\}, \{2\}\}$. which of the following are true?

(i) $1 \in A$ — True

(ii) $\{1\} \in A$ — True

(iii) $\{1\} \subseteq A$ — True

(iv) $\{\{1\}\} \subseteq A$ — True

(v) $\{2\} \in A$ — True

(vi) $\{2\} \subseteq A$ — False

(vii) $\{\{2\}\} \subseteq A$ — True

(viii) $\{\{2\}\} \subset A$ — True

(Q3) which of the following statements are true?

a) $\phi \in \phi$ ~~or~~

Here both are sets

$\therefore$ False

b) $\phi \subset \phi$ ($|\phi| = |\phi| \Rightarrow$ False)

ϕ is not a proper subset to ϕ $\because \phi = \phi$

$\therefore$ False

c) $\phi \subseteq \phi$

Every set is subset to itself

$\therefore$ True

d) $\phi \in \{\phi\}$

This is written as

$\{\} \in \{\{\}\}$

$\therefore$ True

e) $\phi \subset \{\phi\}$

ϕ is proper subset to every set but ϕ

$\therefore$ True

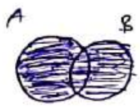
f) $\phi \subseteq \{\phi\}$

ϕ is subset to every set

$\therefore$ True.

Set Operations:

→ $A \cup B = \{x \mid x \in A \vee x \in B\}$



→ $A \cap B = \{x \mid x \in A \wedge x \in B\}$

→ $A \Delta B = \{x \mid x \in A \odot x \in B\} = \{x \mid x \in A \cup B \wedge x \notin A \cap B\}$

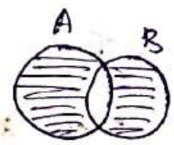
↳ XOR

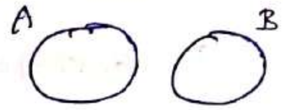
↓

Symmetric difference

It is also written as $A \oplus B$

$$\boxed{\begin{aligned} A \oplus B &= (A-B) \cup (B-A) \\ &= (A \cup B) - (A \cap B) \end{aligned}}$$

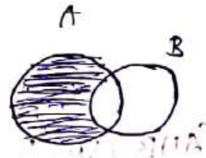




Disjoint set:

Two sets A & B are said to be disjoint iff $A \cap B = \phi$

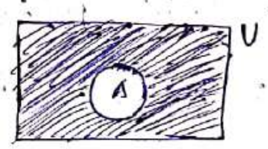
Set difference:



$A - B = \{x \mid x \in A \wedge x \notin B\}$

* $\boxed{A - B = A - (A \cap B) = A \cap \bar{B}}$

Complement ($\bar{A}$) (A^c):



Complement of set 'A'

$\bar{A} = \{x \mid x \in U \wedge x \notin A\}$

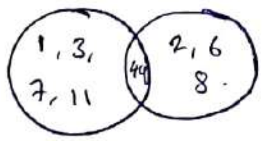
$\bar{A} = U - A = U \cap \bar{A}$

Eg: let $A - B = \{1, 3, 7, 11\}$

$B - A = \{2, 6, 8\}$

$A \cap B = \{49\}$

Find A & B.



$\therefore A = \{1, 3, 7, 11, 49\}$

$B = \{2, 6, 8, 49\}$

Note:

→ $A \cap U = A$, $A \cup \phi = A$ } Identity laws

→ $A \cup U = U$, $A \cap \phi = \phi$ } Domination law

→ $A \cup A = A$, $A \cap A = A$ } Idempotent law

→ $\overline{(\bar{A})} = A$ Complementation law

→ $\left.\begin{array}{l} A \cup B = B \cup A \\ A \cap B = B \cap C \\ A \Delta B = B \Delta A \end{array}\right\}$ Commutative law

→ $\left.\begin{array}{l} A \cup (B \cup C) = (A \cup B) \cup C \\ A \cap (B \cap C) = (A \cap B) \cap C \end{array}\right\}$ Associative law

→ $\left.\begin{array}{l} A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \end{array}\right\}$ Distributive law

→ $\left.\begin{array}{l} \overline{A \cap B} = \bar{A} \cup \bar{B} \\ \overline{A \cup B} = \bar{A} \cap \bar{B} \end{array}\right\}$ Demorgan's law

→ $\left.\begin{array}{l} A \cup (A \cap B) = A \\ A \cap (A \cup B) = A \end{array}\right\}$ Absorption law

→ $\left.\begin{array}{l} A \cup \bar{A} = U \\ A \cap \bar{A} = \phi \end{array}\right\}$ Complement law

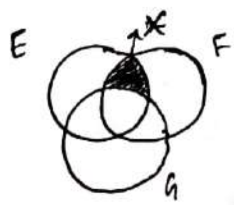
Q4 / G-06 E, F & G are finite sets

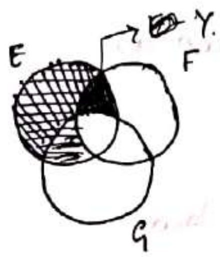
$X = (E \cap F) - (F \cap G)$

$Y = (E - (E \cap G)) - (E - F)$

Find the correct relation

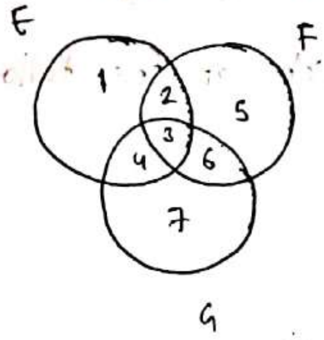
a) $X \subset Y$ b) $X = Y$ c) $X \supset Y$ d) $X - Y \neq \phi$ & $Y - X \neq \phi$





$\therefore X = Y$

Method: 2



$x = (E \cap F) - (F \cap G)$

$= (2,3) - (3,6)$

$= 2$

$Y = (E-(E \cap G)) - (E-F)$

$= ((1,2,3,4)-(3,4)) - ((1,2,3,4)-(2,3,5,6))$

$= (1,2) - (1,4)$

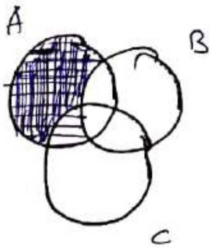
$= 2$

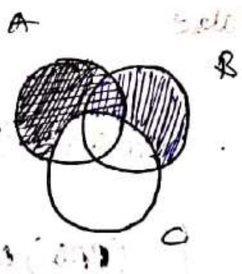
$\therefore x = y$

Q5 / G-05 Let A, B & C be non-empty sets and let $x = (A-B)-C$ and $Y = (A-C)-(B-C)$. Which of the following is true?

a) $x = y$ b) $x \subset y$ c) $y \subset x$ d) none

Sol:





$\therefore x = y$

Finite & Infinite sets:

→ A set with finite no of elements is known as finite set.

→ A set with infinite no of elements is known as infinite set.

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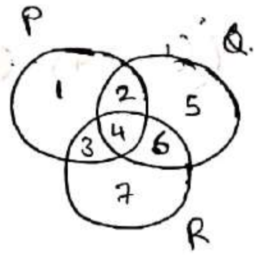
Q6 / G-06

Let P, Q and R be sets. Let Δ denote the symmetric difference operator defined as. $P \Delta Q = (P \cup Q) - (P \cap Q)$. Using Venn diagram, determine which of the following is/are true?

I) $P \Delta (Q \cap R) = (P \Delta Q) \cap (P \Delta R)$

II) $P \cap (Q \Delta R) = (P \cap Q) \Delta (P \cap R)$

I.



$P \Delta (Q \cap R)$

$= (P - (Q \cap R)) - ((Q \cap R) - P)$

$= [(1,2,3,4) - (4,6)] - [(4,6) - (1,2,3,4)]$

$= (1,2,3) - (6)$

$= (1,2,3)$

$(P \Delta Q) \cap (P \Delta R)$

$(1,3,5,6) \cap (1,2,6,7) = (1)$

$P \Delta Q = (1,3,5,6)$

$P \Delta R = (1,2,6,7)$

$\therefore$ False

II. $P \cap (Q \Delta R) = (P \cap Q) \Delta (P \cap R)$

$P \cap (Q \Delta R)$

$P \cap [(Q-R) \cup (R-Q)]$

$P \cap (2,3,5,7)$

$(1,2,3,4) \cap (2,3,5,7)$

$(2,3)$

$(P \cap Q) \Delta (P \cap R)$

$[(P \cap Q) - (P \cap R)] \cup [(P \cap R) - (P \cap Q)]$

$[(2,4) - (3,4)] \cup [(3,4) - (2,4)]$

$(2) \cup (3)$

$(2,3)$

$\therefore$ True

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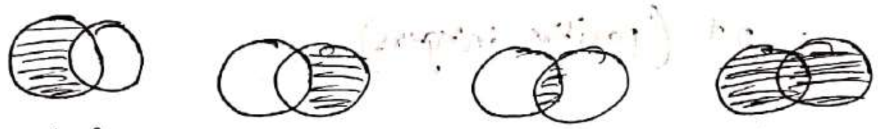
Q7 (G-96) let A and B be sets and let A^c and B^c denote the complements of the set A and B. The $(A-B) \cup (B-A) \cup (A \cap B)$ is equal to

a) $A \cup B$ b) $A^c \cup B^c$ c) $A \cap B$ d) $A^c \cap B^c$



$A-B$ $B-A$ $A \cap B$ $(A-B) \cup (B-A) \cup (A \cap B)$
$= A \cup B$

Q8 Let $A_i = [-i, i]$ $\forall i \geq 1$ and let domain be the set of integers.

Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$

$A_1 = \{-1, 0, 1\}$.

$A_2 = \{-2, -1, 0, 1, 2\}$

$\vdots$

$\bigcup_{i=1}^{\infty} A_i = \{-1, 0, 1\} \cup \{-2, -1, 0, 1, 2\} \cup \ldots \cup \{-\infty \ldots -2, -1, 0, 1, 2 \ldots \infty\}$

$= \{-\infty \ldots -2, -1, 0, 1, 2 \ldots \infty\}$

$= Z$ (Integers)

$\bigcap_{i=1}^{\infty} A_i = \{-1, 0, 1\} \cap \{-2, -1, 0, 1, 2\} \cap \ldots \cap \{-\infty \ldots -2, -1, 0, 1, 2 \ldots \infty\}$

$= \{-1, 0, 1\}$

Q9 Let $A_i = [i, \infty)$ $\forall i \geq 1$ and let domain be set of integers

Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$

$A_1 = \{1, 2 \ldots \infty\}$

$A_2 = \{2, 3, \ldots \infty\}$

$\bigcup_{i=1}^{\infty} A_i = \{1, 2 \ldots \infty\} \cup \{2, 3 \ldots \infty\} \cup \ldots$

$= \{1, 2, \ldots \infty\}$

$= Z^+$ (positive integers)

$\bigcap_{i=1}^{\infty} A_i = \{1, 2 \ldots \infty\} \cap \{2, 3 \ldots \infty\} \cap \{3, 4 \ldots \infty\} \cap \ldots$

$= \phi$

* If $A \subseteq B$ then $\bar{A} \cup B = U$

Proof:

~~if~~ we need to S.T. for every $x \in U$

also $x \in \bar{A} \cup B$

Consider $x \in A \Rightarrow x \in B$

$\Rightarrow x \in (\bar{A} \cup B)$

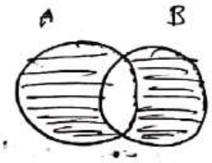
if $x \notin A \Rightarrow x \in \bar{A}$

$\Rightarrow x \in (\bar{A} \cup B)$

$\therefore$ any $x \in \bar{A} \cup B \Rightarrow \bar{A} \cup B = U$

~~Q10 $A_i = \{$~~

P/5 $X = \{1, 2, 3 \ldots 2n\}$



for $A \Delta B = \{2, 4, 6 \ldots 2n\}$

the shaded region should contain all these elements

so we have n elements and each element has two

ways $\Rightarrow$ no of ways $= 2^n$

Now the each element $\{1, 3, 5 \ldots 2n-1\}$

can either fall into intersection part or not

$\Rightarrow 2^n$ ways

total no of ways $= 2^n \times 2^n = 2^{2n}$

P/06 a) $A \oplus B = (A \cup B) - (A \cap B)$

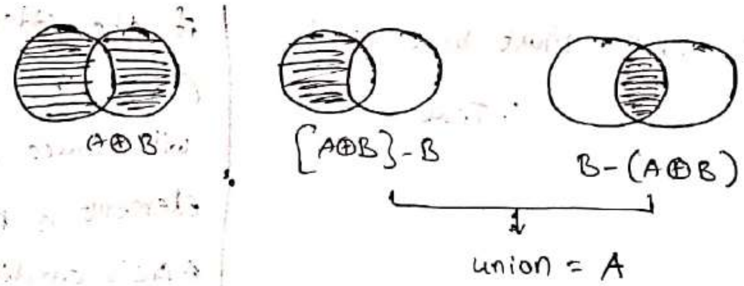
$= (A \cup \phi) - (A \cap \phi)$

$= A - \phi = A$

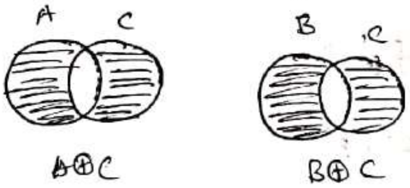
b) $[\cancel{(A \oplus B) - B}]$

$(A \oplus B) \oplus B$

$[(A \oplus B) - B] \cup [B - (A \oplus B)]$



c)



~~From above figure~~

~~A⊕C = B⊕C iff left shaded part is same in~~ both ~~the figures~~

$\Rightarrow \cancel{A - C = B - C}$ ∴ True $\boxed{\text{explained in P/07 (c)}}$

$\Rightarrow \cancel{A = B}$

d) ~~True~~ False

P/07 a) $A = \{1,2,3\}$ $B = \{1,4,5\}$ $C = \{1\}$

$\cancel{A=B}$ $A \cap C = \{1\}$ $B \cap C = \{1\}$

But $A \neq B$

∴ False

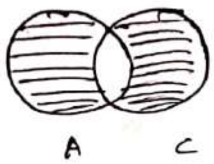
b) $A = \{1,2\}$ $B = \{3,4\}$ $C = \{1,2,3,4\}$

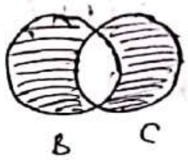
$A \cup C = \{1,2,3,4\}$ $B \cup C = \{1,2,3,4\}$

But $A \neq B$

∴ False

c) $(A \Delta C) = (B \Delta C)$ then $A = B$





$A = B$

Lets say A has one element which is not present in B.

Then this one element will surely change $A-C$ or $C-A$

$\therefore$ A & B must be equal

$\therefore$ True

if the added element is in C also, then $A \Delta C$'s cardinality will reduce, if the added element is not in C, then $A \Delta C$'s cardinarity will raise by 1.

~~d) $A=\{1,2\}$ $B=\{1,5\}$ $C=\{1,3\}$~~

~~$A-C=\{1,2\}$ $B-C=\{5$~~

d) $A = \{1,2\}$ $B = \{2,3\}$ $C = \{1,3\}$

$A - C = \{2\}$ $B - C = \{2\}$

But $A \neq B$

$\therefore$ False

P/01 For every existing subset of n subset we can either add $\{\}$ (or) $\{a\}$ or $\{b\}$ (or) $\{a,b\}$

$\Rightarrow 4 \times n$ new subsets

P/02 S1: $A \in B$ and $B \subseteq C$ then $A \subseteq C$

$\in$: element of B

A: element, C: set

$\therefore$ False

$A \in C$ is Correct

S2: $A \subseteq B$ and $B \in C$ then $A \subseteq C$

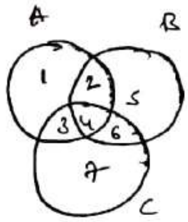
$\downarrow$ set, $\downarrow$ set, $\downarrow$ set

So C must be a set in which set B is an element

So we can't say if $A \subseteq C$ or not

$\therefore$ False

(P/3)



S1 $A \cup (B - C) = (A \cup B) - (C - A)$

$(1,2,3,4) \cup (2,5)$ $\quad$ $(1,2,3,4,5,6) - (6,7)$

$(1,2,3,4,5)$ $\quad$ $(1,2,3,4,5)$

$\therefore$ S1 : True

S2 : $A \cap (B - C) = (A \cap B) - (A \cap C)$

$(1,2,3,4) \cap (2,5)$ $\quad$ $(2,4) - (3,4)$

(2) $\quad$ (2)

$\therefore$ S2 : True

(P/4) If $A = \phi$

B can be choosen in 2^n ways

If $|A| = 1$

B can be choosen in 2^{n-1} ways

we have nC_1 such A's

$\Rightarrow nC_1 \cdot 2^{n-1}$

If $|A| = 2$

B can be choosen in 2^{n-2} ways

we have nC_2 such A's

$\Rightarrow nC_2 \cdot 2^{n-2}$

If $|A|=n$

B can be choosen in 1 way

i.e., $B = S$

$\therefore$ Total no of ways

$$= 2^n + {}^nC_1 . 2^{n-1} + {}^nC_2 2^{n-2} + \dots + {}^nC_k 2^{n-1} + \dots + {}^nC_n 2^{n-n}$$

$$= \sum_{k=0}^{n} {}^nC_k 2^{n-k} = (1+2)^n = 3^n$$

03/06/20

Function: (or) Mapping (or) Transformation

→ A function is a rule that assign each input to exactly one output.

(or)

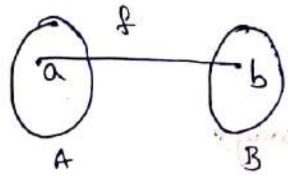
→ A function 'F' form A to B is an assignment of exactly one element of B to each element of A. It is denoted as $F : A \to B$

Here,

Set A is called Domain

Set B is called Co-domain

Let $a \in A$ and $b \in B$



$f(a) = b$

b is called image

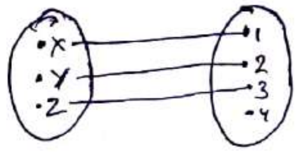
a is called pre-image

Range : Set of all images. It is also called image of the function.

Note :

→ It is not necessary that Range = Co-domain but Range ⊆ Co-domain

Ex: Consider below function



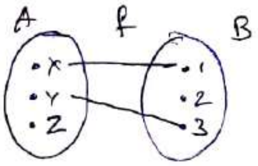
Here domain = {x, y, z}

Co-domain = {1, 2, 3, 4}

Range = {1, 2, 3}

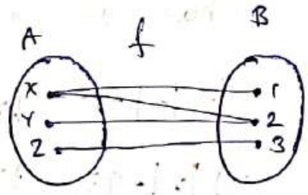
> **Note** :
> If f_1 & f_2 are two functions, then
> $(f_1+f_2)(x) = f_1(x)+f_2(x)$
> $(f_1 f_2)(x) = f_1(x) f_2(x)$
> $\frac{1}{f}(x) = 1/f(x)$

→ For every elemement in domain, there must exist exactly one image. Otherwise, it won't considered a function.



Here z has no image.

∴ It is not a function.



Here x has two image

∴ It is not a function

~~Note elem~~

→ Consider $f: N \to N$, N is set of natural numbers

$f(x) = x/2$

Now for $x = 3$

$f(3) = 1.5$

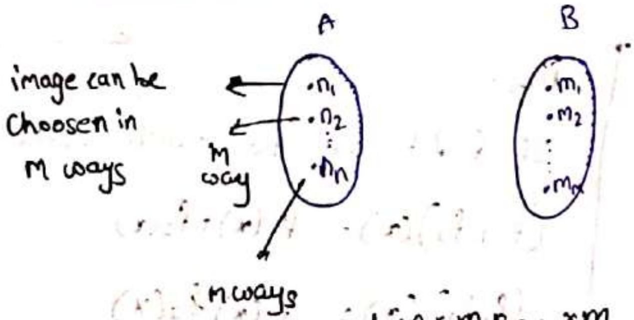
but $1.5 \notin N$

So, ~~for f~~ there is not image for 3 in N.

∴ It is not a function.

> Let $f: A \to B$ be a function.
> Let $S \subseteq A$
> $f(S)$ is set of images of S w.r.to f
> $f(S) = \{t \mid \forall s \in S, f(s) = t\}$
> Note that $f(S)$ is a set.

$$\boxed{\begin{array}{c} |A| = n \quad |B| = m \\ \hline \text{no of function possible} = m^n \end{array}} = (R \cdot S)^{L \cdot S}$$



$\therefore \underbrace{m \times m \times \cdots \times m}_{n \text{ time}} = m^n$

Q10 / G-96 Let $f: X \to Y$ be a function.

Let total no of function possible $= 97$

Choose the correct option

a) $|X| = 1 \quad |Y| = 97$

b) $|X| = 97 \quad |Y| = 1$

c) $|X| = 97 \quad |Y| = 97$

d) none

sol :

no of function $= (|B|)^{|A|} = 97$

$\therefore |B| = 97 \quad |A| = 1$

$\therefore$ opt (a)

Types of functions :

(i) <u>1:1 Function</u> (Injective)

$f: A \to B$ is Injective iff $\forall a \in A$ and $\forall b \in B$

$f(a) = f(b) \Leftrightarrow a = b$

(or)

$f: A \to B$ is Injective iff $\forall a \in A$ and $\forall b \in B$

$a \neq b \Leftrightarrow f(a) \neq f(b)$

$\rightarrow$ In a one-one function, no two elements of the domain should have same image.

