CSCE 5218 & 4930
Deep Learning

Machine Learning Overview

Slides adapted from Adriana Kovashka

Linear algebra review

See http://cs229.stanford.edu/section/cs229-linalg.pdf for more

Vectors and Matrices

- Vectors and matrices are just collections of ordered numbers that represent something: movements in space, scaling factors, word counts, movie ratings, pixel brightnesses, etc.

- We'll define some common uses and standard operations on them.

Vector

- A column vector $\mathbf{v} \in \mathbb{R}^{n \times 1}$ where

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

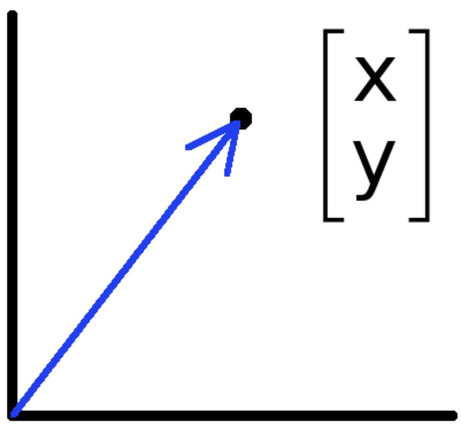
- A row vector $\mathbf{v}^T \in \mathbb{R}^{1 \times n}$ where

$$\mathbf{v}^T = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$$

T denotes the transpose operation

- You need to keep track of orientation

Vectors have two main uses



- Vectors can represent an offset in 2D or 3D space
- Points are just vectors from the origin

- Data can also be treated as a vector
- Such vectors don't have a geometric interpretation, but calculations like "distance" still have value

Matrix

- A matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is an array of numbers with size $m \downarrow$ by $n \rightarrow$, i.e. m rows and n columns.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

- If $m = n$, we say that $\mathbf{A}$ is square.

Matrix Operations

- Addition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a+1 & b+2 \\ c+3 & d+4 \end{bmatrix}$$

  - Can only add a matrix with matching dimensions, or a scalar.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + 7 = \begin{bmatrix} a+7 & b+7 \\ c+7 & d+7 \end{bmatrix}$$

- Scaling

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times 3 = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$$

Inner vs outer vs matrix vs element-wise product

- $\boldsymbol{x}, \boldsymbol{y}$ = column vectors (nx1)
- $\boldsymbol{X}, \boldsymbol{Y}$ = matrices (mxn and nxp)
- x, y = scalars (1x1)

- $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}^T \boldsymbol{y}$ = inner product (1xn x nx1 = scalar)
- $\boldsymbol{x} \otimes \boldsymbol{y} = \boldsymbol{x}\,\boldsymbol{y}^T$ = outer product (nx1 x 1xn = matrix)

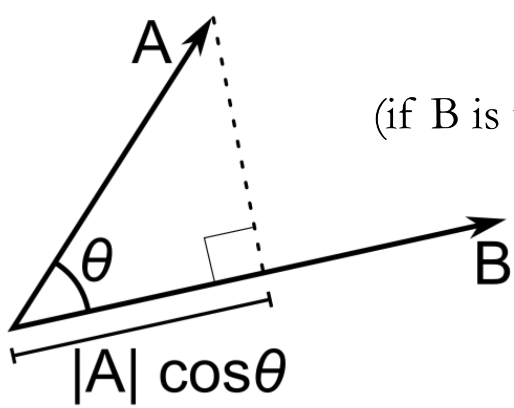
- $\boldsymbol{X} * \boldsymbol{Y}$ = matrix product
  - Watch out: could also be element-wise product in NumPy, if class is array rather than matrix

Inner Product

- Multiply corresponding entries of two vectors and add up the result

$$\mathbf{x}^T\mathbf{y} = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i \qquad \text{(scalar)}$$

- x·y is also |x||y|Cos(*angle between x and y*)
- If B is a unit vector, then A·B gives the length of A which lies in the direction of B (projection)



