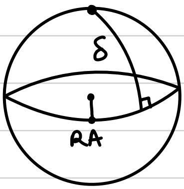
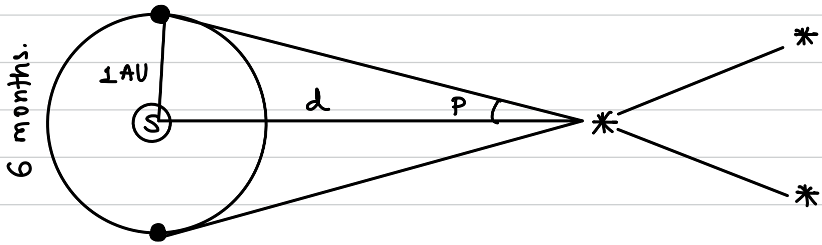
Lecture 1:

① Measuring stars



Earth / any other planets.

RA: right assention
δ : declanation



for very small θ, $\tan\theta \approx \theta$

$$\tan P = \frac{1\,AU}{d} \Rightarrow P = \frac{1\,AU}{d}$$

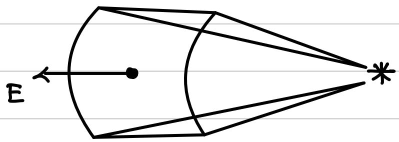
$$1 \text{ arcsec } (1'') = \frac{1}{3600^\circ} \text{ degree} = \frac{1\,AU}{d}$$

$$1 \text{ radian} = \frac{180}{\pi} \times 3600 = 206265''$$

$$p('') = \frac{1\,AU}{d} \times 206265 \Rightarrow 1'' \quad = 206265\ AU$$

$$d = 1 \text{ parsec (pc)}$$

distance (parsec) $= \dfrac{1}{p('')}$ → angle in arcseconds.



$$\text{flux} = \frac{E}{dA\,dT} \quad \text{erg cm}^{-2}\,\text{s}^{-1}$$

for the sun (flux) $= 1.36 \times 10^{6}$ erg $\text{cm}^{-2}\,\text{s}^{-1}$

sirius (brightest star in the sky) $= 1.05 \times 10^{-4}$ erg $\text{cm}^{-2}\,\text{s}^{-1}$

$$\text{flux } (f) = \left(\frac{R}{d}\right)^2 = F$$

R : Radius of the star
d : distance
F : surface flux

Luminosity : $L = 4\pi R^2 F$

$$\left.\begin{array}{l} L = 4\pi R^2 F \\ f = \dfrac{L}{4\pi d^2} \end{array}\right\}$$ relationship between luminosity and flux.

$L_{sun} = 3.828 \times 10^{23}$ erg/s $= 1 L_\odot$ (one solar luminosity)
$L_{sirius} = 22.9\ L_\odot$

② Sorting stars

O B A F G K M

Brightest (O) · sun (G2) · dimmest (M)

$\Delta m = 5 \Rightarrow 100 \times$ brightness difference.

$$\frac{f}{f_0} = \left(\sqrt[5]{100}\right)^m \cong (2.5119)^m$$

$$m = -2.5 \times \log_{10}\left(\frac{f}{f_0}\right)$$

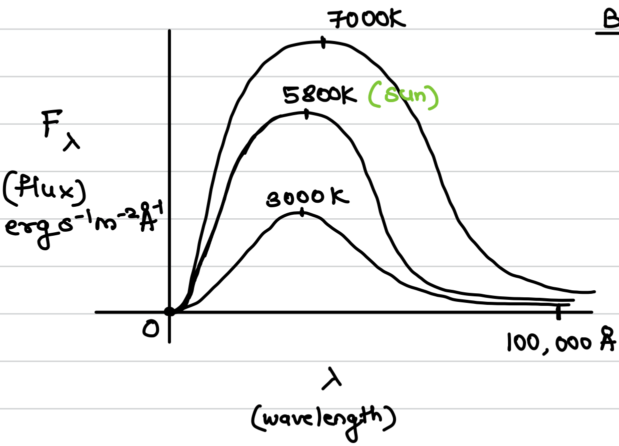
star vega is used for reference.

