DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI
*****MA102 Mathematics-II : Test 4*****
Date: June 24, 2021

Total Time: **90** Minutes (10:00 am to 11:30 am) Total Marks: **25** Marks

Instructions:

- The question paper has **FIVE** questions. Answer **ALL** questions. Answers to all subdivisions/ subparts of a question should appear together.

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1. Find a fundamental set of solutions of the linear homogeneous system $\mathbf{x}' = A\,\mathbf{x}$ where

$$A = \begin{bmatrix} -2 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

(**5** marks)

Solution:
Step 1: Finding the eigenvalues of A

$$|A - \lambda I| = 0 \qquad \Longrightarrow \qquad (-1)(\lambda+2)(\lambda^2 + 4\lambda + 5) = 0\,.$$

Therefore the eigenvalues of A are $\lambda_1 = -2$, $\lambda_2 = -2 + i$ and $\lambda_3 = -2 - i = \overline{\lambda_2}$.
(1 mark)

Step 2: Finding the eigenvectors of A
Corresponding to the eigenvalue $\lambda_1 = -2$, we find the nontrivial solution of $(A - \lambda_1 I)\mathbf{v} = \mathbf{0}$ as follows.

$$(A + 2I)\mathbf{v} = \mathbf{0} \qquad \Longrightarrow \qquad \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

It gives that $x_1 = 0$ and $x_2 = 0$. We choose $x_3 = 1$. Therefore $\mathbf{v}_1 = (0,\ 0,\ 1)^T$ is an eigenvector corresponding to the real eigenvalue $\lambda_1 = -2$. (1 mark)

Corresponding to the eigenvalue $\lambda_2 = -2 + i$, we find the nontrivial solution of $(A - \lambda_2 I)\mathbf{w} = \mathbf{0}$ as follows.

$$(A - (-2+i)I)\mathbf{w} = \mathbf{0} \qquad \Longrightarrow \qquad \begin{bmatrix} -i & 1 & 0 \\ -1 & -i & 0 \\ 0 & 0 & -i \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

It gives that $w_3 = 0$ and $-iw_1 + w_2 = 0$. Choose $w_1 = 1$, we get $w_2 = i$. Therefore $\mathbf{w} = (1,\ i,\ 0)^T$ is an eigenvector corresponding to the complex eigenvalue $\lambda_2 = -2 + i$.
Set

$$\mathbf{u} = \mathrm{Re}\,(\mathbf{w}) = (1,\ 0,\ 0)^T \qquad \text{and} \qquad \mathbf{v} = \mathrm{Im}\,(\mathbf{w}) = (0,\ 1,\ 0)^T\ .$$

(1 mark)

Step 3: Writing the matrices P, D, P^{-1}

The matrix P is given by

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{u} & \mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\ .$$

Therefore

$$P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\ .$$

The matrix D is given by

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & -1 & -2 \end{bmatrix}\ .$$

(1 mark)

Step 3: Computing the matrices e^{Dt} and e^{At}

$$e^{Dt} = \begin{bmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{-2t}\cos t & e^{-2t}\sin t \\ 0 & -e^{-2t}\sin t & e^{-2t}\cos t \end{bmatrix}\ .$$

A fundamental matrix e^{At} is given by

$$\begin{aligned} e^{At} = Pe^{Dt}P^{-1} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{-2t}\cos t & e^{-2t}\sin t \\ 0 & -e^{-2t}\sin t & e^{-2t}\cos t \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\ . \\ &= \begin{bmatrix} e^{-2t}\cos t & e^{-2t}\sin t & 0 \\ -e^{2t}\sin t & e^{-2t}\cos t & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}\ . \end{aligned}$$

(1 mark)

A fundamental set of solutions to $\mathbf{x}' = A\mathbf{x}$ is given by $\{\mathbf{\Phi}_1,\ \mathbf{\Phi}_2,\ \mathbf{\Phi}_3\}$ where

$$\mathbf{\Phi}_1(t) = \begin{bmatrix} e^{-2t}\cos t \\ -e^{-2t}\sin t \\ 0 \end{bmatrix}, \qquad \mathbf{\Phi}_2(t) = \begin{bmatrix} e^{-2t}\sin t \\ e^{-2t}\cos t \\ 0 \end{bmatrix}, \qquad \mathbf{\Phi}_1(t) = \begin{bmatrix} 0 \\ 0 \\ e^{-2t} \end{bmatrix}\ .$$

Note: If some one takes different eigenvectors, then the matrix P will be different and the further calculation will change accordingly. If some one takes the matrix P as $P = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{v}_1 \end{bmatrix}$ then $P = I = P^{-1}$ and hence $D = A$ and $e^{Dt} = e^{At}$.

