1. *Consider a star with a density that varies linearly from $\rho = \rho_c$ at the center ($r = 0$), to $\rho = 0$ at the surface ($r = R$). Using the mass equation [4.1], derive the interior mass function $m(r)$ for this star. Apply this function to obtain an expression for ρ_c in terms of the star's mass M and radius R.*

2 points for setting up $\rho(r)$ expression, **2 points** for integrating to find M, **1 point** for result.

A linearly-varying density can be expressed as

$$\rho(r) = ar + b$$

where a and b are constants. The boundary conditions $\rho(0) = \rho_c$ and $\rho(R) = 0$ fix these constants as $a = -\rho_c/R$ and $b = \rho_c$, and so

$$\rho(r) = \rho_c \left(1 - \frac{r}{R}\right).$$

To determine ρ_c, we use the integral form [4.1] of the mass equation to find the interior mass function as

$$m(r') = \int_0^{r} 4\pi r^2 \rho(r)\, \mathrm{d}r = 4\pi R^3 \rho_c \frac{r'^3}{R^3} \left(\frac{1}{3} - \frac{1}{4}\frac{r'}{R}\right).$$

Setting $r = R$, the total mass is

$$M \equiv m(R) = \frac{\pi R^3 \rho_c}{3};$$

solving for ρ_c then yields the desired result

$$\rho_c = \frac{3M}{\pi R^3}.$$

2. *For the linear-density star in Q1, derive an expression for the gravitational potential energy Ω of the star in terms of M and R. By comparing this against eqn. [4.8], determine the shape factor f_Ω for the star. How does it compare to the shape factor $f_\Omega = 3/5$ for a uniform density star; is the result expected or unexpected?*

2 points for setting up integral for Ω, **2 points** for working through to result, **1 point** for commentary on result.

The gravitational potential energy is defined in eqn. [4.7] as

$$\Omega = -\int_0^M \frac{Gm}{r}\, \mathrm{d}m.$$

Using the mass equation [4.2], we can transform this into an integral over radial coordinate r rather than interior mass m:

$$\Omega = -\int_0^R \frac{Gm}{r} 4\pi r^2 \rho\, \mathrm{d}r.$$

Substituting in the expressions for ρ and m from Q1, this becomes

$$\Omega = -\int_0^R \frac{144GM^2}{R^2}\frac{r^4}{R^4}\left(\frac{1}{3} - \frac{1}{4}\frac{r}{R}\right)\left(1 - \frac{r}{R}\right) \mathrm{d}r = -\frac{26}{35}\frac{GM^2}{R}.$$

Comparing this against eqn. [4.8], we identify the shape factor as $f_\Omega = 26/35$. This is rather larger than the $3/5$ value pertaining to a uniform-density star, because the mass in the linear-density star is concentrated closer to the center (leading to a stronger gravitational binding).

3. *With reference to the equation of hydrostatic equilibrium, explain why the pressure in a star is a monotonic-decreasing function of radial coordinate.*

1 point for establishing that right-hand side of equation is always negative, **1 point** for linking this to the pressure behavior.

Combining eqns. [4.4] and [5.6], the equation of hydrostatic equilibrium is

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho.$$

On the right hand side, each of G, m, r and ρ are positive (physically, we cannot have a negative m or ρ; the radial coordinate r is positive by convention; and G is a positive constant). Therefore, the pressure gradient dP/dr is always negative, meaning that the pressure is a monotonic-decreasing function of r.

4. *A star in hydrostatic equilibrium, with radius R and mass M, is cooled instantaneously down to $T = 0\,\mathrm{K}$. What will happen to the star — and how long will it take?*

1 point for identifying that the star will collapse, **1 point** for recognizing this will occur on the dynamical timescale.

If a star is cooled instantaneously down to absolute zero, then (assuming the stellar material behaves as an ideal gas) the pressure will vanish. Without pressure support, the gravitational force will then cause the star to collapse down to the origin, $r = 0$.

