Homework 3.

1) initial thermal energy of the star $(U_0) = 2 \times 10^{51}$ erg

total stellar energy of the star (E_0) using virial theorem is given by:

$$E_0 \equiv U_0 + \Omega$$, where Ω is the gravitational potential energy.

However, $U_0 = -\dfrac{\Omega}{3(\gamma-1)}$

$\therefore \Omega = -U_0 \times 3(\gamma-1)$

plugging this into the total stellar energy equation, we get:

$$E_0 = U_0 + (-U_0 \times 3(\gamma-1))$$

$$\therefore E_0 = U_0 - U_0 \times 3(\gamma-1)$$

$$\therefore E_0 = U_0(1-3(\gamma-1))$$

$$\therefore E_0 = U_0(4-3\gamma)$$

plugging in $\gamma = 5/3$ for the value of γ, we get:

$$E_0 = U_0\left(4 - \cancel{3} \times \frac{5}{\cancel{3}}\right)$$

$$\therefore E_0 = -U_0 = -2 \times 10^{51} \text{ erg}$$

The total stellar energy of the star comes out to be -2×10^{51} erg.

Now, you add $e = 7 \times 10^{50}$ erg of heat near the center of the star.

The star remains in hydrostatic equilibrium.

The final thermal energy of the star becomes: $U_0 + e$

$$U = (2 \times 10^{51} \text{ erg}) + (7 \times 10^{50} \text{ erg})$$

$$U = 2.7 \times 10^{51} \text{ erg}.$$

This means that the final stellar energy of the star will be:
(after plugging in γ as $5/3$:)

$$E = -U = -2.7 \times 10^{51} \text{ erg}$$

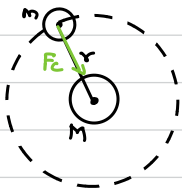
Now, let's find the initial central temperature and the final central temperature of the star to see if the heat made the star hotter or colder.

$$T_{c_i} = \frac{(\gamma - 1) m_H U_0}{k_B M} = \frac{(\frac{5}{3} - 1) m_H \times (2 \times 10^{51} \text{ erg})}{k_B M} = \frac{m_H}{k_B M}(1.33 \times 10^{51})$$
(initial)

$$T_{cf} = \frac{(\gamma - 1) m_H U}{k_B M} = \frac{(\frac{5}{3} - 1) m_H \times (2.7 \times 10^{51} \text{ erg})}{k_B M} = \frac{m_H}{k_B M}(1.8 \times 10^{51})$$
(final)

The central temperature of the star after the addition of heat is higher than before. As the thermal energy of the star goes up, so does the central temperature.

Stars primarily produce energy through nuclear fusion in their cores. As more and more hydrogen nuclei fuses into helium, it releases tremendous amounts of energy and the star's core becomes hotter and denser. This leads to an increase in the thermal energy within the core. The central temperature of a star is a measure of how hot the core of the star is. It is directly proportional to the thermal energy in the core. The higher the thermal energy due to ongoing nuclear fusion, the higher the central temperature of the star.

2) 

Kinetic energy = ?
gravitational potential energy = ?

} in terms of m, M, r.

if you consider the satellite's kinetic energy as its thermal energy, prove the satellite obeys the virial theorem for $\gamma = 5/3$.

As the satellite is orbiting, it is experiencing a centripetal force from the planet, which is essentially the force applied by the planet on the satellite when it is moving in a circular orbit

$$F_c = \frac{mv^2}{r}$$, here v is the velocity of the satellite.

This force is equal to the gravitational attraction force between the planet and the satellite and is given by:

$$F_G = \frac{GMm}{r^2}$$

$$\therefore \quad F_c = F_G$$

$$\frac{mv^2}{\cancel{r}} = \frac{GMm}{r^{\cancel{2}}}$$

multiplying both sides with $1/2$,

$$\frac{1}{2}mv^2 = \frac{GMm}{2r}$$. (Kinetic energy is given by $\frac{1}{2}mv^2$.)

Hence, $KE = \frac{GMm}{2r}$. This will be the kinetic energy of the satellite.