Eg: $f: N \rightarrow N$

$f(x) = x+1$ is a one-one function

Consider two numbers $x_1 \in N$ and $x_2 \in N$ and $x_1 \neq x_2$

To P.T f is not one-one, we need to P.T

$\cancel{f(x_1) \neq f}$ $f(x_1) = f(x_2)$

$x_1 + 1 = x_2 + 1$

$x_1 = x_2$

which is a contradiction

$\therefore f$ is one-one

Rosen Questions:

Q: If A, B & C are sets. Show that

a) $(A \cup B) \subseteq (A \cup B \cup C)$

b) $(A \cap B \cap C) \subseteq (A \cap B)$ (Hint: Use venn diagrams)

c) $(A-B)-C \subseteq A-C$

d) $(A-C) \cap (C-B) = \phi$

e) $(B-A) \cup (C-A) = (B \cup C) - A$

$\rightarrow$ What can you conclude from below statements

a) $A \cup B = A$

b) $A \cap B = A$

c) $A - B = A$

d) $A - B = B - A$

Ans: a) $B \subseteq A$ b) $A \subseteq B$ c) $A \cap B = \phi$ d) $A = B$

04/06/20

Find if given function is one-one or not

Q11 $f: Z \to Z$

$f(x) = x^2 - 5x + 5$

Let $x_1, x_2 \in Z$

$f(x_1) = f(x_2)$

$x_1^2 - 5x_1 + 5 = x_2^2 - 5x_2 + 5$

$x_1^2 - 5x_1 = x_2^2 - 5x_2$

$x_1(x_1 - 5) = x_2(x_2 - 5)$

Put $x_1 = 0$ and $x_2 = 5$

$0(-5) = 5(0)$

$0 = 0$

$\therefore f(x) = x^2 - 5x + 5$ is not one-one

(or)

let $x_1, x_2 \in Z$

$f(x_1) = f(x_2)$

$x_1^2 - 5x_1 + 5 = x_2^2 - 5x_2 + 5$

$x_1^2 - 5x_1 = x_2^2 - 5x_2$

$x_1^2 - x_2^2 = 5x_1 - 5x_2$

$\cancel{(x_1 - x_2)}(x_1 + x_2) = 5\cancel{(x_1 - x_2)}$

$x_1 + x_2 = 5$

we can have infinitely different pairs of x_1, x_2 such that $f(x_1) = f(x_2)$

$\therefore$ not 1-1

Note:

* For a function $F: A \to B$ to be one-one

$|B| \geq |A|$

* let $|B| = n$ and $|A| = m$ and $m \leq n$, then no of 1-1 functions possible from A to B are $\boxed{{}^nP_m = {}^{R.S}P_{L.S}}$

(ii) Onto (surjective) function :

* A function $F: A \to B$ is surjection, if range = codomain
  i.e., ~~ele~~ every element of B has a preimage
  (or)
  $$\forall y \in B \; \exists x \in A \; (f(x) = y)$$

* For a function $F: A \to B$

  $|A| \geq |B|$ for the function to be onto.

Eg: $f: Z \to Z$ & $f(x) = x+1$ is onto

Eg: $f: Z \to Z$ & $f(x) = x^2$ is not onto

because for element '2' in domain, '2' doesn't have preimage.

* Calculating no of onto functions possible

  let $F: A \to B$ be a function & $|A| \geq |B|$

  let $|A| = m$ $|B| = n$

no of onto functions = total no of functions − no of non-onto functions.

* for function ~~has~~ to be non-onto atleast one element in B must not have ~~co-do~~ preimage.

* So no of non-onto functions can be

  = no of non-onto function with 1 element not having ~~co-domain~~ pre-image

  no of non-onto functions with 2 elements not having ~~co-domain~~ pre-image

  ⋮

  no of non-onto functions with n elements not having pre-image

$= \cancel{n^m} \left[\cancel{n_{C_1}} (n-1)^m \right.$

$$= \boxed{nc_0\, n^m - nc_1 (n-1)^m + nc_2 (n-2)^m - nc_3 (n-3)^m + \cdots \cdots + (-1)^n\, nc_n (n-n)^m}$$

$$= \boxed{\sum_{i=0}^{n-1} (-1)^i\, nc_i (n-i)^m}$$

(inclusion-exlusion principle)

(iii) One to One correspondence (or) Bijection:

1:1 Correspondence = 1:1 + onto

$F: A \to B$ ~~that~~ $|A| = m$ $|B| = n$

for 1:1 $|A| \le |B|$

onto $|A| \ge |B|$

1:1 + onto $\Rightarrow |A| = |B|$

* If $F: A \to B$ is one-one and $|A| = |B|$ then F is bijection.
* If $F: A \to B$ is onto and $|A| = |B|$ then F is ~~an~~ bijection.
* No of 1:1 Correspondences $= n!$

Eg: $f: Z \to Z$

$f(x) = x+1$

~~So~~ f is clearly one-one

also for every element co-domain, it has pre-image

So f is onto.

$\therefore$ f is a bijection

(iv) Identity function:

The function $I_A: A \to A$ is called an identity function, if

$I_A(x) = x \;, \forall x \in A$

Identity function is a bijection.

e) Inverse function

For a function $f: A \to B$

inverse function is

$f^{-1}: B \to A$ such that

$f^{-1}(b) = a$ whenever $f(a) = b$

Note:

$f: A \to B$ be a function, $S \subseteq B$

Inverse image of S is defined as subset of A whose elements are precisely all preimages of S.

$f^{-1}(S) = \{a \in A \mid f(a) \in S\}$

$f^{-1}(y)$, $y \in B$ is defined iff f is invertible

$f^{-1}(S)$, $S \subseteq B$ can be defined for any f

* Inverse of a function exists only if the function is bijection. Such functions are known as invertible function.

bijection $\Leftrightarrow$ invertible function.

If $f: A \to A$ is a function and also if A is finite then

f is 1-1 $\Leftrightarrow$ f is onto

However this not necessarily the case when A is infinite

Eg: $f: R \to R$

$f(x) = x^2$

Here

$f(-2) = f(2) = 4$

$\therefore$ not one-one

$\therefore$ It is not invertible

Eg: $f: R \to R$

$f(x) = 2x - 3$

Here

$f(x_1) = f(x_2)$

$2x_1 - 3 = 2x_2 - 3$

$\Rightarrow x_1 = x_2$

$\therefore$ 1-1

Note:

A function $f: A \to B$ is said to be a constant function if $f(x) = c \ \forall x$

Let $f(x) = y$

$y = 2x - 3$

$\Rightarrow x = \dfrac{y+3}{2}$

Thus for every 'x' we can find preimage

$\therefore$ onto and Hence f is invertible

To find inverse

let $f(x) = y \Rightarrow f^{-1}(y) = x$

$2x - 3 = y$

$2x = y + 3$

$x = \frac{y+3}{2} = f^{-1}(y)$

$f^{-1}(y) = \frac{y+3}{2}$

$\Rightarrow f^{-1}(x) = \frac{x+3}{2}$

Eg: find inverse of

$f(x) = \frac{4x}{2x-1}$, $f: R - \{1/2\} \to R$

let $f(x) = y \Rightarrow f^{-1}(y) = x$

$\frac{4x}{2x-1} = y$

$4x = y(2x-1)$

$4x = 2xy - y$

$4x - 2xy = -y$

$x(4-2y) = -y$

$x = \frac{y}{2y-4}$

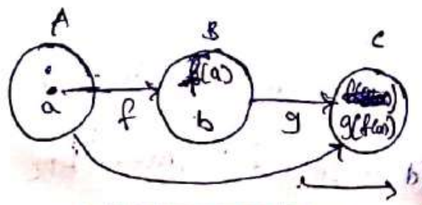
$\Rightarrow f^{-1}(y) = \frac{y}{2y-4}$

$\Rightarrow f^{-1}(x) = \frac{x}{2x-4}$

Composition of a functions:

Consider to functions

$f: A \to B$ and $g: B \to C$ $\quad$ $gof: A \to C$ is called a composition function



$gof()$ or $g.f()$

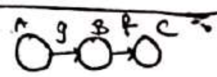
* $gof(x) = g(f(x))$

> In gof,
> domain of gof = domain of f
> range of gof = image of range of f w.r.to g

Eg: $f(x) = 2x+3$ $\quad g(x) = 3x+2$

fog	gof
$fog(x) = f(g(x))$	$gof(x) = g(f(x))$
$= f(3x+2)$	$= g(2x+3)$
$= 2(3x+2)+3$	$= 3(2x+3)+2$
$= 6x+7$	$= 6x+11$

$$\boxed{\therefore gof(x) \neq fog(x)}$$

> Note: 
>
> The composition gof cannot be defined unless range of f is subset of domain of g
> $gof: B \to B$

Note: Composition is Associative i.e., $(fog)oh = fo(goh)$

* let $f: A \to B$ be a invertible then
  $f^{-1}: B \to A$
  $f^{-1}of: A \to A$ is an identity function I_A
  $fof^{-1}: B \to B$ is an idenhity funhion I_B

* Theorem 1:

If $f: A \to B$ & $g: B \to C$ are two 1:1 functions then
~~gof is also~~ $gof: A \to C$ is also 1:1

Theorem: 2:

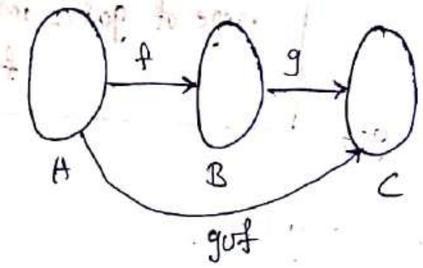
If $f: A \to B$ & $g: B \to C$ are onto, then
$gof: A \to C$ is also onto

Theorem: 3

If $f: A \to B$ & $g: B \to C$ are bijections
$gof: A \to C$ is also a bijection

Note:

1) If $g\circ f$ is onto then g is onto.



$g\circ f$ is onto $\Rightarrow \forall c \in C \; \exists a \in A$ such that $g\circ f(a) = c$

$g(f(a)) = c$

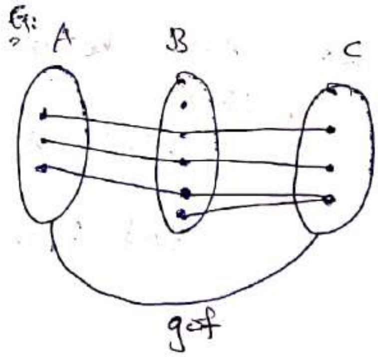
$g(b) = c$ for some $b \in B$

i.e., $\forall c \in C \; \exists b \in B \mid g(b) = c$

2) If $g\circ f$ is onto, then f need not to be onto

$g\circ f$ is onto $\Rightarrow \forall c \in C \; \exists a \in A \mid g\circ f(a) = c$

Eg:



In the figure we can clearly see that $g\circ f$ is onto & f is not onto

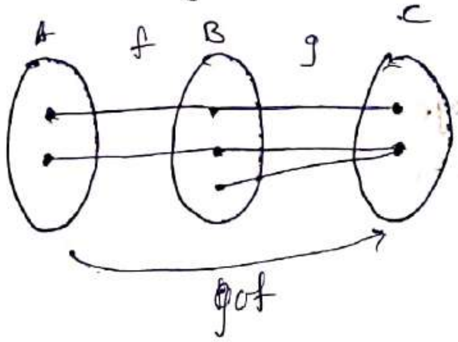
3) If $g\circ f$ is 1:1, then f is 1:1

$g\circ f$ is 1:1 $\Rightarrow \forall a_1 \in A$ & $\forall a_2 \in A$

$g\circ f(a_1) = g\circ f(a_2) \Rightarrow a_1 = a_2$

If we map two element of 'A' to a same element in 'B' then we can never obtain $g\circ f$ such that it is one-one

4) If $g \circ f$ is 1:1 then g need not be 1:1



> **Note:**
> If $g \circ f$ is a bijection, then
> f is onto $\Leftrightarrow$ g is 1-1

Above figure is an example to say that

$g \circ f$ is 1:1 and g need not to be 1:1

*

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend left, "\text{if onto}"] & B \arrow[r, "\text{onto}"] & C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "1:1"] \arrow[rr, bend left, "\text{if } 1:1"] & B \arrow[r, leftrightarrow] & C \end{tikzcd}$$

Q12) Determine if the below functions are one-one &/(or) onto for the two cases $f: \mathbb{Z} \to \mathbb{Z}$ and $f: R \to R$

a) $f(x) = x + 7$

$f: \mathbb{Z} \to \mathbb{Z}$

$x_1 + 7 = x_2 + 7$

$x_1 = x_2$

$\therefore$ 1-1

$\forall y \in \mathbb{Z}$ we can determine $x \in \mathbb{Z}$ such that $y = x + 7$

$\therefore$ onto

Say $f: R \to R$ is also 1-1 & onto

b) $f(x) = 2x - 3$

$\underline{f: \mathbb{Z} \to \mathbb{Z}}$

1-1

Let $f(x) = y$

$2x - 3 = y \Rightarrow x = \frac{y+3}{2}$

Here for $y = 2$

$$x = \frac{2+3}{2} \notin Z$$

$\therefore$ '2' has no preimage.

$\therefore$ not onto

<u>$f: R \to R$</u>

It is clearly 1-1

$f(x) = y$

$$x = \frac{y+3}{2}$$

for any $y \in R$ we can find x.

$\therefore$ onto.

c) $f(x) = -x+5$

$f: Z \to Z$ is 1-1 & onto

$f: R \to R$ is 1-1 & onto

d) $f(x) = x^2$

<u>$f: Z \to Z$</u>

$f(-2) = f(2) = 4$

$\therefore$ not 1-1

$x = 2$ has no preimage in domain

$\therefore$ not onto

<u>$f: R \to R$</u>

$f(-2) = f(2) = 4$

$\therefore$ not 1-1

Let $f(x) = y$

$\forall y < 0$, $\sqrt{y} \notin R$

$\therefore$ not onto

e) $f(x) = x^2 + x$

<u>$f: Z \to Z$</u>

$x_1^2 + x_1 = x_2^2 + x_2$

$x_1^2 - x_2^2 = x_2 - x_1$

$(x_1 - x_2)(x_2 + x_1) = x_2 - x_1$

$\Rightarrow x_2 + x_1 = -1$

$\therefore$ we can find infinite such (x_1, x_2)

$\therefore$ not 1-1

let $y = f(x)$

$x^2 + x = y$

put $y = -1$

$x^2 + x + 1 = 0$

$x = \frac{-1 \pm \sqrt{1-4}}{2}$

$x = \frac{-1 \pm 3i}{2} \notin Z$

$\therefore$ not onto

$f: R \to R$

* not 1-1

* $y = -1$ has its preimage in complex numbers

$\therefore$ not onto

f) $f(x) = x^3$

$f: Z \to Z$

$x_1^3 = x_2^3$

$\Rightarrow x_1 = x_2$

$\therefore$ one-one

$y = 2$ has no preimage

$\therefore$ not onto

$f: R \to R$

$x_1^3 = x_2^3$

$\Rightarrow x_1 = x_2$

$\therefore$ 1-1

$\forall y \in R$ has $\sqrt[3]{y}$ in R

$\therefore$ onto

Q13 / 4-15

Find no of onto functions

$f: X \to Y$

$|X| = m = 4 \quad |Y| = n = 3$

no of onto functions

$= {}^nC_0 n^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-2)^m - \ldots$

$= {}^3C_0 3^4 - {}^3C_1 (2)^4 + {}^3C_2 (1)^4 - {}^3C_3 (0)^4$

$= 3^4 - 3 \cdot 2^4 + 3 \cdot (1) - 0$

$= 81 - 48 + 3 = 84 - 48$

$= 36$

Q14 / G-12 How many onto functions are there from an n-element ($n \geq 2$) set to a 2-element set?

a) 2^n b) $2^n - 1$ c) $2^n - 2$ d) $2(2^n - 2)$

Sol:

$2^n - 2C_1 (2-1)^n$

$2^n - 2\cdot(1)^n = 2^n - 2$

Q15 / G-96 Let R denote the set of real numbers. Let $f: R\times R \to R\times R$ be a bijective function defined by $f(x,y) = (x+y, x-y)$. The inverse function of f is given by

a) $f^{-1}(x,y) = \left(\frac{1}{x+y}, \frac{1}{x-y}\right)$ b) $f^{-1}(x,y) = (x-y, x+y)$

c) $f^{-1}(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$ d) $f^{-1}(x,y) = (2(x-y), 2(x+y))$

Sol:

Let $f(x,y) = (a,b) \Rightarrow f^{-1}(a,b) = (x,y)$

$x+y = a$

$x-y = b$

$\Rightarrow 2x = a+b$

$x = \frac{a+b}{2}$

$\Rightarrow \frac{a+b}{2} + y = a$

$y = \frac{a-b}{2}$

$\therefore f^{-1}(a,b) = \left(\frac{a+b}{2}, \frac{a-b}{2}\right)$

$f^{-1}(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$

Q16 (4-05) Let $f: A \to B$ be a function, $g: B \to C$ be a function and $h: A \to C$ be a function such that, $h(a) = g(f(a))$ $\forall a \in A$.
Which of following statements is always true for all f and g.

a) g is onto $\Rightarrow h$ is onto

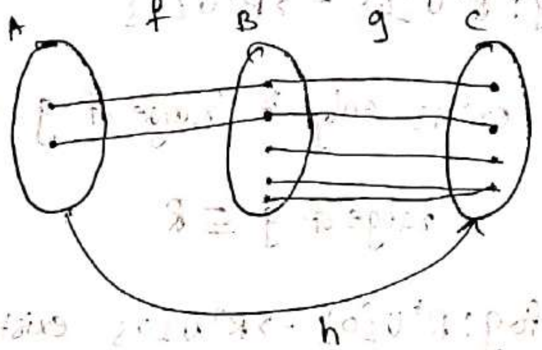
b) h is onto $\Rightarrow f$ is onto

c) h is onto $\Rightarrow g$ is onto

d) h is onto $\Rightarrow f$ and g are onto

Sol:

a)

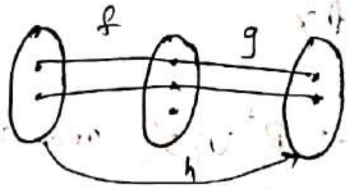


Here g is onto, but h is clearly not

$\therefore$ false

~~Also in the same figure~~

b)



Here h is onto but f is not

$\therefore$ false

Hence we can say (d) is also false

c) true.

(Q17) Let $f: R \to R^+ \cup \{0\}$ with $f(x) = x^2$

$g: R^+ \cup \{0\} \to R$ with $g(x) = \sqrt{x}$ ($\sqrt{x}$ is non-negative square root of x).

what is the function $(f \circ g)(x)$?

Sol :

$f \circ g(x) = f(g(x))$

Domain of $f \circ g$ = Domain of $g = R^+ \cup \{0\}$

Co Domain of $f \circ g$ = Co-domain of $f = R^+ \cup \{0\}$

$$f \circ g : R^+ \cup \{0\} \to R^+ \cup \{0\}$$

Composition $f \circ g$ exists only if (range of $g \subseteq$ domain of f)

$$\text{range of } g \subseteq R$$

$$\therefore f \circ g : R^+ \cup \{0\} \to R^+ \cup \{0\} \text{ exists.}$$

Now

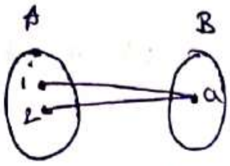
$$f \circ g(x) = f(g(x))$$
$$= f(\sqrt{x})$$
$$= (\sqrt{x})^2 = x$$

$\therefore f \circ g(x): R^+ \cup \{0\} \to R^+ \cup \{0\}$ and $f \circ g(x) = x$ is an identity function.

Note :

* If f is function from A to B, and S, T are subsets of A, then
  - $\to f(S \cup T) = f(S) \cup f(T)$
  - $\to f(S \cap T) \subseteq f(S) \cap f(T)$

Proof



$A=\{1,2\}$ $B=\{a\}$

Let $S=\{1\}$, $T=\{2\}$

$f(S\cup T)=f(\{1,2\})=a$

$f(S)\cup f(T)=\{a\}\cup\{a\}=a$

$$\boxed{\therefore f(S\cup T)=f(S)\cup f(T)}$$

$f(S\cap T)=f(\{1\}\cap\{2\})=f(\phi)=\phi$

$f(S)\cap f(T)=\{a\}\cap\{a\}=\{a\}$

$$\boxed{\therefore f(S\cap T)\subseteq f(S)\cap f(T)}$$

* If f is 1-1 then

$$\boxed{f(S\cap T)=f(S)\cap f(T)}$$

Note:

→ If f is a function from A to B and $S\subseteq B$, $T\subseteq B$ then

$$\boxed{\begin{aligned} f^{-1}(S\cup T)&=f^{-1}(S)\cup f^{-1}(T)\\ f^{-1}(S\cap T)&=f^{-1}(S)\cap f^{-1}(T)\end{aligned}}$$

Proof:

For union it is same as above proof

For intersection

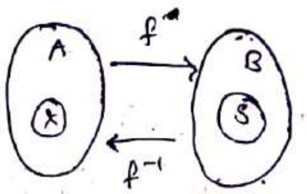
f^{-1} exists only if f is bijection

⇒ f is 1-1 and f^{-1} is 1-1

$\therefore f^{-1}(S\cap T)=f^{-1}(S)\cap f^{-1}(T)$

→ Let f be function from A to B. Let S be a subset of B

$$f^{-1}(\bar{S}) = \overline{f^{-1}(S)}$$



Proof:

f, f^{-1} ~~is~~ are bijection

~~Let x be to~~

If x is correspondence to S

$$f^{-1}(\bar{S}) = \bar{x} = \overline{(x)} = \overline{f^{-1}(S)}$$

$f^{-1}(S) = x$

$f(x) = S$

- Partial Functions:

A partial function $f: A \to B$ is an assignment ~~of~~ to each element a in a subset of A, called the domain definition of f, of a unique element b in B.

We say that f is undefined for elements in A that are not in the domain definition of f. When domain definition of f equals A, ~~we~~ we say that f is a total function.

* Partial functions are also denoted as $f: A \to B$ (like total functions)

Eg: $f: \mathbb{Z} \to \mathbb{R}$ where $f(x) = \sqrt{x}$ is a partial function from Z to R

∴ domain definition of f = set of non-negative integers.