2. Using the **method of variation of parameters**, solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 4\,e^{2t} \\ 4\,e^{4t} \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

(**5** marks)

Solution:
Step 1: Finding a Fundamental Matrix $\mathbf{\Phi}(t)$
Step 1(a): Finding the eigenvalues of A

$$|A - \lambda I| = 0 \qquad \Longrightarrow \qquad (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8 = 0 .$$

Therefore the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 4$. (0.5 mark)
Step 1(b): Finding the eigenvectors of A
Corresponding to the eigenvalue $\lambda_1 = 2$, we find the nontrivial solution of $(A - \lambda_1 I)\mathbf{v} = \mathbf{0}$ as follows.

$$(A - 2I)\mathbf{v} = \mathbf{0} \qquad \Longrightarrow \qquad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

It gives that $x_1 = x_2$. We choose $x_1 = 1$. Therefore $\mathbf{v}_1 = (1,\ 1)^T$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 2$. (0.5 mark)
Corresponding to the eigenvalue $\lambda_2 = 4$, we find the nontrivial solution of $(A - \lambda_2 I)\mathbf{v} = \mathbf{0}$ as follows.

$$(A - 4I)\mathbf{v} = \mathbf{0} \qquad \Longrightarrow \qquad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

It gives that $x_1 = -x_2$. We choose $x_1 = 1$. Therefore $\mathbf{v}_1 = (1,\ -1)^T$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 4$. (0.5 mark)
Step 1(c): Writing a Fundamental Matrix $\mathbf{\Phi}(t)$
A fundamental matrix $\mathbf{\Phi}(t)$ is given by

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{2t} & e^{4t} \\ e^{2t} & -e^{4t} \end{bmatrix} .$$

(0.5 mark)

Step 2: Computing $\mathbf{\Phi}^{-1}(t)$

$$\mathbf{\Phi}^{-1}(t) = \left(\frac{-1}{2e^{6t}}\right)\begin{bmatrix}-e^{4t} & -e^{4t}\\ -e^{2t} & e^{2t}\end{bmatrix} = \left(\frac{1}{2}\right)\begin{bmatrix}e^{-2t} & e^{-2t}\\ e^{-4t} & -e^{-4t}\end{bmatrix}.$$

(0.5 mark)

Step 3: Computing $\int_0^t \mathbf{\Phi}^{-1}(u)F(u)\,du$

$$\int_0^t \mathbf{\Phi}^{-1}(u)F(u)\,du = \left(\frac{1}{2}\right)\int_0^t \begin{bmatrix}e^{-2u} & e^{-2u}\\ e^{-4u} & -e^{-4u}\end{bmatrix}\begin{bmatrix}4e^{2u}\\ 4e^{4u}\end{bmatrix}du$$

$$= \left(\frac{1}{2}\right)\int_0^t \begin{bmatrix}4e^{2u}+4\\ 4e^{-2u}-4\end{bmatrix}du = \left(\frac{1}{2}\right)\begin{bmatrix}2e^{2t}+4t-2\\ -2e^{-2t}-4t+2\end{bmatrix} = \begin{bmatrix}e^{2t}+2t-1\\ -e^{-2t}-2t+1\end{bmatrix}.$$

(1 mark)

