1. *For the solar model introduced in the online Python/*MESA-Web *labs, create a Python plot of* $\log(L/\mathrm{L}_\odot)$ *as a function of* $\log(t/\mathrm{yr})$, *where* t *is the time since the start of the* MESA-Web *calculation. Be sure to label your axes appropriately.*

   Attach this plot to your manuscript, either electronically or by printing and scanning it. (Don't include the full Jupyter notebook. To save a plot from within Python, use the `plt.savefig()` *command. If you get stuck, ask for help on the Python discussion forum in Canvas.).*

   From your plot, estimate by what factor the Sun's luminosity has increased over the period extending from the pre-main sequence luminosity minimum at $\log(t/\mathrm{yr}) \approx 7.10$ *through to the present day.*

   See Fig. 1.1 for the plot. At the luminosity minimum, $\log(L/\mathrm{L}_\odot) = -0.367$. Given that the present-day luminosity is $\log(L/\mathrm{L}_\odot) = 0.00$ (by definition), the luminosity has increased since the minimum by a factor $10^{0.367} = 2.328$.

3 points for plot (partial credit if axes are unlabelled), **2 points** for determining luminosity increase.

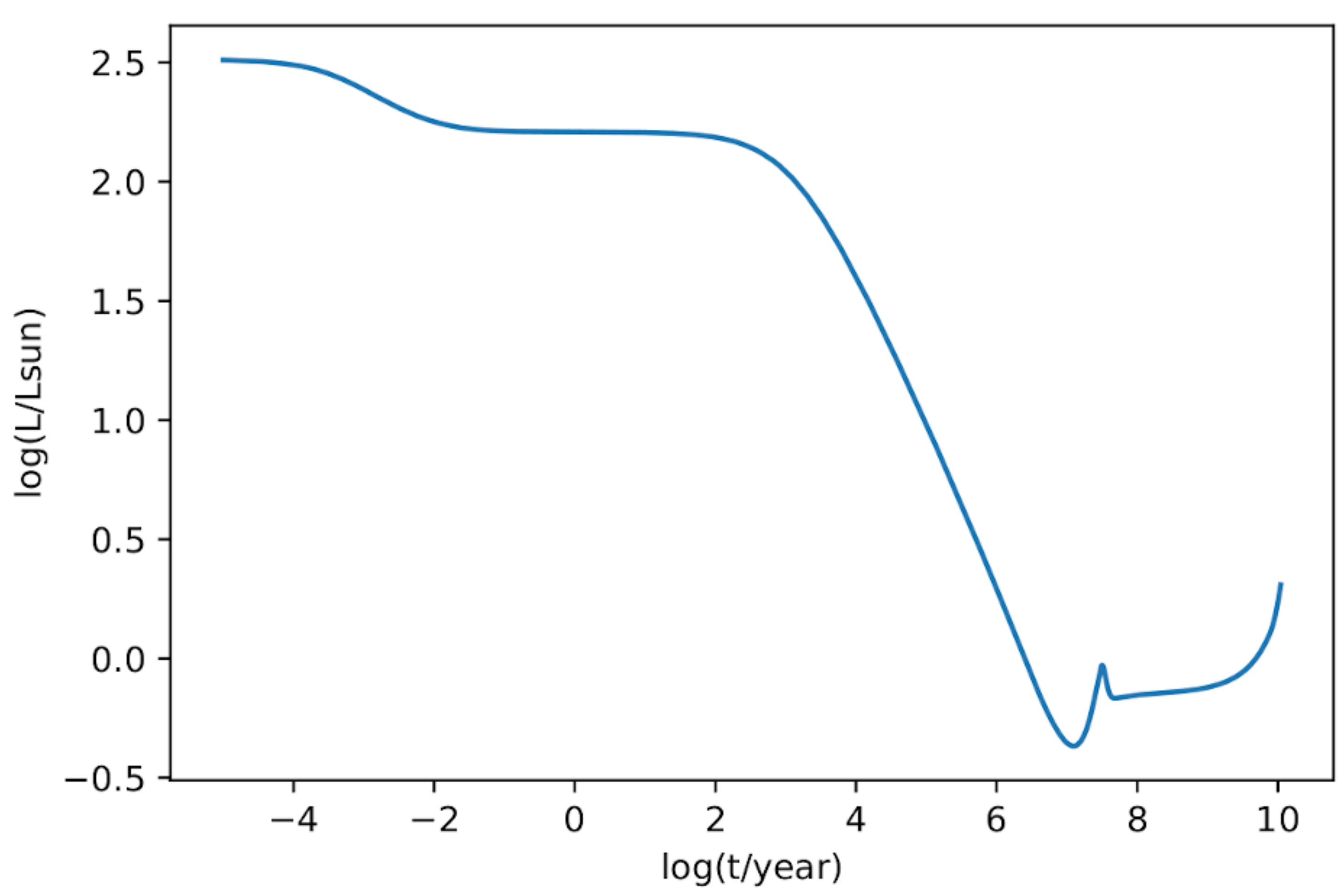

Figure 1.1: Plot of $\log(L/\mathrm{L}_\odot)$ versus $\log(t/\mathrm{yr})$ for the *MESA-Web* solar model.

2. *For a sample of pure hydrogen, write an expression relating the total number density of particles* n *to the number densities of hydrogen atoms/ions* (n_H) *and free electrons* (n_e). *Use this expression, and others appearing in* Handout 17, *to rewrite equation* [17.7] *for the ionization fraction* x *in terms of* n *and* T. *Then, evaluate the ionization fraction at a temperature* 10^4 K *and pressure* 10^5 Ba. *(You may neglect radiation pressure and assume the atoms, ions and electrons behave classically). How does the ionization fraction change if the temperature is doubled? Or if the pressure is doubled?*

   The number density of particles is given simply as the sum of hydrogen atom/ion and free electron number densities:

$$n = n_\mathrm{H} + n_\mathrm{e}.$$

   The referenced equation, a variant of the Saha equation for hydrogen, is