(Q18) Let $g(x) = \lfloor x \rfloor$. Find

a) $g^{-1}(\{0\}) = [0, 1)$

b) $g^{-1}(\{-1, 0, 1\}) = [-1, 2)$

c) $g^{-1}(\{x \mid 0 < x < 1\}) = \phi$

> Note:
> $g(x) = \lfloor x \rfloor$ is not a bijection & hence not invertible.
> But still we can find inverse image of subset of its range

Q19) $f: R \to R$, $f(x) = x^2$. Find

a) $f^{-1}(\{1\})$ b) $f^{-1}(\{x \mid 0 < x < 1\})$ c) $f^{-1}(\{x \mid x > 4\})$

Sol :

$f^{-1}(x) = \sqrt{x}$

a) $f^{-1}(\{1\}) = \{-1, 1\}$

b) $f^{-1}(\{x \mid 0 < x < 1\}) = (-1, 0) \cup (0, 1)$

(or)

$(-1, 1) - \{0\}$

c) $f^{-1}(\{x \mid x > 4\}) = (-\infty, -2) \cup (2, \infty)$

(or)

$(-\infty, \infty) - [-2, 2]$

Cartesian Products :

* It is one of the operations on sets

→ The cartesian product of two sets A & B is

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

* If $|A| = m$, $|B| = n$

$|A \times B| = n$

* Cartision Product is not commutative

$$A \times B \neq B \times A$$

* Carhisian Product is associative

$$\underbrace{A \times B}_{1st} \times C = A \times \underbrace{B \times C}_{2nd}$$

* $(A \times B) \times C \neq A \times B \times C$

Here elements are $((a, b), c)$ — Here elements are (a, b, c)

Slly

$(A \times B) \times (C \times D) \neq A \times (B \times C) \times D$

* Cartisian product of set A with ~~itset~~ itself is denoted as : A^2

$$A^2 = A \times A$$

Slly $A^3 = A \times A \times A$

$A^4 = A \times A \times A \times A$

$\vdots$

Note :

$A \times B = B \times C$

$\Leftrightarrow (A = B \vee A = \phi \vee B = \phi)$

Note :

→ $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Proof :

$A \times (B \cap C) = \{(a,b) \mid a \in A \wedge (b \in B \cap C)\}$

$= \{(a,b) \mid a \in A \wedge b \in B \wedge b \in C\}$

$= \{(a,b) \mid (a \in A \wedge b \in B) \wedge (a \in A \wedge b \in C)\}$

$= \{(a,b) \mid a \in A \wedge b \in B\} \cap \{(a,b) \mid a \in A \cap b \in C\}$

$= (A \times B) \cap (A \times C)$

→ $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Proof :

$A \times (B \cup C) = \{(a,b) \mid a \in A \wedge (b \in B \cup C)\}$

$= \{(a,b) \mid a \in A \wedge (b \in B \vee b \in C)\}$

$= \{(a,b) \mid (a \in A \wedge b \in B) \vee (a \in A \wedge b \in C)\}$

$= \{(a,b) \mid a \in A \wedge b \in B\} \cup \{(a,b) \mid a \in A \wedge b \in C\}$

$= (A \times B) \cup (A \times C)$

→ $(A \cap B) \times C = (A \times C) \cap (B \times C)$
$(A \cup B) \times C = (A \times C) \cup (B \times C)$ } similar proof

→ $\bar{A} \times \bar{B} \neq \overline{A \times B}$

→ $A \times (B - C) = (A \times B) - (A \times C)$

$\bar{A} \times \bar{B} = (U - A) \times (U - B)$

$\overline{A \times B} = (U \times U) - (A \times B)$

$\bar{A} \times \bar{B} \subseteq \overline{A \times B}$

Note :

* If A, B are two ~~sets~~ finite sets

Jaccard Similarity $J(A,B) = \dfrac{|A \cap B|}{|A \cup B|}$

$J(\phi, \phi) = 1$

Jaccard distance $(d_J(A,B)) = 1 - J(A,B)$

05/06/20

Relations:

A binary relation from A to B is a subset of $A \times B$. If $A = B$, we say R is a relation on A.

Let $|A| = m$ $|B| = n$

$|A \times B| = m \times n$

Total no of relation possible = 2^{mn} (no of subsets possible)

*. If $|A| = n$, no of relations possible on $A = 2^{n^2}$

Reflexive relation:

→ A relation on a set A is said to be reflexive $\Leftrightarrow aRa \ \forall a \in A$

Eg: Let $A = \{1, 2, 3\}$

$R_1 = \{(1,1)\ (2,2)\ (3,3)\}$ is a reflexive relation

$R_2 = \{(1,1)\ (2,2)\ (3,3)\ (1,2)\ (2,1)\}$ is a reflexive relation

$R_3 = \{(1,1)\ (2,2)\}$ is not a reflexive relation.

* size of reflexive relation is

$$n \le |R| \le n^2$$

Note:

~~R={}~~

Relation, $R = \{\}$ on set S is reflexive $\Leftrightarrow S = \phi$

Eg: Let $A = \{1, 2, 3, 4\}$

$R = \{(a,b) \mid a+b \le 4 \text{ and } a, b \in A\}$

R is not reflexive because $(3,3) \notin R$

* Let R_1 & R_2 be two relations on set A, ~~then~~

~~$R_1 \cup R_2$ is~~ If R_1, R_2 are reflexive relations, then

$R_1 \cup R_2$ is a reflexive relation

$R_1 \cap R_2$ is a reflexive relation

* Total no of reflexive relations possible on a set ~~set~~ A with $|A|=n$ is: ~~2~~

we need to include every (a,a) $\forall a\in A$ in Relation

we may or may not include rest of the n^2-n ordered pairs.

$\therefore$ $\boxed{\text{no of reflexive relations possible} = 2^{n^2-n}}$

Symmetric Relation:

A relation ~~is~~ ~~set~~ R on set A is said to be symmetric

if xRy then yRx $\forall x,y\in A$

i.e., $\forall x \forall y \left(xRy \rightarrow yRx \right)$

(or)

$\forall x \forall y \left((x,y)\in R \rightarrow (y,x)\in R \right)$

Eg: $R_1 = \{(1,2)\}$ is not a symmetric relation

$R_2 = \{(1,2)\ (2,1)\}$ is a symmetric relation

$R_3 = \{(1,1)\}$ is a symmetric relation

$R_3 = \{\ \}$ is a symmetric relation

Eg: Let $R = \{(a,b) \mid (a+b) \leq 4 \ \& \ a,b\in A\}$ $A = \{1,2,3\}$

This is a symmetric relation because addition is commutative.

Eg: Let $R = \{(a,b) \mid a = b+1\}$

$a = b+1 \Rightarrow a-b = 1$

Subtraction is not commutative

Hence not symmetric.

Note:

In a relation R, Domain of relation $= \{x \mid (x,y)\in R\}$

Range of relation $= \{y \mid (x,y)\in R\}$

* If R_1 & R_2 are symmetric relations on A, then

$R_1 \cup R_2$ is a symmetric relation

$R_1 \cap R_2$ is a symmetric relation

Finding no. of symmetric relation

* If A is a set with n elements

$\forall a \in A$ (a,a) may or may not belong to symmetric relation

Out of remaining elements we get $\frac{n^2-n}{2}$ symmetric pairs.

For each pair we may or may not include it in the relation

$$\therefore \boxed{2^{n+\frac{n^2-n}{2}} = 2^{\frac{n^2+n}{2}}}$$

* No of relations that are both symmetric & Reflexive is

$\forall a \in A$. (a,a) must be include

Remaining elements are considered $\frac{n^2-n}{2}$ symmectric pairs

Out of which each pair may or may not be included

$$\therefore 2^{\frac{n^2-n}{2}}$$

* Cardinality of a ~~refle~~ symmetric relation ranges from 0 to n^2.

Q20 G-06 Let X, Y, Z be sets of sizes x, y and z respectively. Let $W = X \times Y$ and E be set of all subsets of W. The no of functions from Z to E is

a) $2^{2^{xy}}$ b) 2×2^{xy} c) $2^{2^{x+y}}$ d) 2^{xyz}

Sol:

$|X| = x$ $|Y| = y$ $|Z| = z$ $|W| = xy$ $|E| = 2^{xy}$

no of function from Z to E $= (2^{xy})^z = 2^{xyz}$

Q21 / G-15 Let R be a relation on the set of ordered pairs of positive integers such that $((p,q),(r,s)) \in R^2$ if and only if $p-s = q-r$.
which of the following is true about R?

a) Both reflexive & symmertic
b) Reflexive but not symmetric
c) Not reflexive but symmetric
d) Neither reflexive nor ~~sym~~ symmetric

sol:

Consider $a,b \in \mathbb{Z}^+$

~~$\forall a,b \in R$~~ $\forall a,b \ \ ((a,b),(a,b)) \notin R$

$\because a-b \neq b-a$

$\therefore$ not reflexive

Let $a,b,c,d \in \mathbb{Z}^+$

let $((a,b),(c,d)) \in R$

$\Rightarrow a-d = b-c$

Consider $(\underset{p\ q}{(c,d)}, \underset{r\ s}{(a,b)})$

$c-b = d-a$

$\Rightarrow b-c = a-d$

$\therefore$ Symmetric

$\therefore$ R is not reflexive but symmetric.

9/8 X can be choosen in 2^n ways

for $X = \phi$ we can't define (x,x)

for $|X| = 1$ we can define one (x,x)

we have nc_1 such X.

for $|X| = 2$ we can define two (x,x)

we have nc_2 such X

$\therefore$ No. of ways $= nC_1 \cdot (1) + nC_2 \cdot (2) + nC_3 (3) + \dots + nC_n (n)$

$$= \sum_{k=1}^{n} k \cdot nC_k$$

$$= n 2^{n-1} \text{ (By option verification)}$$

$\therefore$ Both I & II

Material questions on functions:

P/43 a) $f(x) = x^2 \quad f: R \to R$

$f(-2) = f(2) = 4$

$\therefore$ not 1-1 $\Rightarrow$ not bijection

b) $g(x) = |x|$

$g(-1) = g(1) = 1$

$\therefore$ not bijection

c) $h(x) = \lfloor x \rfloor$

~~h(~~ $h(2.5) = h(2.3) = 2$

$\therefore$ not bijection

d) ~~$g(x) = x^3$~~ $\phi(x) = x^3$

1-1 & onto

$\therefore$ bijection

P/44 Given $f(A) = g(B) = h(C)$

$f: A \to S \quad g: B \to S \quad h: C \to S$

$|A| = |B| = |C| = k \quad |S| = n, \quad k \le n$

f, g, h are 1-1

given images of A under f = B under g = C under h

since ~~it~~ they are 1-1, $|f(A)| = |g(B)| = |h(C)| = k$

This k from S can be choosen in ^nC_k ways

for function 'f' no of one-one possible: $= k!$

" " 'g' " " " " $= k!$

" " 'h' " " " " $= k!$

$\therefore$ no of ways $= {}^nC_k (k!)^3$

P/46 Here $|A| = |B| = 50$

$f: A \to B$ is one-one

$\Rightarrow$ f must be onto

Hence f is bijection

$\therefore f^{-1}$ exists

P/47 $f: X \to Y$ is bijection

$\Rightarrow |X| = |Y|$

$f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$
$f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$ } Its intuitive

P/48 $f(x,y) = (2x-y, x-2y) = (a,b)$ (say)

$f^{-1}(a,b) = (x,y)$

$2x - y = a$
$x - 2y = b$
$2x - 4y = 2b$

$\Rightarrow -y+4y = 3y = a-2b \Rightarrow y = \frac{a-2b}{3}$

$2x - \frac{a-2b}{3} = a \Rightarrow 2x = a + \frac{a-2b}{3} = \frac{4a-2b}{3}$

$x = \frac{2a-b}{3}$

$\Rightarrow f^{-1}(x,y) = \left(\frac{2x-y}{3}, \frac{x-2y}{3}\right)$

(P/49) let string be 00110

Here position of '0' has 3 values.

∴ 'i' is not function

number 1's in a string gives a unique value

∴ function

(P/50) S1: $f(x) = x^3$

1-1

~~for~~ '2' has no preimage

∴ not onto

∴ True

S2: $f(n) = \left\lceil \frac{n}{2} \right\rceil$

for

$f(5) = f(6) = 3$

∴ not 1-1

for every y we can define its preimage as $2y$

∴ onto

∴ True

(P/51) $f(x_1) = f(x_2)$ $\quad x_1; x_2 \in A$

$\frac{x_1-2}{x_1-3} = \frac{x_2-2}{x_2-3} \Rightarrow \cancel{x_1x_2} - 2x_2 - 3x_1 + \cancel{6} = \cancel{x_1x_2} - 3x_2 - 2x_1 + \cancel{6}$

$\Rightarrow x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2$

∴ 1-1

Let $f(x)=y$

$$\frac{x-2}{x-3}=y \Rightarrow x-2=xy-3y$$

$$\Rightarrow x-xy=2-3y$$

$$\Rightarrow x=\frac{3y-2}{y-1}$$

Thus onto

$\therefore$ Bijection

P/52 $f(x)=\frac{x}{x+1}$ $g(x)=\frac{x}{1-x}$

$$(f o g)^{-1}x=(g^{-1} o f^{-1})x$$

$$=g^{-1}(f^{-1}(x))$$

$$=g^{-1}\left(\frac{x}{1-x}\right)$$

$$=\frac{\frac{x}{1-x}}{1+\frac{x}{1-x}}$$

$$=\frac{\frac{x}{1-x}}{\frac{1}{1-x}}=x$$

$f(x)=\frac{x}{x+1}=y$

$\Rightarrow x=xy+y$

$x=\frac{y}{1-y}$

$f^{-1}(x)=\frac{x}{1-x}$

$g(x)=\frac{x}{1-x}=y$

$\Rightarrow x=y-xy$

$x=\frac{y}{1+y}$

$\Rightarrow g^{-1}(y)$ $g^{-1}(x)=\frac{x}{1+x}$

P/53 a) $f: A\rightarrow B$ $f^{-1}: B\rightarrow A$

$f o f^{-1}=I_A$

Let $f(a)=b$

$f(f^{-1}(b))$

$f(a)=b$

$\therefore f o f^{-1}=I_B$

$\therefore$ False

b) $f^{-1} o f$

$f^{-1}(f(a))$

$f^{-1}(b) = a$

$\therefore f^{-1} o f = I_A$

$\therefore$ True

c) $I_B o f^{-1}$

$I_B(f^{-1}(b))$

$I_B(a)$

not defined

d) $f o I_A$

$f(I_A(a))$

$f(a) = b$

$\therefore f o I_A = f$

$\therefore$ True

$\therefore$ a & c are wrong

(P/55) $f : A \to B \quad g : B \to A$

$gof : I_A$

f is identity

$\Rightarrow A = B$

$\therefore f : A \to A \quad g : A \to B$

$gof : I_A \Rightarrow g = f^{-1}$

f is identity ~~f^{-1} exits~~

$\therefore f$ is one-one & f is onto

(P/53) a) Every function can be represented graphically

$\therefore$ True

b) $f(x) = x \; , \; f(2) = 2$

$g(x) = \sqrt{x^2} \; , \; g(2) = \sqrt{4} = \pm 2$

$\therefore$ false

c) $f(x) = \log x^2 \; , \; f(-2) = \log 4$

$g(x) = 2 \log x \; , \; g(-2) = 2 \log(-2)$ i.e., undefined $\therefore$ false

d) $f(x) = \dfrac{1}{\sqrt{|x|-x}}$

$\sqrt{|x|-x} \neq 0$ & $|x|-x \geq 0$

$|x|-x \neq 0$ $\qquad$ $|x| \geq x$

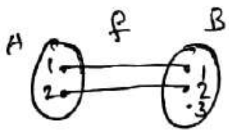
$|x| \neq x$ $\qquad$ $x \in (-\infty, 0]$

$\Rightarrow x \in (-\infty, 0)$

$x \in (-\infty, 0)$

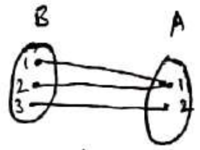
$\therefore$ True

(P/45) Let $A = \{1,2\}$ & $B = \{1,2,3\}$



1−1

$\therefore$ S1: True



onto

$\therefore$ S2: True

06/05/20

Antisymmetric Relation:

$\forall a \forall b \, (aRb \wedge bRa \rightarrow a = b)$

(or)

$\forall a \forall b \, ((a,b) \in R \wedge (b,a) \in R \rightarrow a = b)$

Eg: $R_1 = \{(1,2)\}$ is an antisymmetric relation.

$R_2 = \{(1,2), (2,1)\}$ is not an antisymmetric relation

$R_3 = \{(1,2)\ (1,1)\}$ is an antisymmetric relation

$R_4 = \{\ \}$ is an antisymmetric relation

Eg: $R_1 = \{(a,b) \mid a \leq b$ & $a,b \in Z\}$ is an antisymmetric relation

Eg: $R = \{(a,b) \mid a = b+1\}$ is an antisymmetric relation.

Here a never equals to b

$a \neq b$

$\therefore$ antisymmetric

Eg: $R = \{(a,b) \mid a > b\}$ is antisymmetric}

Eg: $R = \{(a,b) \mid a = b$ (or) $a = -b\}$ is not antisymmetric

because $5R-5$ & $-5R5$

for an integer x

$xR(-x)$ & $(-x)Rx$

$\therefore$ not antisymmetric

Eg: $R_5 = \{(a,b) \mid a+b \leq 3\}$ is not antisymmetric

$2R1$ & $1R2$

* If R_1 & R_2 are two antisymmetric relations

$R_1 \cup R_2$ need not to be an antisymmetric relation

$R_1 \cap R_2$ is an antisymmetric relation.

Proof:

→ Let $R_1 = \{(1,2)\}$ $R_2 = \{(2,1)\}$ are antisymmetric relations

$R_1 \cup R_2 = \{(1,2),(2,1)\}$ is not antisymmetric

→ $R_1 \cap R_2 = \{(x,y) \mid (x,y) \in R_1 \wedge (x,y) \in R_2\}$

for $R_1 \cap R_2$ to be antisymmetric $(y,x) \notin R_1 \cap R_2$

$(x,y) \in R_1$

because R_1 is antisymmetric $(y,x) \notin R_1$

slly $(y,x) \notin R_2$

$\therefore (y,x) \notin R_1 \cap R_2 \Rightarrow R_1 \cap R_2$ is antisymmetric

* Total no of antisymmetric relations possible on set A, with $|A|=n$ is:

~~$\forall a\ (a,a)\in A$~~

$\forall a\in A$ (a,a) may ~~are~~ or may not be included $\Rightarrow 2^n$ ways

$\forall a \forall b \wedge a\neq b$ we may include (a,b) or (b,a) ~~not~~ or not include both $\Rightarrow$ $3^{\frac{n^2-n}{2}}$ ways

$\Rightarrow \boxed{2^n \cdot 3^{\frac{n^2-n}{2}}}$

* No of relation such that it is both symmetric and antisymmetric $=2^n$

* No of relations such that it is both reflexive & antisymmetric $= 3^{\frac{n^2-n}{2}}$

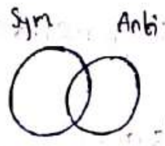
* No of relation such that it is ~~both~~ reflexive, symmetric & antisymmetric is only 1.

i.e. $\{(a_1,a_1),(a_2,a_2)\dots(a_n,a_n)\}$, $a_i\in A$

* Size of an antisymmetric relation ranges ~~from~~ from

0 to $n+\frac{n^2-n}{2}$

* No of relations that are only symmetric but not antisymmetric



~~|Sym| - |Anti|~~ $|\text{Symmetric}| - |\text{Sym} \cap \text{Anti}|$

$= 2^n \cdot 2^{\frac{n^2-n}{2}} - 2^n$

$= 2^n\left(2^{\frac{n^2-n}{2}} - 1\right)$

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Asymmetric Relation:

$\forall a \forall b \, (aRb \rightarrow b \not{R} a)$.

(or)

$\forall a \forall b \, ((a,b) \in R \rightarrow (b,a) \notin R)$

Eg: $R = \{(1,1)(1,2)\}$ is not assymetric relation

$R = \{(1,3)(3,1)\}$ is not assymetric relation

$R = \{(1,2)(2,3)(1,3)\}$ is assymetoric relation

$R = \{\ \}$ is assymetric relation

$R = \{(a,b) \mid a \leq b\}$ is not assymetric relation

$R = \{(a,b) \mid (a+b) \leq 3\}$ is not assymmetric

$R = \{(a,b) \mid a < b\}$ is assymetric relation.

* If R_1, R_2 are two assymmetric relations

$R_1 \cup R_2$ is ~~also asym~~ not assymmetric relation

$R_1 \cap R_2$ is an assymmetric relation

* No of assymmetric relations possible $= 3^{\frac{n^2-n}{2}}$

* Size of an assymmetric relation ranges from

0 to $\frac{n^2-n}{2}$

* Every assymmetric relation is antisymmetric relation

* No of relations that are both symmetric & assymetric $= 1$, i.e., $\{\ \}$

* ~~|Reflexive ∩ Assymmetric| = 1 (relation that are both assymmetric and~~

Irreflexive Relation:

$\forall a \in A \; a \not R a$ (or) $\forall a \in A \; \{(a,a) \notin R\}$

let $A = \{1,2,3\}$

Ex: $R_1 = \{(1,1), (1,2)\}$ is not reflexive and not irreflexive.

$R_2 = \{(1,2)\}$ is irreflexive and not reflexive

$R = \{(1,1)(2,2)(3,3)\}$ is not irreflexive but reflexive

~~$R = \{(1,2)\}$~~ $R = \{\}$ is irreflexive.

* There is no relation such that it is both reflexive & irreflexive
* But there are relations that are ~~no~~ neither reflexive ~~not~~ nor irreflexive.

* No of irreflexive relations possible on set A, $|A| = n$ is

$$2^{n^2-n}$$

* If R_1, R_2 are two irreflexive relations, then

$R_1 \cup R_2$ is irreflexive

$R_1 \cap R_2$ is irreflexive

* Size of an irreflexive ~~element~~ relation ranges from

0 to n^2-n.

* The below table shows no of relations such that both are possible.

