Optimal Shelter Allocation with Risk-Weighted Coverage: Three-Stage Formulation

Riya Parikh, Ayela Chughtai, Rose Dana

15.C57 Optimization Methods Fall 2025

1 Model Overview

We partition the region into 1 km^2 cells and treat each cell as an assignment node. Splitting is allowed: a cell can send fractions of its population to multiple shelters.

To generate a Pareto-style frontier, we vary K, the maximum number of new shelters that may be opened. For each fixed K, we solve a **three-stage problem**:

1. **Stage 1 (Risk-coverage first):** maximize total risk-weighted covered population, yielding $S_r^\star(K)$ and $\alpha_r^\star(K)$.
2. **Stage 1b (Existing-use tie-break):** among (nearly) Stage-1-optimal solutions, maximize the amount of population sent to *existing* shelters, yielding $U_E^\star(K)$.
3. **Stage 2 (Distance tie-break):** among solutions that preserve (nearly) the max risk coverage and (nearly) the max existing-use, minimize total travel distance.

A hard travel cutoff $L_{\max}$ is enforced in *all stages* via a precomputed feasibility matrix.

Sets and Indices

- I: set of 1 km^2 population cells, indexed by i.
- J_E: set of existing shelters, indexed by j.
- J_N: set of candidate new-shelter sites, indexed by j.
- $J = J_E \cup J_N$: set of all shelters.

Parameters

- d_i: population in cell $i \in I$.
- $r_i \geq 0$: cyclone risk score for cell $i \in I$ (higher means more at risk).
- u_j: capacity of shelter $j \in J$.
- c_{ij}: travel distance (or cost) from cell i to shelter j.
- K: maximum number of *new* shelters that may be opened.
- $L_{\max}$: maximum allowed travel distance.
- $M_{ij} \in \{0, 1\}$: precomputed feasibility indicator:

$$M_{ij} = \begin{cases} 1, & c_{ij} \leq L_{\max}, \\ 0, & c_{ij} > L_{\max}. \end{cases}$$

Total population:

$$D := \sum_{i \in I} d_i.$$

Total risk-weighted population:

$$R := \sum_{i \in I} r_i d_i.$$

Tolerance parameters (as in code):

$$\delta_{\text{frac}} > 0, \qquad \delta S_r = \delta_{\text{frac}}\, R, \qquad \delta U_E = \delta_{\text{frac}}\, D.$$

Decision Variables

- $x_{ij} \in [0,1]$ for $i \in I,\ j \in J$: fraction of cell i's population assigned to shelter j.
- $y_j \in \{0,1\}$ for $j \in J_N$:
  - $y_j = 1$ if a new shelter is opened at candidate site j,
  - $y_j = 0$ otherwise.
- For $j \in J_E$, shelters are always open, so $y_j = 1$ is fixed.

Risk-weighted covered population:

$$S_r(x) := \sum_{i \in I} \sum_{j \in J} r_i d_i x_{ij}. \tag{1}$$

Risk-weighted coverage fraction:

$$\alpha_r(x) := \frac{S_r(x)}{R}. \tag{2}$$

Raw coverage (for reporting):

$$S(x) := \sum_{i \in I} \sum_{j \in J} d_i x_{ij}, \qquad \alpha(x) := \frac{S(x)}{D}.$$

Existing-shelter utilization:

$$U_E(x) := \sum_{i \in I} \sum_{j \in J_E} d_i x_{ij}. \tag{3}$$

2 Stage 1: Maximize Risk-Weighted Coverage Given K

Objective (Stage 1)

$$\max_{x,y} \quad S_r(x) = \sum_{i \in I} \sum_{j \in J} r_i d_i x_{ij}. \tag{S1-Obj}$$

Constraints (Stage 1)

(1) Assignment constraint. Each cell can assign at most 100% of its population:

$$\sum_{j \in J} x_{ij} \le 1 \quad \forall i \in I. \tag{S1-1}$$

(2) Capacity constraints. Assignments into shelter j cannot exceed its capacity:

$$\sum_{i \in I} d_i x_{ij} \le u_j y_j \quad \forall j \in J. \tag{S1-2}$$

(3) New-shelter budget. At most K new shelters may be opened:

$$\sum_{j \in J_N} y_j \le K. \tag{S1-3}$$

(4) Hard travel cutoff ($L_{\max}$). Residents may only be assigned within $L_{\max}$ and only to open shelters:

$$x_{ij} \le M_{ij} y_j \quad \forall i \in I,\ j \in J. \tag{S1-4}$$

(5) Domains.

$$x_{ij} \in [0,1]\ \forall i,j, \qquad y_j \in \{0,1\}\ \forall j \in J_N, \qquad y_j = 1\ \forall j \in J_E. \tag{S1-5}$$

Stage 1 Output

Let $S_r^\star(K)$ denote the optimal Stage-1 value:

$$S_r^\star(K) := \max S_r(x).$$

Then

$$\alpha_r^\star(K) := \frac{S_r^\star(K)}{R}. \tag{S1-Out}$$

3 Stage 1b: Maximize Existing-Shelter Use Among Max Risk-Coverage Solutions

Stage 1b selects the solution that uses existing shelters the most *without sacrificing* risk-weighted coverage, up to tolerance δS_r.