The expression for gravitational potential energy is given by:

$$U(r) = -\int_{\infty}^{r} F_G \, dr \quad , \quad F_G = \frac{-GMm}{r^2}$$

The negative sign comes because of the reference point of the energy I have chosen. If I consider infinity to be at 0 gravitational potential, then anything below that will have a value and it would become increasingly more negative as I approach the planet. Meaning, the magnitude of the gravitational potential energy will increase but the sign will be negative because of the reference point.

$$\therefore \; U(r) = -\int_{\infty}^{r} -\frac{GMm}{r^2} \, dr = GMm \int_{\infty}^{r} \frac{1}{r^2} \, dr$$

$$\therefore \; U(r) = GMm \left[\frac{-1}{r} \right]_{\infty}^{r} = -\frac{GMm}{r} = \Omega$$

The gravitational potential energy of the satellite $\Omega = -\dfrac{GMm}{r}$

If we assume that the kinetic energy is equal to the thermal energy, we get $U = \dfrac{GMm}{2r}$. Virial theorem states that: $U = \dfrac{-\Omega}{3(\gamma - 1)}$, $\gamma = 5/3$.

Plugging the given values: $\dfrac{GMm}{2r} = \dfrac{-\left(-\dfrac{GMm}{r}\right)}{3(5/3 - 1)}$

$$\text{RHS} = \frac{GMm}{r} \times \frac{1}{\cancel{3} \times 2/\cancel{3}} = \frac{GMm}{2r} = \text{LHS}$$

This means that the satellite obeys the virial theorem for $\gamma = 5/3$.

3) Time of sun on the main sequence $\approx 10^{10}$ yrs.

$$\approx 10^{10} \times 3.156 \times 10^{7} \text{ s}$$

$$\approx 3.156 \times 10^{17} \text{ s}$$

Luminosity of the sun $L_\odot = 3.83 \times 10^{33}$ erg s^{-1}

Assuming the luminosity remains fixed throughout, the energy radiated into space in this phase will be: $E = L \times t$ (As luminosity is energy per unit time).

$$E = 3.83 \times 10^{33} \text{ erg } \cancel{s^{-1}} \times 3.156 \times 10^{17} \cancel{s}$$

$$E = 1.20875 \times 10^{51} \text{ erg}$$

1.20875×10^{51} erg energy will be radiated out into the space when the sun is on main sequence.

3.8×10^{24} MeV of energy is released for each gram of hydrogen fused into helium.

3.8×10^{24} MeV g^{-1} $= 3.8 \times 10^{24} \times 1.6022 \times 10^{-6}$ erg g^{-1}

energy released by 1g of hydrogen fused into helium $(E_H) = 6.08836 \times 10^{18}$ erg g^{-1}

total mass of hydrogen consumed to provide this energy : $\dfrac{E}{E_H} = \dfrac{1.20875 \times 10^{51} \text{ erg}}{6.08836 \times 10^{18} \text{ erg } g^{-1}}$

$$= 1.9853 \times 10^{32} \text{ g}$$

1.9853×10^{32} g of hydrogen must be consumed for the sun to remain on main sequence for 10^{10} years.

The total mass of the sun is $M_\odot = 1.989 \times 10^{33}$ g
Initial composition of hydrogen in the sun is $X = 0.7$. this means that 70% of the sun's total mass is hydrogen's mass.