	Reflexive	Irreflexive	Symmetric	Antisymmetric	Asymmetric
Asymmetric	0	$3^{\frac{n^2-n}{2}}$	1	$3^{\frac{n^2-n}{2}}$	$3^{\frac{n^2-n}{2}}$
Antisymmetric	$3^{\frac{n^2-n}{2}}$	$3^{\frac{n^2-n}{2}}$	2^n	$2^n 3^{\frac{n^2-n}{2}}$	
Symmetric	$2^{\frac{n^2-n}{2}}$	$2^{\frac{n^2-n}{2}}$	$2^n \cdot 2^{\frac{n^2-n}{2}}$		
Irreflexive	0	2^{n^2-n}			
reflexive	2^{n^2-n}				

(Q22) If R is a relation such that

$R = \{ (A,B) \mid A \cap B = \phi \}$, where A, B are sets

is R reflexive & symmetric?

sol:

for any non-empty set A

$A \cap A = A \neq \phi$

$\therefore$ not reflexive

for any two disjoint sets A, B

$A \cap B = \phi$

$\Rightarrow B \cap A = \phi$

$\therefore$ Symmetric

(Q23) $R = \{(a,b) \mid \gcd(a,b) \neq 1 \wedge a \neq b\}$

Find reflexive, symmetric & antisymmetric nature of R

sol:

$\forall a$, $(a,a) \notin R$

~~becau~~ it is given ~~that~~ if $(a,b) \in R$ then $a \neq b$

$\therefore$ not reflexive

$\forall a,b$, if $a \neq b$,

and $\gcd(a,b) \neq 1$

then $(a,b) \in R$

$\Rightarrow b \neq a$ & $\gcd(b,a) \neq 1$

$\Rightarrow (b,a) \in R$

$\therefore$ symmetric

$\forall a,b$ if $\gcd(a,b) \neq 1 \wedge a \neq b$ then $(a,b) \in R$

then $(b,a) \in R$

$\therefore$ not antisymmetric.

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(Q24) Let $R_1, R_2, R_3, R_4, R_5, R_6$ be relations defined from R to R as follows

$R_1 = \{(a,b) \mid a > b\}$ $R_4 = \{(a,b) \mid a \le b\}$

$R_2 = \{(a,b) \mid a \ge b\}$ $R_5 = \{(a,b) \mid a = b\}$

$R_3 = \{(a,b) \mid a < b\}$ $R_6 = \{(a,b) \mid a \ne b\}$.

Talk about below relations.

a) $R_2 \cup R_4$

$R_2 \cup R_4 = \cancel{\{(a,b) \mid a \ge b\}} = R^2$

i.e., Reflexive & symmetric & Transitive

b) $R_3 \cup R_6$

$R_3 \cup R_6 = \{(a,b) \mid a < b \vee a > b\} = R_6$

i.e., irreflexive, symmetric,

c) $R_3 \cap R_6$

$R_3 \cap R_6 = \{(a,b) \mid a < b\} = R_3$

i.e., irreflexive, antisymmetric, assymmetric, transitive

d) $R_4 \cap R_6$

$R_4 \cap R_6 = \{(a,b) \mid a < b\} = R_3$

i.e., irreflexive, antisymmetric, assymmetric, transitive

e) $R_3 - R_6$

$R_3 - R_6 = \{\ \} = \phi$

i.e., irreflexive, symmetric, antisymmetric, assymmetric, transitive

f) $R_6 - R_3 = \{(a,b) \mid a > b\} = R_1$

i.e., irreflexive, antisymmetric, asymmetric, transive

g) $R_2 \oplus R_6$

$(R_2 \cup R_6) - (R_2 \cap R_6)$

$\{(a,b) \mid \forall a, \forall b\} - \{(a,b) \mid a > b\}$

$\{(a,b) \mid a \leq b\} = R_4$

i.e., Reflexive, Antisymmetric, Transitive

h) $R_3 \oplus R_5$

$(R_3 \cup R_5) - (R_3 \cap R_5)$

$\{(a,b) \mid a \leq b\} - \{\}$

$\{(a,b) \mid a \leq b\} = R_4$

i.e., Reflexive, Antisymmetric, Transitive.

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Transitive Relation:

$\forall a \forall b \forall c\ (aRb \wedge bRc \rightarrow aRc)$

Let $A = \{1,2,3\}$ and the below relations be defined on A.

Eg: $R = \{\ \}$ is a transitive relation

$R = \{(1,1)\ (2,2)\}$ is a transitive relation.

$R = \{(1,2)\}$ is a transitive relation

$R = \{(1,2)\ (2,1)\}$ is not transitive relation

$R = \{(1,2)\ (2,1)\ (1,1)\}$ is a transitive relation

Eg: $R_1 = \{(a,b) \mid a \le b\}$

Consider $a, b, c \in A$

$a \le b$, $b \le c$

$\Rightarrow a \le c$

$\therefore$ transitive

$R_2 = \{(a,b) \mid a = b+1\}$

aRb & bRc

$a = b+1$ $b = c+1$

$a = c+1+1$

~~a=c+1~~ $a = c+2$

~~$\therefore aRc$~~

$\therefore a \not R c$

$\therefore$ not transitive

$R_3 = \{(a,b) \mid a+b \le 3\}$

aRb bRc

$a+b \le 3$ $b+c \le 3$

Eg. $2+1 \le 3$ $1+2 \le 3$ $(a=2, b=1, c=2)$

$a+c = 2+2 = 4$

$\therefore a \not R c$

$\therefore$ not transitive

Eg:

$R_4 = \{(a,b) \mid \gcd(a,b) \ne 1$ & a, b are distinct$\}$

$a = 10$ $b = 2$ $c = 10$

aRb & bRc

$a \not R c$

$\therefore$ not transitive

$R_5 = \{(A,B) \mid A \cap B = \phi\}$

let $A = X$; $B = \bar{X}$; $C = X$

ARB & BRC

but $A \cap C = A \neq \phi$

$\therefore$ not transitive

Q25 G-16

A binary relation R on $N \times N$ is defined as follows:

$(a,b) R (c,d)$ if $a \le c$ or $b \le d$. Consider the following propositions:

P : R is reflexive

Q : R is transitive

which one of the following statements is TRUE?

a) P & Q b) only P c) only Q d) none

sol:

~~$(a,b) R (c,d) \Rightarrow a \le c$ & $b \le d$~~

~~consider~~

~~$(c,d) R (a,b) \Rightarrow c \le a$ & $d \le b$~~

~~clearly not~~

$(a,b) R (a,b) \Rightarrow a \le a$ or $b \le b$

which is true

$\therefore$ reflexive

~~Consider~~

~~$(a,b) R (c,d)$ $(c,d) R (e,f)$~~

~~$a \le c$ & $b \le d$ $(c \le e)$ & $(d \le f)$~~

~~we can conclude $(a \le e)$ & $(b \le f)$~~

~~$\therefore (a,b) R (e,f)$~~

$\therefore$ ~~P & Q are true~~

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Let $(a,b) R (c,d)$ and $a \le c$, $b > d$

Let $(c,d) R (e,f)$ and $c > e$, $d \le f$

$(a,b) R (e,f) \Rightarrow \underbrace{(a \le e) \text{ or } (b \le f)}$

we don't know about these

$\therefore$ not transitive

Eg:

$(1,5) R (3,2)$

Or $(3,2) R (-5,3)$ because $2 \le 3$

but $(1,5) \not R (-5,3)$

$\therefore$ not transitive

So only P is true

(or)

To P.T it is transitive we need to P.T

$(a,b) R (c,d) \wedge (c,d) R (x,y) \longrightarrow (a,b) R (x,y)$ is tautology

i.e., $\underbrace{[(a \le c) \vee (b \le d)]}_{T} \wedge \underbrace{[(c \le x) \vee (d \le y)]}_{T} \rightarrow \underbrace{[(a \le x) \vee (b \le y)]}_{F}$

with $a \le c$: F, $b \le d$: T, $c \le x$: T, $d \le y$: F, $a \le x$: F, $b \le y$: F

$\therefore$ not valid $\Rightarrow$ not transitive

Note:

→ If R_1, R_2 are two transitive relations, then

* $R_1 \cup R_2$ need not to be transitive

Eg: $R_1 = \{(1,2)\}$ is transitive

$R_2 = \{(2,3)\}$ is transitive.

$R_1 \cup R_2 = \{(1,2)\ (2,3)\}$ is not transitive.

* $R_1 \cap R_2$ is transitive

proof:

$R_1 \cap R_2 = \{(a,b) \mid (a,b) \in R_1 \ \& \ (a,b) \in R_2\}$

Now let $(a,b), (b,c) \in R_1 \cap R_2$

$\Rightarrow (a,b) \ \& \ (b,c) \in R_1 \quad \& \quad (a,b) \ \& \ (b,c) \in R_2$

$\Rightarrow (a,c) \in R_1 \qquad\qquad (a,c) \in R_2$

i.e., $(a,c) \in R_1 \cap R_2$

$\therefore$ transitive

Diagonal relation:

Diagonal relation on set A is
$\forall a \forall b \ (aRb \leftrightarrow a = b)$

Eg. $A = \{1,2,3\}$

diagonal relation $\Delta_A = \{(1,1)\ (2,2)\ (3,3)\}$

→ If R is a reflexive relation

$\Delta_A \subseteq R$

Inverse relation: (R^{-1}):

Inverse relation of R from A to B

$R^{-1} = \{(b,a) \mid (a,b) \in R\}$

Eg: $R = \{(a,b), (c,d)\}$

$R^{-1} = \{(b,a), (d,c)\}$

Complementary relation :

If R is a relation ~~A×A A×B~~ from A to B

Complementary relation

$$\bar{R} = \{(a,b) \mid (a,b) \notin R\}$$

i.e., $\bar{R} = \cancel{A \times A - R}\ A \times B - R$

Eg: Let $R = \{(a,b) \mid a \text{ divides } b\}$

$\cancel{R^{-1} = \{(b,a) \mid}$ $R^{-1} = \{(a,b) \mid b \text{ divides } a\}$

$\bar{R} = \{(a,b) \mid a \text{ does not divide } b\}$

Composition of Relation :

Let R be a relation from A to B

and S be a relation from B to C

Composite relation $S \circ R$ is defined as

$$S \circ R = \{(a,c) \mid (a,b) \in R \ \& \ (b,c) \in S\}$$

(Q26) Consider below relations

$R_1 : a > b$ $\quad$ $R_4 : a \leq b$

$R_2 : a \geq b$ $\quad$ $R_5 : a = b$

$R_3 : a < b$ $\quad$ $R_6 : a \neq b$

Find

a) $R_2 \circ R_1$

Let $(a,b) \in R_1$ & $(b,c) \in R_2$

$a > b$ & $b \geq c$

i.e., $a > b \geq c \Rightarrow a > c \ \therefore R_1$

b) $R_3 \circ R_5$

$(a,b) \in R_5 \qquad (b,c) \in R_3$

$a = b \qquad b < c$

i.e., $a < c$

$R_3 \circ R_5 = R_3$

c) $R_5 \circ R_3$

$(a,b) \in R_3 \qquad (b,c) \in R_5$

$a < b \qquad b = c$

i.e., $a < c$

$\therefore R_5 \circ R_3 = R_3$

R_5 is diagonal relation

$R_5 \circ R_3 = R_3$

$R_3 \circ R_5 = R_3$

$\boxed{\text{If } d \text{ is a diagonal relation, and if } R \text{ is any relation} \quad R \circ d = R \quad d \circ R = R}$

Diagonal relation is nothing but an identity function

d) $R_1 \circ R_4$

$(a,b) \in R_4 \qquad (b,c) \in R_1$

$a \leq b \qquad b > c$

$a \leq b < c$

~~In this case~~ sometimes

$\therefore a < c = R_3$

$R_1 \circ R_4 = R_3$

e) $R_3 \circ R_6$

$(a,b) \in R_6 \qquad (b,c) \in R_3$

$a \neq b \qquad b < c$

Here we have 2 cases

$a > b$ or $a < b$

$a > b$ & $b < c$ → any (a,c)

$a < b < c$ → $a < c$

$\therefore R_3 \circ R_6 =$ Domain $\times$ Domain

Note :

* R is symmetric $\Rightarrow$ R^{-1} is symmetric
* R is reflexive $\Rightarrow$ R^{-1} is reflexive
* R is reflexive $\Rightarrow$ $\overline{R}$ is irreflexive
* $R \cup R^{-1}$ is symmetric..
* R_1, R_2 are symmetric then $R_1 o R_2$ need not to be symmetric

If R is transitive, R^{-1} is also transitive

Proof :

$R_1 = \{(1,2)(2,1)\}$ $R_2 = \{(2,3), (3,2)\}$

$R_1 o R_2 = \{(3,1)\}$ is not symmetric

$R_2 o R_1 = \{(1,3)\}$ is not symmetric

Note that if R is symmetric, then R^2 is symmetric

* If R_1, R_2 are antisymmetric then their composition need not to be antisymmetric

Eg:

$R_1 = \{(1,2), (3,2)\}$ $R_2 \{(2,3), (2,1)\}$

$R_2 o R_1 = \{(1,3); \underbrace{(3,3), (3,1)}\}$

$\therefore$ not antisymmetric

If R_1 & R_2 as asymmetric $R_0 R_1$ need not to be asymmetric

* If R_1, R_2 are transitive, then $R_1 o R_2$ need not to be transitive
* If R_1, R_2 are irreflexive, then $R_1 o R_2$ need not to be irreflexive.

Closure.

If R_1 & R_2 on set A are reflexive then $R_1 o R_2$ is reflexive.

If R is a relation then

closure of a property on R is R^+ such that

i) property holds on R

(ii) Contain R

(iii) It is minimal possible

(i) Reflexive closure :

Let $R = \{(1,2)\}$ be a relation on set $A = \{1,2,3\}$ then

reflexive closure of R^* is

R^+ which is Reflexive.
contains R.
minimal

$\therefore$ reflexive closure, $R^+ = \{(1,2)\ (1,1)\ (2,2)\ (3,3)\}$

$\boxed{\text{Reflexive closure : } R^+ = R \cup \Delta_A}$

(ii) Symmetric closure:

symmetric closure of a relation R^* is

R^+ which is symmetric
contains R
minimal

Eg : If $R = \{(1,2)\ (1,1)\}$

$R^+ = \{(1,2)\ (2,1)\ (1,1)\}$

$\boxed{\text{Symmetric closure : } R^+ = R \cup R^{-1}}$

(iii) Transitive closure :

transitive closure of a relation R is

R^+ which is transitive
contains R
minimal

Eg : Let R be relation on

$A = \{1,2,3,4,5\}$

Let $R = \{(1,2)\ (2,3)\ (3,4)\ (5,4)\}$

Draw a diagram as below

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \\ & & 5 \arrow[ur] & \end{tikzcd}$$

Now from each vertex find all reachable vertices

for 1:	for 2:	for 3:	for 5:
(1,2)	(2,3)	(3,4)	(5,4)
(1,3)	(2,4)		
(1,4)			

$\therefore$ Transitive closure $= \{(1,2)(1,3)(1,4)(2,3)(2,4)(3,4)(5,4)\}$

* Let ~~R~~ a relation R be represented by a directed graph in way that whenever aRb a directed edge is drawn from a to b.

Now, if

$(a,c) \in R \circ R$

$\Rightarrow$ there is 2-edge path from a to c.

Slly: for two vertice a & b, there exists an n-length path from a to b iff $(a,b) \in R^n$

Note:

→ If R is a relation,

a composite ~~rel~~ relation from R to itself is denoted as R^2.

Slly $R \circ R \circ R = R^3$

$R \circ R \circ \cdots \circ R$ (n times) $= R^n$

* The standard procedure to calculate transitive closure is

$$R^* = R \cup R^2 \cup R^3 \cup R^4 \cdots \cup R^n$$

where n is a number such that

$$R^n = R^{n+1}$$

$$\boxed{\therefore R^* = \bigcup_{n=1}^{\infty} R^n}$$

Note:

→ If R is a transitive relation, then

$$R^n \subseteq R \quad \forall n \geq 1$$

also R^n is transitive too.

* Irreflexive closure can be defined iff the relation on which it is to be defined is irreflexive.

Also the set itself is the irreflexive closure

* The closure of relation R w.r.to a property P, if exists, is the intersection of all the relations with the property P that contain R.

* when asked to find

Symmetric, reflexive & transitive closure on a relation such that all 3 properties hold at the same time, then the order of finding closures is.

(i) Reflexive
(ii) Symmetric
(iii) Transitive

~~other ord~~ "But"

also

transitive
reflexive
symmetric

works but may produce redundant ~~relation~~ pairs

other order would also work but gives redundant pairs

Q27 Let relation R be defined on real numbers as

$$R = \{(a,b) \mid a = b+1\}$$

what is the transitive closure of R?

Let aRb bRc cRd ......

$a = b+1$ $b = c+1$ $c = d+1$

By finding transitive closure we obtain

(a,b) (a,c) (a,d)....

(b,c) (b,d) ...

Thus, transitive closure $= \{(a,b) \mid a<b\}$

Q28 / G-98 The binary relation

$R = \{(1,1)\ (2,1)\ (2,2)\ (2,3)\ (2,4)\ (3,1)\ (3,2)\ (3,3)\ (3,4)\}$ on the set $A = \{1,2,3,4\}$ is

a) Reflexive, symmetric & transitive

b) Neither reflexive, nor irreflexive but transitive.

c) Irreflexive, symmetric & transitive

d) Irreflexive & antisymmetric.

Sol:

$(4,4) \notin R$

$\therefore$ not reflexive

$(1,1) \in R$

$\therefore$ not irreflexive

Thus we can eliminate opt ⓐ ⓒ ⓓ

$\therefore b$

Q29
G-01

Consider the following relations

R_1: (a,b) iff $(a+b)$ is even over set of integers

R_2 : (a,b) iff $(a+b)$ is odd over set of integers

R_3 : (a,b) iff $a \cdot b > 0$ over set of non-zero rational numbers

R_4 : (a,b) iff $|a-b| \leq 2$ over the set of natural numbers

Which of the following ~~is correct~~? are equivalence relations?

a) R_1 & R_2

b) R_1 & R_3

c) R_1 & R_4

d) all.

sol :

R_1 : $\forall a \in Z$ $\quad a+a$ is even

$\therefore$ reflexive

$\forall a, \forall b \in Z$ $\quad a+b$ is even $\Rightarrow$ $b+a$ is even

$\therefore$ symmetric

let

$a+b$ is even & $b+c$ is even

$\Rightarrow (a+b)+(b+c)$ is also even.

$a+c+\underbrace{2b}_{\text{even}}$ is even

$\therefore$ $a+c$ must be even

$\therefore$ transitive

Hence R_1 is equivalence relation

R_2 : $\forall a \in Z$ $\quad a+a$ is even

$\therefore$ not reflexive

Hence not equivalence relation

R_3: $\forall a \in R \quad a^2 > 0$

$a \neq 0$

$\therefore$ reflexive

$ab > 0 \Rightarrow ba > 0$

$\therefore$ symmetric

$ab > 0$ & $bc > 0$

$\Rightarrow (ab)(bc) > 0$

$(ac)(b^2) > 0$

$\downarrow$

> 0

$\Rightarrow ac > 0$

$\therefore R_3$ is equivalence relation

R_4: $\forall a \in N$

$|a-a| = 0 \leq 2$

$\therefore$ reflexive

$\forall a, b \in N$

$|a-b| \leq 2 \Rightarrow |b-a| \leq 2$

$\therefore$ symmetric

$a, b, c \in N$

let $a = 1 \quad b = 3 \quad c = 5$

$|a-b| \leq 2 \qquad |b-c| \leq 2$

$|a-c| = 4 \nleq 2$

$\therefore$ not transitive

$\therefore R_1$ & R_3 are equivalence relations.

Q30 G-04 Consider the binary relation

$S = \{(x,y) \mid y = x+1$ and $x, y \in \{0,1,2 \ldots\}\}$

The reflexive transitive closure of S is

a) $\{(x,y) \mid y > x$ and $x, y \in \{0,1,2 \ldots\}\}$

b) $\{(x,y) \mid y \geq x$ and $x, y \in \{0,1,2 \ldots\}\}$

c) $\{(x,y) \mid y<x$ and $x,y \in \{0,1,2 \ldots\}\}$

d) $\{(x,y) \mid y \le x$ and $x,y \in \{0,1,2,\ldots\}\}$

Sol:

Finding reflexive closure

$$\forall x \in \{0,1,2 \ldots\}$$
$$(x,x) \in R$$

Consider the graph

$$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow \ldots$$

By applying transitive closure

$$(x,y) \in R \mid y > x$$

$\therefore$ Reflexive transitive closure is

$$\{(x,y) \mid y \ge x \text{ and } x,y \in \{0,1,2 \ldots\}\}$$

Q31 G-04 The number of different $n \times n$ symmetric matrices with each element being either 0 or 1 is

a) 2^n b) 2^{n^2} c) $2^{\frac{n^2+n}{2}}$ d) $2^{\frac{n^2-n}{2}}$

Sol:

If we represent a relation on set A, with n elements, by a matrix such that if xRy then the corresponding cell of matrix is 1 and 0 otherwise.