Objective (Stage 1b)

$$\max_{x,y} \quad U_E(x) = \sum_{i\in I} \sum_{j\in J_E} d_i x_{ij}. \tag{S1b-Obj}$$

Constraints (Stage 1b)

$$\sum_{j\in J} x_{ij} \le 1 \qquad \forall i \in I \qquad \text{(assignment)} \tag{S1b-1}$$

$$\sum_{i\in I} d_i x_{ij} \le u_j y_j \qquad \forall j \in J \qquad \text{(capacity)} \tag{S1b-2}$$

$$\sum_{j\in J_N} y_j \le K \qquad \text{(new-shelter budget)} \tag{S1b-3}$$

$$x_{ij} \le M_{ij} y_j \qquad \forall i \in I,\ j \in J \qquad \text{(hard travel cutoff)} \tag{S1b-4}$$

$$\sum_{i\in I} \sum_{j\in J} r_i d_i x_{ij} \ge S_r^\star(K) - \delta S_r \qquad \text{(preserve max risk-coverage)} \tag{S1b-5}$$

$$x_{ij} \in [0,1]\ \forall i,j, \qquad y_j \in \{0,1\}\ \forall j \in J_N, \qquad y_j = 1\ \forall j \in J_E. \tag{S1b-6}$$

Stage 1b Output

Let $U_E^\star(K)$ denote the optimal existing-use value:

$$U_E^\star(K) := \max U_E(x) \text{ subject to (S1b-1)–(S1b-6).}$$

4 Stage 2: Minimize Distance Among Max Risk-Coverage and Max Existing-Use Solutions

Stage 2 selects the minimum-distance plan among all solutions that preserve:
- risk-weighted coverage within δS_r of $S_r^\star(K)$, and
- existing-use within δU_E of $U_E^\star(K)$.

Objective (Stage 2)

$$\min_{x,y} \quad Z(x) = \sum_{i\in I} \sum_{j\in J} c_{ij} d_i x_{ij}. \tag{S2-Obj}$$

Constraints (Stage 2)

$$\sum_{j\in J} x_{ij} \le 1 \qquad \forall i \in I \qquad \text{(assignment)} \tag{S2-1}$$

$$\sum_{i\in I} d_i x_{ij} \le u_j y_j \qquad \forall j \in J \qquad \text{(capacity)} \tag{S2-2}$$

$$\sum_{j\in J_N} y_j \le K \qquad \text{(new-shelter budget)} \tag{S2-3}$$

$$x_{ij} \le M_{ij} y_j \qquad \forall i \in I,\ j \in J \qquad \text{(hard travel cutoff)} \tag{S2-4}$$

$$\sum_{i\in I}\sum_{j\in J} r_i d_i x_{ij} \ge S_r^\star(K) - \delta S_r \qquad \text{(preserve max risk-coverage)} \tag{S2-5}$$

$$\sum_{i\in I}\sum_{j\in J_E} d_i x_{ij} \ge U_E^\star(K) - \delta U_E \qquad \text{(preserve max existing-use)} \tag{S2-6}$$

$$x_{ij} \in [0,1]\ \forall i,j, \qquad y_j \in \{0,1\}\ \forall j \in J_N, \qquad y_j = 1\ \forall j \in J_E. \tag{S2-7}$$

Reported Outputs for Each K

Let $(x^\star(K), y^\star(K))$ be the Stage-2 solution. We report:

- Risk-weighted coverage:

$$S_r^{\text{real}}(K) = S_r(x^\star(K)), \qquad \alpha_r(K) = \frac{S_r^{\text{real}}(K)}{R}.$$

- Raw coverage:

$$S^{\text{real}}(K) = S(x^\star(K)), \qquad \alpha(K) = \frac{S^{\text{real}}(K)}{D}.$$

- Existing-use realized:

$$U_E^{\text{real}}(K) = U_E(x^\star(K)).$$

- Minimum total distance:

$$Z^\star(K) = Z(x^\star(K)).$$

- Open shelters $y^\star(K)$ and assignments $x^\star(K)$.