Optimal Shelter Allocation: Reformulated Model

1 Revised Deterministic Shelter-Allocation Model

Sets and Indices

- I: set of population nodes (e.g., unions), indexed by i.
- J_E: set of existing shelters, indexed by j.
- J_N: set of candidate new-shelter locations, indexed by j.
- $J = J_E + J_N$: all shelters (existing + candidate).

Parameters

- d_i: population at node $i \in I$.
- u_j: capacity (number of people that can be served) at shelter $j \in J$.
- c_{ij}: travel distance (our cost) from node i to shelter j.
- K: maximum number of *new* shelters allowed to be built (budget constraint on J_N).
- $\alpha \in (0, 1]$: minimum fraction of the total population that must be covered (e.g., $\alpha = 0.75$ if we want 75 percent of the population covered).

Decision Variables

- $x_{ij} \in [0, 1]$ for all $i \in I, j \in J$: fraction of population at node i that is assigned to shelter j.
- $y_j \in \{0, 1\}$ for all $j \in J_N$:
  - $y_j = 1$ if a new shelter is built at candidate site j,
  - $y_j = 0$ otherwise.
- For existing shelters $j \in J_E$, we take $y_j = 1$ as a fixed parameter (we wouldn't remove an existing shelter ever because it can serve people to some capacity).

Objective Function

We now try to minimize the total population-weighted travel distance, subject to limits on the number of new shelters and a required coverage levels:

$$\min_{x,y} \quad \sum_{i \in I} \sum_{j \in J} c_{ij}\, d_i\, x_{ij}. \tag{1}$$

Constraints

(1) Assignment constraint (fractions sum to at most 1). Each population node i can assign at most 100% of its population across shelters:

$$\sum_{j \in J} x_{ij} \;\leq\; 1 \quad \forall i \in I. \tag{2}$$

So the fractions x_{ij} represent the share of node i's population going to each shelter while not being able to assign more than the total population at node i.

(2) Shelter capacity constraints. The total number of people assigned to shelter j can't exceed its capacity:

$$\sum_{i \in I} d_i\, x_{ij} \;\leq\; u_j\, y_j \quad \forall j \in J. \tag{3}$$

So if $y_j = 0$ for a candidate site, the rhs is zero and no one can be assigned there. For known shelters $j \in J_E$, we set $y_j = 1$.

(3) New-shelter budget constraint. We limit the number of new shelters that can be opened:

$$\sum_{j \in J_N} y_j \;\leq\; K. \tag{4}$$

K is the design parameter that controls how many new shelters we're allowed to build. By solving the model for different values of K, we can get the pareto frontier for distance and number of new shelters.

(4) Coverage constraint. We require that at least a fraction α of the total population is assigned to some shelter:

$$\sum_{i \in I} \sum_{j \in J} d_i \, x_{ij} \;\geq\; \alpha \sum_{i \in I} d_i. \tag{5}$$

Intuition:
- LHS = total number of people actually assigned to any shelter.
- RHS = α times the total population across all nodes i=1..I.

This prevents the model from "saving" distance by leaving large portions of the population unassigned; instead, it must cover at least an α fraction of people, and then minimize distance subject to that requirement.

Combining the objective and constraints, the full deterministic model is:

$$\begin{aligned}
\min_{x,y} \quad & \sum_{i \in I} \sum_{j \in J} c_{ij} \, d_i \, x_{ij} & & & (6)\\
\text{s.t.} \quad & \sum_{j \in J} x_{ij} \leq 1 & \forall i \in I & \quad \text{(assignment)} & (7)\\
& \sum_{i \in I} d_i \, x_{ij} \leq u_j \, y_j & \forall j \in J & \quad \text{(capacity)} & (8)\\
& \sum_{j \in J_N} y_j \leq K & & \quad \text{(new-shelter budget)} & (9)\\
& \sum_{i \in I} \sum_{j \in J} d_i \, x_{ij} \geq \alpha \sum_{i \in I} d_i & & \quad \text{(coverage requirement)} & (10)\\
& x_{ij} \in [0,1] & \forall i \in I, j \in J & & (11)\\
& y_j \in \{0,1\} & \forall j \in J_N, & & (12)\\
& y_j = 1 & \forall j \in J_E. & & (13)
\end{aligned}$$

Summary

- **Objective:** minimize total distance traveled by all assigned residents.
- **Altering K:** how many new shelters we are allowed to build.
- **Altering α:** the minimum coverage level we insist on (fraction of population assigned to shelters).
- **Trade-off:** by varying K (and possibly α), we can examine how total distance and coverage respond to different infrastructure investment levels.

Other Recommendation/Possible Extension: Maximum Travel Distance via Big-M

To reflect that in the event of a weather emergency, it may be worse for residents to travel beyond a certain amount - whether that be due to incoming weather conditions or lack of resources - we can also look into enforcing a max travel distance $L_{\max}$.

Additional Parameter

- $L_{\max}$: maximum allowed travel distance from any population node i to its assigned shelter.
- Define M_{ij} as a (precomputed) parameter:

$$M_{ij} = \begin{cases} 0, & \text{if } c_{ij} > L_{\max}, \\ 1, & \text{if } c_{ij} \leq L_{\max}. \end{cases}$$

Additional Big-M Constraint

We then add the following constraint:

$$x_{ij} \;\le\; M_{ij}\, y_j \quad \forall i \in I,\; j \in J. \tag{14}$$

Interpretation:

- If $c_{ij} > L_{\max}$, then $M_{ij} = 0$ and (14) forces $x_{ij} \le 0$, i.e., no assignment from i to j is allowed because the shelter is too far.
- If $c_{ij} \le L_{\max}$, then $M_{ij} = 1$ and (14) reduces to $x_{ij} \le y_j$:
  - If $y_j = 0$ (shelter not opened), then $x_{ij} = 0$ (cannot assign).
  - If $y_j = 1$, then x_{ij} is free up to the other constraints (capacity, coverage, etc.).

So, residents are only assigned to shelters within $L_{\max}$ distance, while the model still chooses locations and assignments to minimize distance, satisfy capacity, and meet the coverage and shelter-budget constraints.