$$\frac{x^2}{1-x} = \frac{(2\pi m_\mathrm{e} k_\mathrm{B} T)^{3/2}}{n_\mathrm{H} h^3} \exp[-\chi_\mathrm{H}/(k_\mathrm{B}T)].$$

1 point for the expression for n, **2 points** for obtaining ionization fraction in terms of n and T, **3 points** for the ionization fraction at the given T and P, at double T, and at double P.

To eliminate the $n_{\rm H}$ term, we can use the above expression for n together with the definition

$$x \equiv \frac{n_{\rm H,1}}{n_{\rm H}}$$

of the ionization fraction x, and the charge conservation relation

$$n_{\rm H,1} = n_{\rm e}.$$

Combining these equations, we find that

$$n_{\rm H} = \frac{n}{1+x},$$

and so the Saha equation becomes

$$\frac{x^2}{(1-x)(1+x)} = \frac{(2\pi m_{\rm e} k_{\rm B} T)^{3/2}}{n h^3} \exp[-\chi_{\rm H}/(k_{\rm B}T)].$$

To evaluate the ionization fraction at a temperature 10^4 K and pressure 10^5 Ba, we can use the fact that the ideal gas law applies (because we're in the classical limit and radiation pressure is negligible); hence,

$$P = n k_{\rm B} T,$$

and so $n = 7.24 \times 10^{16}\,\mathrm{cm}^{-3}$. Then, evaluating the Saha equation

$$\frac{x^2}{(1-x)(1+x)} = 0.00469.$$

Solving for the ionization fraction,

$$x = 0.0683.$$

If the temperature is doubled, the ionization fraction is $x = 0.993$ — the hydrogen transitions from barely ionized to almost completely ionized. Conversely, if the pressure is doubled, the ionization fraction drops to $x = 0.0483$ — the hydrogen becomes slightly more neutral, because with more particles per unit volume (n) recombinations become more common.

3. *Consider an ideal-gas star whose pressure and density follow the polytropic relation*

$$P = K\rho^{(n+1)/n}$$

(in this and subsequent equations, n is the polytropic index, not the number density).

(*a*) *Derive an expression for the temperature in terms of the pressure P and other constant quantities (K, n, μ, etc.).*

(*b*) *Evaluate the dimensionless temperature gradient ∇_T.*

(*c*) *Assuming that $\gamma = 5/3$, show that the star is convectively stable everywhere if $n > 1.5$, and convectively unstable everywhere if $n < 1.5$.*

2 points for deriving expression for T, **1 point** for evaluating ∇_T, **3 points** for applying Schwarzschild criterion to obtain result.