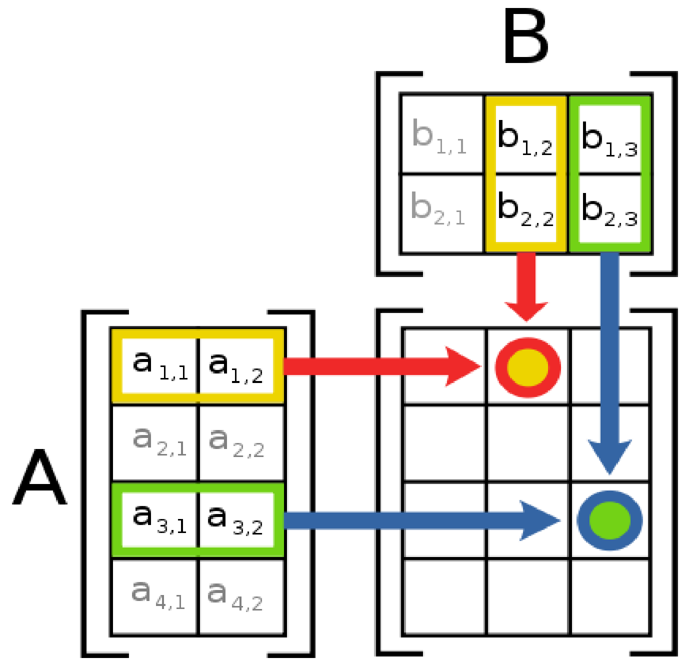
(if B is unit-length hence norm is 1)

Matrix Multiplication

- Let X be an axb matrix, Y be an bxc matrix
- Then Z = X*Y is an axc matrix
- Second dimension of first matrix, and first dimension of second matrix have to be the same, for matrix multiplication to be possible
- Practice: Let X be an 10x5 matrix. Let's factorize it into 3 matrices…

Matrix Multiplication

- The product AB is:



- Each entry in the result is (that row of A) dot product with (that column of B)

Matrix Multiplication

- Example:

A x B

$$\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} & 14 \\ & \end{bmatrix}$$

$$0 \cdot 3 + 2 \cdot 7 = 14$$

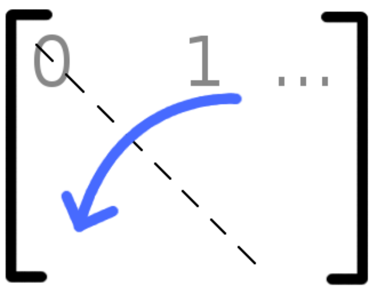
  – Each entry of the matrix product is made by taking the dot product of the corresponding row in the left matrix, with the corresponding column in the right one.

Matrix Operation Properties

- Matrix addition is commutative and associative
  - $A + B = B + A$
  - $A + (B + C) = (A + B) + C$
- Matrix multiplication is associative and distributive but *not* commutative
  - $A(B*C) = (A*B)C$
  - $A(B + C) = A*B + A*C$
  - $A*B != B*A$

Matrix Operations

- Transpose – flip matrix, so row 1 becomes column 1



$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix}$$

- A useful identity:

$$(ABC)^T = C^T B^T A^T$$

Inverse

- Given a matrix **A**, its inverse $\mathbf{A}^{-1}$ is a matrix such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$
- E.g. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$
- Inverse does not always exist. If $\mathbf{A}^{-1}$ exists, **A** is *invertible* or *non-singular*. Otherwise, it's *singular*.

Special Matrices

- Identity matrix **I**
  - Square matrix, 1's along diagonal, 0's elsewhere
  - **I** · [another matrix] = [that matrix]

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Diagonal matrix
  - Square matrix with numbers along diagonal, 0's elsewhere
  - A diagonal · [another matrix] scales the rows of that matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$$

Special Matrices

- Symmetric matrix

$$\mathbf{A}^T = \mathbf{A} \qquad \begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 7 \\ 5 & 7 & 1 \end{bmatrix}$$

Norms

- L1 norm

$$\|\boldsymbol{x}\|_1 := \sum_{i=1}^{n} |x_i|$$

- L2 norm

$$\|\boldsymbol{x}\| := \sqrt{x_1^2 + \cdots + x_n^2}$$

- L*p* norm (for real numbers $p \geq 1$)

$$\|\mathbf{x}\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

System of Linear Equations

- MATLAB example // linalg.solve or lingalg.lstsq in Pvthon

$$AX = B$$

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

```
>> x = A\B
x =
    1.0000
   -0.5000
```

Matrix Rank

- Column/row rank

$$\text{col-rank}(\mathbf{A}) = \text{ the maximum number of linearly independent column vectors of } \mathbf{A}$$
$$\text{row-rank}(\mathbf{A}) = \text{ the maximum number of linearly independent row vectors of } \mathbf{A}$$

