Astronomy 310 Homework 8

Riya Kore
University of Wisconsin-Madison

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Problem 1

The Sun is often modeled using an n = 3 polytrope model. Here, we will compute an interior model for the Sun using an n = 3 polytrope and then compare the results to the output from the Sun-like MESA-Web model you ran a few weeks ago for a previous homework. For all the steps below, recall that Table 19.1 in the handouts gives critical values for the n = 3 polytrope, and that you can divide $(-z^2 dw/dz)$ by factors of z_s to compute the necessary derivative terms. You may assume the Sun has M = 1.989 x 10^{33} g, R = 6.696 x 10^{10} cm, and X = 0.7 and Y = 0.3.

(a) Calculate the central pressure at the core of the Sun (in Ba).

(b) Calculate the central density at the core of the Sun (in g cm^{-3}).

(c) Assuming the Sun is a fully-ionized ideal gas, estimate the central temperature in the Sun's core (in K) using the composition given above.

Solution:

For an n = 3 polytrope, we have been given the following values:

$$z_s = 6.89685 \tag{1}$$

$$\left(-z^2 \frac{dw}{dz}\right)_{z_s} = 2.01824$$

$$-(6.89685)^2 * \frac{dw}{dz} = 2.01824$$

$$\frac{dw}{dz} = \frac{-2.01824}{(6.89685)^2} = -0.04243 \tag{2}$$

We can use these values to solve the question.
(a) To find the central pressure at the core of the Sun, we can use the following expression:

$$P_c = \frac{G}{4\pi(n+1)} \frac{M^2}{R^4} \left(-\frac{dw}{dz}\right)^{-2}_{z=z_s}$$

$$P_c = \frac{6.67 * 10^{-8}}{4\pi(3+1)} * \frac{(1.989 * 10^{33})^2}{(6.696 * 10^{10})^4} * (0.04243)^{-2}$$

$$P_c = 1.4505 * 10^{17}\ Ba$$

(b) To find the central density at the core of the Sun, we can use the following expression:

$$\rho_c = \frac{1}{4\pi} \frac{M}{R^3} \left(\frac{-1}{z} \frac{dw}{dz}\right)^{-1}_{z=z_s}$$

$$\rho_c = \frac{1}{4\pi} * \frac{1.989 * 10^{33}}{(6.696 * 10^{10})^3} * \left(\frac{-1}{6.89685} * -0.04243 \right)^{-1}$$
$$\rho_c = 85.6951 \ g \ cm^{-3}$$

(c) If we assume the Sun to be a fully-ionized ideal gas, we can use the ideal gas equation of states to find the central temperature.

$$P = nk_BT$$
$$P_c = \left[\sum_j \frac{(1 + \mathscr{L}_j)\chi_j}{\mathcal{A}_j} \right] \frac{\rho_c k_B T}{m_H}$$

Here, χ_j is the mass fraction, $\mathcal{A}_j$ is the mass number, and $\mathscr{L}_j$ is the atomic number of the isotope j.

$$1.4505 * 10^{17} = \left[\frac{2 * 0.7}{1} + \frac{3 * 0.3}{4} \right] * \frac{(85.6951) * (1.3807 * 10^{-16}) * T_c}{(1.67 * 10^{-24})}$$
$$1.4505 * 10^{17} = (1.625) * \frac{(85.6951) * (1.3807 * 10^{-16}) * T_c}{(1.67 * 10^{-24})}$$
$$T_c = \frac{(1.4505 * 10^{17}) * (1.67 * 10^{-24})}{1.625 * 85.6951 * (1.3807 * 10^{-16})} = 12598693.9$$
$$T_c \approx 1.26 * 10^7 \ K$$

Problem 2

Using Poly-Web (http://user.astro.wisc.edu/ townsend/ static.php?ref=poly-web), create an n = 3 polytrope model using 500 grid points. The output will be a text file with three columns. The three columns are: z, $w(z)$, and dw/dz. Using Equation [19.4] and the output from this Poly-Web calculation, create a plot of the Log10 density of the Sun (in Log10[g cm^{-3}]) as a function of radius r, where r = 1 corresponds to the Solar surface. Recall that $z/z_s = r/R$.

Solution:
The central density for a polytrope is given by the expression:

$$\rho_c = \frac{1}{4\pi} \frac{M}{R^3} \left(\frac{-1}{z} \frac{dw}{dz} \right)^{-1}_{z=z_s}$$

Equation 19.4 from the handout gives us:

$$w = \left(\frac{\rho}{\rho_c} \right)^{1/n}$$
$$\rho = w^n * \rho_c = w^n * \frac{1}{4\pi} \frac{M}{R^3} \left(\frac{-1}{z} \frac{dw}{dz} \right)^{-1}_{z=z_s}$$
$$\rho = w^3 * 85.69551$$
$$\log(\rho) = 3 * \log(w) + \log(85.69551) \qquad (1)$$

Equation (1) is basically what we will use to plot the y-axis.
From the question, we also know that:

$$\frac{z}{z_s} = \frac{r}{R}$$
$$r = \frac{z}{6.89685} \qquad (2)$$

We will use equation (2) to plot the x-axis.

