$(1-f)^2 = (1-2f) + f^2$ $(1-f)(1-2f) = 1 - 2f - f + 2f^2$
$= 1 - 3f + 2f^2$

ASTRON 330 - Galaxies:
The Virial Theorem
In-Class 09 Exercise: Feb 24, 2025
Due as Homework 4: March 3, 2025

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at t_0, M_0, R_0, σ_0 (eq.) virial $\longrightarrow$ at t_1, $M_1 = M_0(1-f), \ldots$ $\longrightarrow$ at t_2, M_1, R_2, σ_2 (eq) virial

A young star cluster of mass M_0, radius R_0, and velocity dispersion σ_0 is initially in equilibrium at time t_0. Suppose that a fraction, f, of the cluster's mass is suddenly removed by stars exploding as supernovae at time t_1, leaving behind a mass, $M_1 = M_0(1-f)$. By time t_2, the cluster has returned to an equilibrium state, with new values of radius, R_2, and velocity dispersion, σ_2.

The cluster's potential energy (PE) and kinetic energy (KE) can be expressed as follows:

$$PE = -\frac{3\pi}{32}\frac{GM^2}{R} \qquad KE = \frac{3}{2}M\sigma^2$$

Recall that the total energy is $E = KE + PE$. The Virial Theorem, $2KE + PE = 0$, will apply when the cluster is in equilibrium.

1. Show that the initial energy of the cluster is $E_0 = PE_0/2$, where PE_0 is the initial potential energy of the cluster. [3 pts]

$PE_0 = -\frac{3\pi}{32} \cdot \frac{GM_0^2}{R_0}$

$KE_0 = \frac{3}{2} M_0 \sigma_0^2$

$E_0 = PE_0 + KE_0$

$2KE_0 + PE_0 = 0$

$2KE_0 = -PE_0$

$\longrightarrow KE_0 = -\frac{PE_0}{2}$

$\therefore E_0 = PE_0 - \frac{PE_0}{2} = \frac{PE_0}{2}$

2. At time, t_1, in the immediate aftermath of the mass expulsion, the radius and velocity dispersion will not have had time to change from their initial values. Use this fact to show that the new energy of the system is $E_1 = (1-f)(1-2f)E_0$. [5 pts]

$E_1 = KE_1 + PE_1$

$KE_1 = \frac{3}{2} M_1 \sigma_0^2 = \frac{3}{2} M_0 (1-f) \sigma_0^2$

$KE_1 = (1-f) KE_0$ — (1)

$PE_1 = -\frac{3\pi}{32} \frac{GM_0^2 (1-f)^2}{R_0}$

$PE_1 = (1-f)^2 PE_0$ — (2)

$E_0 = -KE_0$ — (3)

$E_1 = (1-f) KE_0 + (1-f)^2 PE_0$

$E_1 = (1-f) KE_0 - (1-f)^2 2KE_0$

$E_1 = (1-f) KE_0 (1 - (1-f)2)$

$E_1 = (1-f) KE_0 (1 - 2 + 2f)$

$E_1 = KE_0 (1-f)(-1+2f)$

$E_1 = -E_0 (1-f)(-1+2f)$

$E_1 = E_0 (1-f)(1-2f)$

Hence, proved.

3. The cluster will become unbound if $E_1 > 0$ (e.g., $KE > -PE$). At what value of f does this occur? (Hint: start with the answer from question 2). [4 pts]

$E_1 = (1-f)(1-2f)E_0$

we want to find the value of f when $E_1 > 0$

$(1-f)(1-2f)E_0 > 0$

so, $(1-f)(1-2f) > 0$

E_0 is negative because potential energy dominates kinetic energy.

$(1-f)(1-2f)$ must be positive

$1-f=0 \qquad 1-2f=0$

$1=f \qquad f=\frac{1}{2}$

at $f=1$, all the mass is removed, so no mass will be left. (this option doesn't make sense)

$f<\frac{1}{2}$, both terms are positive, so E_1 remains negative

$f>\frac{1}{2}$, $(1-2f)$ becomes negative, so E_1 becomes positive.

$\boxed{f=\frac{1}{2}}$

4. Show that when the cluster reaches equilibrium again, at time t_2, its new radius will be: $R_2 = R_0\,(1-f)/(1-2f)$. [4 pts]

$E_2 = PE_2 + KE_2 = \frac{PE_2}{2} = E_1 = E_0(1-f)(1-2f)$

$2KE_2 + PE_2 = 0$

$KE_2 = -\frac{PE_2}{2}$

$-\frac{3\pi}{32} \times \frac{GM_0^2(1-f)^2}{2R_2} = \frac{PE_0}{2}(1-f)(1-2f)$

$-\frac{\cancel{3\pi}}{\cancel{32}} \times \frac{\cancel{GM_0^2}(1-f)^{\cancel{2}}}{\cancel{2}R_2} = -\frac{\cancel{3\pi}}{\cancel{32}}\frac{\cancel{GM_0^2}}{R_0} \times \frac{1}{\cancel{2}}\cancel{(1-f)}(1-2f)$

$\frac{(1-f)}{R_2} = \frac{(1-2f)}{R_0}$

$R_2 = R_0 \times \frac{(1-f)}{(1-2f)}$ Hence, proved.

5. How does the velocity dispersion, σ, change? (i.e., $\sigma_2/\sigma_0 =$??) [4 pts]

$E_2 = PE_2 + KE_2$

$2KE_2 + PE_2 = 0$

$PE_2 = -2KE_2$

$E_2 = -2KE_2 + KE_2$

$E_2 = -KE_2$

$E_1 = -KE_2$

$E_0(1-f)(1-2f) = -KE_2$

$\cancel{-}KE_0(1-f)(1-2f) = \cancel{-}KE_2$

$\frac{\cancel{3}}{\cancel{2}}M_0\sigma_0^2\cancel{(1-f)}(1-2f) = \frac{\cancel{3}}{\cancel{2}}M_0\cancel{(1-f)}\sigma_2^2$

$\sigma_0^2(1-2f) = \sigma_2^2$

$\frac{\sigma_2^2}{\sigma_0^2} = (1-2f)$

$\left(\frac{\sigma_2}{\sigma_0}\right)^2 = 1-2f$

$\boxed{\frac{\sigma_2}{\sigma_0} = \sqrt{1-2f}}$