(observed magnitude)

absolute magnitude (M) : $M = -2.5 \log\left(\frac{L}{4\pi (10\ pc)^2 f_0}\right)$

combining the absolute and apparent / observed magnitude : $\underbrace{m - M}_{\text{distance modulus}} = \underbrace{5 \log\left(\frac{d}{10\ pc}\right)}_{\text{used to usually figure out the distance.}}$

Lecture 2:



Blackbody radiation.

$$B_\lambda(T) = \frac{2cKT}{\lambda^4}$$ Rayleigh-Jeans equation (only for large wavelengths)

denominator approaches ∞ for wavelengths close to ultraviolet

$$B_\lambda(T) = \frac{a}{\lambda^5}\,\frac{1}{e^{b/\lambda T}}$$ Wien's Law

combining the two laws:

$$B_\lambda(T) = \frac{a}{\lambda^5} \times \frac{1}{(e^{b/\lambda T}-1)}$$

planck's constant = 6.626×10^{-34} joule second.

$E_\lambda = nh\nu = \frac{nhc}{\lambda}$; for the minimum orbit energy $n=1$ $E_\lambda = \frac{hc}{\lambda}$

$$B_\lambda(T) = \frac{2\pi hc^2}{\lambda^5} \times \frac{1}{(e^{hc/\lambda KT}-1)}$$ final equation for the Blackbody function.

$$B_\lambda \cdot d\lambda = \frac{2\pi hc^2}{\lambda^5} \times \frac{1}{(e^{hc/\lambda KT}-1)} \cdot d\lambda$$ planck's law.

① Take derivative of $B_\lambda(T)$
② equate to zero
③ solve for maximum.

$$\lambda_{max} = \frac{2.898 \times 10^7 \text{ Å K}}{T \text{ (temperature)}}$$ As things get hotter, they get bluer.

$5000 \text{ Å} = \lambda_{max\,\odot}$
(for the sun)

$$\int_0^\infty B_\lambda \, d\lambda = F = \sigma T^4$$
(total flux measured)

$$\sigma = \frac{2\pi^5 k^4}{15 h^3 c^2}$$ Stefan-Boltzman Law.

effective temperature (T_{eff}) $= \sqrt[4]{\frac{F}{\sigma}}$

form of way you talk about temperature of stars. you find the equivalent blackbody temperature.

$L = 4\pi R^2 \sigma (T_{eff})^4$
(luminosity)

radius of star

find L using distance
T_{eff} using spectroscopy.

Classification of stars based on temperature.

sun G2

O B A F G K M

hot cool

$\{0 \rightarrow 9\}$ number classification for each letter.

giant stars
(helium fusion)
red giants / yellow giants.

main sequence stars
(fusing hydrogen)

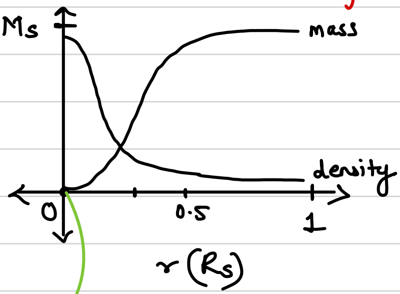
white dwarfs
(stellar core)

dwarf stars

HR diagram
(Hertzsprung - Russell diagram)

- hotter stars are on the main sequence for a very short time

Lecture 3:



center of the star.

integrating

$dm = 4\pi \rho(r) r^2 dr$

Ms : total mass of the star.

$$m = \int_0^r 4\pi r^2 \rho \, dr$$

$\rho(r)$ } density is also a function of the radius.

$$\frac{\partial m}{\partial r} = 4\pi r^2 \rho(r)$$

time is also an important factor. The mass and density graph of a star changes over time.

how much. vector field pointing away

Gravity of the star: Gravitational field

$$\nabla \vec{g} = 4\pi G \rho.$$ } divergence operator.

$$\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

$$\vec{g} = \nabla V$$

$$\nabla \cdot (-\nabla V) = -4\pi G \rho$$

$$-\nabla^2 V = -4\pi G \rho$$ } Place's equation.

$$V = \Phi$$

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) = 4\pi G \rho$$

multiplying by r^2 and integrating on both sides.

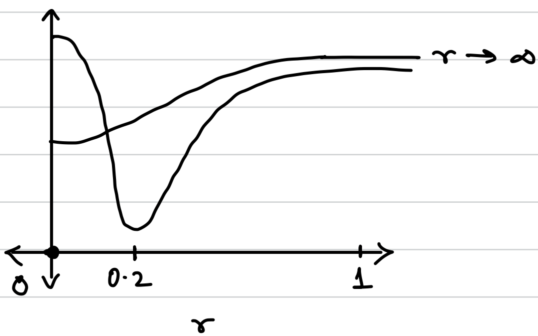
$$g = \frac{-\partial \Phi}{\partial r} = \frac{-Gm}{r^2}$$

gravitational acc is the partial derivative of gravitational potential.