Thus the question reduces to no of symmetric relations possible on set with n elements

i.e., $2^n \cdot 2^{\frac{n^2-n}{2}} = 2^{\frac{n^2+n}{2}}$

P/13 Let $R = \{(1,2)(2,1)\}$. ~~not~~ (irreflexive)

$R^* = \{(1,1),(2,2)\}$ (not irreflexive)

$\therefore S_1$ is false

Let $(a,c) \in$ transitive closure of R

then

$(a,b) \in R$ $(b,c) \in R$ for some $b \in$ Domain

$\Rightarrow$ $(b,a) \in R$ $(c,b) \in R$

$\Rightarrow$ $(c,a) \in$ transitive closure of R

$\therefore S_2$ is true

P/14 xRy iff $|x-y|<1$

$\forall x$ xRx

$\therefore$ Reflexive

$\forall x \forall y$ $|x-y|<1 \Rightarrow |y-x|<1$

$\therefore$ symmetric

let $a=1$ $b=1.5$ $c=2$

$|a-b| = 0.5$ $|b-c|=0.5$

$\therefore aRb$ bRc

$|a-c|=1$

$\therefore a \not R c$

$\therefore$ not transitive

$\therefore$ not an equivalence relation

$\therefore$ option (a)

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(P/15) No of relations such that

symmetric & Anti symmetric is 2^n

This kind of relation includes on (x, x)

$\therefore$ It is also transitive

$\therefore 2^n$ ways

(P/16) $\forall a \quad \frac{a}{a} = 1 = 2^0$

$\therefore$ reflexive

$\forall a \; \forall b \quad a \neq b$

$\frac{a}{b} = 2^i$

$\therefore \quad i > 0 \quad (\because a \neq b)$

Now $\frac{b}{a} = 2^{-i} \neq 2^i$

$\therefore$ antisymmetric

and not symmetric

$\frac{a}{b} = 2^i \qquad \frac{b}{c} = 2^j$

$\frac{a}{c} = \frac{a}{b} \times \frac{b}{c} = 2^{i+j}$

$\therefore$ transitive

$\therefore$ opt (d)

(P/22) S1: $(R \cap R^{-1}) \subseteq \Delta_A$

if R was symmetric then R^{-1} ~~contains~~ is also symmetric which contains also symmetric pairs

if R is symmetric

$R = R^{-1}$

$\Rightarrow R \cap R^{-1} = R$

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But it is given that

$$(R \cap R^{-1}) \subseteq \Delta_A$$

it means for
$(a,b) \notin R \cap R^{-1}$ $a \neq b$

$\Rightarrow$ (a,b) either belongs to R or R^{-1} but not both
for $a \neq b$

i.e., antisymmetric

$\therefore$ S1 is true

S2: R is transitive

~~RoR~~: Let $(a,c) \in R \circ R$

$\Rightarrow (a,b) \in R$ & $(b,c) \in R$ for some $b \in$ Domain

Since R is transitive
$(a,c) \in R$

$\therefore$ ~~$\forall a \forall c\ ((a,c) \in R \circ R \rightarrow$~~ a

$\therefore \forall a \forall c\ ((a,c) \in R \circ R \rightarrow (a,c) \in R)$

$\therefore R \circ R \subseteq R$

$\therefore$ S2 is true

P/23 $R = \{(a,b) \mid a \text{ divides } b\}$

$R^{-1} = \{(a,b) \mid b \text{ divides } a\}$

Symmetric closure $R \cup R^{-1} = \{(a,b) \mid a \text{ divides } b \text{ (or) } b \text{ divides } a\}$

10/06/20

Equivalence Relation :

A relation R is equivalence relation if it is Reflexive, symmetric, transitive.

Eg: $R = \{(P_1, P_2) \mid P_1, P_2 \text{ born in same month}\}$

P_1 & P_1 born in same month $\therefore$ reflexive

$P_1RP_2 \Rightarrow P_2RP_1$ $\therefore$ symmetric.

$P_1RP_2 \wedge P_2RP_3 \rightarrow P_1RP_3$ $\therefore$ transitive

$\therefore$ equivalence relation

Eg: $R = \{(a,b) \mid a \equiv b \pmod{n}\}$ is equivalen relation

* Equivalence relation creates partition in domain. These partitions are called equivalence classes

Eg: $R = \{(a,b) \mid a \equiv b \pmod{4}\}$

Now this is an equivalence relation and we obtain 4 equivalence classes.

0, 4, 8, 12 ... | 1, 5, 9, ... | 2, 6, 10 ... | 3, 7, 11 ...

→ Every element of same class is related to each other

→ No element is related to an element of different class.

Eg: $R = \{(a,b) \mid a+b \text{ is even}\}$ is an equivalence relation and it forms two equivalence ~~classes~~ classes.

0, 2, 4, 6, 8, ... | 1, 3, 5, 7, ...

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Eg: Let $A = \{0,1,2,3,4,5\}$

Let R be an equivalence relation which partions A into 3 equivalence ~~cla~~ classes

$\{0\}$ $\{1,2\}$ $\{3,4,5\}$.

Find R.

$R = \{(0,0)\ (1,1)\ (2,2)\ (1,2)(2,1)\ (3,3)(4,4)(5,5)\ (3,4)(4,3)\ (3,5)$
$(5,3)\ (4,5)\ (5,4)\}$

* Let an equivalence relation parthioned set A, as

$A_1, A_2 \ldots A_n$, then

* $A_1 \cup A_2 \cup \cdots \cup A_n = A$
* $A_1 \cap A_2 \cap \cdots \cap A_n = \emptyset$
* $x \not R y \quad \forall x \in A_i \quad \forall y \in A_j \quad i \neq j$

A partion on a set A is defined as $A_1, A_2 \ldots A_n$ such that
$A_1 \cup A_2 \cup \ldots \cup A_n = A$
$A_1 \cap A_2 \cap \ldots \cap A_n = \phi$
Thus equivalent classes forms a partition

* An equivalence class A_i is represent using one of its element

A_i is represent as

$[a]_R$ (or) $[a]$ for $a \in A_i$

$\boxed{[a]_R = \{s \mid (a,s) \in R\}}$

Eg.

$R = \{(a,b) \mid a \equiv b \pmod 4\}$

The equivalence classes are

$[0] = [0, 4, 8, 12 \ldots\}$
$[1] = [1, 5, 9 \ldots\}$
$[2] = \{2, 6, 10 \ldots\}$
$[3] = \{3, 7, 11 \ldots\}$

In an equivalent relation if aRb then we say a is equivalent to b. It is denoted as $a \sim b$

In an equivalence relation the below 3 stmts are ~~true~~ same
(i) aRb (ii) $[a]=[b]$ (iii) $[a] \cap [b] \neq 0$

Refinement :

→ Partition P_1 is called refinement of partition P_2 if every set P_1 is a subset one of set in P_2

Eg:

mod 6 (R_1)

$[0]_6 = \{0, 6, 12 \ldots\}$

$[1]_6 = \{1, 7, 13 \ldots\}$

$[2]_6 = \{2, 8, 14 \ldots\}$

$[3]_6 = \{3, 9, 15 \ldots\}$

$[4]_6 = \{4, 10, 16 \ldots\}$

$[5]_6 = \{5, 11, 17 \ldots\}$

mod 3 (R_2)

$[0]_3 = \{0, 3, 6, 9 \ldots\}$

$[1]_3 = \{1, 4, 7, 10 \ldots\}$

$[2]_3 = \{2, 5, 8, 11 \ldots\}$

Here

$[0]_6 \subseteq [0]_3$ $\quad [1]_6 \subseteq [1]_3$ $\quad [2]_6 \subseteq [2]_3$

$[3]_6 \subseteq [0]_3$ $\quad [4]_6 \subseteq [1]_3$ $\quad [5]_6 \subseteq [2]_3$

$\therefore$ we say mod 6 is a refinement of mod 3

~~R_1 is a refine~~

i.e., partition formed from congruence class mod 6 is a refinement of partition formed from congruence class mod 3.

* If R_1 & R_2 are two equivalence relations

$R_1 \cup R_1$ is ~~equival~~ not an equivalence relation ($\because$ union of 2 transitive relation is not transitive)

$R_1 \cap R_2$ is an equivalence relation

Finding no of equivalence relations possible:

→ Here we can find no of way we can partition. This will be equal to no of equivalent classes

Let $A = \{1, 2, 3\}$

different types of ~~equiva~~ partition are

(123)

(12) (3)

(13) (2)

23   1

1   2   3

$\therefore$ 5 equivalence relations.

This no of equivalence relations is called bell number.

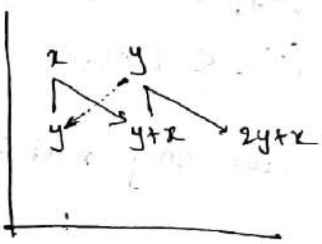
$B_3 = 5$

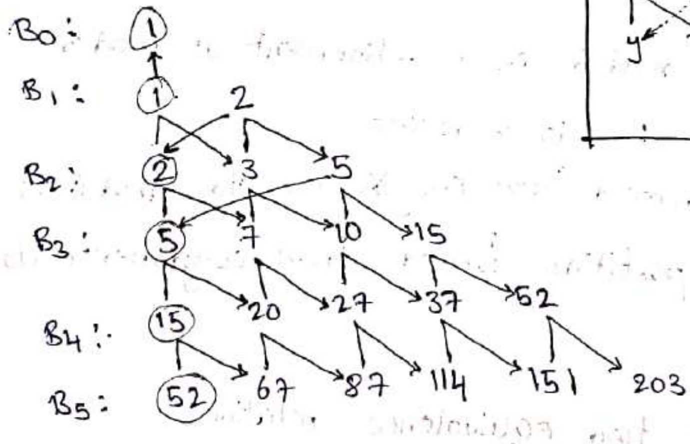
Bell Number :

Bell number

B_n = no of possible partitions of a set with n elements.

finding bell number (shortcut)





Thus from figure

$B_0 - 1 \quad B_1 - 1 \quad B_2 - 2 \quad B_3 = 5$

$B_4 - 15 \quad B_5 = 52 \quad B_6 - 203$

Proof (This is not needed for gate) :

We calculate no of equivalence relations of $A = \{a, b, c, d\}$ by considering no of onto ~~fun~~ function from A to $B = \{1, 2, 3\}$

This will give us no of equivalence relation we 3 partions.

consider a mapping to 1 means

a falls in first partition

slly 2 & 3 are 2nd & 3rd partitions

But we get duplicates as shown below.

$[ab]_1 \quad [c]_2 \quad [d]_3$
$[ab]_1 \quad [d]_2 \quad [c]_3$
$[c]_1 \quad [ab]_2 \quad [d]_3$
$[c]_1 \quad [d]_2 \quad [ab]_3$
$[d]_1 \quad [ab]_2 \quad [c]_3$
$[d]_1 \quad [c]_2 \quad [ab]_3$

These are calculated as 6 onto function. but they actual mean the same. So we divide by 3!

$\therefore$ no of equivalence relation on $A = \{a, b, c, d\}$ with 3 partition is

$$\frac{\text{no of onto function for } A \text{ to } B(\text{size } 3)}{3!}$$

$$= \frac{36}{6} = 6$$

(This is called sterling 2nd kind number)

Sterling 2nd kind number

$$S(m, n) = \frac{\text{no of onto function from } A \text{ to } B}{n!} \quad (|A| = m \quad |B| = n)$$

$$= \frac{1}{n!} \sum_{i=0}^{n} (-1)^i \, {}^nC_i \, (n-i)^m$$

* To calculate total number of equivalence relation on A, we need to find

no of equivalence relation with 4 partitions $\quad S(4,4)$
$+ \quad\quad +$
no of equivalence relations with 3 partition i.e., $\quad S(4,3)$
$+ \quad\quad +$
- - - - - - with 2 partitions $\quad S(4,2)$
$+ \quad\quad +$
- - - - - - with 1 partition $\quad S(4,1)$

$$S(4,2) = \frac{1}{2!} \sum_{i=0}^{2} (-1)^i \, {}^nC_i \, (n-i)^m$$

$$= \frac{1}{2!} \left(2^4 - 2C_1 (2-1)^4 \right)$$

$$= \frac{16-2}{2} = 7$$

Slly $S(4,1) = 1$

$S(4,4) = 1$

$\therefore$ no of equivalence relations on $A = \{a, b, c, d\}$ is

$$S(4,1) + S(4,2) + S(4,3) + S(4,4)$$

$$1 + 7 + 6 + 1$$

$$= 15$$

$\therefore$ No of equivalence relations on ~~S~~ a set with n elements is

Bell number $B_n = S(n,1) + S(n,2) + \dots + S(n,n)$

$$\boxed{B_n = \sum_{k=1}^{n} S(n,k)}$$

where

$$\boxed{S(m,n) = \frac{1}{n!} \sum_{i=0}^{n} (-1)^i \, {}^nC_i \, (n-i)^m}$$

Partial Order Relation:

- The relation which is reflexive
  antisymmetric
  transitive

  is called a partial order relation

Eg: $R_1 = \{(a,b) \mid a \le b\}$ is a partial order relation

$R_2 = \{(a,b) \mid a < b\}$ is not a partial order relation.

$R_3 = \{(a,b) \mid a \text{ divides } b\}$ on domain Z^+ or $Z - \{0\}$

$a|a$ $\therefore$ reflexive

$a|b \wedge b|a \rightarrow a=b$ $\therefore$ antisymmetric

$a|b \wedge b|c \rightarrow a|c$ $\therefore$ transitive

$\therefore$ partial order relation

R_3 on Z is not a partial order relation

because

for 0

$0/0$ is not true

$\therefore$ not reflexive

$\therefore$ not a partial order relation.

> If (S,R) is a poset, (S,R^{-1}) is called dual of the poset. (S,R^{-1}) is also a poset

* If R is a relation and bRa or aRb then we say a,b are comparable

* $R = \{(a,b) \mid a \text{ divides } b\}$ on Z

$3 \not R 2$ $\quad 2 \not R 3$

$\therefore 3, 2$ are not comparable.

Total ordered set or Linear ordered set:

* If relation R is such that every ~~per~~ pair of elements are comparable, then it is called total ordered ~~set~~ ~~so~~ set.

Eg: $R = \{(a,b) \mid a \le b\}$ on Z

This relation is called total ordered ~~rest~~ set relation.

* Some elements are comparable ⇒ partial order set and relation is called partial order relation.

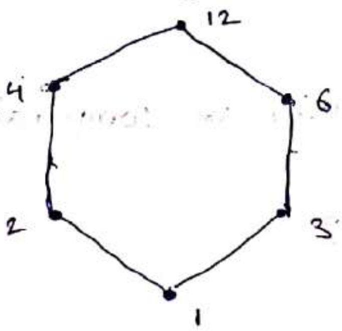
* Total ~~ordered~~ ordered set is called toset
  Partial ordered set is called poset.
  Every toset is poset.

* If R_1, R_2 are two partial order relations,
  $R_1 \cup R_2$ is not ~~a part~~ need to be a partial order relation
  $R_1 \cap R_2$ is a partial order relation.

Hasse Diagrams:

Let $A = \{1, 2, 3, 4, 6, 12\}$
let R be a relation on A
$R = \{(a, b) \mid a \text{ divides } b\}$

* This represented by Hasse diagram



$1 R 2$ is represented as

Reflexivity is not represent in hasse diagram
transitivity is not represented in hasse diagram.
However, we can obtain transitive pairs from the diagram.

Let $(S, \leq)$ be a poset, we say $y \in S$ cover an element $x \in S$ if $x < y$ and there is no element $z \in S$ such that $x < z < y$. The set of pairs $x < y$ such that y covers x is called the covering relation of $(S, \leq)$. Infact, applying reflexive-transitive closure of its covering relation, we can obtain the corresponding Poset. The edges in Hasse diagram are elements from the covering relation

* The set of Divisors of a number n is represented as D_n.

Eg: $D_{12} = \{1, 2, 3, 4, 6, 12\}$

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→ ~~Every relation Dn~~

→ $(D_n, /)$ is always a poset.

(Q32) $R = \{ (f,g) \mid$ for some $c \in Z$, $\forall x \in Z$, $f(x) - g(x) = c \}$

where f, g are functions.

is above realation an equivalence relation?

Sol:

In other

$(f,g) \in R$, if $\forall x \in Z$ ~~keep~~ difference b/w $f(x)$ and $g(x)$

is ~~constant~~ integer

let $(g,h) \in R$

$\therefore$ difference b/w $g(x)$ & $h(x)$ is ~~constant~~ integer for all x

$\Rightarrow$ difference b/w $f(x)$ & $h(x)$ is ~~constant~~ integer for all x

$\therefore$ transitive.

Also it is clearly symmetric & reflexive

$\therefore$ Equivalence relation.

* In a poset if aRb we denote it as $a \leq b$

NOTE that '$\leq$' doesn't mean ~~less~~ logical comparion $\leq$

The notation $a < b$ is used when $a \leq b$ but $a \neq b$

* If A is a set and R is partial order set, then set A along with R is called poset. It is denoted as (A, R)

P/9

1 5, 2,3 6 4

$\therefore$ 3 equivalence class

P/10 R is an antisymmetric relation

i.e., $(aRb \wedge bRa) \to a = b$

$R \cap S = \{ (a,b) \mid (a,b) \in R \ \& \ (a,b) \in S \}$

$\downarrow$

$\therefore$ antisymmetry holds

$R - S = R \cap \bar{S}$

$\therefore$ antisymmetric

R^{-1} is also antisymmetric

Let $A = \{1, 2\}$

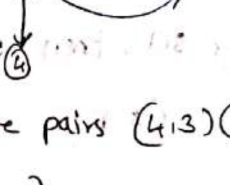
$R = \{\}$ is antisymmetric

$\bar{R} = A \times A$ is not antisymmetric

P/11 find symmetric closure

$\{ (1,1) \ (2,2) \ (4,4) \ (2,3) \ (3,2) \ (4,2) \ (2,4) \}$

finding transitive closure



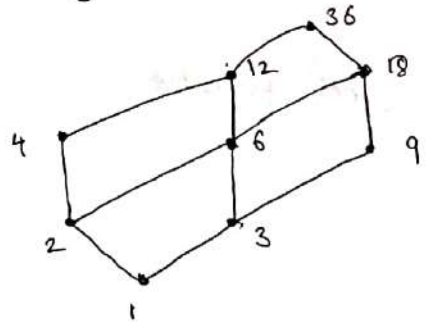
$\to$ Here we can also obtain (3,3)

So the pairs (4,3) (3,4) can be added

$\therefore \{ (1,1) \ (2,2) \ (3,3) \ (4,4) \ (2,3) (3,2) \ (4,2) (2,4) \ (4,3) (3,4) \}$

$\therefore$ ~~4 pairs~~ & 10 pairs

(P/12) ~~D~~ $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$



$\therefore$ 12 edges

(P/17) ~~Let~~ n is no of elements

no of pairs in equivalence relation is

no of pairs due to reflexivity

+

no of pairs due to symmetricity

+

no of pairs due to transitivity

no of reflexive pairs = n

~~no of symm~~ no of rest of pairs = even (cuz they all follow symmetry)

$\therefore$ if n is odd, then no of pairs = odd

even - - - - even

$\therefore$ Both S_1 & S_2 are true

(P/18) S_1: Reflexive $\cup$ anything is reflexive

S_2: if $(a,b) \in R \cup S$

$(a,b) \in R \vee (a,b) \in S$

$\Rightarrow (b,a) \in R \vee (b,a) \in S$

$\Rightarrow (b,a) \in R \cup S$

$\therefore$ Symmetric

S3: $\{(1,2)\} \cup \{(2,3)\}$

transitive   transitive

$\cancel{\{(1,3)\}}$ $\{(1,2)\ (2,3)\}$ is not transitive

$\therefore S_1$ & S_2

P/19

$P_1 \cup P_2 \cup \cdots P_n = S$

$P_1 \cap P_2 \cap \cdots P_n = \phi$

P/20

Same as P/19

P/21

$\{\{1,2,3\}\ \{4,5\}\ \{6\}\}$

no of equivalence relations possible on 3 elements $B_3 = 5$

no of equivalence relations on 2 elements $B_2 = 2$

$B_1 = 1$

$\therefore$ no of refinements $= 5 \times 2 \times 1$

$= 10$

P/24 ~~We need to find~~

R

12   34   5

S

1   2   3   4,5

$\{\{1,2\}\ \{3,4\}\ \{5\}\}$ $\{\{1\}\ \{2\}\ \{3\}\ \{4,5\}\}$

to find smallest equivalence relation which contains R & S, that relation should have both R & S as its refinements

$\therefore \quad \{\{1,2\}\ \{3,4,5\}\}$

P/25 $[x] = [y] \Rightarrow xRy$

$[x] \cap [y] = \{\} \Rightarrow x \not R y$

$\therefore$ either of them must be true

P/26 It means we need to satisfy both antisymmetric and symmetric property at the same time.

It possible only when aRb iff a=b

$\therefore$ $R = \{(1,1)(2,2)(3,3)\}$ is the only relation

P/27

$\forall x.\ x-x=0$ is even

$\therefore$ reflexive

$\forall x \in Z\ \forall y \in Z$

if $x-y$ = even then $y-x$ is also even

$\therefore$ symmetric

$\forall x, \forall y, \forall z \in Z$

$x-y$ = even & $y-z$ = even

$x-y+y-z$ = even + even

$x-z$ = even

i.e., transitive

$\therefore$ Equivalence

$\therefore$ only S1

P/28 not reflexive $\Rightarrow \exists x\ (x,x) \notin R$

not irreflexive $\Rightarrow \exists x\ (x,x) \in R$

$\therefore$ no of way for choosing (x,x) in Relation is $2^n - 2$

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Since relation is symmetric

no of way for rest of the elements is $2^{\frac{n^2-n}{2}}$

$\therefore$ total req no of ways $= 2^{\cancel{n}} \cdot (2^n - 2)\, 2^{\frac{n^2-n}{2}}$

$= (8-2)\, 2^3$

$= 6 \times 8 = 48$

(P/29) Δ_A is reflexive
antisymmetric
transitive
$\therefore$ partial order

for Δ_A $\forall a,b$ $(a \neq b \Rightarrow a \not R b)$
$\therefore$ not total order

Δ_A is reflexive
Symmetric
transitive
$\therefore$ equivalence

$\therefore S_1 \& S_3$

(P/30) Reflexive $\cup$ Reflexive = Reflexive
Reflexive $\cap$ Reflexive

Antisymmetric $\cup$ Antisymmetric may or may not be antisymmetric

Antisymmetric $\cap$ Antisymmetric = Antisymmetric

transitive $\cup$ transitive may or may not be transitive

transitive $\cap$ transitive is transitive

$\therefore R \cup S$ need not to be partial order

$R \cap S$ is partial order.

$\therefore$ opt (b)

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(P/31) (1,1) (2,2) must be included to be reflexive

if we include (1,2) we must include (2,1)

as (1,1) & (2,2) are present transitivity holds

$\therefore$ (1,2) $\notin$ R (2,1) $\notin$ R

$\therefore$ ~~the only relation is~~ {(1,1) (2,2)}

$\therefore$ Now this is also transitive

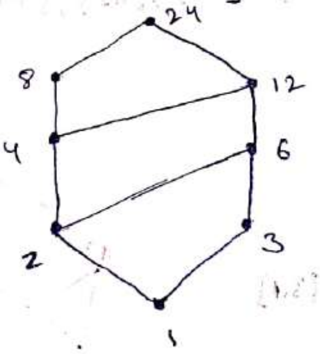
~~To P, not transitive we need~~ to

$\therefore$ no of relations possible = 0.