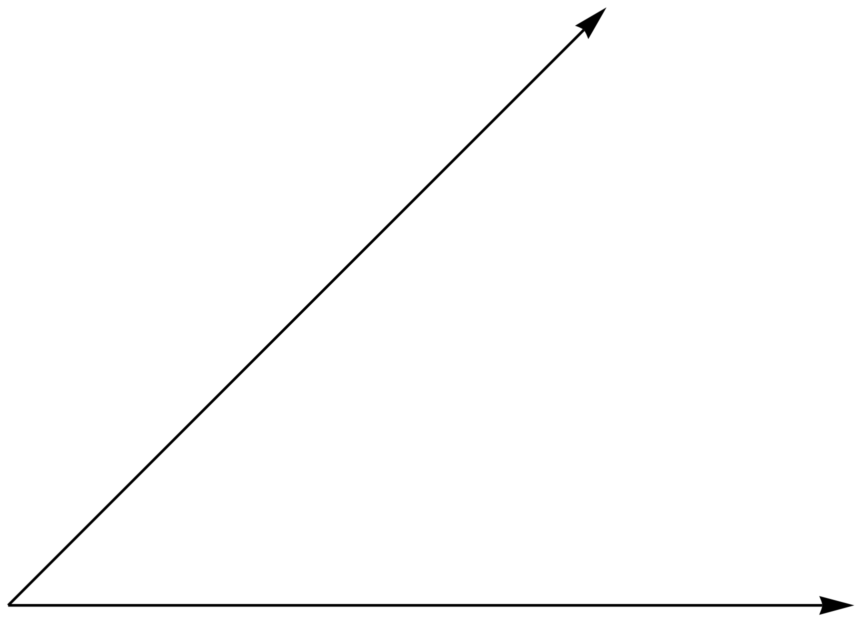
- Column rank always equals row rank
- Matrix rank $\text{rank}(\mathbf{A}) \triangleq \text{col-rank}(\mathbf{A}) = \text{row-rank}(\mathbf{A})$
- If a matrix is not full rank, inverse doesn't exist
  - Inverse also doesn't exist for non-square matrices

Linear independence

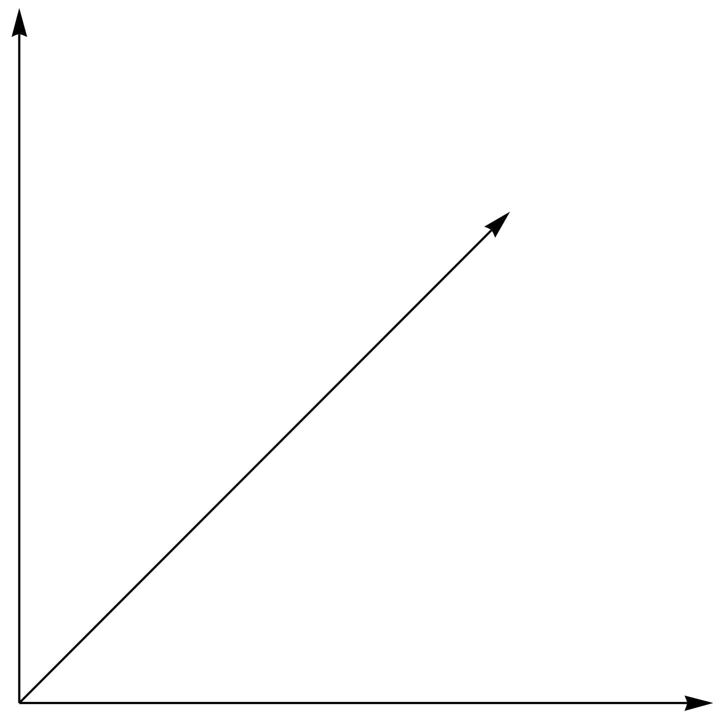
- Suppose we have a set of vectors $\mathbf{v_1, \ldots, v_n}$
- If we can express $\mathbf{v}_1$ as a linear combination of the other vectors $\mathbf{v}_2 \ldots \mathbf{v}_n$, then $\mathbf{v}_1$ is linearly *dependent* on the other vectors.
  - The direction $\mathbf{v}_1$ can be expressed as a combination of the directions $\mathbf{v}_2 \ldots \mathbf{v}_n$. (E.g. $\mathbf{v}_1 = .7\ \mathbf{v}_2 - .5\ \mathbf{v}_4$)
- If no vector is linearly dependent on the rest of the set, the set is linearly *independent*.
  - Common case: a set of vectors $\mathbf{v_1, \ldots, v_n}$ is always linearly independent if each vector is perpendicular to every other vector (and non-zero)

Linear independence

Linearly independent set



Not linearly independent



Singular Value Decomposition (SVD)

- There are several computer algorithms that can "factor" a matrix, representing it as the product of some other matrices
- The most useful of these is the Singular Value Decomposition
- Represents any matrix **A** as a product of three matrices: $\mathbf{U\Sigma V^T}$