Φ } gravitational potential.

most of the gravity for a star is centrally concentrated.

$$\Phi(r) = \int_{r'}^{\infty} \frac{Gm}{r^2}\, dr + c$$



Ω : gravitational potential energy.

Ω } total amount of energy inside the star (total potential energy in the star).

moving from outer to inner.

$$d\Omega = -\frac{Gm}{r}\, dm$$ } Bringing packets of mass in, builds up energy.

$$\int d\Omega = -\int_{0}^{M} \frac{GM}{r}\, dm$$

M → stellar mass

} total energy gained doing this

$$\Omega = -f_{\Omega} \frac{GM^2}{R}$$

As mass adds up, radius also increases.

gravitational binding energy of the star.

shape factor

for sun = 1.3 (f_{Ω})

$$\Omega_\odot = -6.14 \times 10^{48} \text{ ergs}$$

if the sun explodes, it would release $\Omega_\odot$ amount of energy.

typical supernova : $SN_{(\Omega)} = 10^{51}$ ergs.

$$\frac{6.14 \times 10^{48} \text{ erg}}{3.8 \times 10^{33} \text{ erg/s}} = 1.6 \times 10^{15} \text{ s}$$

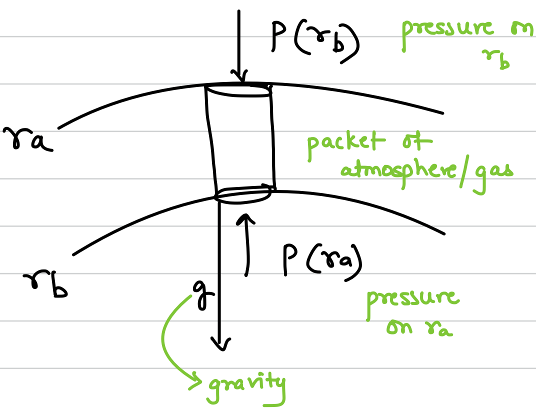
(3.8×10^{33} erg/s: luminosity of the sun)

$(\pi \times 10^7)$ no. of seconds in a year.

$= 5 \times 10^7$ yrs.

50 million years } very less.

Hydrostatic equilibrium!



$$f = \frac{dp}{dr}$$

(p: momentum.)

$$p = \int_{r_a}^{r_b} \rho V_r \, dr \, dA.$$

$$f = [P(r_a) - P(r_b)] \, dA. + \int_{r_a}^{r_b} g \rho \, dr \, dA.$$

$$f = -\int_{r_a}^{r_b} \frac{\partial P}{\partial r} \, dr \, \cancel{dA} + \int_{r_a}^{r_b} g\rho \, dr \, \cancel{dA}$$

$$= -\int_{r_a}^{r_b} \left[\frac{\partial P}{\partial r} - g\rho \right] dr = \frac{d}{dt} \int_{r_a}^{r_b} \rho V_r \, dr$$

} momentum equation for specific part of a star.

force experienced by that area is the change of momentum.

radial changes but no temp changes.

$=0$

$$\frac{\partial P}{\partial r} = g\,\underbrace{\rho(r)}_{\text{density.}}$$

equal to gravity pulling the gas down.

derivative of pressure in the atmosphere.

$$P_{center} = \int_0^M \frac{Gm}{4\pi r^4}\, dm$$

$$P_{center} \geq \underbrace{\frac{GM^2}{8\pi R^4}}_{\text{minimum value.}}$$

To find pressure at the center of the star.

$$\underbrace{P_{center,\odot} \geq 9\times10^{14} \text{ bar}}_{\text{conditions for nuclear fusion to occur.}}$$

$$P_{surface} - P_{center} = \underbrace{\int_0^R g\rho\, dr}_{\text{Radius}} = \underbrace{-\int_0^M \frac{Gm}{4\pi r^4}\, dm}_{\text{Mass.}}$$

(pressure)

Lecture 4:

gravity at the surface of the star (g_s) $= \dfrac{GM}{R^2}$

$GM \to$ mass of star

$R^2 \to$ radius of star

Let's assume acceleration is constant

$$t_{dyn} = \sqrt{\frac{2R}{g_s}} = \sqrt{\frac{2R^2}{GM}}$$

for the Sun $t_{dyn} = 1200\,s$ (about 20 mins)

$\to$ amount of time sun would take to collapse in on itself without fission pressure.