Step 4: Computing $\mathbf{x}_p(t)$

$$\mathbf{x}_p(t) = \mathbf{\Phi}(t)\int_0^t \mathbf{\Phi}^{-1}(u)F(u)\,du = \begin{bmatrix}e^{2t} & e^{4t}\\ e^{2t} & -e^{4t}\end{bmatrix}\begin{bmatrix}e^{2t}+2t-1\\ -e^{-2t}-2t+1\end{bmatrix}$$

$$= \begin{bmatrix}-2te^{4t}+2e^{4t}+2te^{2t}-2e^{2t}\\ 2te^{4t}+2te^{2t}\end{bmatrix}$$

$$\mathbf{x}_p(t) = \begin{bmatrix}-2\\ 2\end{bmatrix}te^{4t} + \begin{bmatrix}2\\ 0\end{bmatrix}e^{4t} + \begin{bmatrix}2\\ 2\end{bmatrix}te^{2t} + \begin{bmatrix}-2\\ 0\end{bmatrix}e^{2t}.$$

(0.5 mark)

Step 5: Computing $\mathbf{x}_h(t)$

$$\mathbf{x}_h(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(0)\mathbf{x}_0 = \begin{bmatrix}e^{2t} & e^{4t}\\ e^{2t} & -e^{4t}\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{-1}{2}\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\mathbf{x}_h(t) = \begin{bmatrix}e^{2t}\\ e^{2t}\end{bmatrix}.$$

(0.5 mark)

Step 6: Writing the unique solution $\mathbf{x}(t)$ to the given IVP

$$\mathbf{x}(t) = \mathbf{x}_h(t)+\mathbf{x}_p(t) = \begin{bmatrix}e^{2t}\\ e^{2t}\end{bmatrix} + \begin{bmatrix}-2\\ 2\end{bmatrix}te^{4t} + \begin{bmatrix}2\\ 0\end{bmatrix}e^{4t} + \begin{bmatrix}2\\ 2\end{bmatrix}te^{2t} + \begin{bmatrix}-2\\ 0\end{bmatrix}e^{2t}$$

$$\mathbf{x}(t) = \begin{bmatrix}-2\\ 2\end{bmatrix}te^{4t} + \begin{bmatrix}2\\ 0\end{bmatrix}e^{4t} + \begin{bmatrix}2\\ 2\end{bmatrix}te^{2t} + \begin{bmatrix}-1\\ 1\end{bmatrix}e^{2t}.$$

(0.5 mark)

Note: If a student computes the particular solution **without using the method of variation of parameters** then award **zero** marks.

3. Find the general solution in the neighborhood of the ordinary point $x_0 = 0$ of

$$(1 + x^2)\, y'' \; + \; 2x\, y' \; - \; 2y \; = \; 0\,.$$

(**5** marks)

Solution:

Step 1: Writing Form of the Series Solution, Substituting it in the ODE, Shifting the indices and Gathering all terms of same power of x

Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$ for $|x| < R$, where $R > 0$.

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\,.$$

Substituting in the given ODE, we get

$$(1 + x^2)\; \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \; + \; 2x \sum_{n=1}^{\infty} n a_n x^{n-1} \; - \; 2 \sum_{n=0}^{\infty} a_n x^n \; = \; 0\,.$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \; + \; \sum_{n=2}^{\infty} n(n-1) a_n x^n \; + \; 2 \sum_{n=1}^{\infty} n a_n x^n \; - \; 2 \sum_{n=0}^{\infty} a_n x^n \; = \; 0\,.$$

(0.5 mark)

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n \; + \; \sum_{n=2}^{\infty} n(n-1) a_n x^n \; + \; 2 \sum_{n=1}^{\infty} n a_n x^n \; - \; 2 \sum_{n=0}^{\infty} a_n x^n \; = \; 0\,.$$

$$(2a_2 - 2a_0) + 6a_3\, x + \sum_{n=2}^{\infty} \left[(n+1)(n+2) a_{n+2} + n(n-1) a_n + 2n a_n - 2a_n \right]\, x^n \; = \; 0\,.$$

(0.5 mark)

Step 2: Equating each coefficient to zero

$$2a_2 - 2a_0 = 0 \qquad \Longrightarrow \qquad a_2 = a_0\,.$$

$$6a_3 = 0 \qquad \Longrightarrow \qquad a_3 = 0\,.$$

(0.5 mark)

For $n \geq 2$,

$$(n+1)(n+2) a_{n+2} + n(n-1) a_n + 2n a_n - 2a_n = 0 \qquad \Longrightarrow \qquad a_{n+2} \; = \; \frac{(-1)(n-1)}{(n+1)}\, a_n\,.$$