Singular Value Decomposition (SVD)

$$\mathbf{U\Sigma V^T = A}$$

- Where **U** and **V** are rotation matrices, and **Σ** is a scaling matrix. For example:

$$\begin{array}{ccccccc} U & & \Sigma & & V^T & & A \\ \begin{bmatrix} -.40 & .916 \\ .916 & .40 \end{bmatrix} & \times & \begin{bmatrix} 5.39 & 0 \\ 0 & 3.154 \end{bmatrix} & \times & \begin{bmatrix} -.05 & .999 \\ .999 & .05 \end{bmatrix} & = & \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix} \end{array}$$

Singular Value Decomposition (SVD)

- In general, if **A** is *m* x *n*, then **U** will be *m* x *m*, **Σ** will be *m* x *n*, and $\mathbf{V^T}$ will be *n* x *n*.

$$\underset{U}{\begin{bmatrix} -.39 & -.92 \\ -.92 & .39 \end{bmatrix}} \times \underset{\Sigma}{\begin{bmatrix} 9.51 & 0 & 0 \\ 0 & .77 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

Singular Value Decomposition (SVD)

- **U** and **V** are always rotation matrices.
  - Geometric rotation may not be an applicable concept, depending on the matrix. So we call them "unitary" matrices – each column is a unit vector.
- **Σ** is a diagonal matrix
  - The number of nonzero entries = rank of **A**
  - The algorithm always sorts the entries high to low

$$\underset{U}{\begin{bmatrix} -.39 & -.92 \\ -.92 & .39 \end{bmatrix}} \times \underset{\Sigma}{\begin{bmatrix} 9.51 & 0 & 0 \\ 0 & .77 & 0 \end{bmatrix}} \times \underset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

Singular Value Decomposition (SVD)

$$\mathbf{M} = \mathbf{U\Sigma V^T}$$

Illustration from Wikipedia

Calculus review

Differentiation

The derivative provides us information about the rate of change of a function.

The derivative of a function is also a function.

Example:

The derivative of the rate function is the acceleration function.