Equations of State:

$P = P(\rho, T, \mathcal{C})$ (pressure)

ρ : density

T : temperature

$\mathcal{C}$: composition.

$$\left.\frac{\partial P}{\partial r} = g\rho \right\}$$ Hydrostatic equilibrium equation.

Ideal Gas Law for equations of state:

$P = n k_B T$, $n = \dfrac{N}{V}$

$N \to$ no of particles

$V \to$ volume of gas.

k_B - Boltzmann const.

mean molecular weight : average mass of a particle

$$\mu = \frac{\rho}{n\,(m_H)}$$ for fully ionized hydrogen $= 1/2$; for non-ionized hydrogen $= 1$ $\rightarrow n = \frac{\rho}{\mu}$

The ideal gas equation of state becomes:

$$P = \frac{\rho\, k_B T}{\mu\,(m_H)}$$

ρ → density

μ → mean molecular weight (depends on composition)

k_B : Boltzmann constant

T : temperature (Kelvin)

Hydrogen	H^X	
Helium	He^Y	
metals	Z	

X, Y, Z : mass fractions.

metals ↔ metallicity } everything except H, He

$[X + Y + Z = 1]$ → mass fractions (content) has to add upto 1.

Stars: usually 2% metal. (0.02 mass fraction)

$$Z = 1 - X - Y$$

Sun:

80% H

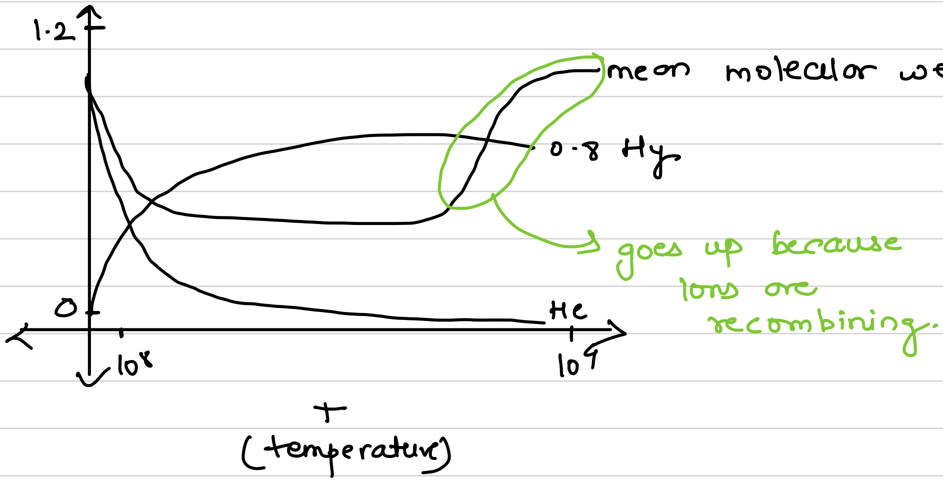
18% He

2% Z

$$\mu = \left[X + \frac{Y}{4} + \frac{Z}{12}\right]^{-1}$$

↑ mean molecular weight for neutral gas.

$$\mu = \left[2X + \frac{3Y}{4} + \frac{Z}{2}\right]^{-1}$$ } mean molecular weight for fully ionized gas.



for an ideal gas:

① isothermal. – temperature remains constant

$P = K_{iso} \rho$

↳ some constant for that temperature.

② adiabatic. – pressure changes so fast, but temp doesnt have time to change.

$P = K_{ad} \rho^{\gamma}$ γ: adiabatic constant.

always $\gamma > 1$ for gas.

60-80 craft break
↳ below that stars rotate very slow.

$\gamma = 5/3$ neutral/fully ionized

$\gamma = 1$ partialy ionized.

thermal energy:

$$u = \frac{1}{\gamma - 1} \times \frac{kT}{\mu m_H}$$

equation for hydrostatic eq:

$$\frac{\partial P}{\partial r} = \rho g = \underbrace{-\frac{GM}{r^2}}_{g} \rho$$

multiplying both sides by $4\pi r^3$

$$\int_0^R \frac{\partial P}{\partial r} \times 4\pi r^3 \, dr = -\int_0^R \frac{Gm}{r} 4\pi r^2 \rho \, dr$$

gravity changes because of radius so mass also changes.

$$\text{RHS} = -\int_0^R \frac{Gm}{r} \times \underbrace{4\pi r^2 \rho}_{\frac{\partial m}{\partial r}} \, dr$$

rate of change of mass as a fxn of radius.