(0.5 mark)

Step 3: Expressing a_n for $n \geq 2$ in terms of a_0 and a_1

$$
\begin{aligned}
a_4 &= \frac{(-1)(2-1)}{(2+1)} a_2 = \frac{-a_2}{3} = -\frac{1}{3} a_0 \, . \\
a_5 &= \frac{(-1)(3-1)}{(3+1)} a_3 = \frac{-2a_3}{4} = 0 \, . \\
a_6 &= \frac{(-1)(4-1)}{(4+1)} a_4 = \frac{-3a_4}{5} = \frac{1}{5} a_0 \, . \\
a_7 &= \frac{(-1)(5-1)}{(5+1)} a_5 = \frac{-4a_5}{6} = 0 \, .
\end{aligned}
$$

In general,

$$a_{2n+1} = 0 \qquad \text{for} \qquad n = 1,\, 2,\, 3,\, 4,\, \ldots \, .$$

(0.5 mark)

$$a_{2n} = \frac{(-1)^{n+1}\, a_0}{(2n-1)} \qquad \text{for} \qquad n = 1,\, 2,\, 3, \ldots \, .$$

(0.5 mark)

Step 3: Writing Two LI Solutions

We get two linearly independent solutions as

$$f_1(x) = a_0 + a_0 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} x^{2n} \qquad \text{for} \qquad |x| < 1 \, ,$$

(0.5 mark)

$$f_2(x) = a_1 x \qquad \text{for} \qquad |x| < \infty \, .$$

(0.5 mark)

Step 4: Writing the General Solution

The general solution to the given ODE is given by

$$y(x) = A\, y_1(x) + B\, y_2(x) \qquad \text{for} \qquad |x| < 1 \, ,$$

where

$$
\begin{aligned}
y_1(x) &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} x^{2n} \qquad \text{for} \qquad |x| < 1 \, , \\
y_2(x) &= x \qquad \text{for} \qquad |x| < \infty \, ,
\end{aligned}
$$

and A and B are arbitrary real constants. (1 mark)

4. Find a series solution $y_1(x)$ corresponding to the root r_1 about the singular point $x_0 = 0$ of the differential equation

$$4x^2\, y'' \;-\; 8x^2\, y' \;+\; (4x^2+1)\, y \;=\; 0$$

and write $y_1(x)$ in a closed form. (**5** marks)

Solution:

Step 1: Writing Form of the Series Solution, Substituting it in the ODE, Shifting the indices and Gathering all terms of same power of x

Let $y(x) = x^r \displaystyle\sum_{n=0}^{\infty} a_n x^n$ for $0 < x < R$, where $a_0 \neq 0$ and for some $R > 0$.

$$y'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}, \qquad y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \,.$$

Substituting in the given ODE, we get

$$4\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - 8\sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1} + 4\sum_{n=0}^{\infty} a_n x^{n+r+2} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \,.$$

(0.5 mark)

$$4\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - 8\sum_{n=1}^{\infty}(n+r-1)a_{n-1} x^{n+r} + 4\sum_{n=2}^{\infty} a_{n-2} x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \,.$$

$$(4r(r-1)+1)a_0\, x^r \;+\; [4r(1+r)a_1 - 8ra_0 + a_1]\;\, x^{r+1}$$

$$+\; \sum_{n=2}^{\infty} [4(n+r)(n+r-1)a_n - 8(n+r-1)a_{n-1} + 4a_{n-2} + a_n]\;\, x^{n+r} = 0 \,.$$

(1 mark)

Step 2: Equating the coefficient of the lowest power of x to zero to get Indicial equation and its roots

Equating the coefficient of x^r to zero, we get the indicial equation as

$$4r(r-1) + 1 = 4r^2 - 4r + 1 = 0 \,.$$

(0.5 mark)

Its roots are

$$\textit{Exponents}: \qquad r_1 = r_2 = \frac{1}{2} \,.$$

Since $r_1 = r_2$, it follows that $r_1 - r_2 = 0$ is a nonnegative integer and we get only one series solution corresponding to $r_1 = 1/2$. (0.5 mark)

