Homework 4.

1) Given: interior luminosity in a star: $l(m) = L\left(2\left(\frac{m}{M}\right) - \left(\frac{m}{M}\right)^2\right)$

The star is in thermal equilibrium, and the neutrino losses are negligible.

Hydrostatic equilibrium equation: $\frac{\partial P}{\partial r} = \rho g = -\frac{GM}{R^2}\rho$

Thermal equilibrium equation: $\frac{\partial l}{\partial r} = 4\pi r^2 \rho \left(\epsilon_{nuc} - \epsilon_{\nu}\right)$

To find: nuclear energy generation rate (ϵ_{nuc}) = ?
(in terms of m, M, L).

Solution:

$$\frac{\partial P}{\partial r} = -\frac{GM}{R^2}\rho \quad \text{--- (hydrostatic equilibrium equation)}$$

multiplying both sides by $4\pi r^3$, and then integrating, you get:

$$\int_0^R \frac{\partial P}{\partial r} 4\pi r^3 \, dr = -\int_0^R \frac{GM}{r} 4\pi r^2 \rho \, dr = -\int_0^M \frac{Gm}{r} dm$$

$-\int_0^M \frac{Gm}{r} dm$: you get this expression when you substitute from the interior mass equation $m(r) = \int_0^r 4\pi r^2 \rho \, dr$

solving this, you get the relation $\frac{\partial m}{\partial r} = 4\pi r^2 \rho$, which is the differential form of the interior mass equation.

$$\frac{\partial m}{\partial r} = 4\pi r^2 \rho \quad - (1)$$

using the thermal equilibrium equation,

$$\frac{\partial l}{\partial r} = 4\pi r^2 \rho \left(\epsilon_{nuc} - \epsilon_\nu\right)$$

$\epsilon_\nu = 0$ as neutrino losses are negligible.

$$\therefore \frac{\partial l}{\partial r} = 4\pi r^2 \rho \, \epsilon_{nuc}$$

$$\therefore \frac{\partial l}{\partial m} \times \frac{\partial m}{\partial r} = 4\pi r^2 \rho \, \epsilon_{nuc}$$ ... (multiplying the left side by $\frac{\partial m}{\partial m}$)

$$\frac{\partial l}{\partial m} \times \cancel{4\pi r^2 \rho} = \cancel{4\pi r^2 \rho} \, \epsilon_{nuc}$$

$$\therefore \frac{\partial l}{\partial m} = \epsilon_{nuc}.$$

$$\therefore \frac{\partial}{\partial m}\left(L\left(2\left(\frac{m}{M}\right) - \left(\frac{m}{M}\right)^2\right)\right) = \epsilon_{nuc}$$

$$\therefore \frac{\partial}{\partial m}\left(\frac{2Lm}{M} - \frac{Lm^2}{M^2}\right) = \epsilon_{nuc}$$

$$\frac{2L}{M} - \frac{2Lm}{M^2} = \epsilon_{nuc}$$

$$L\left(\frac{2}{M} - \frac{2m}{M^2}\right) = \epsilon_{nuc}$$

This is the expression of nuclear energy generation rate (ϵ_{nuc}) in terms of m, M, and L.

2) ~~Bon~~ · Bound-free absorption is where a photon is absorbed by a bound electron, freeing the electron from the atom. (It is the same process as ionization). In this process, the electron needs to be bounded to the nucleaus when the photon gets absorbed. In a sample of fully ionized hydrogen, none of the electrons are bounded to the nucleus. This means that the bound-free opacity of a sample of fully ionized hydrogen is 0.

$$K_H = 0$$

(K_H: opacity of hydrogen (fully ionized))

Electron scattering is where a photon is scattered by a free electron. This implies that the electron should not be bounded to a nucleus. In a sample of completely neutral Helium, all of the electrons are tightly bounded to the Helium nucleus, producing no free electrons in the gas sample. This means that the electron-scattering opacity of a sample of completely neutral helium is 0.

$$K_{He} = 0$$

(K_{He}: opacity of helium (completely neutral))

3) Given: In a sample of stellar material,
average cross sectional per particle (σ) = 10^{-23} cm^2
mean molecular weight (μ) = 2.5
density (ρ) = 20 $g\,cm^{-3}$