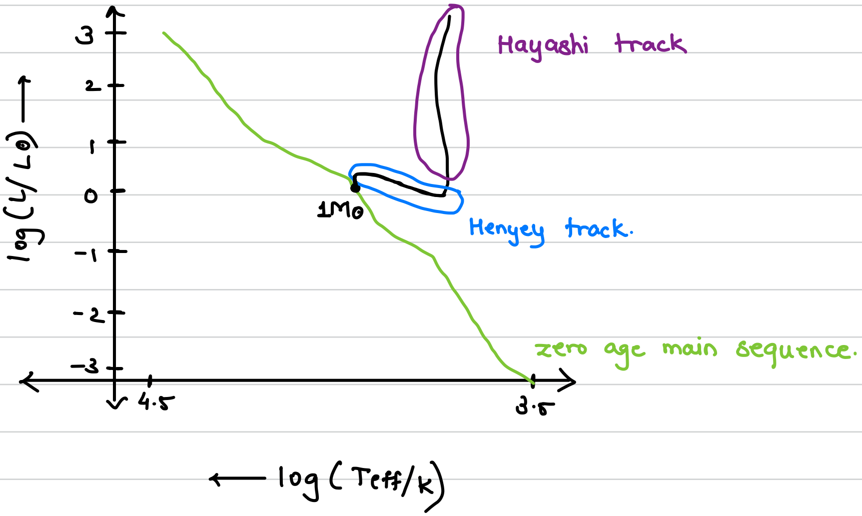
$$\text{initial hydrogen mass} = \frac{70}{100} \times 1.989 \times 10^{33} \text{ g} = 1.3923 \times 10^{33} \text{ g}.$$

Out of this, 1.9853×10^{32} g of hydrogen gets consumed during the main sequence phase.

$$\therefore \quad \frac{1.9853 \times 10^{32} \text{ g}}{1.3923 \times 10^{33} \text{ g}} \times 100 = 14.2591\,\%$$

This tells us that from the Sun's initial hydrogen content, 14.2591% or $\frac{0.142591}{1}$ fraction of hydrogen mass gets converted to helium during the main sequence phase.

4)



5) The late phases of pre-main sequence evolution assume that a star with mass M moves along the Henyey track at constant Luminosity L.

$$L = 4\pi R^2 \sigma T^4$$

$$\therefore \; R^2 = \frac{LT^{-4}}{4\pi\sigma}$$

$$\therefore \; R = \sqrt{\frac{LT^{-4}}{4\pi\sigma}} = T^{-2}\sqrt{\frac{L}{4\pi\sigma}}$$

differenciating both sides with respect to time,

$$\frac{dR}{dt} = -2T^{-3}\sqrt{\frac{L}{4\pi\sigma}}\,\frac{dT}{dt} \quad \text{— eq ①}$$

From the equation giving the Kelvin - Helmholtz Contraction, we get:

$$\frac{dR}{dt} = \frac{-2R^2L}{f_\Omega GM^2}$$

plugging in the value for luminosity found previously, we get:

$$\frac{dR}{dt} = \frac{-2L^2T^{-4}}{f_\Omega GM^2 4\pi\sigma} \quad \text{— eq 2}$$

The left side of both 1 and 2 is the same, lets equate the right side

$$\therefore \frac{-\cancel{2}L^2T^{-4}}{f_\Omega GM^2 4\pi\sigma} = -\cancel{2}T^{-3}\sqrt{\frac{L}{4\pi\sigma}}\frac{dT}{dt}$$

$$\therefore \frac{L^2}{T^4 f_\Omega GM^2 4\pi\sigma} = \frac{1}{T^3}\frac{(L)^{1/2}}{2(\pi)^{1/2}(\sigma)^{1/2}}\frac{dT}{dt}$$

$$\frac{dT}{dt} = \frac{L^2 \cancel{T^3}\cancel{2}(\pi)^{1/2}(\sigma)^{1/2}}{L^{1/2}T^{\cancel{4}} f_\Omega GM^2 \underset{2}{\cancel{4}}\pi\sigma}$$

$$\frac{dT}{dt} = \frac{L^{3/2}}{2T} \times \frac{1}{f_\Omega GM^2} \times \frac{1}{(\pi\sigma)^{1/2}}$$

$$\therefore T\,dT = \frac{L^{3/2}}{2f_\Omega GM^2} \times \frac{1}{(\pi\sigma)^{1/2}}\,dt$$

Integrating both sides of the equation,

$$\int_{T_{eff}^a}^{T_{eff}^b} T\,dT = \int \frac{L^{3/2}}{2f_\Omega GM^2} \times \frac{1}{(\pi\sigma)^{1/2}}\,dt$$

$$\therefore \left[\frac{T^2}{2}\right]_{T_{eff}^a}^{T_{eff}^b} = \frac{L^{3/2}}{2 f_\Omega G M^2} \times \frac{1}{(\pi\sigma)^{1/2}} \int dt$$

$$\therefore \frac{(T_{eff}^b)^2}{2} - \frac{(T_{eff}^a)^2}{2} = \frac{L^{3/2}}{2 f_\Omega G M^2} \times \frac{1}{(\pi\sigma)^{1/2}} \times t$$

Here, t represents the time taken for the effective temperature to increase to T_{eff}^b starting from the value T_{eff}^a when the star moves along the Henyey track during its late phases of pre-main sequence evolution.