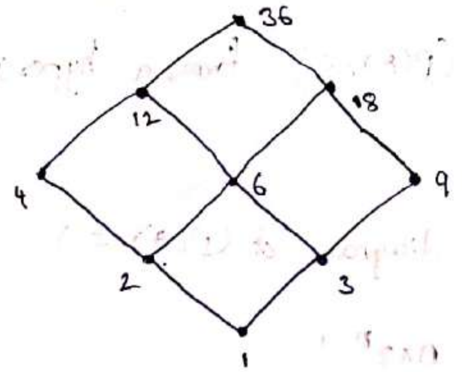
11/06/20

$\rightarrow (D_{24}, |)$

$D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$

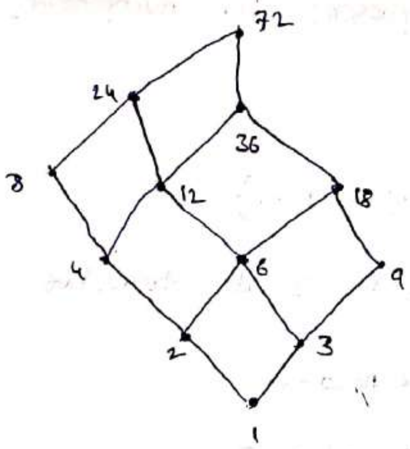


$\rightarrow (D_{36}, |)$ $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$



$\rightarrow (D_{72}, \mid)$

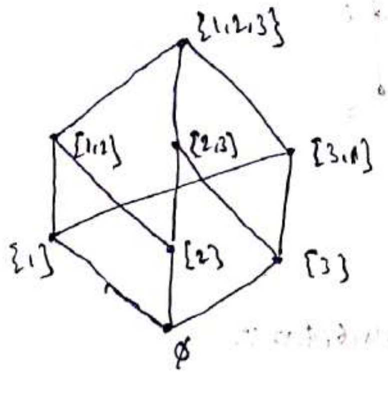
$D_{72} = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72\}$

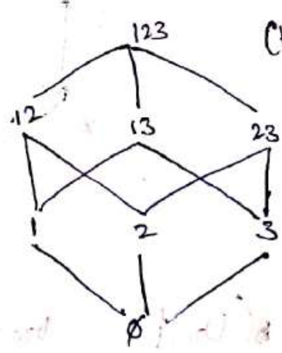


$\rightarrow A = \{1, 2, 3\}$

draw Hasse diagram $(P(A), \subseteq)$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$





(Hypercube Q_3)

The hasse diagram of any $(P(A), \subseteq)$ forms a hypercube.

Thus if $|A| = n$

no of edges in hasse diagram of $(P(A), \subseteq)$

is $\dfrac{2^n \times n}{2} = n \times 2^{n-1}$

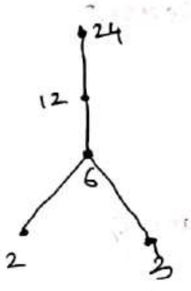
for $n = 3$

$3 \times 2^{3-1} = 12$ edges.

Q33
G-96

Let $X = \{2, 3, 6, 12, 24\}$. Let $\le$ be the partial order defined by $x \le y$ if x divides y. The number of edges in Hasse diagram of $(X, \le)$ is

a) 3 b) 4 c) 9 d) None



∴ ~~5 edges~~ ∴ 4 edges

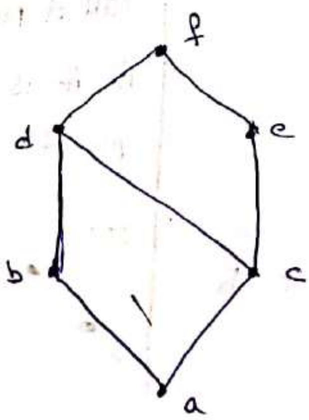
* Hasse diagram of a toset is a chain.

Eg.: $(\{1,2,3,4\}, \le)$

Greatest element (or) Maximum element:

x is called greatest element ~~if $a \le x$~~ of poset $(A, \le)$ if $a \le x \ \forall a \in A$

Eg:



Here f is the greatest element

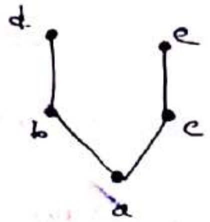
Maximal element:

An element of poset is called maximal element if it is less than no other element

Eg:

e, d are maximal elements

Eg:



This diagram has no greatest element

~~case~~ for d: $c \nleq d$ for e: $d \nleq e$

$\therefore$ no greatest element

__Note:__

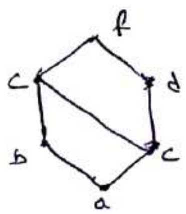
→ Thus, it is not neccessary that a poset has greatest element. But if there is a greatest element then it is unique.

Least element or Minimimum element:

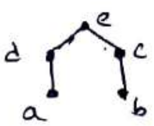
In a poset (A, R)

$x \in A$ is ~~a~~ called a least element if ~~$a \leq x \ \forall a \in A$~~ $x \leq a \ \forall a \in A$

Eg:



Here 'a' is the least element.



Here we don't have any least element.

__Note:__

→ Least element need not to exist every time. If exists, it is unique.

Upper bound (UB)

If (A, R) is poset and $B \subseteq A$

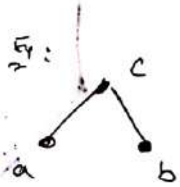
$x \in A$ ~~a~~ is called upper bound of B

$\forall b \in B \ \ b \leq x$

Minimal element

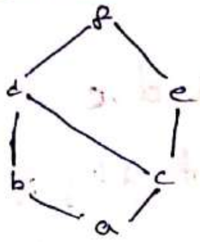
An element of a poset is called minimal element if it is greater than no other element.

Eg:



a, b are minimal elements

Eg:



. Let $B = \{b, c\}$

for d

$b \leq d$ & $c \leq d$

$\therefore$ d is upper bound of B

for f

$b \leq f$ & $c \leq f$

$\therefore$ f is upper bound of B

$b \nleq e$ & $c \leq e$

$\therefore$ e is not upper bound of B

Let $B = \{a, b\}$

b is also upperbound of B

d, f are also upper bounds of B

Every finite non-empty poset has atleast one minimal and atleast one maximal element

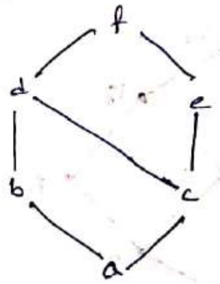
Lower Bound (LB)

If (A, R) is a poset and $B \subseteq A$

$x \in A$ is called lower bound of B if

$\forall b \in B \quad x \leq b$

Eg:



Lower bounds of $\{d, c\}$ is c, a

Lower bounds of $\{f, e\}$ are e, c, a

Greatest Lower Bound (GLB):

(A, R) is poset & $B \subseteq A$

let X be ~~lower bo~~ set of lower bounds of B

then GLB, g is $g \in B$ & $\forall x \in X \quad x \leq g$

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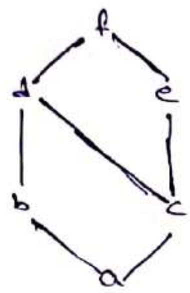
Eg:



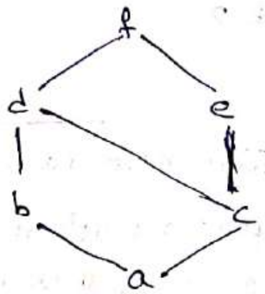
Lower bounds of {d,f} are d, b, a, c

But the greatest lowerbound of {d,f} is d

Least Upper Bound: (LUB)

it is lowest among all the upper bounds.

Eg:



for {b,c}

~~lower~~ upper bounds are d, f

The least upper bound is d.

for {a,b}

upper bounds are b, d, f

the least upper bound is b

Eg: In $(D_{36}, |)$

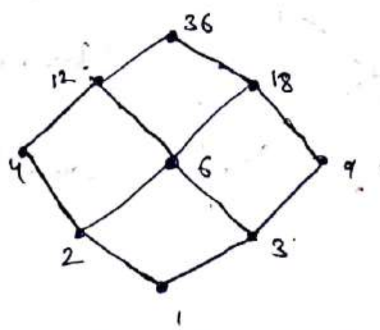
find a) glb(3,9)

b) glb(4,12)

c) lub(3,9)

b) lub(4,12)

$D_{36} = \{1,2,3,4,6,9,12,18,36\}$



a) ~~3|9 ∴ glb(3,9) = 9~~

a) glb(3,9) = 9

b) glb(4,12) = 12

c) lub(3,9) = 3

d) glb(4,12) = 4

Since the relation is divides
lub is nothing but L.C.M
glb is nothing but G.C.D

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→ Let ~~$(A, \subseteq)$~~ A be a set and

$(P(A), \subseteq)$ be a powerset.

~~let~~ ~~$p \in P(A)$~~ Let $x \in P(A)$, $y \in P(A)$

$glb(x,y)$ is $x \cap y$

$lub(x,y)$ is $x \cup y$

slly $P_1 \in P(A)$, $P_2 \in P(A) \ldots P_n \in P(A)$

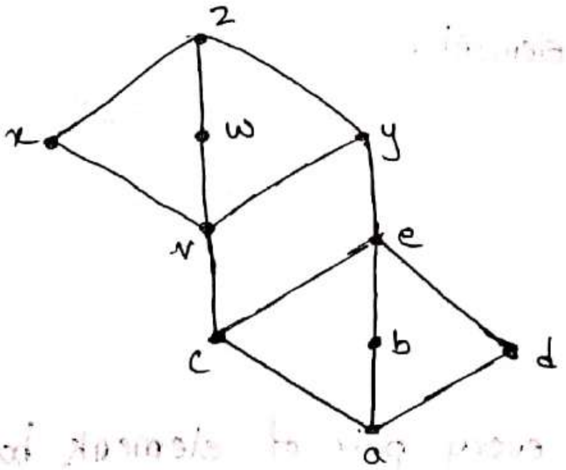
$glb(P_1, P_2, \ldots P_n) = P_1 \cap P_2 \cap \ldots \cap P_n$

$lub(P_1, P_2 \ldots P_n) = P_1 \cup P_2 \cup \ldots \cup P_n$.

→ Thus glb is also represented by operator $\wedge$ (or) $\cdot$

lub is also represented by operator $\vee$ (or) $+$

Q34) Consider below hasse diagram



Find a) $glb(b,c)$ b) $glb(b,w)$ c) $glb(e,x)$

d) $lub(c,b)$ e) $lub(d,x)$ f) $lub(c,e)$ g) $lub(a,x)$

a) $glb(b,c) = a$

b) $glb(b,w) = a$

c) $glb(e,x) = c$

d) $lub(c,b) = e$

e) $lub(d,x) = z$

f) $lub(c,e) = e$

g) $lub(a,x) = x$

<u>Note:</u>

→ If $a \leq b$

$lub(a,b) = b$

$glb(a,b) = a$

→ lub(greatest element , a) = greatest element

glb (greatest element , a) = a

lub(least element , a) = a

glb (least element , a) = least element

where a is any element.

<u>12/06/20</u>

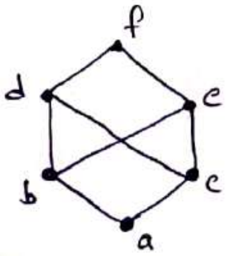
<u>Lattice (L):</u>

Lattice is a poset in which every pair of elements has both glb and lub

$lub(a,b) = a \vee b$ is called join of a,b

$glb(a,b) = a \wedge b$ is called meet of a,b

Eg:



~~This is a lattice because lub and glb exists~~ for ~~every pair of elements~~

upperbounds of b, c are d, e, f

but. {d, e, f} has not least element

Thus {b, c} has no lub

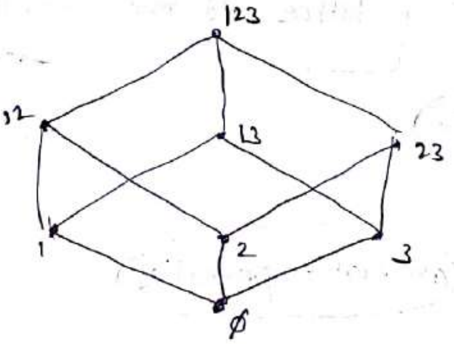
∴ This is not a lattice.

A Lattice is generally represented as {L, ∨, ∧}

Eg: $A = \{1, 2, 3\}$

$(P(A), \subseteq)$ is a poset.

find whether the poset is lattice or not.



This is a lattice because every pair has lub & glb

Eg: Determine whether (Dn, |) is lattice or not.

sol:

Every finite lattice has a greatest element and a least element

we know that

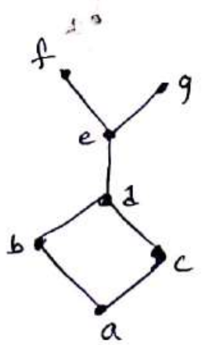
glb is gcd

lub is lcm

gcd, lcm always exists and hence it is always a lattice.

Eg:

Lattice



not lattice

lub(f, g) not exists

Lattice

Note:

→ $a \vee b = b \Leftrightarrow a \leq b$

$lub(a,b) = b$

→ $a \wedge b = a \Leftrightarrow a \leq b$

→ $a \wedge b = a \Leftrightarrow a \vee b = b$

Join Semi Lattice:

A poset $[A; R]$ in which each pair of element a & b of A have a least upper bound is called join semi lattice.

Meet Semi Lattice:

A poset $[A; R]$ in which each pair of element a & b of A have a glb is called meet semi lattice.

Thus a lattice is both join semi lattice and meet semi lattice

Properties of Lattices:

Let L be a lattice

1) $a \vee a = a$
$a \wedge a = a$ Idempotent

2) $a \vee b = b \vee a$
$a \wedge b = b \wedge a$ Commutative

3) $a \vee (b \vee c) = (a \vee b) \vee c$
$a \wedge (b \wedge c) = (a \wedge b) \wedge c$ associative properties

4) $a \vee (a \wedge b) = a$
$a \wedge (a \vee b) = a$ Absorption law

5) If a, b, c are elements in L

(i) $a \leq b \Rightarrow a \vee c \leq b \vee c$
$a \leq b \Rightarrow a \wedge c \leq b \wedge c$ → verify with this example

(ii) $(a \leq c)$ and $(b \leq c)$
~~then~~ iff
$(a \vee b) \leq c$

(iii) $c \leq a$ and $c \leq b$
~~then~~ iff
$c \leq (a \wedge b)$

(iv) If $a \leq b$ and $c \leq d$

$a \vee c \leq b \vee d$

$a \wedge c \leq b \wedge d$

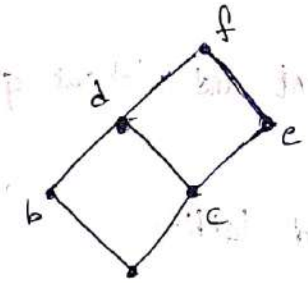
Sublattice:

B is called sublattice of a lattice A when it satisfies

(i) $B \subseteq A$

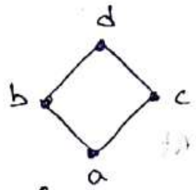
(ii) B is also a lattice.

Eg:



A is a lattice

consider



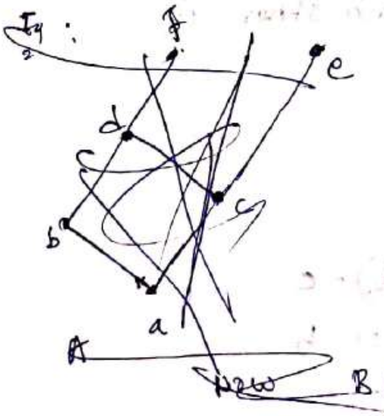
B

every pair in B has lub & glb

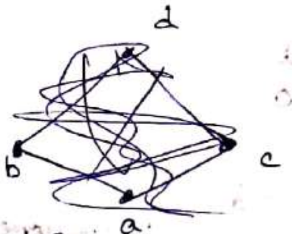
$\therefore$ B and $B \subseteq A$.

$\therefore$ B is sublattice of A.

Eg:



A



B

~~Now $B \subseteq A$~~

~~for every pair in B, lub & glb exists~~

Bounded lattice:

Every lattice which has a greatest element (1) and a least element (0) is called bounded lattice.

Eg:  is a bounded lattice

Note:

* $(Z^+, |)$ is a lattice with least element 1 and greatest element undefined.
* $(Z, \leq)$ is a lattice without least element and without greatest element.

* Thus, every finite lattice is a bounded lattice.

$\forall a \in A$; $0 \leq a \leq 1$

0 - least element

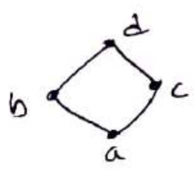
1 - greatest element

Complement Lattice:

Lattice A is said to be complement lattice if every element has a complement.

a & b are said to be complements of each other if

$a \vee b = 1$
$a \wedge b = 0$

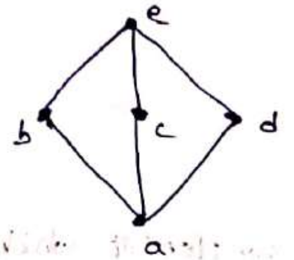


$b \vee c = d$
$b \wedge c = a$ $\therefore$ Complement(b) = c
complement(c) = b

$a \vee d = d$
$a \wedge d = a$ $\bar{a} = d$, $\bar{d} = a$

$\therefore$ The above lattice is a complement lattice

Eg:



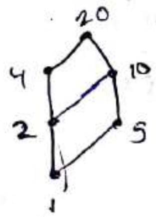
$a' = e$ $e' = a$

$b' = e$ $c' = b$, $b' = d$, $d' = b$, $d' = c$, $c' = d$

∴ Bounded lattices.

Eg: $(D_{20}, |)$

$D_{20} = \{1, 2, 4, 5, 10, 20\}$



~~2' = 5, 5' = 2~~

~~4' = 10, 10' = 4~~

~~1' = 20, 20' = 1~~

~~∴ Complement lattice~~

$1' = 20$, $20' = 1$

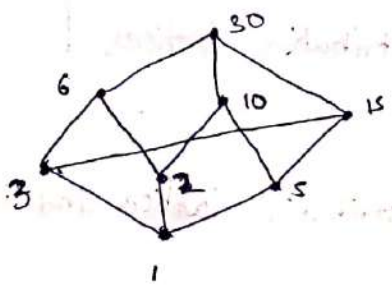
$4' = 5$, $5' = 4$

But we don't have complement for 2 & 10

∴ not a complement lattice.

Eg: $(D_{30}, |)$

~~$D_{12} = \{1, 2, 3, 4, 6, 12\}$~~ $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$



$1' = 30$

$3' = 10$

$5' = 6$

$2' = 15$

∴ Complement lattice.

Eg: $(P(S), \subseteq)$

Since glb is ~~union~~ intersection and lub is union

For every sub subset of A (say B), we can find its complement which is ~~B-A~~ A-B

$\cancel{A \cup A =}$ $\cancel{B \cup (B-A)}$ $B \cup (A-B) = A$

$B \cap (A-B) = \emptyset$

∴ Complement lattice.

Distributive lattice:

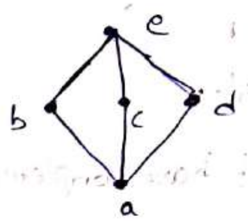
A lattice L is called distributive lattice, if

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

are satisfied $\forall a, b, c \in L$

Eg:



Now consider

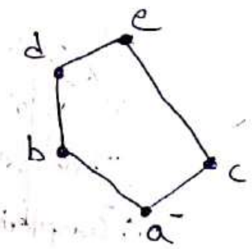
$b \vee (c \wedge d) = (b \vee c) \wedge (b \vee d)$

$b \vee (a) = e \wedge e$

$b = e$

∴ not true

∴ not a distributive lattices



$b \vee (c \wedge d) = (b \vee c) \wedge (b \vee d)$

$b \vee (a) = e \wedge d$

$b = d$

∴ not a distributive lattice

Note:

If L is a bounded distributive lattice and if complement exists the it must be unique.

Eg: $[D_n ; |]$ is a distributive lattice.

Eg: $(P(S), \subseteq)$

In boolean algebra every element has exactly one complement

~~Here every element has a unique complement~~

∴ ~~distributive~~

$glb \rightarrow \cap \quad lub \rightarrow \cup$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

∴ It is a distributive lattice.

In a distributive lattice each element can have atmost one complement

Thus if you find an element with more than one complement then it is not a distributive lattice

But this is only a sufficient condition.

Boolean Algebra:

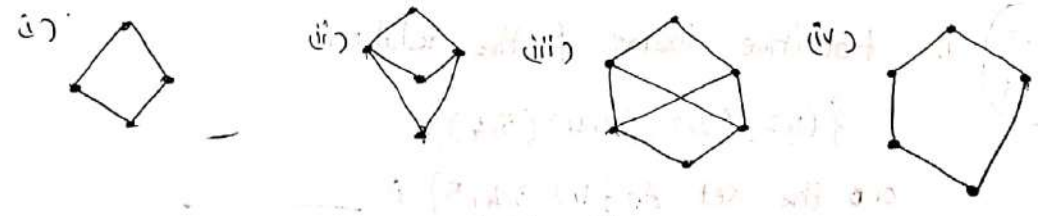
A lattice which is both distributive and complement is known as boolean algebra.

Boolean lattice contains 2^n elements for $n \geq 0$. This lattice is isomorphic to poset $[P(S); \subseteq]$ where S is a set with n element. This lattice is a hypercube Q_n

→ It is called boolean algebra because when it is both distributive & complement, it satisfies all the properties of a boolean algebra.

→ If n is a square free number, $[D_n ; |]$ is a boolean algebra. For $x \in D_n$ $\bar{x} = \frac{n}{x}$

Q35 (9-08) Consider the following hasse diagrams.