To find: opacity of the sample (κ) = ?
mean free path for photons ($\langle l \rangle$) = ?

solution:

$$\kappa = \frac{\sigma}{\mu m_H} \quad \cdots \text{(opacity of the matter)}$$

$$\therefore \kappa = \frac{10^{-23}\ cm^2}{2.5 \times 1.67 \times 10^{-24}\ g}$$

$$\therefore \kappa = 2.395\ cm^2\ g^{-1}$$

The opacity of the sample is $2.395\ cm^2\ g^{-1}$.

$$\langle l \rangle = \frac{1}{\kappa \rho} \quad \cdots \text{(mean free path for a photons)}$$

$$\langle l \rangle = \frac{1}{2.395 \times 20\ cm^2 g^{-1}\ cm^{-3} g}$$

$$\langle l \rangle = 0.02088\ cm$$

The mean free path of photons in the sample is 0.02088 cm.

4) <u>Given:</u> For the outer layers of the star,
The opacity is uniform. $\kappa = 20\ cm^2\ g^{-1}$
density $(\rho) = 10^{-9}\ g\ cm^{-3}$
optical depth $(\tau) = 25$

<u>To find:</u>

The physical depth below the surface corresponding to the optical depth =?
total distance travelled by a photon in order to random-walk its way from this depth to the surface. =?

<u>Solution:</u>

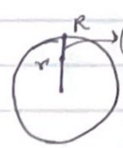
The optical depth is given as:

$$\tau(r') = \int_{r'}^{R} \kappa\rho\, dr$$

The density and the opacity of the outer layers of the star is constant

$$\tau(r') = \kappa\rho \int_{r'}^{R} dr = \kappa\rho\, [r]_{r'}^{R} = \kappa\rho\,(R - r')$$

$$\tau(r') = \kappa\rho\,(R - r')$$

For a star,  $(R-r')$ is the physical depth below the surface for that optical depth.

$$(R-r') = \frac{\tau}{\kappa\rho} = \frac{25}{20 \times 10^{-9}\ cm^2 g^{-1}\ g\,cm^{-3}}$$

$$(R-r') = 1.25 \times 10^{9}\ cm$$

The optical depth $\tau = 25$ is found at ~~the~~ a depth of 1.25×10^{-9} cm from the surface.

The total distance that a photon travels in order to random-walk its way to the surface is given by:

$$\Delta = N\langle l \rangle \approx (R-r)\tau$$

$$\Delta \approx (1.25 \times 10^{9}) \times 25$$

$$\Delta \approx 3.125 \times 10^{10}\ cm$$

$\therefore$ The total distance is 3.125×10^{10} cm.

5) Given: For the Eddington standard model,
let's assume hydrostatic equilibrium, radiative transport of energy with a constant opacity κ.
pressure is given by: $P = \dfrac{aT^4}{3(1-\beta)}$

here, ~~$0 \le \beta \le$~~ $0 \le \beta < 1$ is an arbitrary constant
a is the radiation constant.

To find: derive an expression for the constant of proportionality in terms of β, κ, and physical constants.

Solution:

Using the hydrostatic equilibrium equation:

$$\frac{\partial P}{\partial r} = \rho g = -\frac{GM}{R^2}\rho$$

$$\therefore \frac{\partial}{\partial r}\left(\frac{aT^4}{3(1-\beta)}\right) = \frac{a}{3(1-\beta)}\frac{\partial (T^4)}{\partial r}$$ (plugged in the value for Pressure)

$$\therefore -\frac{GM}{R^2}\rho = \frac{a}{3(1-\beta)} 4T^3 \frac{\partial T}{\partial r}$$

$$\therefore \frac{\partial T}{\partial r} = -\frac{\rho GM \, 3(1-\beta)}{aR^2 \, 4T^3} \quad \text{— ①}$$