$$= \int_0^M \frac{Gm}{r} \, dm \equiv -\Omega$$

gravitational potential energy

$$\text{LHS} = \underbrace{\int_0^R \frac{\partial P}{\partial r} 4\pi r^3 \, dr}_{\text{integral by parts}} = \left[\cancelto{0}{4\pi r^3 P}\right]\Bigg|_0^R - 3\int_0^R 4\pi r^2 P \, dr$$

P = 0 for outside.

inside r = 0

(virial theorem)

As we know $\frac{\partial m}{\partial r} = 4\pi r^2 \rho$,

$$\text{LHS} = 0 - 3\int_0^M \frac{P}{\rho}\, dm \quad , \quad P = \frac{\rho k_B T}{\mu m_H}$$

$$= -3\int_0^M (\gamma - 1)\, u\, dm \qquad \underbrace{u = \frac{1}{\gamma - 1} \times \frac{k_B T}{\mu m_H}}$$

$$= -3(\gamma - 1)U \qquad \frac{P}{\rho} = \frac{k_B T}{\mu m_H} = (\gamma - 1)u$$

$U \equiv \int_0^M u\, dm$

adiabatic constant

for solving this,

we are assuming γ (adiabatic constant) to be constant and not change as R, M changes

$$U = \frac{-\Omega}{3(\gamma - 1)}$$

$$U = -\frac{\Omega}{2}$$

why does gravity relate to thermal energy ?

total energy (E_{tot}) $= U + \Omega$

(thermal) (gravitational potential)

γ: compressibility of the gas (star in general)

$$= -(3\gamma - 4)U$$

$$= \frac{3\gamma - 4}{3\gamma - 3}\Omega \qquad \text{or} \qquad \left(\gamma > \frac{4}{3}\right)$$

always.

if the star shrinks :

$$E = U + \Omega$$

stays the same

thermal energy goes up

gravitational potential energy goes down.

Lecture 5:

$$\Omega = -f_\Omega \frac{GM_c^2}{R_c}$$

mass of cloud

shape factor.

radius shape of cloud

assuming density is uniform.

$$= -\frac{3}{5}\frac{GM_c^2}{R_c}$$

Jeans Collapse

$u = -\frac{1}{2}\Omega$ ↳ gravitational potential energy.

(thermal energy)

$$[2u + \Omega = 0]$$

if $2u + \Omega \leq 0$ } gas collapses

if its greater, it difuses.

$$2u + \Omega \leq 0$$

for gas cloud, $u = \frac{3}{2}NKT$, $N = \frac{M_c}{\mu m_H}$

for the cloud to collapse, $\frac{3M_ckT}{\mu m_H} < \frac{3}{5}\frac{GM_c^2}{R_c}$

thermal energy < gravitational potential energy.

$$R_c = \left(\frac{3M_c}{4\pi\rho_c}\right)^{1/3}$$

At what M_c will the gas cloud start to collapse? (How big does R_c have to be?)

$$M_c > \underbrace{\left(\frac{5KT}{G\mu m_H}\right)^{3/2}\left(\frac{3}{4\pi\rho_c}\right)^{1/2}}_{M_J \text{ Jeans mass.}}$$

when M_c exceeds jeans mass, it starts collapsing

$$R_c > \left(\frac{15kT}{4\pi G\mu m_H \rho_c}\right)^{1/2}$$ R_J Jeans Radius.

when R_c exceeds this limit, it collapses.

for ISM: $T = 50K$

$\rho = 8.4 \times 10^{-22}\ g/cm^3$

$\mu \sim 1$ (mean molecular weight)

$M_J \sim 1500\ M_\odot$

(Jeans mass)

core of a giant molecular cloud (GMC): The material that collapses and forms stars.

$T = 10K$

$\rho = 3 \times 10^{-20}\ g/cm^3$

$\mu = 2$

sheilding makes the core cold. (H_2) instead of H = μ

$M_J \sim 8 M_\odot$

energy in the cloud: (Pre main sequence, no fusion till now).

$$\frac{dE}{dt} = -L$$ as gas collapses, it radiates energy out into space.

L → luminosity.

using virial theorem: γ → adiabatic constant of the gas.

$$E = \frac{3\gamma - 4}{3\gamma - 3}\,\Omega$$

$$\Omega = -f\frac{GM^2}{R}$$

$$\frac{d}{dt}\left(\frac{3\gamma-4}{3\gamma-3}\, f\,\frac{GM^2}{R}\right) = L \quad , \quad \gamma = 5/3$$

only R changes over time.

R is -ve because its shrinking and collapsing.