(a) Combining the polytropic relation (above) with the ideal-gas EOS [6.4], the density can be eliminated to obtain

$$P = K\left(\frac{P\mu m_{\mathrm{H}}}{k_{\mathrm{B}}T}\right)^{(n+1)/n}.$$

Solving for the temperature,

$$T = \frac{\mu m_{\mathrm{H}}}{k_{\mathrm{B}}} K^{n/(n+1)} P^{1/(n+1)}.$$

(b) Using eqn. [12.6], the dimensionless temperature gradient evaluates as

$$\nabla_T \equiv \frac{\partial \ln T}{\partial \ln P} = \frac{1}{n+1},$$

independent of position within the star.

(c) The Schwarzschild criterion for convective stability is

$$\nabla_T < \nabla_{\mathrm{ad}}$$

(see eqn. [13.9]). For an ideal gas with $\gamma = 5/3$, then eqn. [13.7] gives $\nabla_{\mathrm{ad}} = 2/5$. Using this, and the above expression for ∇_T, the stability criterion becomes

$$\frac{1}{n+1} < \frac{2}{5},$$

or

$$n > \frac{3}{2}.$$

Therefore, the star is convectively stable if $n > 1.5$, independent of position (i.e., everywhere). Conversely, the star is convectively unstable if $n < 1.5$, independent of position.

4. *Consider an $n = 1$ polytrope with mass M and radius R.*

(*a*) *Write down an expression for the central density ρ_{c}, in terms of M and R.* (Hint: apply eqn. [19.11], using the analytic solution for $n = 1$ to evaluate the term in parentheses.)

(*b*) *Evaluate the ratio $\rho_{\mathrm{c}}/\bar{\rho}$, where $\bar{\rho} = 3M/(4\pi R^3)$ is the mean density. Is this ratio larger or smaller than for an $n = 0$ (constant density) polytrope?*

(*c*) *Derive an expression for the interior mass m, in terms of M, R and r.*

(*d*) *Use your $m(r)$ expression to confirm that*

$$m(R/2) = \frac{M}{\pi}.$$

1 point for central density, **1 point** for density comparison, **2 points** for $m(r)$ derivation, **1 point** for $m(R/2)$ result.

(a) The central density of an arbitrary polytrope is

$$\rho_{\mathrm{c}} = \frac{1}{4\pi}\frac{M}{R^3}\left(-\frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-1}_{z=z_{\mathrm{s}}}.$$

(see eqn. [19.11]). When $n = 1$,

$$w(z) = \frac{\sin z}{z} \qquad w'(z) = \frac{\cos z}{z} - \frac{\sin z}{z^2},$$

and $z_s = \pi$. Therefore,

$$\rho_c = \frac{\pi}{4}\frac{M}{R^3}.$$

(b) The ratio is

$$\frac{\rho_c}{\bar{\rho}} = \frac{\pi^2}{3}$$

This is larger than for a constant-density polytrope (which has a ratio of unity), indicating that the $n = 1$ polytrope is more centrally concentrated.

(c) Equation [19.9] gives the interior mass as

$$m(r) = 4\pi \rho_c r^3 \left(-\frac{1}{z}\frac{dw}{dz}\right).$$

Using the above expressions for ρ_c and w', together with the linear relationship

$$\frac{r}{R} = \frac{z}{z_s} = \frac{z}{\pi}$$

between r and z, leads to an interior mass

$$m(r) = \pi^2 \frac{M}{R^3} r^3 \left[\frac{\sin(\pi r/R)}{(\pi r/R)^3} - \frac{\cos(\pi r/R)}{(\pi r/R)^2}\right].$$

(d) Setting $r = R/2$ in this expression leads to

$$m(R/2) = \pi^2 \frac{M}{R^3}\frac{R^3}{2^3}\left[\frac{1}{(\pi/2)^3}\right] = \frac{M}{\pi}.$$

5. *Use* Poly-Web *to create models for $n = 0$ and $n = 1$ polytropes. For each model, plot a graph of w versus z/z_s, and (using a dashed line-style) also show the corresponding analytic solution. Do the* Poly-Web *and analytic solutions agree?*

   See Fig. 1.1 for an example plot. The analytic and numerical solutions do agree nicely!