The radiative diffusion equation is given by:

$$l_{rad} = \frac{-16\pi r^2 acT^3}{3\kappa\rho}\frac{\partial T}{\partial r}$$

plugging in the value found in ~~question~~ equation ① in the above expression, you get:

$$l_{rad} = \frac{-16\pi \cancel{R^2} \cancel{a} c \cancel{T^3}}{\cancel{3} \kappa \cancel{\rho}} \times \frac{-\cancel{\rho} GM \cancel{3}(1-\beta)}{\cancel{a} \cancel{R^2} 4 \cancel{T^3}}$$

$$l_{rad} = \frac{+\cancel{16}^{4}\pi}{\cancel{4}} \times \frac{c}{\kappa \cancel{\rho}} \times \cancel{+}GM \times (1-\beta)$$

$$l_{rad} = \frac{4\pi c}{\kappa} \times G(1-\beta) \times M$$

This expression tells us that $l_{rad} \propto M$ as the radiative luminosity is directly proportional to the mass.

$$l_{rad} \propto M$$

$$\therefore l_{rad} = K M$$

from the expression of l_{rad}, you see the constant of proportionality K to be:

$$K = \frac{4\pi c G(1-\beta)}{\kappa}$$

6) Given: In the cores of red giant stars, nuclear reactions do not take place but there are strong non-nuclear neutrino losses.

assuming thermal equilibrium : $\frac{\partial l}{\partial r} = 4\pi r^2 \rho \, (\epsilon_{nuc} - \epsilon_\nu)$

radiative transport of energy : $l_{rad} = \frac{-16\pi r^2 acT^3}{3k\rho} \frac{\partial T}{\partial r}$

To find: create a sketch of $l(m)$ in the core.
use the sketch and the radiative diffusion equation to prove/argue that the dimensionless temperature gradient ∇_r must be negative throughout the core.

Solution:
using the thermal equilibrium equation

$$\frac{\partial l}{\partial r} = 4\pi r^2 \rho \, (\epsilon_{nuc} - \epsilon_\nu)$$

as the nuclear reactions do not take place, $\epsilon_{nuc} = 0$, so the equation becomes.

$$\frac{\partial l}{\partial r} = 4\pi r^2 \rho \, (-\epsilon_\nu)$$

using the differential form of the interior mass equation given by $\frac{\partial m}{\partial r} = 4\pi r^2 \rho$,

$$\frac{\partial l}{\partial m} \times \frac{\partial m}{\partial r} = 4\pi r^2 \rho \, (-\epsilon_\nu)$$

$$\frac{\partial l}{\partial m} \times \cancel{4\pi r^2} \rho = \cancel{4\pi r^2} \rho \, (-\epsilon_\nu)$$

$$\therefore \frac{\partial l}{\partial m} = -\epsilon_v \quad - (1)$$

This means that the slope of the luminosity vs mass graph will ~~have a slope~~ be $-\epsilon_v$.
When the radius is zero, the mass of the star becomes 0.

If we were to find a function (linearly varying) of the luminosity vs mass, the function would have the form of

$$y = mx + c$$

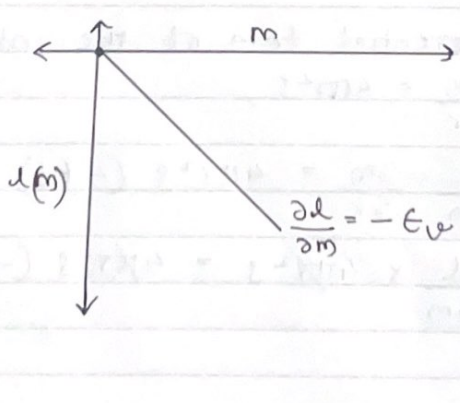
comparing it to the slope we found in (1), to the equation,

$$m = \frac{\partial l}{\partial m} = -\epsilon_v$$

In the equation, $y = l$ (luminosity) and $x = m$ (mass)
The constant term here is the y-intercept. In this case, as $r = 0$, mass is also equal to zero.

$$l = -\epsilon_v(m) + 0$$

This will be the function. The ~~l~~ graph will look like this:



The graph suggests that in the core of red giant stars, the value of radiative luminosity always remains negative.

the temperature gradient ∇_T is given by:

$$\nabla_T = \frac{3}{16\pi a c G} \frac{\kappa\, l_{rad}\, P}{m T^4}.$$

All the values in this expression are positive, so the sign of ∇_T depends only on the sign of l_{rad}.
In the core of red giant stars, l_{rad} is always negative, seeing from the sketch.
So, the temperature gradient ∇_T must be negative throughout the core.