Optimal Shelter Allocation: Coverage-First Lexicographic Formulation

1 Model Overview

We generate a Pareto-style frontier by varying K, the maximum number of new shelters that may be opened. For each fixed K:

1. **Stage 1 (Coverage-first):** maximize the total covered population, yielding $\alpha^*(K)$.
2. **Stage 2 (Distance tie-break):** among all solutions achieving $\alpha^*(K)$, minimize total travel distance.

Modeling Note

We partition the region into 1 km^2 cells and treat each cell as an assignment node. Splitting is allowed: a cell can send fractions of its population to multiple shelters.

Sets and Indices

- I: set of 1 km^2 population cells, indexed by i.
- J_E: set of existing shelters, indexed by j.
- J_N: set of candidate new-shelter sites, indexed by j.
- $J = J_E \cup J_N$: set of all shelters.

Parameters

- d_i: population in cell $i \in I$.
- u_j: capacity of shelter $j \in J$.
- c_{ij}: travel distance (or cost) from cell i to shelter j.
- K: maximum number of *new* shelters that may be opened.

  Total population:

$$D := \sum_{i \in I} d_i.$$

Decision Variables

- $x_{ij} \in [0, 1]$ for $i \in I,\ j \in J$: fraction of cell i's population assigned to shelter j.
- $y_j \in \{0, 1\}$ for $j \in J_N$:
  - $y_j = 1$ if a new shelter is opened at candidate site j,
  - $y_j = 0$ otherwise.
- For $j \in J_E$, shelters are always open, so $y_j = 1$ is fixed.

  Covered population:

$$S(x) := \sum_{i \in I} \sum_{j \in J} d_i x_{ij}. \tag{1}$$

Coverage fraction:

$$\alpha(x) := \frac{S(x)}{D}. \tag{2}$$

2 Stage 1: Maximize Feasible Coverage Given K

Objective (Stage 1)

For fixed K, prioritize covering as many residents as possible:

$$\max_{x,y} \quad S(x) = \sum_{i \in I} \sum_{j \in J} d_i x_{ij}. \tag{S1-Obj}$$

Constraints (Stage 1)

(1) Assignment constraint. Each cell can assign at most 100% of its population:

$$\sum_{j\in J} x_{ij} \le 1 \quad \forall i \in I. \tag{S1-1}$$

(2) Capacity constraints. Assignments into shelter j cannot exceed its capacity:

$$\sum_{i\in I} d_i x_{ij} \le u_j y_j \quad \forall j \in J. \tag{S1-2}$$

(3) New-shelter budget. At most K new shelters may be opened:

$$\sum_{j\in J_N} y_j \le K. \tag{S1-3}$$

(4) Domains.

$$x_{ij} \in [0,1]\ \forall i,j, \qquad y_j \in \{0,1\}\ \forall j \in J_N, \qquad y_j = 1\ \forall j \in J_E. \tag{S1-4}$$

Stage 1 Output

Let $S^*(K)$ denote the optimal covered population from Stage 1:

$$S^*(K) := \max S(x).$$

Then the maximum feasible coverage fraction at budget K is:

$$\alpha^*(K) := \frac{S^*(K)}{D}. \tag{S1-Out}$$

3 Stage 2: Minimize Distance Among Max-Coverage Solutions

Stage 2 selects the minimum-distance plan *among all solutions that achieve the maximum coverage* $S^*(K)$.

Objective (Stage 2)

$$\min_{x,y} \quad \sum_{i\in I}\sum_{j\in J} c_{ij} d_i x_{ij}. \tag{S2-Obj}$$

Constraints (Stage 2)

$$\sum_{j\in J} x_{ij} \le 1 \qquad \forall i \in I \qquad \text{(assignment)} \tag{S2-1}$$

$$\sum_{i\in I} d_i x_{ij} \le u_j y_j \qquad \forall j \in J \qquad \text{(capacity)} \tag{S2-2}$$

$$\sum_{j\in J_N} y_j \le K \qquad \text{(new-shelter budget)} \tag{S2-3}$$

$$\sum_{i\in I}\sum_{j\in J} d_i x_{ij} \ge S^*(K) \qquad \text{(preserve max coverage)} \tag{S2-4}$$

$$x_{ij} \in [0,1]\ \forall i,j, \qquad y_j \in \{0,1\}\ \forall j \in J_N, \qquad y_j = 1\ \forall j \in J_E. \tag{S2-5}$$

Interpretation

- Stage 1 ensures we cover as many residents as physically possible for each K.
- Stage 2 ensures that, while keeping that maximum coverage, each cell is assigned to the closest feasible open shelters whenever possible (subject to capacity).

Reported Outputs for Each K

- Maximum feasible coverage $\alpha^*(K)$ (Stage 1),
- Minimum total distance under max coverage (Stage 2),
- Shelter locations $y^*(K)$ and assignments $x^*(K)$ (Stage 2).

Optional Extension: Maximum Travel Distance via Big-M

To reflect the possibility that residents cannot travel beyond a fixed distance $L_{\max}$ during evacuation, we consider an assignment cutoff.

Additional Parameter

- $L_{\max}$: maximum allowed travel distance from any cell i to an assigned shelter.
- Define a precomputed indicator:

$$M_{ij} = \begin{cases} 0, & c_{ij} > L_{\max}, \\ 1, & c_{ij} \le L_{\max}. \end{cases}$$

Additional Constraint

Add to both Stage 1 and Stage 2:

$$x_{ij} \le M_{ij} y_j \quad \forall i \in I,\ j \in J. \tag{BM}$$

Interpretation:

- If $c_{ij} > L_{\max}$ then $M_{ij} = 0$ and assignment from i to j is forbidden ($x_{ij} = 0$).
- If $c_{ij} \le L_{\max}$ then $M_{ij} = 1$ and assignment is allowed only if shelter j is open.