Home Work 2

1) Linearly varying density : $\rho = \rho_c$ (at the center)

$\rho = 0$ (at the surface)

The mass equation is given as:

$$dm = 4\pi \rho(r)\, r^2\, dr$$

dm : small mass element of volume $4\pi r^2 dr$

$\rho(r)$: density function given.

$$\rho(r) = \rho_c \left(1 - \frac{r}{R}\right)$$

The mass function can be obtained by simply integrating the mass equation with respect to r.

$$\int_0^{m(r)} dm = \int_0^{r} 4\pi \rho(r) r^2\, dr$$

$$m(r) = 4\pi \int_0^{r} \rho_c \left(1 - \frac{r}{R}\right) r^2\, dr$$

$$\therefore m(r) = 4\pi \rho_c \int_0^{r} \left(r^2 - \frac{r^3}{R}\right) dr = 4\pi \rho_c \left[\frac{r^3}{3} - \frac{1}{R} \times \frac{r^4}{4}\right]_0^{r}$$

$$\therefore m(r) = 4\pi \rho_c r^3 \left(\frac{1}{3} - \frac{r}{4R}\right)$$

To find the total mass of the star:

$$m(r) = 4\pi \rho_c r^3 \left(\frac{1}{3} - \frac{r}{4R}\right)$$

$$m(r) = M = 4\pi \rho_c R^3 \left(\frac{1}{3} - \frac{R}{4R}\right)$$

$$\therefore \quad M = \frac{1}{3}\pi \rho_c R^3$$

$$\therefore \quad \rho_c = \frac{3M}{\pi R^3}$$

Hence, we have derived the mass function and the expression for ρ_c.

2) We need to derive an expression for the gravitational potential energy of a linear-density star in terms of mass (M) and radius (R), we'll first calculate the gravitational potential energy of a point mass and then integrate it across the entire star.

$$\Omega = -\int_0^M \frac{Gm}{r}\, dm = -\int_0^M \frac{G}{r} \pi r^3 \rho_c \left(\frac{4}{3} - \frac{r}{R}\right) dm$$

As we have our equation for mass in terms of radius, we should do a variable conversion to change this:

$$\frac{dm}{dr} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3 \rho_c - \frac{\pi r^4 \rho_c}{R}\right) = 4\pi r^2 \rho_c - \frac{4\pi r^3 \rho_c}{R}$$

$$dm = 4\pi r^2 \rho_c \left(1 - \frac{r}{R}\right) dr$$

Let's substitute this for dm into the integral and change the bounds.

$$\Omega = -\int_0^R G\pi r^2 \rho_c \left(\frac{4}{3} - \frac{r}{R}\right) 4\pi r^2 \rho_c \left(1 - \frac{r}{R}\right) dr$$

$$\therefore \Omega = -4G\pi^2 \rho_c^2 \int_0^R r^4\left(\frac{4}{3} - \frac{7r}{3R} + \frac{r^2}{R^2}\right) dr = -4G\pi^2\rho_c^2 \int_0^R \frac{4r^4}{3} - \frac{7r^5}{3R} + \frac{r^6}{R^2}\, dr$$

$$\therefore \Omega = -4G\pi^2\rho_c^2 \left[\frac{4r^5}{15} - \frac{7r^6}{18R} + \frac{r^7}{7R^2}\right]_0^R = -4G\pi^2\rho_c^2\left[\frac{4}{15} - \frac{7}{18} + \frac{1}{7}\right]R^5$$

$$\therefore \Omega = \frac{-26}{35} G\pi^2 \rho_c^2 R^5$$

Let's substitute in ρ_c that we found in Part 1.

$$\rho_c = \frac{3M}{\pi R^3} \longrightarrow \rho_c^2 = \frac{9M^2}{\pi^2 R^6}$$

$$\Omega = \frac{-26}{35} \times \frac{G\pi^2 R^5 M^2}{\pi^2 R^6} = \frac{-26}{35} \times \frac{GM^2}{R}$$

When comparing this with the equation 4.8, you get $f_\Omega = 26/35$, which is larger than $3/5$. This does make sense for the numbers to be different because in physical stars, the density would not increase linearly. We would also expect the gravitational potential energy to increase because more material is closer to the surface. So this increase is expected.