Step 3: Equating the coefficients of higher powers of x to zero

Equating the coefficient of x^{r+1} to zero by taking $r = 1/2$, we get

$$4r(1+r)a_1 - 8ra_0 + a_1 = 0 \qquad \Longrightarrow \qquad a_1 = a_0 \ .$$

(0.5 mark)

Equating the coefficient of x^{n+r} to zero, we get

$$4(n+r)(n+r-1)a_n - 8(n+r-1)a_{n-1} + 4a_{n-2} + a_n = 0 \qquad \text{for } n \geq 2 \ .$$

Putting $r = 1/2$, we get

$$a_n = \frac{8\left(n - \frac{1}{2}\right) a_{n-1} - 4a_{n-2}}{4n^2} = \frac{(2n-1)\, a_{n-1} \ - \ a_{n-2}}{n^2} \qquad \text{for } n \geq 2 \ .$$

(0.5 mark)

Step 4: Expressing a_n for $n \geq 1$ in terms of a_0

Now,

$$\begin{aligned} a_2 &= \frac{3a_1 - a_0}{2^2} = \frac{a_0}{2} = \frac{a_0}{2!} \ . \\ a_3 &= \frac{5a_2 - a_1}{3^2} = \frac{a_0}{6} = \frac{a_0}{3!} \ . \\ a_4 &= \frac{7a_3 - a_2}{4^2} = \frac{a_0}{24} = \frac{a_0}{4!} \ . \end{aligned}$$

In general,

$$a_n = \frac{a_0}{n!} \qquad \text{for} \qquad n \geq 2 \ .$$

(0.5 mark)

Step 5: Writing the series solution $y_1(x)$ along with its domain of convergence

$$y_1(x) = x^{1/2} \left\{ a_0 + a_0 x + \sum_{n=2}^{\infty} \frac{a_0}{n!} x^n \right\} = x^{1/2} \left\{ a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} \right\} \text{ for } 0 < x < \infty \ .$$

$$y_1(x) = a_0 \ |x|^{1/2} \left\{ \sum_{n=0}^{\infty} \frac{x^n}{n!} \right\} \text{ for } 0 < |x| < \infty \ .$$

(0.5 mark)

Step 6: Writing $y_1(x)$ in closed form

$$y_1(x) = = a_0 \ |x|^{1/2} \ e^x \qquad \text{for} \qquad 0 < |x| < \infty \ ,$$

where a_0 is an arbitrary real constant. (0.5 mark)

5. (a) Write the system that is equivalent to the equation $\dfrac{d^2x}{dt^2} + 4x = 0$. Then, locate the critical point of the system and determine its type and stability.

(b) Determine the type and stability of the critical point $(0,0)$ of the following nonlinear autonomous system.

$$\begin{aligned} \frac{dx}{dt} &= x + x^2 - 4xy, \\ \frac{dy}{dt} &= -2x + y + 8y^2. \end{aligned}$$

(c) Let $P_n(x)$ denote the Legendre polynomial of degree $n = 0,\ 1,\ 2,\ \ldots$. Compute the value of the integral $\displaystyle\int_{-1}^{1} x\, P_9(x)\, P_{10}(x)\, dx$.

(1.5 + 1.5 + 2 = **5** marks)

Solution:

5(a) The Linear Autonomous System that is equivalent to the differential equation $\dfrac{d^2x}{dt^2} + 4x = 0$ is

$$\begin{aligned} \frac{dx}{dt} &= y\ , \\ \frac{dy}{dt} &= -4x\ . \end{aligned}$$

The critical point of this system is $(0,0)$. (0.5 mark)

Now, we find the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$. The characteristic polynomial of A is $\lambda^2 + 4 = 0$. So, $\lambda_1 = 2i$ and $\lambda_2 = -2i$ are the eigenvalues of A. (0.5 mark)

Since the eigenvalues are complex and pure imaginary, the critical point $(0,0)$ is a **center** and **stable**. (0.5 mark)

Note: If somebody writes just the critical point $(0,0)$ is a center and stable **without computing the eigenvalues of** A, then award only 0.5 mark.

5(b) Consider the corresponding linear autonomous system

$$\begin{aligned} \frac{dx}{dt} &= x\ , \\ \frac{dy}{dt} &= -2x + y\ . \end{aligned}$$

Now, we find the eigenvalues of $A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$. The characteristic polynomial of A is $(1-\lambda)^2 = 0$ and hence $\lambda = 1$ is the repeated real eigenvalue of A.