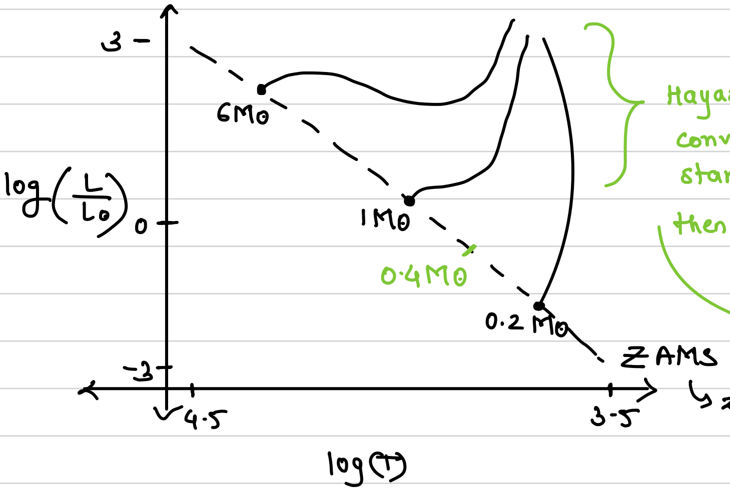
$$\frac{dR}{dt} = \frac{-2R^2L}{f G M^2}$$ Kelvin - Helmholtz contraction.

$$\tau_{KH} = \frac{GM^2}{RL}$$ Kelvin-Helmholtz time scale (how much time it takes to collapse).

Ω

$\tau_\odot \approx 50{,}000$ years.

HR diagram:



Hayashi track. convection transports energy to the star initially. then it switches to radiation.

Henyey tracks

↳ zero age main sequence

Radius decreases but temperature goes up.

0.2 M☉ } doesn't have a Henyey track

Stars below 0.4 M☉ only use convection, no radiation.

$L = 4\pi R^2 \sigma T^4$ — eq

$$\frac{dR}{dt} = \frac{-2R^2L}{fGM^2} = \frac{-8\pi R^4 \sigma T^4}{fGM^2}$$

$$\frac{1}{R^3} = \frac{24\pi\sigma T^4}{fGM^2} t$$ → time since the cloud starts collapsing.

$$\frac{dE}{dt} = -L + L_H$$

L_H → hydrogen fusion

L ↳ luminosity

eventually Luminosity equals fusion luminosity.

when $L = L_H$ } hydrostatic equillibrium is achieved.

Pressure for gas cloud $(P_c) \approx \frac{GM^2}{8\pi R^4}$

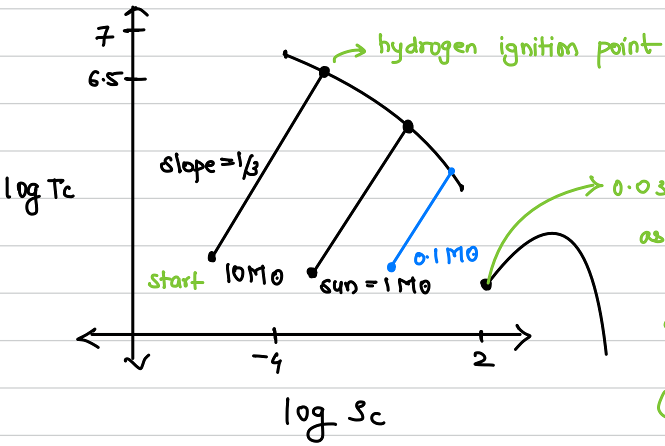
fusion ignition temp also depends on the stars core's temperature.

for ideal gas : $P = \frac{\rho KT}{\mu m_H}$

$$\therefore \frac{\rho_c KT}{\mu m_H} = \frac{GM^2}{8\pi R^4}$$

$$T_c = \frac{\mu m_H G}{8\pi K} \times M^2 \times R^{-4} \times \rho^{-1} = \frac{\mu m_H G}{8\pi K} M^{2/3} \rho^{1/3}$$

$$\log(T_c) = \frac{1}{3}\log(\rho_c) + \frac{2}{3}\log(M) + c$$



as density is very high for a brown dwarf.

Quantum mechanical effects halts the contraction.

(Pauli exclusion principle)

fusion in sun : proton - proton fusion (p-p fusion).

$$4H \longrightarrow {}^4He + 2e^- + \underbrace{2\nu_e}_{\text{neutrinos}} + N\gamma$$

$N\gamma$ → photons.

