1. *Suppose the interior luminosity in a star with mass M and luminosity L is given by*

$$\ell(m) = L\left[2\left(\frac{m}{M}\right) - \left(\frac{m}{M}\right)^2\right].$$

Assuming that the star is in thermal equilibrium, and that neutrino losses are negligible, derive an expression for the nuclear energy generation rate $\epsilon_{\rm nuc}$ *in terms of m, M and L. (Hint: start by combining the hydrostatic equilibrium equation* [5.6] *with the thermal equilibrium condition* [10.4].*)*

1 point for invoking energy conservation equation, **1 point** for invoking mass equation, **1 point** for $\mathrm{d}\ell\,/\,\mathrm{d}m$ equation, **2 points** for result.

For a star in thermal equilibrium with negligible neutrino losses ($\epsilon_\nu = 0$), energy conservation requires that

$$\frac{\partial \ell}{\partial r} = 4\pi r^2 \rho \epsilon_{\rm nuc}$$

(see eqn. [10.4]). It's useful to convert the derivative on the left-hand side derivative from radial coordinate r to interior mass m. Therefore, we divide through by the mass equation [4.2]

$$\frac{\partial m}{\partial r} = 4\pi r^2 \rho$$

to yield

$$\frac{\partial \ell}{\partial m} = \epsilon_{\rm nuc}$$

Substituting in the provided expression for $\ell(m)$ leads to the result

$$\epsilon_{\rm nuc} = 2\frac{L}{M}\left[1 - \frac{m}{M}\right].$$

2. *What is the bound-free opacity of a sample of fully ionized hydrogen? What is the electron-scattering opacity of a sample of completely neutral helium? Be sure to explain your answers.*

2 points for bound-free opacity result and explanation, **2 points** for electron-scattering opacity result and explanation.

In fully ionized hydrogen (or any other element), there are no bound electrons; hence, the bound-free opacity (which requires bound electrons) must be zero. Likewise, in completely neutral helium (or any other element), there are no free electrons; hence, the electron-scattering opacity (which requires free electrons) must also be zero.

3. *A sample of stellar material has an average cross section per particle of* $10^{-23}\,\mathrm{cm}^2$*, and a mean molecular weight of 2.5. Calculate the opacity of the sample. If the density is* $20\,\mathrm{g\,cm^{-3}}$*, calculate the mean free path for photons.*

1 point for opacity, **1 point** for mean free path.

Substituting the provided values into eqn. [11.2],

$$\kappa = \frac{\sigma}{\mu\, m_{\rm H}},$$

gives the result $\kappa = 2.39\,\mathrm{cm^2\,g^{-1}}$. Likewise, evaluating the mean free path via eqn. [11.5],

$$\langle l \rangle = \frac{1}{\kappa\,\rho},$$

gives the result $\langle l \rangle = 0.0209\,\mathrm{cm}$.

4. *Suppose the outer layers of the star have a uniform opacity* $20\,\mathrm{cm}^2\,\mathrm{g}^{-1}$ *and density* $10^{-9}\,\mathrm{g\,cm}^{-3}$. *What physical depth below the surface corresponds to an optical depth* $\tau = 25$? *What total distance must a photon travel in order to random-walk its way from this depth to the surface?*

2 points for depth, **2 points** for total distance photon must travel.

For uniform opacity and density, the optical depth is related to the physical depth via

$$\tau = \int_r^R \kappa\rho\;\mathrm{d}r = \kappa\rho\int_r^R \mathrm{d}r = \kappa\rho(R-r).$$

(see eqn. [11.6]). Solving for the physical depth,

$$R - r = \frac{\tau}{\kappa\rho};$$

for the supplied values of τ, κ and ρ we find a depth $R - r = 1.25\times10^9$ cm.

As discussed in *Handout 11*, the distance a photon must travel in order to escape from optical depth τ is a factor τ larger than the direct distance. Therefore, the total distance must be $25\times1.25\times10^9$ cm $= 3.125\times10^{10}$ cm.

