

Optimal Shelter Allocation with Risk-Weighted Coverage: Lexicographic Formulation

1 Model Overview

We partition the region into 1 km^2 cells and treat each cell as an assignment node. Splitting is allowed: a cell can send fractions of its population to multiple shelters.

We generate a Pareto-style frontier by varying K , the maximum number of new shelters that may be opened. For each fixed K :

1. **Stage 1 (Risk-coverage first):** maximize risk-weighted covered population, yielding $\alpha_r^*(K)$.
2. **Stage 2 (Distance tie-break):** among all Stage-1-optimal solutions, minimize total travel distance.

Sets and Indices

- I : set of 1 km^2 population cells, indexed by i .
- J_E : set of existing shelters, indexed by j .
- J_N : set of candidate new-shelter sites, indexed by j .
- $J = J_E \cup J_N$: set of all shelters.

Parameters

- d_i : population in cell $i \in I$.
- $r_i \geq 0$: cyclone risk score for cell $i \in I$ (higher means more at risk).
- u_j : capacity of shelter $j \in J$.
- c_{ij} : travel distance (or cost) from cell i to shelter j .
- K : maximum number of *new* shelters that may be opened.

Total population:

$$D := \sum_{i \in I} d_i.$$

Total risk-weighted population:

$$R := \sum_{i \in I} r_i d_i.$$

Decision Variables

- $x_{ij} \in [0, 1]$ for $i \in I, j \in J$: fraction of cell i 's population assigned to shelter j .
- $y_j \in \{0, 1\}$ for $j \in J_N$:
 - $y_j = 1$ if a new shelter is opened at candidate site j ,
 - $y_j = 0$ otherwise.
- For $j \in J_E$, shelters are always open, so $y_j = 1$ is fixed.

Risk-weighted covered population:

$$S_r(x) := \sum_{i \in I} \sum_{j \in J} r_i d_i x_{ij}. \tag{1}$$

Risk-weighted coverage fraction:

$$\alpha_r(x) := \frac{S_r(x)}{R}. \tag{2}$$

(For reporting, we also compute raw coverage: $S(x) = \sum_{i,j} d_i x_{ij}$ and $\alpha(x) = S(x)/D$.)

2 Stage 1: Maximize Risk-Weighted Coverage Given K

Objective (Stage 1)

For fixed K , prioritize protecting high-risk residents:

$$\max_{x,y} S_r(x) = \sum_{i \in I} \sum_{j \in J} r_i d_i x_{ij}. \quad (\text{S1-Obj})$$

Constraints (Stage 1)

(1) **Assignment constraint.** Each cell can assign at most 100% of its population:

$$\sum_{j \in J} x_{ij} \leq 1 \quad \forall i \in I. \quad (\text{S1-1})$$

(2) **Capacity constraints.** Assignments into shelter j cannot exceed its capacity:

$$\sum_{i \in I} d_i x_{ij} \leq u_j y_j \quad \forall j \in J. \quad (\text{S1-2})$$

(3) **New-shelter budget.** At most K new shelters may be opened:

$$\sum_{j \in J_N} y_j \leq K. \quad (\text{S1-3})$$

(4) **Domains.**

$$x_{ij} \in [0, 1] \quad \forall i, j, \quad y_j \in \{0, 1\} \quad \forall j \in J_N, \quad y_j = 1 \quad \forall j \in J_E. \quad (\text{S1-4})$$

Stage 1 Output

Let $S_r^*(K)$ denote the optimal Stage-1 value:

$$S_r^*(K) := \max S_r(x).$$

Then the maximum feasible risk-weighted coverage fraction at budget K is:

$$\alpha_r^*(K) := \frac{S_r^*(K)}{R}. \quad (\text{S1-Out})$$

3 Stage 2: Minimize Distance Among Max Risk-Coverage Solutions

Stage 2 selects the minimum-distance plan among all solutions that preserve the maximum risk-weighted coverage $S_r^*(K)$.

Objective (Stage 2)

$$\min_{x,y} \sum_{i \in I} \sum_{j \in J} c_{ij} d_i x_{ij}. \quad (\text{S2-Obj})$$

Constraints (Stage 2)

$$\sum_{j \in J} x_{ij} \leq 1 \quad \forall i \in I \quad (\text{assignment}) \quad (\text{S2-1})$$

$$\sum_{i \in I} d_i x_{ij} \leq u_j y_j \quad \forall j \in J \quad (\text{capacity}) \quad (\text{S2-2})$$

$$\sum_{j \in J_N} y_j \leq K \quad (\text{new-shelter budget}) \quad (\text{S2-3})$$

$$\sum_{i \in I} \sum_{j \in J} r_i d_i x_{ij} \geq S_r^*(K) \quad (\text{preserve max risk-coverage}) \quad (\text{S2-4})$$

$$x_{ij} \in [0, 1] \quad \forall i, j, \quad y_j \in \{0, 1\} \quad \forall j \in J_N, \quad y_j = 1 \quad \forall j \in J_E. \quad (\text{S2-5})$$

Reported Outputs for Each K

- Maximum feasible risk-weighted coverage $\alpha_r^*(K)$ (Stage 1).
- Minimum total distance under max risk-coverage (Stage 2).
- Shelter sites $y^*(K)$ and assignments $x^*(K)$ (Stage 2).
- Raw population coverage $\alpha^*(K) = S(x^*(K))/D$ for interpretability.

Optional Extension: Maximum Travel Distance via Big-M

To reflect a hard evacuation travel limit, we may enforce that residents cannot be assigned beyond L_{\max} .

Additional Parameters

- L_{\max} : maximum allowed travel distance from any cell i to an assigned shelter.
- Precompute:

$$M_{ij} = \begin{cases} 0, & c_{ij} > L_{\max}, \\ 1, & c_{ij} \leq L_{\max}. \end{cases}$$

Additional Constraint

Add to both Stage 1 and Stage 2:

$$x_{ij} \leq M_{ij}y_j \quad \forall i \in I, j \in J. \quad (\text{BM})$$

Interpretation:

- If $c_{ij} > L_{\max}$ then $M_{ij} = 0$ and assignment from i to j is forbidden ($x_{ij} = 0$).
- If $c_{ij} \leq L_{\max}$ then $M_{ij} = 1$ and assignment is allowed only if shelter j is open.