(0.5 mark)

Since the eigenvalues are real, equal and positive, the critical point $(0, 0)$ is a **node** and **unstable** for the Linear Autonomous System. (0.5 mark)

Since the eigenvalues are real, equal and positive; and A is **non-diagonal**, we conclude that $(0, 0)$ is a **node** and **unstable** for the given Nonlinear Autonomous System. (0.5 mark)

Note: If somebody writes just the critical point $(0, 0)$ is a node and unstable for the given Nonlinear Autonomous System **without computing the eigenvalues of** A, then award only 0.5 mark.

5(c) We know that the Bonnet's three term recurrence relation given by

$$(n+1)\ P_{n+1}(x) - (2n+1)x\ P_n(x) + n\ P_{n-1}(x) = 0 \qquad \text{for } n \in \mathbb{N}\ .$$

Take $n = 10$ in the above expression and we get

$$11\ P_{11}(x) - 21\ x\ P_{10}(x) +\ 10\ P_9(x) = 0\ .$$

Multiplying the above equation by $P_9(x)$, we get

$$x\ P_{10}(x)P_9(x)\ =\ \frac{10}{21}\ P_9(x)P_9(x)\ +\ \frac{11}{21}\ P_{11}(x)\ P_9(x)\ .$$

(1 mark)

$$\int_{-1}^{1} x\ P_{10}(x)P_9(x)\ dx\ =\ \int_{-1}^{1} \frac{10}{21}\ P_9(x)P_9(x)\ dx\ +\ \int_{-1}^{1} \frac{11}{21}\ P_{11}(x)\ P_9(x)\ dx\ .$$

We know that

$$\int_{-1}^{1} P_m(x)P_n(x)\ dx\ =\ \begin{cases} 0 & \text{if}\quad m \neq n\ , \\ \frac{2}{2n+1} & \text{if}\quad m = n\ . \end{cases}$$

Therefore,

$$\int_{-1}^{1} x\ P_{10}(x)P_9(x)\ dx\ =\ \frac{10}{21} \times \frac{2}{19}\ =\ \frac{20}{399}\ .$$

(1 mark)

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We know that the Bonnet's three term recurrence relation given by

$$(n+1)\ P_{n+1}(x) - (2n+1)x\ P_n(x) + n\ P_{n-1}(x) = 0 \qquad \text{for } n \in \mathbb{N}\ .$$

Take $n = 9$ in the above expression and we get

$$10\ P_{10}(x) - 19x\ P_9(x) +\ 9\ P_8(x) = 0\ .$$

Multiplying the above equation by $P_{10}(x)$, we get

$$x\ P_9(x)P_{10}(x) \ = \ \frac{10}{19}P_{10}(x)P_{10}(x) \ + \ \frac{9}{19}\ P_8(x)P_{10}(x)\ .$$

(1 mark)

$$\int_{-1}^{1} x\ P_9(x)P_{10}(x)\ dx \ = \ \int_{-1}^{1} \frac{10}{19}P_{10}(x)P_{10}(x)\ dx \ + \ \int_{-1}^{1} \frac{9}{19}\ P_8(x)P_{10}(x)\ dx\ .$$

We know that

$$\int_{-1}^{1} P_m(x)P_n(x)\ dx \ = \ \begin{cases} 0 & \text{if} \ \ m \neq n\ , \\ \frac{2}{2n+1} & \text{if} \ \ m = n\ . \end{cases}$$

Therefore,

$$\int_{-1}^{1} x\ P_9(x)P_{10}(x)\ dx \ = \ \frac{10}{19} \times \frac{2}{21} \ = \ \frac{20}{399}\ .$$

(1 mark)

Note: One can **derive** in same way the following

$$\int_{-1}^{1} x\ P_n(x)\ P_{n-1}(x)\ dx \ = \ \frac{2n}{4n^2-1}$$

and substitute the value for n to get the desired answer. If a student derives the above expression and substitutes for n, then he/she gets 2 marks. If a student does not derive the above expression, but just uses this expression, then he/she gets only 0.5 mark.

Paper Ends