$E = \Delta mc^2$ } amount of energy released in the fusion process.

$$= [4m_H - m_{He}]\, c^2 = 0.028\ Mc^2 = 26.73\ MeV.$$

efficiency : $\dfrac{0.028}{4} = 0.007$ } same as the efficiency of H bomb.

Lecture 6:

$^{A}_{Z}X$ A · atomic number
Z : atomic mass number.

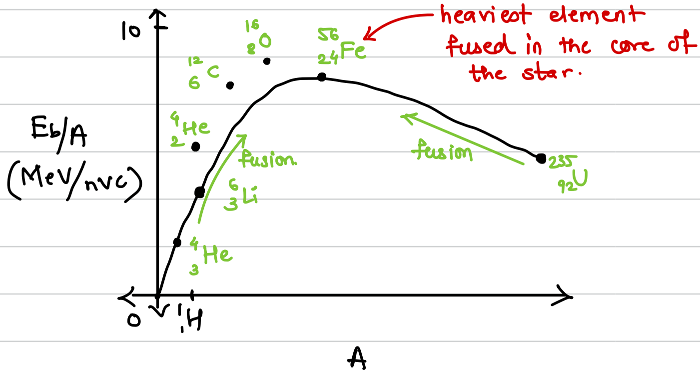
$4H \longrightarrow He$

$^{4}_{2}He$

$E_b = \Delta m c^2$

$E_b = (Z m_p + (A-Z) m_n - m_{nucleus}) c^2$
(binding energy)

$^{1}_{1}H$ } hydrogen
$^{2}_{1}H$ } deuterium
$^{3}_{1}H$ } tritium



Thermal time scale:

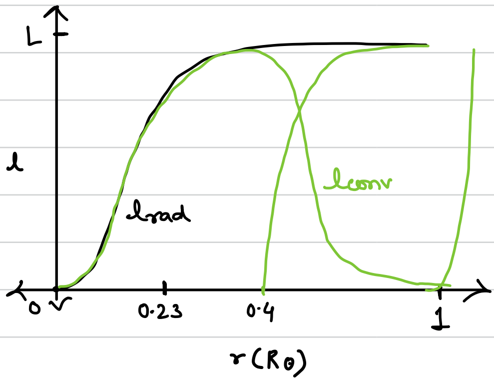
$$\tau_{nuc} = \frac{E_n}{L} = 3 \times 10^{8} s \sim 10^{11} yrs.$$

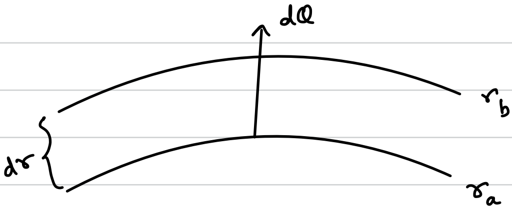
$\epsilon_N = 0.007$
$Q_N = 6.3 \times 10^{18}$ erg/s
$E_N = Q_N$
for sun, $M_\odot = 1.25 \times 10^{52}$ erg

$\tau_{nuc} \gg \tau_{KH} \gg \tau_{dyn}$
(thermal) (Kelvin Helmholtz) (dynamic)

$$l = l_{radiative} + l_{conventional}$$

(energy transfer or luminosity)





$$dQ = \underbrace{\left(l(r_a) - l(r_b)\right)}_{\text{luminosity falloff}} + \underbrace{\int_{r_a}^{r_b} 4\pi r^3 \rho\, \varepsilon_{nuc}\, dr}_{\text{nuclear reactions.}} - \underbrace{\int_{r_a}^{r_b} 4\pi r^2 \rho\, \varepsilon_\nu\, dr}_{\text{neutrinos}}$$

(you could also generate neutrinos)

when $\underset{\text{photon}}{e^- + \gamma} \rightarrow e^- + \underbrace{\nu_e + \bar{\nu}_e}_{\text{neutrinos}}$)

$$dQ = \int_{r_a}^{r_b} \left[-\frac{\partial l}{\partial r} + 4\pi r^2 \rho \left(\varepsilon_{nuc} - \varepsilon_\nu \right) \right] dr$$

This represents thermal equilibrium. A star reaches thermal equilibrium very rapidly, so, dQ becomes 0.

