1. *A star initially has a thermal energy of* 2×10^{51} erg. *What is its total energy? Suppose that* 7×10^{50} erg *of heat is added near the center of the star. Assuming the star remains in hydrostatic equilibrium, what is its final total energy? Its final thermal energy? If the central temperature of the star is related to its thermal energy U by*

$$T_{\mathrm{c}} = \frac{(\gamma - 1) m_{\mathrm{H}}}{k_{\mathrm{B}} M} U,$$

(*where the symbols have their usual meanings*), *then did the addition of heat make the center of the star hotter or colder?*

1 point for invoking the virial theorem, **1 point** each for the initial total energy, final total energy and final thermal energy, **1 point** for commenting that the center becomes cooler.

Assuming $\gamma = 5/3$, we can apply the virial theorem to find a relationship between total energy E and thermal energy U of the star:

$$E = -U$$

(see eqn. [7.10]). With an initial thermal energy $U_{\mathrm{i}} = 2 \times 10^{51}$ erg, the initial total energy is $E_{\mathrm{i}} = -U_{\mathrm{i}} = -2 \times 10^{51}$ erg.

The addition of 7×10^{50} erg of heat then changes the total energy to a final value $E_{\mathrm{f}} = E_{\mathrm{i}} + 7 \times 10^{50}$ erg $= -1.3 \times 10^{51}$ erg. The final thermal energy is thus $U_{\mathrm{f}} = -E_{\mathrm{f}} = 1.3 \times 10^{51}$ erg — smaller than the initial thermal energy.

Using the expression for T_{c}, it's clear that the center of the star will cool down as a consequence of the addition of heat — an apparently paradoxical outcome. The key to understanding this outcome is to note that although the energy is added to the star as heat, the star rapidly converts this heat (together with some of its own heat energy) into gravitational potential energy, in order to remain in hydrostatic equilibrium. During this conversion, the star expands and cools.

2. *Consider a satellite with mass m in a circular orbit around a planet with mass M. Derive expressions for the kinetic and gravitational potential energies of the satellite, in terms of m, M and the satellite's orbital radius r. Treating the satellite's kinetic energy as its thermal energy, show that the satellite obeys the virial theorem with* $\gamma = 5/3$.

1 point for setting centripetal force to equal gravity (or some equivalent), **1 point** for correct kinetic energy, **1 point** for correct potential energy, **1 point** for relating to virial theorem.

For a satellite in a circular orbit, the centripetal force (necessary to maintain the orbit) is provided by the gravity of the planet, and so

$$\frac{mv^2}{r} = \frac{GMm}{r^2},$$

where v is the satellite's orbital velocity. Its kinetic energy is therefore found as

$$K = \frac{mv^2}{2} = \frac{GMm}{2r}.$$

The gravitational potential energy of the satellite is the product of its mass and the gravitational potential of the planet:

$$\Omega = m\Phi = -\frac{GMm}{r}.$$

Treating K as U, we therefore find that $U = -\Omega/2$, which is the virial theorem when $\gamma = 5/3$.

3. *The Sun will spend approximately* 10^{10} yr *on the main sequence. Assuming that its luminosity remains fixed at the present-day value* $L = L_\odot$*, how much energy will it radiate into space during this phase? Given that* 3.8×10^{24} MeV *of energy is released for each gram of hydrogen fused into helium, what total mass of hydrogen must be consumed to provide this energy? From this mass, determine what fraction of the Sun's initial hydrogen content will be converted into helium during the main-sequence phase (assume an initial composition* $X = 0.7$*, and take* $M = M_\odot$*).*

   If the Sun shines with a luminosity $L = 1\,\mathrm{L}_\odot$ for a time $\Delta t = 10^{10}$ yr, the total energy radiated is $\Delta E = L\Delta t = 1.2 \times 10^{51}$ erg.

   The mass of hydrogen consumed is given by $M_\mathrm{H} = \Delta E / 3.8 \times 10^{24}$ MeV g^{-1} $= 2.0 \times 10^{32}$ g.

   The initial mass of hydrogen in the Sun is $M_\mathrm{H,init} = 0.7 \times 1\,\mathrm{M}_\odot = 1.4 \times 10^{33}$ g. Therefore, the fraction of hydrogen converted into helium by the end of the main sequence is ≈ 0.14.

1 point for obtaining energy released, **2 points** for mass of hydrogen burned, **2 points** for fraction of hydrogen burned.

4. *Sketch a Hertzsprung-Russell diagram (with appropriate axes), showing the main sequence. Draw the track followed by the Sun during its pre-main sequence phase, labeling the Hayashi and Henyey portions of the track.*

   See Fig. 0.1.

1 point for a correctly labeled HR diagram showing the main sequence. **1 point** for the Hayashi track, **1 point** for the Henyey track.