2 points for each plot (partial credit if axis/line labels are missing, or if axes are flipped).

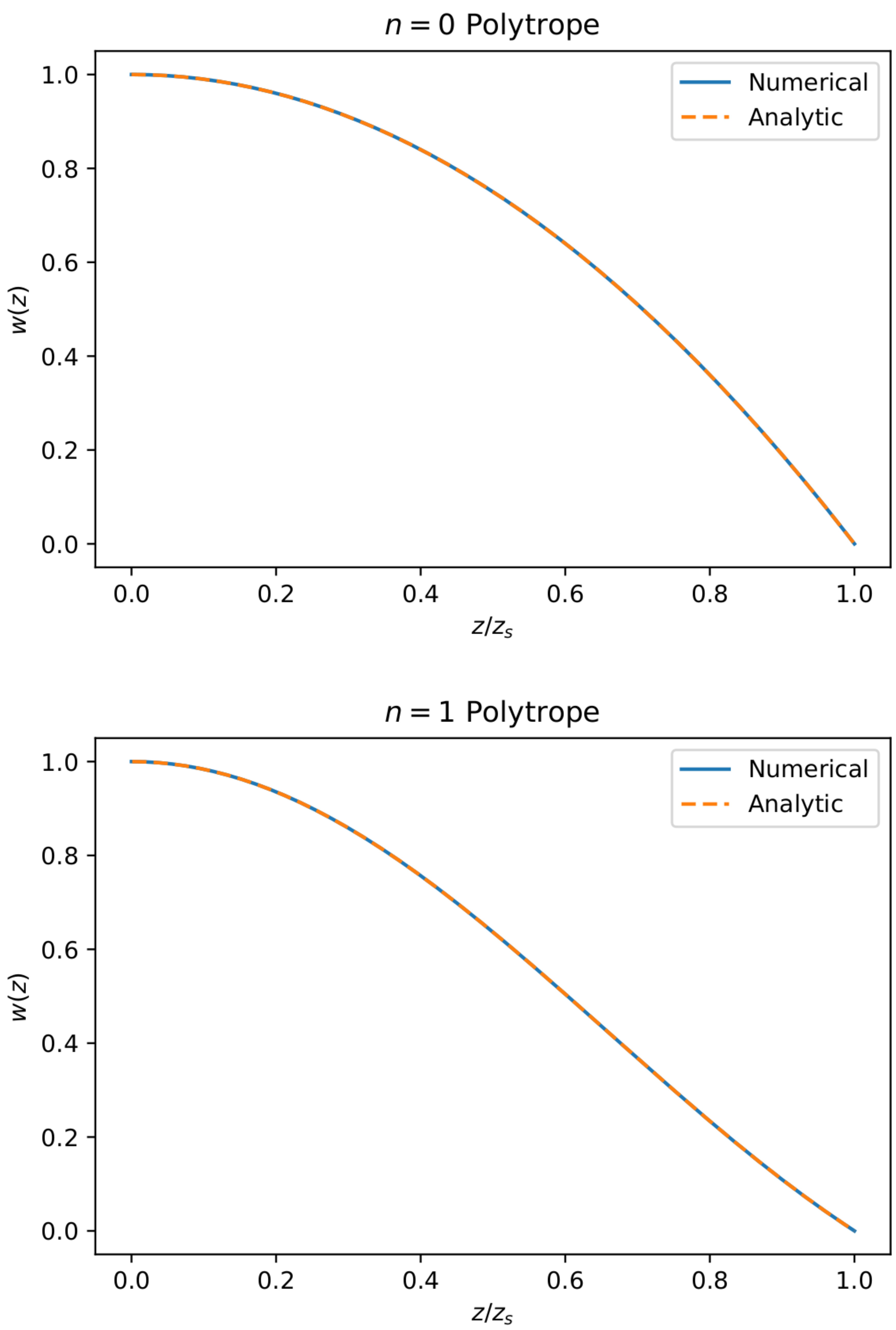

Figure 1.1: Plots of the $n = 0$ and $n = 1$ polytrope solutions.