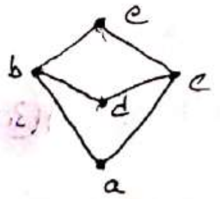
which of the above are lattices?

a) (i) and (iv) b) (ii) & (iii) c) (iii) only d) (i) (ii) and (iv)

Sol:

(i) is clearly a lattice

(ii)

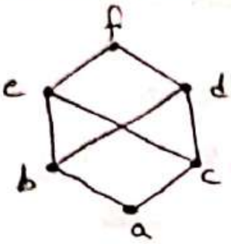


lower bounds of $\{d, a\} = \phi$

∴ d, a has not glb

∴ not a lattice

also glb(b, c) doesn't exist.

(iii) 

upper bounds of $\{b, c\}$: c, d, f

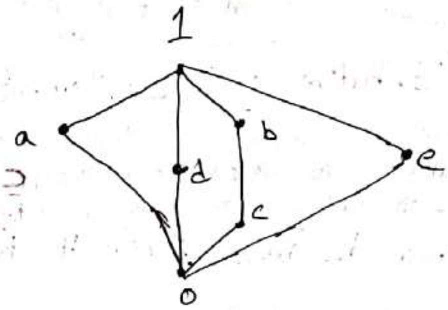
but (b, c) has no lub

$\therefore$ not a lattice.

(iv) is clearly a lattice

$\therefore$ opt (a)

(Q36 / G-88) the complement(s) of the element 'a' in the lattice shown in the fig is (are):



$a \vee e = 1$	$a \vee d = 1$	$a \vee b = 1$	$a \vee c = 1$
$a \wedge e = 0$	$a \wedge d = 0$	$a \wedge b = 0$	$a \wedge c = 0$

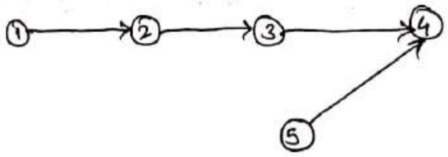
$\therefore$ Complements of 'a' are ; b, c, d, e

(Q37 / G-89) The transitive closure of the relation

$\{(1,2)(2,3)(3,4)(5,4)\}$.

on the set $A = \{1,2,3,4,5\}$ is ______.

sol :



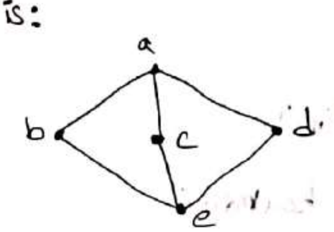
1:			
(1,2)	(2,3)	(3,4)	(5,4)
(1,3)	(2,4)		
(1,4)			

$\therefore$ transitive closure is

$\{(1,2)(1,3)(1,4)(2,3)(2,4)(3,4)(5,4)\}$

Q38 / G-05 The following is the hasse diagram of the poset $[\{a,b,c,d,e\}, <]$

The poset is:



a) not a lattice

b) a lattice but not a distributive lattice

c) a distributive lattice but not a boolean algebra

d) a boolean algebra

Sol :

Every pair has lub & glb

$\therefore$ lattice

Consider

$b \vee (c \wedge d) = (b \vee c) \wedge (b \vee d)$

$b \vee (e) = a \wedge a$

$b = a$ (false)

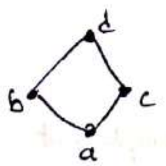
$\therefore$ not a distributive lattice

$\therefore$ opt (b)

Note:

→ Sublattice of a complement lattice need not to be a complement

proof:

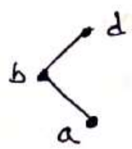


is complement lattice in whice

$a' = d$ & $d' = a$

$b' = c$ & $c' = b$

Now



is a sublattice in which 'b' has no complement.

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→ Sublattice of a bounded lattice is also bounded

proof:

Bounded lattice is finite

∴ Its sublattice is also finite

and every finite lattice is bounded

∴ The sublattice is also bounded

→ Any linear order is a distributive lattice

proof:

let a, b, c belong to the poset.

let $a \leq b \leq c$

Now

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$a \vee (b) = b \wedge c$

$b = b$

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

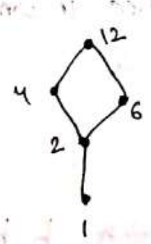
$a \wedge c = a \vee a$

$a = a$

∴ It is a distributive lattice.

P/32 Given $(D_{12} ; |)$

$(D_n ; |)$ is always a distributive lattice



2 has not complement

Lattice is bounded.

∴ opt (b)

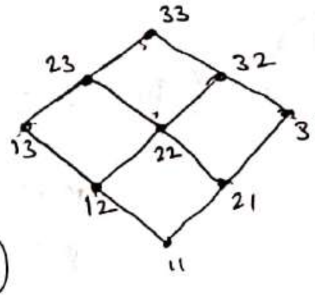
(P/33) Consider $(a,b) \in A \times A$

$(a,b)\ R\ (a,b) \Leftrightarrow (a \le a \wedge b \le b)$

this is true and hence reflexive

It is also clear that it is antisymmetric & transitive

$\therefore$ partial order.



Also for (a,b) & $(c,d) \in A \times A$

every pair has lub $= (\max(a,c), \max(b,d))$

glb $= (\min(a,c), \min(b,d))$

Thus it is a lattice

$\therefore$ opt (c)

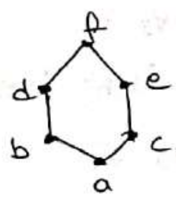
(P/34) a) ~~$2 \vee 18$ = lcm(2,18) = 18~~

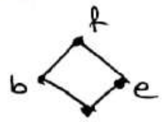
~~$\therefore$ True~~

~~b)~~

Note:

For lattic





is not a sublattice

(P/34)

a) $2 \vee 18 = \text{lcm}(2,18) = 18 \neq 36$

$\therefore$ false

b) $3 \vee 12 = \text{lcm}(3,12) = 12 \neq 36$

$\therefore$ false

c) $4 \vee 9 = \text{lcm}(4,9) = 36$

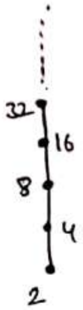
$4 \wedge 9 = \gcd(4,9) = 1$

$\therefore$ True

d) $6 \vee 1 = 6$

$6 \wedge 1 = 1$

P/35



It is a distributive lattice

Since it is infinite, we don't have greatest element. Hence not a bounded lattice.

P/36

$(P(A), \subseteq)$ is the poset with

- meet as intersection and
- join as union.

$\therefore$ $B = \{2, 3, 5, 7\}$

$\bar{B} = A - B = \{1, 4, 6, 8, 9, 10\}$

P/37 Every non-empty subset of S has a minimum element

$\Rightarrow$ for every subset of 2 elements we have minimum element

i.e, $\forall a, \forall b$

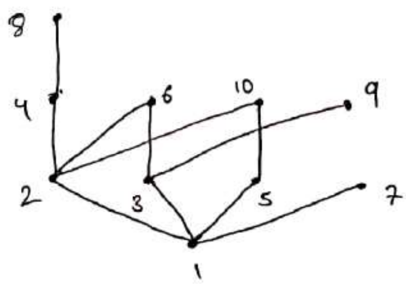
$\{a, b\}$ has minimum

i.e., $\forall a$ $\forall b$

a & b are comparable

i.e, total ordered set

P/38



$\therefore$ no of edges $= 11$

P/39 Given $R \cup R^{-1} = A \times A$

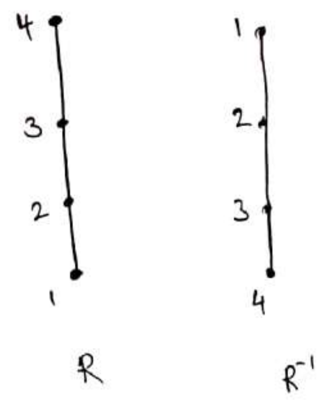
Given R is partial ordering

$\therefore \forall (a,b) \in R \Rightarrow \cancel{(a,b) \notin R}\ (b,a) \notin R$

Thus $(b,a) \in R^{-1}$

Since $R \cup R^{-1} = A \times A$

we can say the partial ordering is a toset



clearly $[A : R]$ is a distributive lattice

Complement doesnot exist for 2,3

$\therefore$ not a Complemented lattice.

P/40 • Distributive $\Rightarrow$ atmost 1 complement

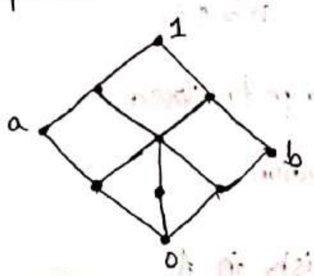
atmost 1 complement $\not\Rightarrow$ Distributive.

$\therefore$ the statement is false.

Proof:

Here we prove this with an example.

In this example sublattice is included to make it not distributive.
Also we design the example such a way that few elements have no complement.

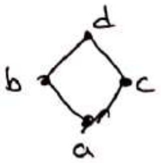


Think about this example for a while

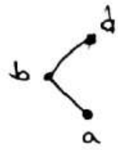
$\therefore$ S1: false

Here the only complement pairs are $(0,1)$ & (a,b)
So the complement of every element is atmost one and still not a distributive lattice.

S2 :



is a complemented lattice



is not a complemented lattice.

$\therefore$ false

(P/41) $(x \wedge y) \vee y = y$

$\because$ let $a = x \wedge y$

Now aRy

$\therefore a \vee y = y$

(P/42) $x \vee (y \wedge z) = (x \vee y) \wedge z$

$x \vee (0) = I \wedge z$

$x = z$

$\therefore S_1$ is false

S2 is false because the lattice is not distributive

13/06/20

Groups:

$(G, *)$ is called group when it satisfies following properties.

(i) $a \in G$, & $b \in G$ then $a * b \in G$

we call this property closed.

$*$ is any ~~operation~~ binary operator

(ii) $*$ is associative operation

(iii) Identity element exists in A.

(iv) Every element has inverse. $\forall a \in A$.

Eg: Integers are closed under addition, subtraction & multiplication

Integers are not closed under division.

→ If 'e' is an identity element, then

$\forall a \in G \quad a * e = a$

and identity element, if exists, is unique.

→ In k is inverse of a

$a \times k = e$

where e is identity element

we denote inverse as

$a^{-1} = k$

also $k^{-1} = a$

Eg: Determine whether $(Z,+)$ is a group or not

$(Z,+)$

(i) $a \in Z, b \in Z \Rightarrow a+b \in Z$

$\therefore$ closed

(ii) $a+(b+c) = (a+b)+c$

$\therefore$ associative

(iii) $\forall a \in A \quad a+0 = a$

$\therefore$ Identity element exists.

(iv) $\forall a \in Z$

$a + (-a) = 0$

$\therefore$ Every element has inverse

$\therefore (Z,+)$ is a group.

Eg : $(G, *)$, where $a*b = \frac{ab}{2}$

Find whethe it is group or not if G is set of real number.

1) $a \in G,\ b \in G \Rightarrow \frac{ab}{2} \in G$

$\therefore$ closed

2) $a*(b*c) = (a*b)*c$

$a*\left(\frac{bc}{2}\right) = \left(\frac{ab}{2}\right)*c$

$\frac{abc}{4} = \frac{abc}{4}$

$\therefore$ Associative

3) $\forall a \in R$

$a*2 = \frac{a(2)}{2} = a$

$\therefore$ 2 is identity element

4) $\forall a \in R$

let b be inverse

$a*b = 2$

$\frac{ab}{2} = 2$

$b = \frac{4}{a} \in R$

$\therefore a^{-1} = \frac{4}{a}$

$\therefore$ Every element has inverse

$\therefore (G, *)$ where $a*b = \frac{ab}{2}$ and G is ~~set~~ real number is a group.

Eg: Consider $(Z, \times)$ $\times$ is multiplication

(i) closed

(ii) Associative

(iii) 1 is identity

(iv) $5^{-1} = \frac{1}{5} \notin Z \rightarrow \therefore$ not a ~~group~~ group

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Eg: Let Q be rational numbers

$(Q, \times)$

(i) closed

(ii) Associative

(iii) 1 is identity element

(iv) $a \times \frac{1}{a} = 1$

But for 0 inverse doesn't exist

$\therefore (Q, \times)$ is not a group

Sly: $(R, \times)$ is not a group.

But $(Q - \{0\}, \times)$ is a group.

$(R - \{0\}, \times)$ is also a group.

→ Now let us see about finite groups

Eg: ~~$(\{2,4,6,8\}, gcd)$~~ $(\{0,1,2,3,4,5\}, \oplus_6)$

↳ Addition modulo m, $\oplus_m$

Finite groups can be represented with a table (cayley's table)

$\oplus_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

For every finite group, If we draw its cayley's table, every element is present in every row and column exactly once

(i) closed

(ii) $2 \oplus (3 \oplus 5) = (2 \oplus 3) \oplus 5$

$2 \oplus (2) = 5 \oplus 5$

$4 = 4$

Slly for any other elements

$\therefore$ associative.

(iii) we have column under '0' as 0 1 2 3 4 5

$\therefore$ 0 is identity element

(iv) Every row has 0

$\therefore$ every element has inverse.

$\therefore$ group.

Eg: $5 \oplus 1 = 0$

$4 \oplus 2 = 0$

$3 \oplus 3 = 0$

Q39 / G

The set $\{1, 2, 4, 7, 8, 11, 13, 14\}$ is a group under multiplication mod 15. what is inverse of 4 & 7 respectively?

a) 3, 13 b) 2, 11 c) 4, 13 d) 8, 14

Sol:

It is clear that '1' is identity element.

opt a: ~~4×13~~ $4 \times 3 = 12 \bmod 15 = 12$

$\therefore 4^{-1} \neq 3$

opt b: $4 \times 2 = 8 \bmod 15 = 8$ ✗

opt c: $4 \times 4 = 16 \bmod 15 = 1$

$7 \times 13 = 91 \bmod 15 = 1$

$\therefore$ opt (c)

Eg : $(\{e, b, a\}, *)$ be a group

let e be the identity element.

Now find the table

	e	b	a
e	e	b	a
b	b	a	e
a	a	e	b

row and column with e can be filled easily

Consider the cell [b,a]

it must be fill with e beacause we have a & b in it cloumn and row

Now cell [b,b] has only 1 possible i.e., a

(same like sudoko)

Q40
9-04

Consider ~~Fill the~~ below table which is commutative

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b				
c				

a) c a e b

b) c b a e

c) c b e a

d) c e a b

• what will be the ~~last~~ last row.

Sol:

observing table we can say 'e' is identity element.

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

filling like sudoko

given $a \cdot b = c$

$\therefore b \cdot a = c$

also ~~$b \cdot c = e$~~ $a \cdot c = e$

$\therefore$ ~~e~~ $c \cdot a = e$

$\therefore$ opt (d) is true

because it starts with ce

Subgroup:

H is called subgroup of G when it satisfies ~~prop~~ below properties:

i) $H \subseteq G$

ii) H should also be a group.

Eg: Consider $(\{0,1,2,3,4,5\}, \oplus_6)$ is a group G

H be $(\{1,3,5\}, \oplus_6)$ is not sub group of G.

i) $H \subseteq G$

ii) $1 * 3 = 1+3 = 4 \mod 6$

$= 4 \notin H$

~~∴ not a subg~~

∴ H is not a group

∴ It is not a subgroup

Note:

→ ~~Every~~ If G is a group with identity element e.

then for any H which subgroup of G, $e \in H$.

i.e., Every subgroup contains identity element of its original group.

Eg: Now consider $H = \{0,3,5\}$

i) $H \subseteq G$

ii) H should be group

a) $0 \oplus_6 3 = 3$, $0 \oplus_6 5 = 5$

$3 \oplus_6 5 = 8 \mod 6 = 2 \notin H$

∴ not closed

∴ not group, Hence not subgroup.

Eg: Consider $H = \{0, 2, 4\}$

	0	2	4
0	0	2	4
2	2	4	0
4	4	0	2

It is clear that $H \subseteq G$

(ii) Closed
associative
0 is identity.

Every row has 0.
$\therefore$ every element has inverse

$0^{-1} = 0$, $2^{-1} = 4$, $4^{-1} = 2$

$\therefore$ group

$\therefore$ H is subgroup of G.

Note:

* Every group consits of 2 trivial subgrougps:

(i) $\{e\}$ is a subgroup of G where e is identity element of G.

(ii) G is also a subgroup of G.

Lagrange's theorem:

If H is a subgroup of G, then $|H|$ divides $|G|$.
(reverse need not to be true).

Eg: $(\{0,1,2,3,4,5\}, \oplus_6)$ is a group G, $|G| = 6$

$H = (\{0, 2, 4\}, \oplus_6)$ is a ~~sup~~ subgroup of G., $|H| = 3$

$\therefore 3 | 6$

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Order of a group:

* Cardinality of a group is also known as order of a group.

Eg: Let G be a group.

let $|G|=84$.

What is the maximum size of proper subgroup of G.

Sol:

~~Here~~ $H \subset G$

$|H| \neq 84$

$|H| \,/\, |G|$

$\therefore |H| = 42$ (maximum size possible)

Eg: Let G be a group with 15 elements. let L be a subgroup of G and $L \neq G$. Also size of L is atleast 4. Find size of L.

Sol:

$|G|=15$ $\quad L \neq G$

$|L| \geq 4$

$5/15$ & $15/15$

$\therefore |L| = 5$ $(\because |L| \geq 4)$

Exponential:

→ $(G,*)$ is a group and let $a \in G$

$a^1 = a$

$a^2 = a * a$

$a^3 = a^2 * a$

$\forall n \geq 1.\ a^n \in G$.

Eg: Let $(\{0,1,2,3,4,5\}, \oplus_6)$ be a group

$3^1 = 3$

$3^2 = 3 \oplus_6 3 = 0$

$3^3 = 3^2 \oplus_6 3 = 0 \oplus_6 3 = 3$

$\vdots$

Now exponential of 3 generates {0,3}

we write it as

$\langle 3 \rangle = \{0,3\}$

Consider $\langle 2 \rangle$

$2^1 = 2$

$2^2 = 2 \oplus_6 2 = 4$

$2^3 = 4 \oplus_6 2 = 0$

$\vdots$

$\langle 2 \rangle = \{0,2,4\}$

Consider $\langle 1 \rangle$

$1^1 = 1$

$1^2 = 1 \oplus_6 1 = 2$

$1^3 = 2 \oplus_6 1 = 3$

$1^4 = 4$

$1^5 = 5$

$1^6 = 0$

$\vdots$

$\therefore \langle 1 \rangle = \{0,1,2,3,4,5\}$

Consider $\langle 5 \rangle$

$5^1 = 5$

$5^2 = 5 \oplus_6 5 = 4$

$5^3 = 4 \oplus_6 5 = 3$

$5^4 = 3 \oplus_6 5 = 2$

$5^5 = 2 \oplus_6 5 = 1$

$5^6 = 1 \oplus_6 5 = 0$

$\therefore \langle 5 \rangle = \{0,1,2,3,4,5\}$

→ Here 1, 5 have generated all the ~~gener~~ elements of the group. Such elements are called generators.

Here 1, 5 are generators of the group

Cyclic group:

A group with atleast ~~given~~ one generator element is called a cyclic group.

Thus the whole group can be represented with powers of generator element.

Eg: $(\{0, 1, 2, 3, 4, 5\}, \oplus_6)$

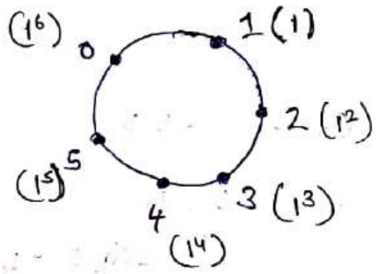
5, 1 are generators

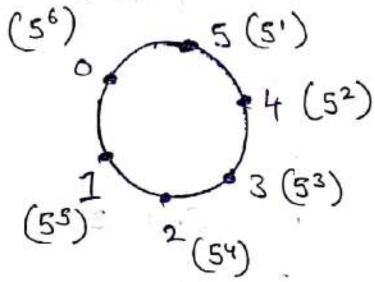
$(\{1^6, 1, 1^2, 1^3, 1^4, 1^5\}, \oplus_6)$

(or)

$(\{5^6, 5^5, 5^4, 5^3, 5^2, 5^1\}, \oplus_6)$

we can represent the both with cycles



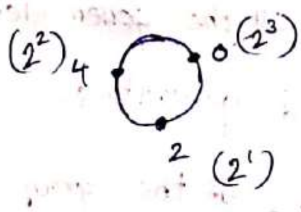


we can see that both the cycles are exactly in reverse directions. It is because 5 and 1 are inverse to each other.

Note:

Thus if 'a' is a generator, then a^{-1} is also a generator.

$\langle 2 \rangle = \{0, 2, 4\}$



Note :

→ Subgroup of a cyclic group is also a cyclic group.

→ A set and operation following

a) closed then binary structure or binary ~~ob~~ operation

b) closed & associative is called semigroup.

c) ~~b~~ closed & associative & identity is called monoid

d) closed, associative, identity, inverse is called group.

→ A group whose operation is commutative is called abelian group.

Groups Revised:

Binary operation (closed operation):

The binary operator $*$ is said to be a binary operation on a non empty set A, if $(a*b) \in A \quad \forall a,b \in A$.

Algebraic Structure (or) Binary structure:

A non empty set 'A' is called a algebraic structure with respect to a binary operation $*$

if $(a*b) \in A \quad \forall a,b \in A$

It is denoted as $(A,*)$

Eg: $(N,+)$, $(N,\times)$, $(Z,+)$ $(Z,\times)$, $(Z,-)$ are algebraic structures

$(N,-)$, $(N,\div)$, $\underbrace{(Z,\div), (Q,\div)}_{1/0 \text{ are not defined}}$ are not algebraic structures

→ $(Q^*, /)$ is an algebraic structure

Q^* is set of non-zero rational numbers.

Semi Group:

An algebraic system $(A, *)$ is said to be a semi group if

1. $*$ is closed operation on A.
2. $*$ is an associative property.