3) The hydrostatic equilibrium equation is given by:

$$\frac{\partial P}{\partial r} = \rho g$$

Here, the outward force due to the pressure gradient must balance the inward force due to gravity.

① $\frac{\partial P}{\partial r}$ is the pressure gradient. It tells us how quickly the pressure changes as we move radially inward or outward from a point within a star.

② ρ (mass density) represents the amount of matter present at a given radial co-ordinate. Generally, as we move deeper into the interior of a star (as r decreases), the mass density increases because there is more mass compressed into a smaller volume due to the star's self gravity.

③ g (gravitational acceleration) represents the strength of the gravitational field at a given radial co-ordinate. Inside a star, gravity is pulling matter inward, and its strength typically increases as you move deeper into the star. This is because there is more mass located closer to the center of the star, resulting in stronger gravitational forces.

Looking at the equation $\frac{\partial P}{\partial r} = \rho g$, as you move deeper into the star (decreasing r), both ρ and g tend to increase.

If ρ and g are both increasing, then their product (ρg) is also increasing.
Since $\frac{\partial P}{\partial r}$ is proportional to ρg, this means that $\frac{\partial P}{\partial r}$ becomes increasingly positive as you move towards the center of the star. The pressure gradient becomes more positive, indicating that the pressure increases more rapidly as you move inward.
Consequently, the pressure in the star is a monotonic-decreasing function of the radial co-ordinate, with the pressure decreasing as you move outward from the star's core towards its surface (increasing radius).

4) When an ideal-gas star in hydrostatic equilibrium with radius R and mass M, is cooled instantaneously down to T = 0K, the thermal pressure keeping the star in hydrostatic equilibrium and stopping it from collapsing under its own gravity goes away.
The star will collapse at the rate of the gravitational collapse, which will make the initial acceleration of the surface layers to be:

$$g_s = \frac{-GM}{R^2}$$

Assuming that this acceleration remains constant, the time taken for these surface layers to collapse down to the origin is:

$$t_{dyn} = \sqrt{\frac{2R^3}{GM}} \sim 1200 \text{ s} \quad (\text{for the sun})$$

So, it will take $\sqrt{\frac{2R^3}{GM}}$ time for the star to collapse.

(The Kelvin-Helmholtz timescale is in the order of 10^4 years, which is very slow to take effect).

5) The equation for hydrostatic equilibrium is given by:

$$\frac{\partial P}{\partial r} = \rho g$$

Integrating the equation, we get:

$$P_s - P_c = \int_0^R g \rho \, dr$$

P_s: pressure at the surface
P_c: pressure at the center.

from Q1, you get the expression for ρ (linear-density function of star). Plugging than in the above equation and solving, you get the integral in terms of mass. This can be written as:

$$P_s - P_c = - \int_0^M \frac{Gm}{4\pi r^4} \, dm$$

The -ve sign comes in because radius and mass are inversly proportional. As you go deeper into the star, radius decreases but the mass increases.

Now, we are assuming that the surface pressure vanishes (as it is negligible compared to the pressure at the center of the star)

$\therefore$ $P_s = 0$, So the above equation becomes:

$$\cancel{-} P_c = \cancel{-} \int_0^M \frac{Gm}{4\pi r^4} \, dm$$

$$\therefore \quad P_c = \frac{G}{4\pi r^4} \int_0^M m \, dm = \frac{G}{4\pi r^4} \left[\frac{m^2}{2} \right]_0^M$$

$$\therefore \quad P_c = \frac{G}{4\pi r^4} \times \frac{M^2}{2}$$

$$\therefore \quad P_c = \frac{GM^2}{8\pi r^4}$$

This expression will be the lower limit of the pressure at the center of the star.

5) density (ρ) = 50 g/cm^3 for a fully ionized helium gas.
Fully ionized helium consists only of 2 protons and 2 neutrons.
So, the mean molecular weight (μ) for fully ionized helium is:
μ = (2 protons + 2 neutrons) / (avogadr's number).

$$\mu = \frac{(2+2)}{6.022 \times 10^{23}} \approx 0.6667$$

The number density of particles (n) is defined as the number of particles per unit volume.