5. *The Eddington standard model is a simplified model for stellar structure, which assumes hydrostatic equilibrium, radiative transport of energy with a constant opacity* κ, *and a pressure that follows the law*

$$P = \frac{aT^4}{3(1-\beta)},$$

where $0 \leq \beta < 1$ *is an arbitrary constant and* a *is the radiation constant. Show that* $\ell \propto m$ *in this model, and derive an expression for the constant of proportionality in terms of* β, κ *and physical constants. (Hint: start by substituting the above expression for* P *into the hydrostatic equilibrium equation* [5.6], *and then use the radiative diffusion equation* [12.4] *to eliminate the resulting* $\mathrm{d}T/\mathrm{d}r$ *term.)*

1 point for invoking hydrostatic equilibrium equation, **1 point** for applying radiative equilibrium equation, **3 points** for correct results.

The equation of hydrostatic equilibrium [5.6] is

$$\frac{\partial P}{\partial r} = -\frac{Gm}{r^2}\rho.$$

Substituting in the given expression for P, this becomes

$$\frac{4aT^3}{3(1-\beta)}\frac{\partial T}{\partial r} = -\frac{Gm}{r^2}\rho.$$

To eliminate the temperature gradient, we rearrange the radiative diffusion equation [12.4] as

$$\frac{\partial T}{\partial r} = -\frac{3\kappa\rho\ell}{16\pi r^2 acT^3}.$$

Combining these two equations,

$$-\frac{4aT^3}{3(1-\beta)}\frac{3\kappa\rho\ell}{16\pi r^2 acT^3} = -\frac{Gm}{r^2}\rho.$$

A number of terms cancel out, and this simplifies after a little rearrangement to

$$\ell = \frac{4\pi Gc(1-\beta)}{\kappa} m.$$

Given that β and κ are constant, we therefore find that $\ell \propto m$ in the Eddington standard model. The constant of proportionality is $4\pi Gc(1-\beta)/\kappa$.

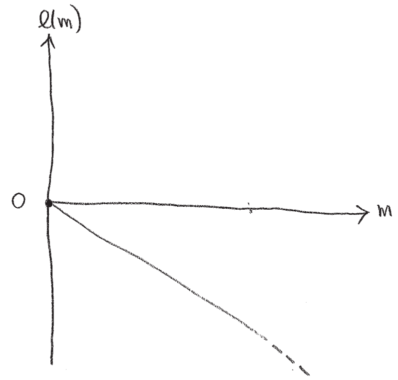

Figure 1.1: Interior luminosity sketch for Q6. The $\ell(m)$ profile has two important properties: $\ell(0) = 0$, and $\partial\ell/\partial m < 0$.

6. *In the cores of red giant stars, nuclear reactions do not take place but there are strong non-nuclear neutrino losses. Assuming thermal equilibrium and radiative transport of energy, create as sketch of $\ell(m)$ in the core. (Hint: to create the sketch, consider the value of ℓ at $r = 0$, and use the thermal equilibrium condition [10.4] to determine the slope of ℓ in the vicinity of $r = 0$). Use this sketch together with the radiative diffusion equation [12.4] to argue that the dimensionless temperature gradient ∇_T must be negative throughout the core.*

3 points for correct sketch, **2 points** for arguing that $\nabla_T < 0$.

For a star in thermal equilibrium with no nuclear reactions ($\epsilon_{\rm nuc} = 0$) but strong neutrino losses ($\epsilon_\nu > 0$), energy conservation requires that

$$\frac{\partial \ell}{\partial m} = -\epsilon_\nu.$$

(here, as in Q1, we've used the mass equation to transform to a derivative with respect to m). This establishes that $\mathrm{d}\ell/\mathrm{d}m < 0$ in the cores of red giant stars. Coupled with the boundary condition $\ell = 0$ at the origin ($m = 0$), it's then clear that $\ell(m) < 0$ at some point sufficiently close to the center (see Fig. 1.1).

The temperature gradient ∇_T can be evaluated from the radiative interior luminosity $\ell_{\rm rad}$ via

$$\nabla_T = \frac{3}{16\pi acG}\frac{\kappa \ell_{\rm rad} P}{mT^4}.$$

(see eqn. [12.7]). If energy transport is by radiation, then $\ell_{\rm rad} = \ell < 0$, which implies that $\nabla_T < 0$ since all other quantities on the right-hand side of this expression are positive.