Monoid:

An algebraic system $(A, *)$ is said to be a monoid if:

1) $*$ is closed operation
2) $*$ is an associative property
3) Identity element exists in A.

Group:

An algebraic system $(A, *)$ is said to be a group if the following conditions are satisfied

1) $*$ is a closed operation
2) $*$ is an associative property
3) Identity element exists in A.
4) Every element of A has inverse.

Eg: $(N, \times)$ is a monoid but not a group

$(R-\{0\}, \times)$ is a group i.e., $(R^*, \times)$ is a group.

slly $(Q^*, \times)$ is a group.

Finite group:

A group containing finite number of elements is known as a finite group.

Eg: $(\{0\}, +)$ $(\{1\}, \times)$ $(\{-1, 1\}, \times)$ are examples of finite groups.

→ The only finite group of real numbers with respect to addition is $\{0\}$

→ The only finite group of real numbers w.r.to multiplication is $\{1\}$, $\{-1, 1\}$

→ Cube roots of unity is also a group of order 3. w.r.to multiplication

Note:

	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$1^{-1} = 1$

$\omega^{-1} = \omega^2$

$(\omega^2)^{-1} = \omega$

→ In any group (G) of order 2

$$a^{-1} = a, \ \forall a \in G$$

→ Set $S = \{0, 1, 2 \cdots, m-1\}$ is a group w.r.to addition modulo m

i.e., $(\{0, 1, 2 \cdots m-1\}, \oplus_m)$ is a group

→ Set S_n be set of positive integers which are less than n and relatively prime to n. then

$(S_n, \otimes_n)$ is group

where $\otimes_n$ is multiplication modulo n.

$S_6 = (\{1, 5\}, \otimes_6)$ is a group

→ For the previous statement we can conclude that, if p is a prime, then

$$(\{1,2,3 \ldots p-1\}, \otimes_p) \text{ is a group.}$$

Abelian Group:

A group $(G,*)$ is said to be abelian or commutative if $a*b = b*a \ \forall a,b \in G$

Eg: $(Z,+)$, $(R^*, \times)$ are abelian groups

Properties of groups:

→ The identity element of a group is unique.

proof:

Let us assume there are 2 identity elements e_1, e_2.

$$\underbrace{e_2 = e_1}_{\text{since } e_1 \text{ is identity}} * \underbrace{e_2 = e_1}_{\because e_2 \text{ is identity}}$$

$$\therefore e_1 = e_2$$

→ Inverse of every element is unique.

Proof:

Let $a,b,c \in G$ and

b,c are 2 different inverse of a.

now $b = b \cdot e$ (e is identity element)

$= b \cdot (ac)$ ($\because c = a^{-1}$)

$= (ba)c$

$= e \cdot c = c \Rightarrow \boxed{b = c}$

→ Left cancellation property

let $a, b, c \in G$, then

$$\boxed{ab = ac \Rightarrow b = c}$$

Proof :

$$ab = ac$$

$$a^{-1}(ab) = a^{-1}(ac)$$

$$(a^{-1}a)b = (a^{-1}a)c$$

$$e \cdot b = e \cdot c$$

$$b = c$$

→ Right cancellation property

$$\boxed{ba = ca \Rightarrow b = c}$$

→ Due to right cancellation & left cancellation properties, every row & column in cayley table do not contain repeatition of any element.

In every row & column all elements are appeared exactly once.

→ If $(G, \circ)$ & $(H, *)$ are two groups.

$(G \times H, \cdot)$ is a group where '$\cdot$' is defined as

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 \circ g_2, h_1 * h_2)$$

This group is known as direct product of G and H.

Identity element : (e_1, e_2) where e_1 is identity element of G
e_2 is identity element of H.

Inverse of element (g, h) is (g^{-1}, h^{-1})

Grimaldi Questions :

① P.T $G = \{q \in \mathbb{Q} \mid q \neq -1\}$ with respect to binary operation $\circ$ on G by $x \circ y = x + y + xy$ is an abelian group.

② P.T set Z is not a group under subtraction.

③ P.T set of all one-one functions $g : A \to A$, where $A = \{1, 2, 3, 4\}$ under function composition is a group.

Note :

→ If G is a group, $a, b \in G$

then

$$\boxed{(ab)^{-1} = b^{-1}a^{-1}}$$

Proof :

Let $\cancel{(ab)^{-1} b^{-1} a^{-1}} = x$

$\cancel{(ab)^{-1} b^{-1} a^{-1} a} = x$

$\cancel{(ab)^{-1}(ab) = x}$

$(ab)\cancel{^{-1} ab b^{-1} = x} b^{-1}$

$\cancel{ab (ba)^{-1} =}$

$ab(ab)^{-1} = e$

$a^{-1}ab(ab)^{-1} = a \cdot e$

$b(ab)^{-1} = a^{-1}$

$b^{-1}b(ab)^{-1} = b^{-1}a^{-1}$

$(ab)^{-1} = b^{-1}a^{-1}$

∴ true.

→ Thus a group is said to be abelian if

$(ab)^{-1} = a^{-1}b^{-1} \quad \forall a, b \in G$

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Order of a group:

It is cardinality of the group.

Order of an element:

The smallest positive integer n ~~such~~ such that $a^n = e$ is called order of 'a'.

Note:

- $\rightarrow a^0 = e$
- $\rightarrow a^n = a * a * a * \ldots * a$ (n times)
- $\rightarrow$ The order of an element of a finite group is a divisor of the order of the group.
- $\rightarrow$ If no n exists such that $a^n = e$, we say order of 'a' is infinite.
- $\rightarrow$ order of identity element, $e = 1$
- $\rightarrow a^{-n} = (a^{-1})^n = b^n$ where b is inverse of a.
- $\rightarrow$ order(a) = order(a^{-1})

Proof:

let order of $a = n$

we need to P.T

$$(a^{-1})^n = e$$

$$a^n (a^{-1})^n = e \cdot a^n$$

$$(aa \cdots a)(a^{-1} a^{-1} \ldots a^{-1}) = e \cdot a^n$$

$$e = e \cdot e$$

$$e = e$$

Subgroup:

A non-empty subset H of a group $(G,*)$ is called subgroup of G, if $(H,*)$ is a group.

Eg:. for group $(\{0,1,2,3,4,5\}, \oplus_6)$

$(\{0,2,4\}, \oplus_6)$ is a subgroup

→ For every group G, $\{e\}$ and G are called trivial subgroups
→ Others are called non-trivial or proper subgroups.
→ Every subgroup contain identity element of its parent group

Properties of subgroups:

→ If $H \subseteq G$, then H is called subgroup of G iff

a) $(a*b) \in H \quad \forall a,b \in H$ (H is non-empty)

b) $a^{-1} \in H \quad \forall a,b \in H$

Proof:

for H to be group.

(i) $(a*b) \in H \quad \forall a,b \in H$

(ii) Associative (since G is group, binary operation of H which is same as G is also associative)

(iii) $a^{-1} \in H \quad \forall a \in H$

Since $a^{-1} \in H$ and $a \in H$ & $a*b \in H \;\forall a,b$

$a^{-1}a = e \in H$

∴ Identity element exists.

∴ Having $(a*b) \in H \quad \forall a,b$
$a^{-1} \in H \quad \forall a \in H$
is enough to say H is subgroup

(Associativity is inherited from G
e's existence is proved by 2nd stmt)

→ If G is a group and $H \subseteq G$ $(H \neq 0)$

If H is ~~fin~~ finite, then

H is subgroup of G iff

$(a*b) \in H \quad \forall a, b \in H$

This is theorem 16.3 in Grimaldi. The proof involves cosets.

Note that this applies only if H is finite.

Eg: $(\mathbb{Z}, +)$ is a group

Yet $(\mathbb{N}, +)$ is not a subgroup

because $\mathbb{N}$ is not finite.

→ If $(G, *)$ is a group then

$\langle a \rangle$ is a subgroup where $a \in G$ and order(a) is finite

Eg: Consider $(\{0,1,2,3,4,5\}, \oplus_6)$

$\langle 1 \rangle$ is calculated as

$1^1 = 1$, $1^2 = 1 \oplus_6 1 = 2$, $1^3 = 1^2 \oplus 1 = 2 \oplus_6 1 = 3$, $1^4 = 4$, $1^5 = 5$, $1^6 = 0$, $1^7 = 1$

$\therefore \langle 1 \rangle = \{0,1,2,3,4,5\}$

$= \{1^0, 1^1, 1^2, 1^3, 1^4, 1^5\}$

$\langle 2 \rangle$

$2^1 = 2 \quad 2^2 = 2 \oplus_6 2 = 4 \quad 2^3 = 2^2 \oplus_6 2 = 0 \quad 2^4 = 2$

$\langle 2 \rangle = \{0,2,4\}$.

$\langle 3 \rangle$

$3^1 = 3$, $3^2 = 0$, $3^3 = 3 \ldots$

$\langle 3 \rangle = \{0,3\}$

Here $\langle 0 \rangle$, $\langle 2 \rangle$, $\langle 3 \rangle$, $\langle 4 \rangle$, $\langle 5 \rangle$ are ~~called~~ subgroups.

Eg: $\langle 2 \rangle = \{0,2,4\}$ is a subgroup

$\langle 3 \rangle = \{0,3\}$ is a subgroup

$\vdots$

→ The subgroups of form $\langle a \rangle$ are called generating sets.
But these are not the only possible subgroups.

→ If generating set of 'a' $\langle a \rangle$ is equal to the group G, where $a \in G$, then a is called generator.

Eg: $(\{0,1,2,3,4,5\}, \oplus_6)$

For above group 1,5 are generators.

→ If 'a' is a generator, then a^{-1} is also a generator.

Eg: In the previous example 1,5 are inverses to each other.

→ If H is a subgroup of a group G, then

$|H| \mid |G|$ Lagrange's Theorem

The converse of lagranges theorem holds for abelian group

→ If G is a group with $|G| = n$, then number of ~~subgroups~~ generators possible to G are $\phi(n)$

where ϕ is euler phi function.

$$\phi(n) = n \cdot \frac{(P_1-1)(P_2-1)\cdots(P_k-1)}{P_1 \cdot P_2 \cdots P_k}$$

where $P_1, P_2 \ldots P_k$ are distinct prime divisors of n.

~~Cyclic group~~:

→ If H and K are two subgroups of G, then

$H \cap K$ is also a subgroup

$H \cup K$ need not to be a subgroup.

proof:

(i) $(a*b) \in H \cap K \quad \forall a,b \in H \cap K$

$a \in H$ & $b \in H$ | $a \in K$ & $b \in K$

$a*b \in H$ | $a*b \in K$

$\therefore a*b \in H \cap K$

(ii) $a^{-1} \in H \cap K \quad \forall a \in H \cap K$

$a \in H \qquad a \in K$

$a^{-1} \in H \qquad a^{-1} \in K$

since a^{-1} is unique for a

$a^{-1} \in H \cap K$

$\therefore H \cap K$ is subgroup

since we are proving $H \cap K$ is subgroup,
proving,
$(a*b) \in H \cap K, \forall a,b \in H \cap K$
$a^{-1} \in H \cap K \quad \forall a \in H \cap K$
is enough.

Cyclic groups:

A group G is called cyclic if $\exists a \in G$ such that every element of G can be written as an integral power of a.

i.e., G is called cyclic group if it contains a generator element.

Eg: $G = \{1, \omega, \omega^2\}$ is a cyclic group w.r.to multiplication

ω, ω^2 are generators.

$G = \{-1, 1, i, -i\}$ is a cyclic group w.r.to multiplication

$i, -i$ are generators.

→ $(Z_n, \oplus_n)$ is a cyclic group $\forall n \geq 2$

$1, n-1$ ~~is~~ are generators

→ All subgroups of a cyclic group are cyclic.

→ If G is a cyclic group of g with generator a, then for every divisor d of $|G|$ there exists exactly one subgroup of order d.

This subgroup is generated by $a^{|G|/d}$.

Eg: $G = \{0,1,2,3,4,5\}$ is group under $\oplus_6$

$|G| = 6$

3 is divisor of 6 ($|G|$)

∴ we have a unique subgroup of order 3 which is generated by $a^{|G|/d} = a^{6/3} = a^2$.

Here a is 1 or 5 (generators)

∴ The subgroup is generated by 1^2.

$(1^2)^1 = 1^2 = 2$

$(1^2)^2 = 1^4 = 4$

$(1^2)^3 = 1^6 = 0$

$(1^2)^4 = 1^8 = 2$

⋮

$\langle 1^2 \rangle = \{0,2,4\}$ is a unique subgroup of size 3.

→ If G is a group of composite order, then G has a non-trivial subgroups.

Homomorphism:

→ If $(G, *)$ and $(G', \oplus)$ are two groups than a function $f: G \to G'$ is called a homomorphism if $f(a*b) = f(a) \oplus f(b)$

If f is a bijection, then the homomorphism f is called an isomorphism between G and G'.

we write it as $G \cong G'$

i.e., G is isomorphic to G'.

→ If two groups are isomorphic, it means that both the group are same with different names.

Eg: $(\{0,1,2,3\}, \oplus_4)$ is isomorphic to $(\{1,-1,i,-i\}, \times)$

The isomorphis f is defined as

$f(0) = 1$

$f(1) = i$

$f(2) = -1$

$f(3) = -i$

we can see that cayley table's for both are same and have one to one correspondence.

$\oplus_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\times$	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

Here generators are 1 & 3

Here generator are $i, -i$

i.e., $f(1)$ & $f(3)$

Problems on Groups:

P/57 $A * B = A \oplus B$

$A \in P(S)$ $B \in P(S)$

$\Rightarrow A \subseteq S$, $B \subseteq S$

$\therefore A \oplus B \subseteq S$ i.e., closed

$(A \oplus B) \oplus C \equiv A \oplus (B \oplus C)$ i.e., Associative

$\forall A \in P(S)$

$A \oplus \phi = A$

$\therefore \phi$ is identity element

$\forall A \in P(S)$

$A \oplus A = \phi$

$\therefore A^{-1} = A$

i.e., Every element has inverse

$\therefore$ group

P/58 $a * b = \frac{ab}{2}$ $(R - \{0\}, *)$

let e be identity element

$\frac{ae}{2} = a$

$e = 2$

to find invere of 4

$4^{-1} = x$

$4 * x = \frac{4x}{2} = 2$

$4x = e$

~~$4x = e \Rightarrow x = 2/4$~~ $= 1/2$

$2x = 2 \Rightarrow x = 1$

$\therefore 4^{-1} = 1$

(P/59) $a * b = 2ab$

Let e be identity element

$a * e = a$

$2ae = a$

$e = 1/2$

to find inverse of 2/3

$\left(\frac{2}{3}\right)^{-1} = x$

$\frac{2}{3} * x = e$

$2 \frac{2}{3} x = \frac{1}{2}$

$x = \frac{3}{8}$

$\therefore \left(\frac{2}{3}\right)^{-1} = \frac{3}{8}$

(P/60) Given binary operation

$a * b = a + b + ab$

a) finding identity

$a * e = a$

$a + e + ae = a$

$e + ae = 0$

$e = 0$

$\therefore$ 0 is the identity

b) finding invere

let $a^{-1} = b$

$a * b = e$

$a + b + ab = 0 \longrightarrow \boxed{b = \frac{-a}{a+1} \quad \text{for } a = -1 \quad b = \frac{1}{0}}$

for $a = -1$, $-1 + b - b = 0$

$-1 = 0$ $\therefore$ inverse ~~does~~ doesn't exists

$\therefore b$ is false

c) $a * b = 0$ $\quad a^{-1} = b$

$a + b + ab = 0$

$b = \frac{-a}{a+1}$

true

d) ~~As~~ As -1 has no inverse

R is not a group

(P/61) a) Let group of order 3 be $\{e, a, b\}$

possible groups are

(i) $e^{-1} = e \quad a^{-1} = a \quad b^{-1} = b$

(ii) $e^{-1} = e \quad a^{-1} = b \quad b^{-1} = a$

(i)

	e	a	b
e	e	a	b
a	a	e	b
b	b	a	e

Here

$ab = b$

$ba = a$

$\therefore$ ~~not commutative~~

$\therefore$ ~~not abelian~~

(ii)

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Here $ab = e$

$ba = e$

It is commutative

Every group of order less than or equal to 5 is abelian

Here (i) itself is not a group because associativity doesn't hold

$(ab)a = a(ba)$

$(b)a = a(a)$

$a = e$

$\therefore$ not associative

$\therefore$ every group of order 3 is abelian

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b) $\{1,-1\}$ under $\times$ is a group

c) let $(\{a,b\}, *)$ be group

let 'a' be identity element

Now $a^{-1} = a$

since ~~eq~~ every element has inverse

b^{-1} must be b

$\therefore$ true

d) $(\{1,-1,i,-i\}, \times)$ is a group under multiplication.

$\therefore$ false

(P/62) a) $a * a = e$

also w.k.T $a * e = e$

$a * a = a * e$

$a = e$ ($\because$ left cancellation property)

b) let $a, b \in G$

$a^{-1} = a \quad b^{-1} = b \quad , \quad ab \in G$

$(ab)^{-1} = ab$

Consider $ab = (ab)^{-1}$

$= b^{-1}a^{-1}$

$= ba$

i.e., $ab = ba$

$\therefore$ abelian group

c) Let $(\{e, a_1, a_2, a_3, a_4, a_5\}, *)$ be a group

Now we have

$e^{-1} = e$

Let use assume that

$a_1^{-1} = a_2$.

$a_3^{-1} = a_4$

Now a_5^{-1} must be a_5 only

$\therefore$ true

d) from problem 61 option a

it is false.

P/63 $2 \oplus_6 4 = 6 \equiv 0$

$3 \oplus 3 = 6 \equiv 0$

$5 \oplus 2 = 7 \equiv 1$

$1 \oplus 5 = 6 \equiv 0$

$\therefore$ opt c

P/64 $(-i)^2 = i^2 = -1$

$(-i)^3 = (-1)(-i) = i$

$(-i)^4 = i(-i) = 1$ (identity element)

$\therefore O(-i) = 4$

P/65 a & b are already proveed

a is false.

c) It is set of multiples of k

$nk + mk = (n+m)k$ $\therefore$ close

Addition is associative

0 is identity

for nk, $-nk$ is inverse

$\therefore$ group

d) Sub group of abelion group is abelion because commutativity always holds.

(P/68) a) ~~Ex~~ If 'a' is generator, every element can be represented as a^k.

consider

$$bc = cb$$

$$a^m . a^n = a^n . a^m$$

$$a^{m+n} = a^{m+n}$$

$\therefore$ commutative

Hence abelian

b) If G is a group of order n and 'a' is a generator

$$a^k = a^k . e$$

$$= a^k . e^{-1}$$

$$= a^k . (a^n)^{-1}$$

$$= a^k . a^{-n}$$

$$= a^{k-n}$$

$$= (a^{-1})^{n-k}$$

$\therefore$ a^{-1} is also a generating element

c) If order of generation element is not n then it cannot generate all the elements ($\because a^k = e$, $k < n$)

d) Subgroup of a cycle group is always cyclic

$\therefore$ ~~tr~~ false

P/67

a) Every element of a prime order group is a generator.

proof:

let G be a group of prime order

Let $a \in G$

Let $\langle a \rangle = H$

Now H is subgroup of G

$|H| / |G|$

Since $|G|$ is prime

$|H| = 1$ (or) $|H| = |G|$

for elements other than identity

$\langle a \rangle = H$ then $|H| \geq 2$

because H contains both a & e

$\therefore |H| = |G|$

Hence the group is cyclic.

$\therefore$ true.

b) Same as P/62-(b)

c) True (every group of order less than or equal 5 is abelian)

d)

$2^1 = 2$

$2^2 = 4$

$2^3 = 8 \equiv 1$

$2^4 = 16 \equiv 2$

2 is not a generator.

$\therefore$ opt-(d) is false

P/68 a) $4 \oplus_6 4 \equiv 8 \equiv 2 \notin \{0,4\}$

$\therefore$ not a subgroup

b) $\{1,5\}$ has no identity

$\therefore$ not a subgroup

c) $\{0,2,4\}$ is a subgroup

d) $\{1,3,5\}$ has no identity element

$\therefore$ not subgroup

P/69 order of Group = 10

no of generators = $\phi(10)$ $10 = 2 \times 5$

$= \phi(10) = 10 \dfrac{(2-1)(5-1)}{2 \cdot 5}$

$= 4$

P/70

$\otimes_{10}$	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

i) from table it is closed

ii) Multiplication is associative

iii) 6 is identity

(iv) $2^{-1} = 8$ $4^{-1} = 4$ $6^{-1} = 6$
$8^{-1} = 2$

$2^1 = 2$ | $8^1 = 8$

$2^2 = 4$ | $8^2 = 64 \equiv 4$

$2^3 = 8$ | $8^3 = 512 \equiv 2$

$2^4 = 16 \equiv 6$ | $8^4 = 4096 \equiv 6$

$\therefore$ 2, 4 are generators.

$\therefore$ (d) is false

(P/71)

*	a	b	c	d
a	b	d	a	c
b	d	c	b	a
c	a	b		
d	c	a		

every 4-element group is abelian

From above figure only 'c' has chance to be identity element thus $[c,c]=c$ $[c,d]=d$ and from that

$$[d,c]=d \quad [d,d]=b$$

*	a	b	c	d
a	b	d	a	c
b	d	c	b	a
c	a	b	c	d
d	c	a	d	b

identity is c

$a^{-1}=d$ $\quad b^{-1}=b$ $\quad c^{-1}=c$
$d^{-1}=a$

$\therefore$ opt (a) is true

Questions :

1) S.T in any boolean algebra, for any a and b, $a\leq b$ iff $\bar{b}\leq\bar{a}$

2) S.T in a boolean algebra, for any a and b

$$a=b \Leftrightarrow (a\wedge\bar{b})\vee(\bar{a}\wedge b)=0$$

3) S.T in any boolean algebra for any a, b, c

i) $a\leq b \Rightarrow a\vee c\leq b\vee c$

ii) $a\leq b \Rightarrow a\wedge c\leq b\wedge c$