1. *Calculate the slope* $\mathrm{d}\log L/\,\mathrm{d}\log T_{\mathrm{eff}}$ *of the Hayashi line for fully convective stars in which the opacity is* independent *of pressure and temperature.*

1 point for slope equation, **2 points** for determining a and b, **1 point** for result.

Equation [20.10] gives the slope as

$$\frac{\mathrm{d}\log L}{\mathrm{d}\log T_{\mathrm{eff}}} = \frac{\mathrm{d}\ln L}{\mathrm{d}\ln T_{\mathrm{eff}}} = \frac{6 + 22a + 4b}{3a - 1}.$$

If the opacity is independent of pressure and temperature, then by eqn. [20.5] we must have $a = b = 0$. Hence, the slope becomes

$$\frac{\mathrm{d}\log L}{\mathrm{d}\log T_{\mathrm{eff}}} = -6.$$

2. *In very massive stars* ($M \gtrsim 50\,\mathrm{M}_\odot$) *the radiation pressure is much greater than the gas pressure, and energy is transported primarily by radiation. For such stars,*

(*a*) *Show that the central temperature follows a scaling* $T_{\mathrm{c}} \sim M^{1/2}/R$ (hint: assume the scaling relations $\rho_{\mathrm{c}} \propto M/R^3$ and $P_{\mathrm{c}} \propto M^2/R^4$ that apply to polytropes).

(*b*) *Show that* $L \propto M$ (hint: follow a similar process to that outlined in *Handout 21*).

(*c*) *With this mass-luminosity relation, argue that the main-sequence lifetime of very massive stars is independent of mass.*

2 points for deriving T_{c} scaling, **2 points** for deriving L scaling, **2 points** for showing lifetime is independent of mass.

(a) If the radiation pressure is much greater than the gas pressure, then the equation of state becomes

$$P \approx P_{\mathrm{rad}} = \frac{aT^4}{3}.$$

Applying this at the center of the star,

$$P_{\mathrm{c}} \propto T_{\mathrm{c}}^4.$$

Next combining with the given scaling for P_{c}, this becomes

$$\frac{M^2}{R^4} \propto T_{\mathrm{c}}^4,$$

from whence it follows that

$$T_{\mathrm{c}} \propto \frac{M^{1/2}}{R},$$

the desired result.

(b) From *Handout 21*, the radiative diffusion equation leads to the scaling

$$L \propto \frac{RT_{\mathrm{c}}^4}{\kappa\,\rho_{\mathrm{c}}}.$$

Assuming κ is constant throughout the star, and with the given/derived scalings for ρ_{c} and T_{c}, this simplifies to

$$L \propto \frac{R^4 M^2}{R^4 M} \propto M,$$

which is the desired result.

(c) From *Handout 21*, the main-sequence lifetime can be approximated as

$$\tau_{\rm MS} = \frac{f M e}{L}.$$

With L proportional to M, we therefore find that the dependence of $\tau_{\rm MS}$ on M cancels out — the lifetime of very massive stars is independent of mass.

3. *For each of the following short-hand reactions comprising the CNO cycle, write out the reaction in full, making sure to conserve baryon number, lepton number and charge, and to have zero net charge on each side of the reaction. Then calculate the total energy released by the reaction (whether as gamma rays, positrons, neutrinos or kinetic energy), in units of* MeV. *Confirm that the total energy released by the cycle adds up to the* 26.73 MeV *for fusion of four hydrogen atoms into one helium atom.*

2 points for each reaction (partial credit if charge, lepton number or baryon number is not conserved, or energy is wrong; no need to label atoms as I have done, however), **3 points** for demonstrating individual energies add up to total energy.

