1. *Calculate the mean molecular weight for a pure hydrogen/helium mixture with $X = 0.4$, assuming the hydrogen and helium are both 50% ionized (i.e., $\mathcal{N}_j = 0.5 \times \mathcal{Z}_j$, where $\mathcal{N}_j$ and $\mathcal{Z}_j$ are defined in* Handout 15).

   4 points, with partial credit for progress toward the answer. (There are a number of different ways to approach the problem).

   Consider a sample of mass m. The mass of hydrogen in this sample is $Xm = 0.4\,m$, and the mass of helium is therefore $0.6\,m$. The number of hydrogen atoms in the sample is $0.4\,m/m_{\rm H}$, and the number of helium atoms is $0.6\,m/4m_{\rm H} = 0.15\,m/m_{\rm H}$ (assuming the mass of a helium atom is $\approx 4\,m_{\rm H}$); hence, the total number of nuclei in the sample is $0.55\,m/m_{\rm H}$.

   At 50% ionization, each hydrogen atom is associated with half a free electron; and each helium atom is associated with one free electron. Hence, the total number of free electrons in the sample is $0.35\,m/m_{\rm H}$.

   Now counting nuclei and free electrons together, there are $0.9\,m/m_{\rm H}$ particles in total. The mean mass per particle is therefore $0.9^{-1} m_{\rm H} = 1.11\,m_{\rm H}$, and so the mean molecular weight is $\mu = 1.11$.

2. *By combining equations* [6.4] *and* [6.8], *derive the dependence of temperature T on pressure P for an ideal gas undergoing an adiabatic change. Use this expression to prove equation* [13.7].

   Equation [6.4] gives the equation of state for an ideal gas,

$$P = \frac{\rho k_{\rm B} T}{\mu m_{\rm H}},$$

   while equation [6.8] gives the relationship that holds between pressure and density during an adiabatic change,

$$P = K_{\rm ad}\,\rho^{\gamma}.$$

   Rewriting the EOS as

$$\rho = \frac{P \mu m_{\rm H}}{k_{\rm B} T},$$

   and using this to eliminate ρ from the adiabatic relationship, leads to

$$P = K_{\rm ad} \left(\frac{P \mu m_{\rm H}}{k_{\rm B} T} \right)^{\gamma}.$$

   This can be rearranged to give the temperature as a function of pressure for an ideal gas undergoing an adiabatic change,

$$T = \frac{\mu m_{\rm H}}{k_{\rm B}} K_{\rm ad}^{1/\gamma} P^{(\gamma-1)/\gamma}.$$

   Taking the natural log of this expression,

$$\ln T = \frac{\gamma - 1}{\gamma} \ln P + \frac{1}{\gamma} \ln K_{\rm ad} + \ln \left(\frac{\mu m_{\rm H}}{k_{\rm B}} \right).$$

   Differentiating with respect to $\ln P$ then leads to

$$\frac{{\rm d}\ln T}{{\rm d}\ln P} = \frac{\gamma - 1}{\gamma}.$$

Since this only applies to adiabatic changes, it's more correctly written as

$$\left(\frac{\partial \ln T}{\partial \ln P}\right)_{\rm ad} \equiv \nabla_{\rm ad} = \frac{\gamma - 1}{\gamma},$$

which proves eqn. [13.7].

3 points for deriving dependence of T on P, **1 point** for using it to prove equation [13.7].

3. *Consider a location within a star where the interior mass and luminosity are given by $m = 0.3\,\mathrm{M}_\odot$ and $\ell = 5\,\mathrm{L}_\odot$, respectively; the pressure and temperature are $P = 10^{17}\,\mathrm{Ba}$ and $T = 10^7\,\mathrm{K}$, respectively; and the opacity is $\kappa = 1\,\mathrm{cm^2\,g^{-1}}$.*

   - *Evaluate the radiative temperature gradient $\nabla_{\rm rad}$.*
   - *Assuming the stellar material behaves as an ideal gas with $\gamma = 1.4$, evaluate the adiabatic temperature gradient $\nabla_{\rm ad}$.*
   - *By applying the algorithm given in* Handout 14*, explain why convection will occur at this location.*
   - *Assuming a convective efficiency $\varphi_{\rm conv} = 0.5$, evaluate the dimensionless temperature gradient ∇_T at the location.*
   - *Evaluate the convective ($\ell_{\rm conv}$) and radiative ($\ell_{\rm rad}$) interior luminosities, in $\mathrm{L}_\odot$.*

   2 points for $\nabla_{\rm rad}$, **1 point** for $\nabla_{\rm ad}$, **1 point** for explaining why convection will occur, **1 point** for ∇_T, **2 points** for interior luminosities.

   - The radiative temperature gradient $\nabla_{\rm rad}$ is given by eqn. [12.7] as

     $$\nabla_{\rm rad} = \frac{3}{16\pi acG}\frac{\kappa \ell P}{mT^4}$$

     Plugging in the supplied values gives $\nabla_{\rm rad} = 1.27$.
   - Using equation [13.7] with $\gamma = 1.4$, we obtain $\nabla_{\rm ad} = 0.4/1.4 = 0.286$.
   - Convection will occur at this location because $\nabla_{\rm rad} > \nabla_{\rm ad}$.
   - With a convective efficiency $\varphi_{\rm conv} = 0.5$, we can apply eqn. [14.1] to obtain $\nabla_T = 0.778$
   - Using eqn. [14.2], the convective luminosity evaluates to $\ell_{\rm conv} = 0.388\,\ell = 1.94\,\mathrm{L}_\odot$, while the radiative luminosity evaluates to $\ell_{\rm rad} = 0.621\,\ell = 3.06\,\mathrm{L}_\odot$.