Derivative = rate of change

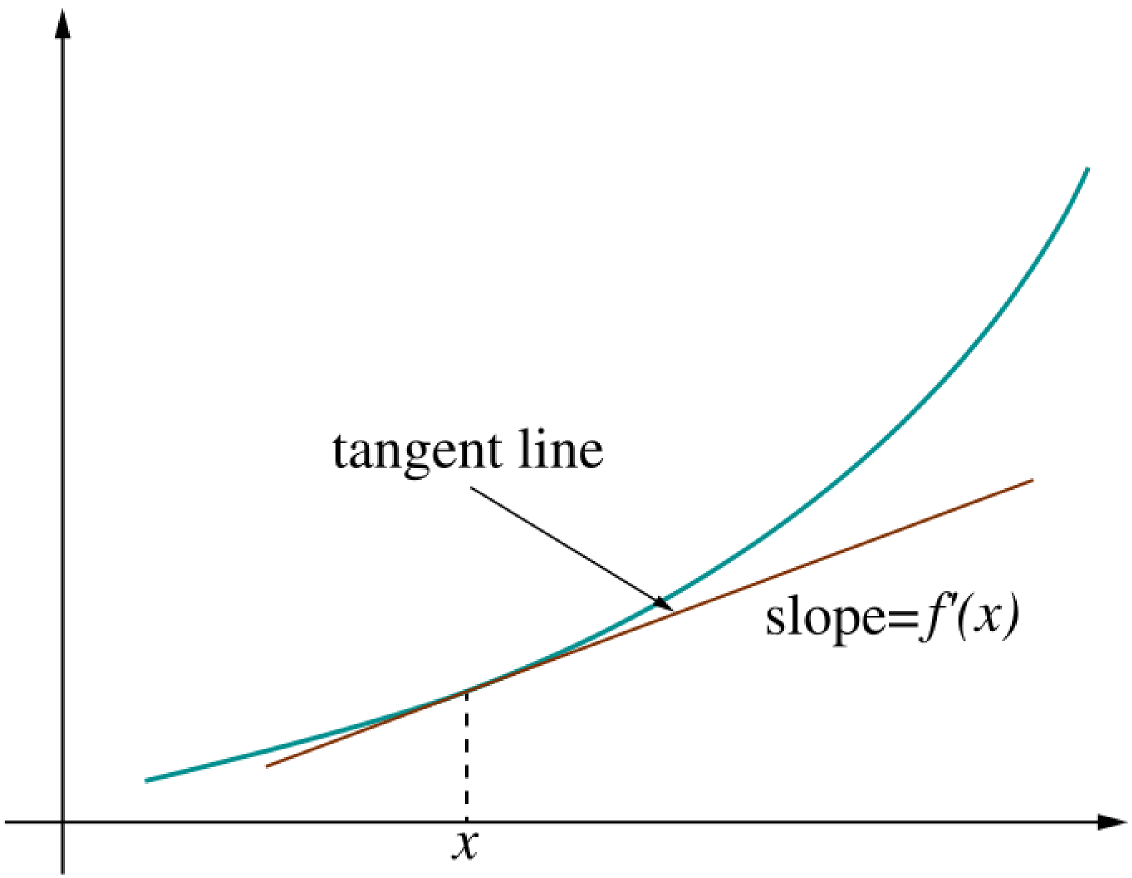


Image: Wikipedia

Derivative = rate of change

- Linear function y = mx + b
- Slope $\boldsymbol{m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}},$

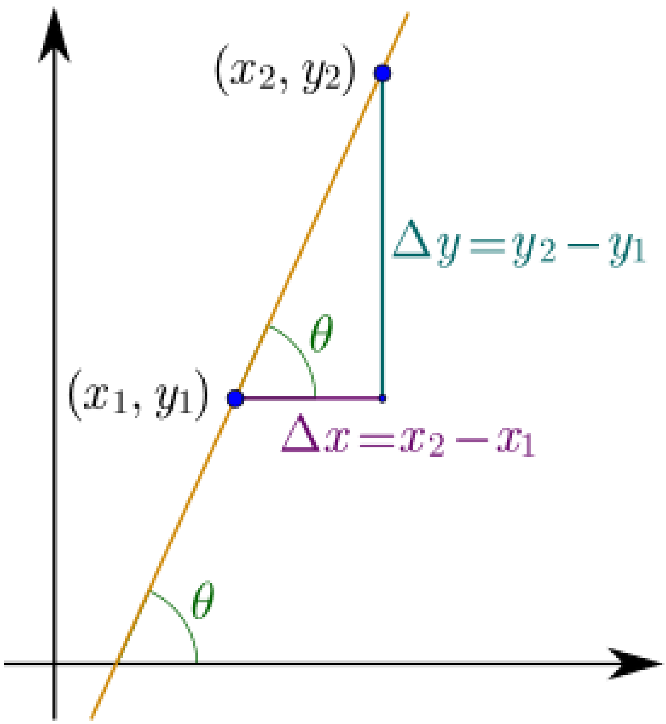


Image: Wikipedia

Ways to Write the Derivative

Given the function f(x), we can write its derivative in the following ways:

- $f'(x)$

- $\frac{d}{dx}f(x)$

The derivative of x is commonly written dx.

Differentiation Formulas

The following are common differentiation formulas:

- The derivative of a constant is 0.

$$\frac{d}{du}c = 0$$

- The derivative of a sum is the sum of the derivatives.

$$\frac{d}{du}(f(u) + g(u)) = f'(u) + g'(u)$$

Examples

- The derivative of a constant is 0.

$$\frac{d}{du}7 =$$

- The derivative of a sum is the sum of the derivatives.

$$\frac{d}{dt}(t+4)=$$

More Formulas

- The derivative of *u* to a constant power:

$$\frac{d}{du}u^n = n*u^{n-1}du$$

- The derivative of *e*:

$$\frac{d}{du}e^u = e^u du$$

- The derivative of *log*:

$$\frac{d}{du}\log(u) = \frac{1}{u}du$$

Product and Quotient

The product rule and quotient rules are commonly used in differentiation.

- Product rule:

$$\frac{d}{du}(f(u)*g(u)) = f(u)g'(u) + g(u)f'(u)$$

- Quotient rule:

$$\frac{d}{du}\left(\frac{f(u)}{g(u)}\right) = \frac{g(u)f'(u) - f(u)g'(u)}{(g(u))^2}$$

Chain Rule

The chain rule allows you to combine any of the differentiation rules we have already covered.

- First, do the derivative of the outside and then do the derivative of the inside.

$$\frac{d}{du} f(g(u)) = f'(g(u)) * g'(u) * du$$

Try These

$$f(z) = z + 11$$

$$s(y) = 4ye^{2y}$$

$$g(y) = 4y^3 + 2y$$

$$p(x) = \frac{\log(x^2)}{x}$$

$$h(x) = e^{3x}$$

$$q(z) = (e^z - z)^3$$

Solutions

$$f'(z) = 1$$

$$s'(y) = 8ye^{2y} + 4e^{2y}$$

$$g'(y) = 12y^2 + 2$$

$$p'(x) = \frac{2 - \log(x^2)}{x^2}$$

$$h'(x) = 3e^{3x}$$

$$q'(z) = 3(e^z - z)^2(e^z - 1)$$