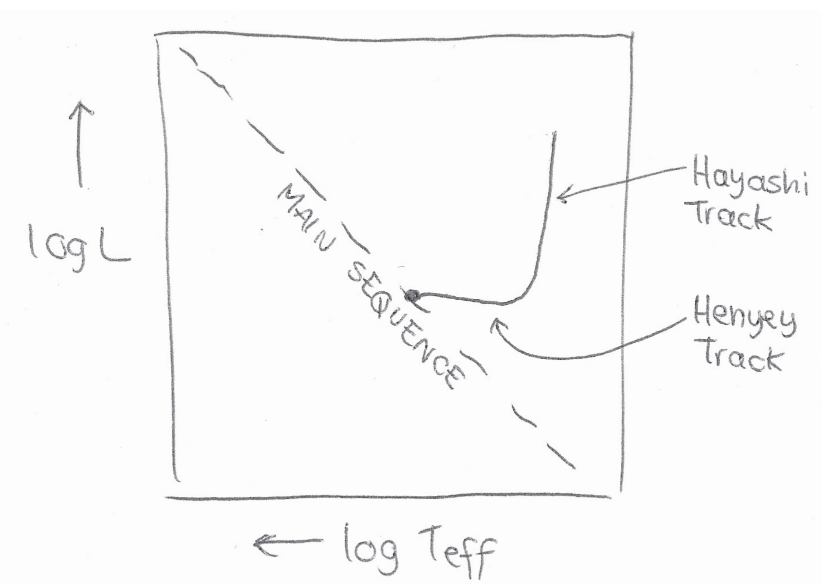

Figure 0.1: HR diagram sketch for Q4.

5. *As a simple model for the late phases of pre-main sequence evolution, assume that a star with mass M moves along the Henyey track at constant luminosity L. Derive an expression for the time taken for the effective*

temperature to increase to $T_{\rm eff}^{\rm b}$, starting from the value $T_{\rm eff}^{\rm a}$. (You can assume the virial theorem applies with $\gamma = 5/3$, and that the shape factor f_Ω remains constant). Apply your formula to the Sun (with $L \approx L_\odot$) to estimate how long the Sun spent on the Henyey track, assuming $T_{\rm eff}^{\rm a} \approx$ 4300 K, $T_{\rm eff}^{\rm b} \approx 5600$ K, and $f_\Omega \approx 1.7$.

1 point for invoking the radius equation for the pre-main sequence, **1 point** for eliminating the radius using the Stefan-Boltzmann equation, **1 point** for integrating the differential equation, **1 point** for the correct result.

The equation for radius evolution on the pre-main sequence, assuming $\langle\gamma\rangle = 5/3$ and constant f_Ω, is

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -\frac{2R^2 L}{f_\Omega G M^2}$$

(see eqn. [8.3]). The Stefan-Boltzmann formula relates the luminosity, radius and effective temperature via

$$L = 4\pi R^2 \sigma T_{\rm eff}^4.$$

Combining these two equations to eliminate R, while assuming L is fixed because we're on the Henyey track, we arrive after some rearranging at the differential equation

$$T_{\rm eff}\frac{\mathrm{d}T_{\rm eff}}{\mathrm{d}t} = \sqrt{\frac{L^3}{4\pi\sigma}}\frac{1}{f_\Omega G M^2}.$$

Integrating with respect to time from 'a' to 'b' (note that the right-hand side is constant), this becomes

$$\frac{1}{2}\left[\left(T_{\rm eff}^{\rm b}\right)^2 - \left(T_{\rm eff}^{\rm a}\right)^2\right] = \sqrt{\frac{L^3}{4\pi\sigma}}\frac{1}{f_\Omega G M^2}\left(t_{\rm b} - t_{\rm a}\right).$$

Solving for the time taken,

$$t_{\rm b} - t_{\rm a} = f_\Omega G M^2 \sqrt{\frac{\pi\sigma}{L^3}}\left[\left(T_{\rm eff}^{\rm b}\right)^2 - \left(T_{\rm eff}^{\rm a}\right)^2\right].$$

Applying this expression to the Sun, with the supplied values, we find $t = 3.3 \times 10^{14}$ s = 10 Myr.

6. *Stars with masses $M \gtrsim 0.8\,M_\odot$ will, at some point in their lives, burn helium in their cores into carbon via the* triple alpha *process*

$$3\,{}^4\mathrm{He} \rightarrow {}^{12}\mathrm{C}$$

The rest mass of a helium atom is 4.0026 u, *and a carbon atom is* 12.0000 u. *Calculate the percentage of rest mass converted to energy by this reaction. Compare this percentage against the corresponding value for hydrogen burning (the mass of a hydrogen atom is* 1.0079 u) — *which is the more efficient fuel?*

1 point for calculating the energy release (or rest mass converted, or equivalent) from the reaction, **1 point** for the percentage converted to energy, **1 point** for comparing against the value for hydrogen burning **1 point** for identifying hydrogen as the more efficient fuel.

Written with electrons included, the triple alpha process is

$$3\left({}^4\mathrm{He} + 2e^-\right) \rightarrow {}^{12}\mathrm{C} + 6e^-.$$

Hence, 3 helium atoms are converted into one carbon atom. The energy released is equivalent to the difference in the rest masses:

$$\Delta\mathscr{E} = \left[3m_{\rm He} - m_{\rm C}\right]c^2 = 0.007\,80\,\mathrm{u}\,c^2$$

Therefore, the fraction of rest mass converted to energy is $0.0078/12 = 0.00065$. This compares against a fraction $0.029/4.0 = 0.0073$ for hydrogen burning. The fractional energy yield from helium burning is around 10 times smaller than from hydrogen; therefore, hydrogen is (by a large margin) the more-efficient fuel.