4. *The central temperature and density of ZAMS stars can be approximated over the stellar mass interval $0.1\,\mathrm{M}_\odot \lesssim M \lesssim 30\,\mathrm{M}_\odot$ by the fits*

   $$\log\left(T_{\rm c}/\mathrm{K}\right) \approx 7.10 + 0.38\,(M/\mathrm{M}_\odot)\,,$$
   $$\log\left(\rho_{\rm c}/\mathrm{g\,cm^{-3}}\right) \approx 1.77 - 0.77\,(M/\mathrm{M}_\odot)\,.$$

   Derive corresponding expressions for the gas pressure $P_{\rm gas} \equiv P_{\rm ion} + P_{\rm e}$ (assuming an ideal gas with $\mu \approx 0.62$) and the radiation pressure $P_{\rm rad}$ at the center. At what stellar mass does radiation pressure begin to exceed gas pressure?

   2 points for determining an expression for $P_{\rm gas,c}$ as a function of stellar mass, **2 points** likewise for $P_{\rm rad,c}$, **1 point** for finding stellar mass where $P_{\rm rad,c} > P_{\rm gas,c}$.

   Assuming an ideal gas, the equation of state governing the gas pressure is

   $$P_{\rm gas} = \frac{\rho k_{\rm B} T}{\mu m_{\rm H}}.$$

Taking the log of this expression, we obtain

$$\log\left(\frac{P_{\rm gas}}{\rm Ba}\right) = \log\left(\frac{\rho}{\rm g\,cm^{-3}}\right) + \log\left(\frac{T}{\rm K}\right) + \log\left(\frac{k_{\rm B}}{\mu m_{\rm H}}\frac{\rm K\,g\,cm^{-3}}{\rm Ba}\right).$$

Evaluating this expression at the center, and using the given approximations for $T_{\rm c}$ and $\rho_{\rm c}$, this becomes

$$\log\left(\frac{P_{\rm gas,c}}{\rm Ba}\right) \approx 16.99 - 0.39\left(\frac{M}{\rm M_\odot}\right).$$

For the radiation pressure, the equation of state is

$$P_{\rm rad} = \frac{aT^4}{3}.$$

Following the same procedure as with $P_{\rm gas}$, the resulting expression for the central value is

$$\log\left(\frac{P_{\rm rad,c}}{\rm Ba}\right) \approx 13.80 + 1.52\left(\frac{M}{\rm M_\odot}\right).$$

The different slopes of these relations mean that $P_{\rm rad}$ will exceed $P_{\rm gas}$ at large masses. To find the mass where it first begins to exceed, we can set the two pressures equal:

$$16.99 - 0.39\left(\frac{M}{\rm M_\odot}\right) \approx 13.80 + 1.52\left(\frac{M}{\rm M_\odot}\right).$$

Solving for the mass,

$$\frac{M}{\rm M_\odot} \approx 1.67.$$

Note: I made a mistake when writing this question — the $(M/{\rm M_\odot})$ terms should have been $\log(M/{\rm M_\odot})$. With this error corrected, the switch-over mass is much higher, at $M \approx 10^{1.67} M_\odot = 46.8\,{\rm M_\odot}$.

5. *For a free electron gas in the completely degenerate limit, use your knowledge of the momentum distribution function $f_{\rm e}(p)$ to determine what fraction of the electrons have momenta less than half of the Fermi momentum $p_{\rm F}$, and what fraction have momenta more than double the Fermi momentum.*

1 point for writing down momentum distribution function for degenerate limit, **2 points** for fraction less than half $p_{\rm F}$, **1 point** for fraction greater than $2p_{\rm F}$.

In the completely degenerate limit, the momentum distribution function for free electrons is

$$f_{\rm e}(p) = \begin{cases} \frac{8\pi p^2}{h^3} & p < p_{\rm F}, \\ 0 & p > p_{\rm F}. \end{cases}$$

where $p_{\rm F}$ is the Fermi momentum (see eqn. [16.8]). The number density of electrons with a momentum less than half $p_{\rm F}$ is

$$n(p < p_{\rm F}/2) = \int_0^{p_{\rm F}/2} f_{\rm e}(p)\;{\rm d}p = \left[\frac{8\pi p^3}{3h^3}\right]_0^{p_{\rm F}/2} = \frac{\pi p_{\rm F}^3}{3h^3};$$

while the number density of *all* electrons is

$$n = \int_0^\infty f_{\mathrm{e}}(p)\ \mathrm{d}p = \left[\frac{8\pi p^3}{3h^3}\right]_0^{p_{\mathrm{F}}} = \frac{8\pi p_{\mathrm{F}}^3}{3h^3}.$$

Therefore, the fraction of electrons with $p < p_{\mathrm{F}}/2$ is $1/8$.

Similarly, the number of electrons with momentum more than double p_{F} is

$$n(p > 2p_{\mathrm{F}}) = \int_{2p_{\mathrm{F}}}^\infty f_{\mathrm{e}}(p)\ \mathrm{d}p = [0]_{2p_{\mathrm{F}}}^\infty = 0;$$

thus, the fraction of electrons with $p > 2p_{\mathrm{F}}$ is zero (this is because, in the completely degenerate limit, there are no electrons with momentum above p_{F}).