For the sun, $L_\odot = 3.83 \times 10^{33}$ erg s^{-1}, $T_{eff}^a = 4300\,K$, $T_{eff}^b = 5600\,K$

$f_\Omega \approx 1.7$ (shape factor)

Let's plug all of this in the expression above. We get:

$$\frac{(5600)^2}{2} - \frac{(4300)^2}{2} = \frac{(3.83 \times 10^{33})^{3/2}}{2 \times 1.7 \times 6.67 \times 10^{-8} \times (1.989 \times 10^{33}\,g)^2} \times \frac{1}{(3.14 \times 5.67 \times 10^{-5})^{1/2}} \times t$$

$$6.435 \times 10^6 = \frac{2.3702 \times 10^{50}}{8.9717 \times 10^{59}} \times \frac{1}{0.01335} \times t$$

$$\frac{7.70734 \times 10^{64}}{2.3702 \times 10^{50}} = t$$

$$\therefore t = 3.2518 \times 10^{14}\,s = 3.2518 \times 10^{14}\,s \times \frac{1\,\text{year}}{3.156 \times 10^7\,s} = 10{,}303{,}548.8 \text{ yrs}$$

So, the sun is estimated to spend 10,303,548.8 years on the Henyey track.

5) To calculate the percentage of rest mass converted to energy by the triple alpha process $3\,^4He \longrightarrow {}^{12}C$, we can use $E = \Delta m c^2$.

change in mass for the triple alpha process (Δm):

total initial mass $= 3 \times 4.0026\,u = 12.0078\,u$.

total final mass $= 12.0000\,u$

$\therefore \Delta m =$ initial mass $-$ final mass

$= 12.0078\,u - 12.0000\,u$

$\therefore \Delta m = 0.0078\,u$

Energy produced (E) $= \Delta m c^2$

$= 0.0078\,u \times (3 \times 10^8\, m/s)^2$

$E = 0.0078\,u \times 1.6605 \times 10^{-24}\, g/u \times (2.998 \times 10^{10}\, cm/s)^2$

$E = 1.1641 \times 10^{-5}$ erg

percentage of rest mass converted to energy:

$$\text{percent} = \frac{0.0078}{12.0078} \times 100$$

percent $\approx 0.06496\,\%$

So, approximately 0.065% of the rest mass is converted to energy in the triple alpha process.

Now, for Hydrogen burning, we consider the fusion of 4H atoms (protons) into one helium nucleus (2 protons and 2 neutrons). The mass of a hydrogen atom is $1.0079\,u$.

initial mass of 4 hydrogen atoms $= 4 \times 1.0079\,u = 4.0316\,u$
final mass of 1 helium nucleus $= 4.0026\,u$

$\Delta m =$ initial mass $-$ final mass.
$\Delta m = 4.0316\,u - 4.0026\,u$
$\Delta m = 0.029\,u$

Energy produced in this process:

$$E = \Delta m c^2$$
$$= 0.029 \times 1.6605 \times 10^{-24} \times (2.998 \times 10^{10}\ \text{cm/s})^2$$
$$= 4.3281 \times 10^{-5}\ \text{erg}$$

percentage of rest mass converted to energy:

$$\text{percent} = \frac{0.029}{4.0316} \times 100$$

$$\text{percent} \approx 0.7193\ \%$$

This means, approximately $0.7193\ \%$ of the rest mass is converted to energy in hydrogen burning. For the triple alpha process, this percent is much smaller. So, the hydrogen burning process is more efficient at converting mass into energy compared to the triple alpha process.