(i) ${}^{12}\mathrm{C}(p,\gamma){}^{13}\mathrm{N}$

$$\underbrace{{}^{12}\mathrm{C} + 6e^-}_{\text{carbon-12 atom}} + \underbrace{{}^{1}\mathrm{H} + e^-}_{\text{hydrogen atom}} \longrightarrow \underbrace{{}^{13}\mathrm{N} + 7e^-}_{\text{nitrogen-13 atom}} + \gamma$$

The energy release is $\Delta\mathcal{E} = [m({}^{12}\mathrm{C}) + m({}^{1}\mathrm{H}) - m({}^{13}\mathrm{N})] = 1.94\,\mathrm{MeV}$.

(ii) ${}^{13}\mathrm{N}(\cdot, e^+){}^{13}\mathrm{C}$

$$\underbrace{{}^{13}\mathrm{N} + 7e^-}_{\text{nitrogen-13 atom}} \longrightarrow \underbrace{{}^{13}\mathrm{C} + 6e^-}_{\text{carbon-13 atom}} + e^+ + e^- + \nu_e$$

The energy release is $\Delta\mathcal{E} = [m({}^{13}\mathrm{N}) - m({}^{13}\mathrm{C})] = 2.22\,\mathrm{MeV}$.

(iii) ${}^{13}\mathrm{C}(p,\gamma){}^{14}\mathrm{N}$

$$\underbrace{{}^{13}\mathrm{C} + 6e^-}_{\text{carbon-13 atom}} + \underbrace{{}^{1}\mathrm{H} + e^-}_{\text{hydrogen atom}} \longrightarrow \underbrace{{}^{14}\mathrm{N} + 7e^-}_{\text{nitrogen-14 atom}} + \gamma$$

The energy release is $\Delta\mathcal{E} = [m({}^{13}\mathrm{C}) + m({}^{1}\mathrm{H}) - m({}^{14}\mathrm{N})] = 7.55\,\mathrm{MeV}$.

(iv) ${}^{14}\mathrm{N}(p,\gamma){}^{15}\mathrm{O}$

$$\underbrace{{}^{14}\mathrm{N} + 7e^-}_{\text{nitrogen-14 atom}} + \underbrace{{}^{1}\mathrm{H} + e^-}_{\text{hydrogen atom}} \longrightarrow \underbrace{{}^{15}\mathrm{O} + 8e^-}_{\text{oxygen-15 atom}} + \gamma$$

The energy release is $\Delta\mathcal{E} = [m({}^{14}\mathrm{N}) + m({}^{1}\mathrm{H}) - m({}^{15}\mathrm{O})] = 7.30\,\mathrm{MeV}$.

(v) ${}^{15}\mathrm{O}(\cdot, e^+){}^{15}\mathrm{N}$

$$\underbrace{{}^{15}\mathrm{O} + 8e^-}_{\text{oxygen-15 atom}} \longrightarrow \underbrace{{}^{15}\mathrm{N} + 7e^-}_{\text{nitrogen-15 atom}} + e^+ + e^- + \nu_e$$

The energy release is $\Delta\mathcal{E} = [m({}^{15}\mathrm{O}) - m({}^{15}\mathrm{N})] = 2.75\,\mathrm{MeV}$.

(vi) $^{15}\mathrm{N}(p, \alpha)^{12}\mathrm{C}$

$$\underbrace{^{15}\mathrm{N} + 7e^-}_{\text{nitrogen-15 atom}} + \underbrace{^{1}\mathrm{H} + e^-}_{\text{hydrogen atom}} \longrightarrow \underbrace{^{12}\mathrm{C} + 6e^-}_{\text{carbon-12 atom}} + \underbrace{^{4}\mathrm{He} + 2e^-}_{\text{helium-4 atom}}$$

The energy release (as kinetic energy) is $\Delta\mathscr{E} = [m(^{15}\mathrm{N}) + m(^{1}\mathrm{H}) - m(^{12}\mathrm{C}) - m(^{4}\mathrm{He})] = 4.97\,\mathrm{MeV}$.

The energy release from the whole cycle is $\Delta\mathscr{E} = (1.94 + 2.22 + 7.55 + 7.30 + 2.75 + 4.97)\,\mathrm{MeV} = 26.73\,\mathrm{MeV}$, confirming the expected result.