For an ideal gas, it is given by:

$$n = \frac{\rho}{m \times \mu}$$

ρ: density of the gas
m: molar mass of the gas (atomic mass number of the gas in g/mol.)
μ: mean molecular weight

$$n = \frac{50}{4 \times 0.6667}$$

$$n \approx 18.75 \text{ cm}^{-3}$$

7) In adiabatic compression, there is no heat exchange with the surroundings, so the relationship between pressure (P), density (ρ) and temperature (T) for an ideal gas is given by:

$$PV^{\gamma} = \text{constant}$$

γ: the ratio of specific heats (C_P/C_V) for the gas, for this gas $\gamma = \frac{5}{3}$.

For the initial state of the gas before compression,
Pressure (P_1), volume (V_1), density (ρ_1), temperature (T_1).

For the final state of the gas after compression,
volume gets reduced to $\frac{1}{3}$ of its initial volume.

$V_2 = \frac{1}{3}V_1$, when volume reduces pressure increases. P_1 will be thrice of P_2.

Using the adiabatic equation for the initial and final states:

① For the initial state:

$$P_1V_1^{\gamma} = \text{constant}.$$

② For final state:

$$P_2V_2^{\gamma} = \text{constant}.$$

In the adiabatic compression, PV^{γ} should always remain constant.
Lets equate the initial and final states:

$$P_1V_1^{\gamma} = P_2V_2^{\gamma}$$

We can relate the initial and final states:

$$P_1 (V_1)^{5/3} = P_2 \left(\frac{1}{3} V_1\right)^{5/3}$$

$$P_1 (V_1)^{5/3} = \frac{1}{243} P_2 (V_1)^{5/3}$$

$$\therefore P_1 = \frac{1}{243} P_2 \Rightarrow \frac{P_1}{P_2} = \frac{1}{243}$$

Using the ideal gas law to relate pressure, density and temperature,

$$P_1 = \rho_1 R T_1$$

$$P_2 = \rho_2 R T_2$$

$$\therefore \frac{P_1}{P_2} = \frac{\rho_1 \cancel{R} T_1}{\rho_2 \cancel{R} T_2}$$

$$\therefore \frac{1}{243} = \frac{\rho_1 T_1}{\rho_2 T_2} \Rightarrow \frac{\rho_2}{\rho_1} = \frac{243 T_1}{T_2}$$

As $V_2/V_1 = 1/3$, and because the gas is ideal, we get:

$$\frac{V_2}{V_1} = \frac{\rho_1}{\rho_2}$$

$$\frac{1}{3} = \frac{\rho_1}{\rho_2}$$

When we plug in the expression for ρ_1/ρ_2 in the density-temperature equation, we get:

$$\frac{3}{1} = \frac{243 T_1}{T_2} \Rightarrow \frac{T_2}{T_1} = \frac{243}{3} = \frac{81}{1}$$

So, by compressing the ideal gas adiabatically to one third of its volume, the density increased by a factor of 3, the pressure increased by a factor of 243 and the temperature also increased by a factor of 81.

8) Mass of the star $(M) = 4 M_\odot = 4 \times (1.989 \times 10^{30} \text{ kg})$

hydrogen mass fraction $(X) = 0.6$

metal mass fraction $(Z) = 0.15$

no. of helium nuclei $(n) = ?$

We can use the information we know about mass fractions of elements in a star. Stars are mostly made up of hydrogen and helium, but it also contains small quantities of metals. Let the mass fraction of Helium be called Y. For a star, we know that:

$$X + Y + Z = 1$$
$$0.6 + Y + 0.15 = 1$$
$$Y = 1 - 0.6 - 0.15$$
$$Y = 0.25$$

Mass of Helium in the star becomes:

$$M_{He} = 4 M_\odot \times Y = 4 \times 1.989 \times 10^{30} \times 0.25$$
$$M_{He} = 1.989 \times 10^{30} \text{ kg}.$$

$$\text{No. of Helium nuclei } (N_{He}) = \frac{\text{Mass of Helium in the star}}{\text{Mass of one Helium nuclei}} = \frac{1.989 \times 10^{30}}{4 \times 1.673 \times 10^{-27}}$$

$$N_{He} = 2.9722 \times 10^{56}$$