Because this collapse is a response to the loss of hydrostatic equilibrium, the time taken for the collapse will be (approximately) the star's dynamical timescale

$$\tau_{\mathrm{dyn}} = \sqrt{\frac{R^3}{GM}}.$$

(for those that are interested, the precise result can be obtained by solving the equation of motion for the outer layers, and has the value $\sqrt{\pi/8}\,\tau_{\mathrm{dyn}}$)

5. *Consider the linear-density star from Q1. Assuming the surface pressure vanishes, i.e. $P(R) = 0$, integrate the equation of hydrostatic equilibrium [5.6] to find the central pressure P_c in terms of M and R.*

1 point for setting up the expression for dP/dr, **1 point** for integrating it, **1 point** for obtaining P_c.

From Q3, the hydrostatic equilibrium equation is

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho.$$

Substituting in the expressions for ρ and m from Q1, this becomes

$$\frac{dP}{dr} = -\frac{36GM^2}{R^5}\frac{r}{R}\left(\frac{1}{3} - \frac{1}{4}\frac{r}{R}\right)\left(1 - \frac{r}{R}\right).$$

Integrating from $r = 0$ to $r = R$, this becomes

$$P_s - P_c = -\int_0^R \frac{36GM^2}{R^5}\frac{r}{R}\left(\frac{1}{3} - \frac{1}{4}\frac{r}{R}\right)\left(1 - \frac{r}{R}\right) dr = -\frac{5GM^2}{4\pi R^4}.$$

Setting the surface pressure P_s to zero, in accordance with the question, the central pressure is then found as

$$P_c = \frac{5GM^2}{4\pi R^4}.$$

6. *A ideal-gas sample of fully ionized helium has a density $\rho = 50\,\mathrm{g\,cm^{-3}}$. What is the mean molecular weight? The number density of particles n?*

1 point for μ, **1 point** for total number density.

For fully ionized material,

$$\mu \approx \left[2X + \frac{3Y}{4} + \frac{Z}{2}\right]^{-1}.$$

(see eqn. [6.6]). Setting $X = 0$, $Y = 1$ and $Z = 0$ (as appropriate for helium), we find $\mu = 4/3$.

To find n, we rearrange eqn. [6.3] as

$$n = \frac{\rho}{\mu\, m_{\mathrm{H}}}$$

Substituting μ and the given density into this expression we find the total number density of particles as $2.24 \times 10^{25}\,\mathrm{cm^{-3}}$.

7. *An ideal gas with $\gamma = 5/3$ is compressed adiabatically to one third its volume. By what factor does its density change? Its pressure? Its temperature?*

1 point each for the change factor for density, pressure and temperature.

If the gas is compressed to one third its volume, its density must increase by a factor 3.

For adiabatic changes, P and ρ are related by

$$P = K_{\mathrm{ad}}\,\rho^{\gamma}$$

where K_{ad} and γ are constants. With $\gamma = 5/3$, then a factor-3 increase in density will lead to a factor $3^{5/3} = 6.24$ increase in pressure.

Since the gas is ideal, it follows that

$$\frac{P_{\mathrm{f}}}{P_{\mathrm{i}}} = \frac{\rho_{\mathrm{f}}}{\rho_{\mathrm{i}}}\frac{T_{\mathrm{f}}}{T_{\mathrm{i}}},$$

where the subscripts ‘i’ and ‘f’ denote initial and final. With $\rho_{\mathrm{f}}/\rho_{\mathrm{i}} = 3$ and $P_{\mathrm{f}}/P_{\mathrm{i}} = 6.24$, it follows that $T_{\mathrm{f}}/T_{\mathrm{i}} = 6.24/3 = 2.08$. Hence, the temperature increases by a factor 2.08.

8. *A star with a mass $4\,M_{\odot}$ has a hydrogen mass fraction $X = 0.6$ and a metal mass fraction $Z = 0.15$. How many helium nuclei does the star contain?*

1 point for determining the mass fraction of helium, **1 point** for the calculating mass of helium in the star, **1 point** for calculating how many helium nuclei the star contains.

The mass fraction of helium in the star is $Y = 1 - X - Y = 0.25$, and so the helium content of the star by mass is $0.25 \times 4\,M_{\odot} = 1\,M_{\odot}$. The number of helium nuclei N_{He} is them found by dividing this mass by the mass of a helium atom, $m_{\mathrm{He}} = 4.002\,602\,\mathrm{u} = 6.65 \times 10^{-24}\,\mathrm{g}$, giving $N_{\mathrm{He}} = 3.00 \times 10^{56}$.