The plot looks like this:

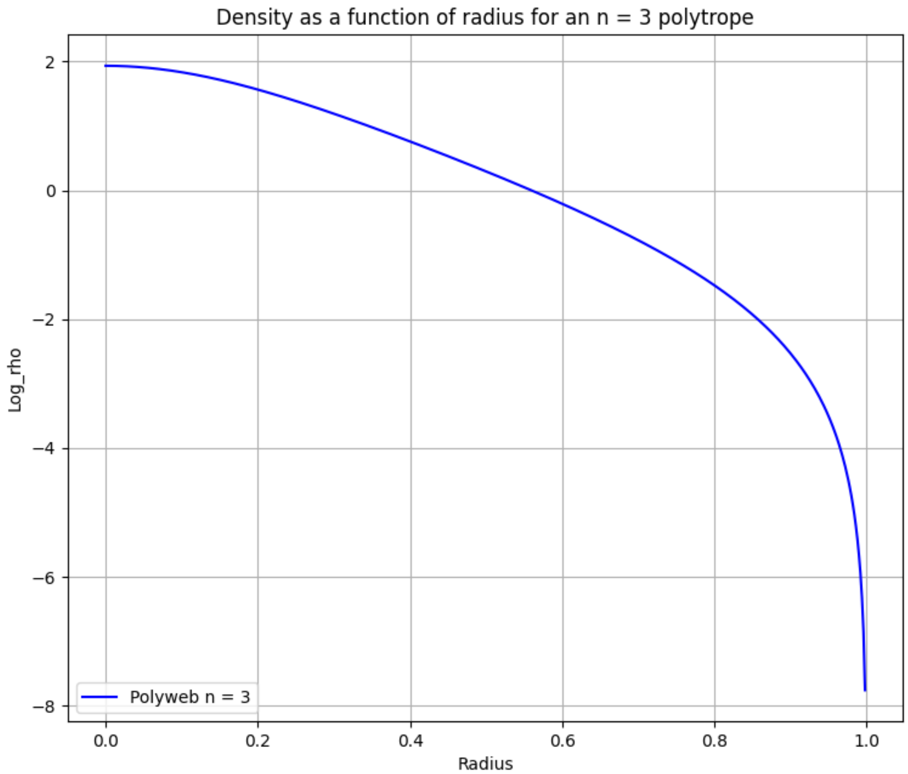

Figure 1: Density as a function of radius for n = 3 polytrope

Problem 3

Use either the Sun-like MESA-Web model you created a few weeks age, or create a new model using the defaults on the MESA-Web submission page (http://user.astro.wisc.edu/ townsend/ static.php?ref=mesa-web-submit). The profile with a model stellar age close to the Sun's 4.5 Gyr should be profile8.data. Read in the columns for radius and $\log(\rho)$ from this file and over-plot them on your polytrope solution above. Submit this combined plot with your homework. Briefly comment on the agreement (or not) of the n = 3 polytrope and the full MESA model. Is there a radius where the predicted density of the two models diverge? What do you hypothesize is the cause of this divergence?

Solution:
From the profile8.data file, if we plot the $\log(\rho)$ vs radius, we get:

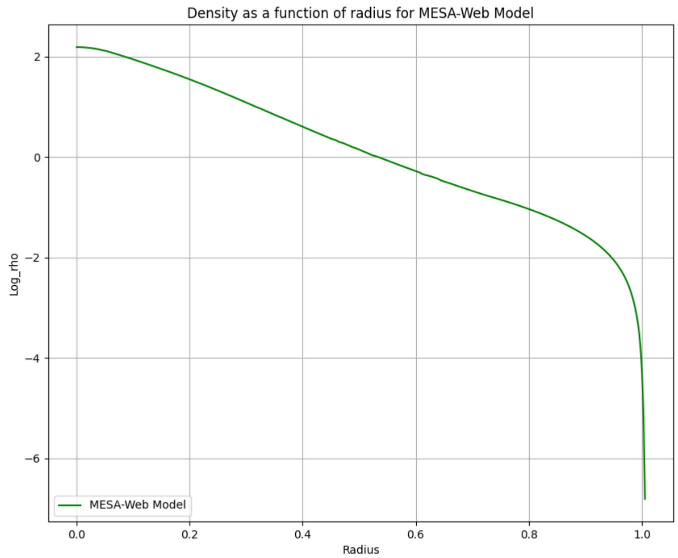

Figure 2: Density as a function of radius for MESA-Web Model

When we combine the two graphs, it looks like this:

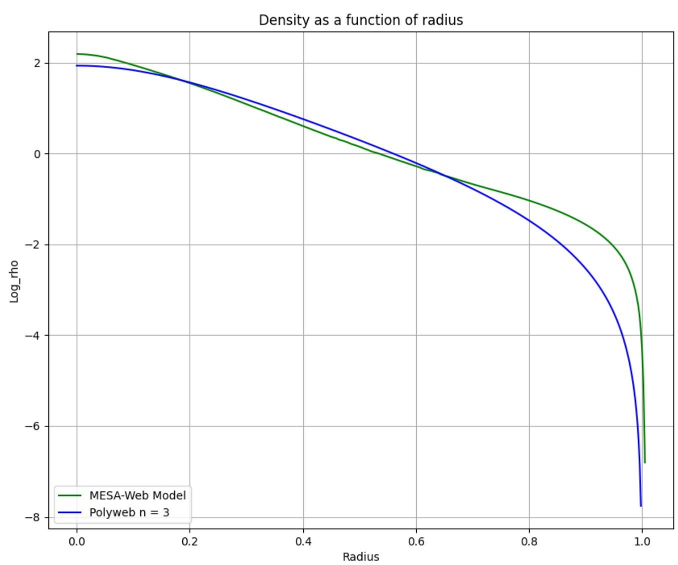

Figure 3: Density as a function of radius for the combined graph

The n = 3 polytrope and the MESA-Web Model agree to some extent. When r = $0.7R_{\odot}$ approximately, the predicted density of the two models diverges. This means that the density is greater than what's predicted by the n = 3 polytrope model. An explanation for this could be that at r = $0.7R_{\odot}$, the sun's convective zone starts and the radiative zone ends. The Polyweb model assumes the whole star to be in hydrostatic equilibrium, but the MESA-Web model takes into account this shift from radiative to convective zone.