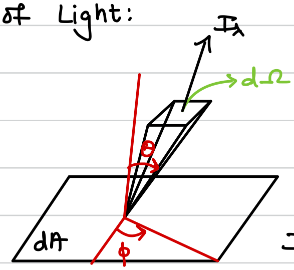
$$\left.\frac{\partial l}{\partial r} = 4\pi r^2 \rho \left(\varepsilon_{nuc} - \varepsilon_\nu \right) \right\} \text{thermal equilibrium.}$$

$$\frac{\partial u}{\partial t} = \frac{1}{4\pi r^2 \rho} \left[-\frac{\partial l}{\partial r} + 4\pi r^2 \rho \left(\varepsilon_{nuc} - \varepsilon_\nu \right) \right] + \frac{P}{\rho^2} \frac{\partial \rho}{\partial t}$$

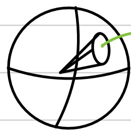
Kelvin Helmholtz contraction happening when star has not reached thermal equilibrium.

$$\tau_{thermal} = \tau_{KH} = \frac{GM^2}{RL}$$

Intensity of Light:



$I_\lambda = \sin\theta \, d\theta \, d\phi$



Ω (steradians)
sr

one sphere has 4π sr

$$dE = I_\lambda \, d\lambda \, dt \, dA \cos\theta \, d\Omega$$

$I_\lambda \rightarrow$ erg $cm^{-2}\, s^{-1}\, sr^{-1}\, Å^{-1}$

$I_\lambda \longrightarrow$ intensity (is always conserved = constant)

$F_\lambda \longrightarrow$ flux $\left(\frac{1}{r^2}\right)$ } you get flux by integrating the intensity.

$L_\lambda \longrightarrow$ luminosity (constant)

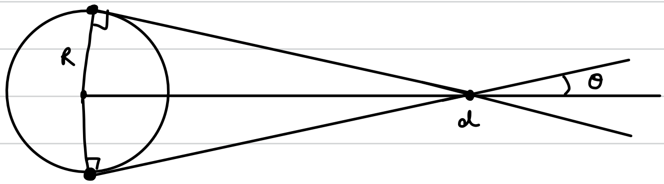
$$F_\lambda = \int I_\lambda \cos\theta \, d\Omega = I_\lambda \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$F_\lambda = \pi I_\lambda$

(surface flux)

looking at the limits of this integral, you are integrating only on half the sphere, because you can only see half the star in one time.

mean intensity $\langle I_\lambda \rangle = J_\lambda = \frac{1}{4\pi} \int I_\lambda \, d\Omega$



$$F_\lambda = \int I_\lambda \cos\theta \, d\Omega$$

$$= I_\lambda \int_0^{2\pi} \int_0^{\sin^{-1}(R/d)} \cos\theta \sin\theta \, d\theta \, d\phi$$

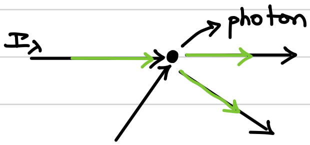
$$= 2\pi I_\lambda \int_0^{R/d} x \, dx$$

$$= \pi I_\lambda \times \left(\frac{R}{d}\right)^2$$

Lecture 7:

Intensity of a ray of light: intensity is conserved (I_λ)
unit for intensity: ergs s^{-1} cm^{-2} sr^{-1} A^{-1}

intensity will change when passing through a gas cloud



when light ray passes through a gas cloud, it collides with material and photons are releases. This decreases intensity.

absorbed → intensity reduces.
scattered → intensity increases

length / distance travelled by photon.

opacity (higher the opacity, more the absorption).

$$\underline{P}(\ell + d\ell) = P(\ell)\,[\,1 - k\rho\, d\ell\,]$$

probability.

$$k = \frac{\sigma}{\mu m_H}$$

(opacity)

σ → cross-sectional area (cm2)

$$\lim_{\ell \to 0} \frac{P(\ell + d\ell) - P(\ell)}{d\ell} = \frac{dP}{d\ell} = -k\rho P$$

$$P(\ell) = e^{-k\rho\ell}$$

$$dI_\lambda = -k\rho I_\lambda\, d\ell$$

$$I_\lambda = \underbrace{I_{\lambda 0}}_{\text{absorption}}\, e^{-k\rho\ell}$$

→ principle scatter.

mean free path (avg dist photon travels before it gets absorbed/scattered).

$$\langle l \rangle = \int_0^\infty l\, I_\lambda\, K \rho\, dl = \frac{1}{K\rho}$$

optical depth (τ) :

$$\tau = \int_r^R K\rho\, dr$$

$r \to R$: the distance we have travellel.

$$I_\lambda = I_{\lambda 0}\, e^{-\tau_\lambda}$$

} intensity falls of as a function of the optical depth.

If $\tau >> 1$: optically thick : no light gets to us, optical depth is very large.

If $\tau << 1$: optically thin : almost all of the light gets to us : optical depth is very small.

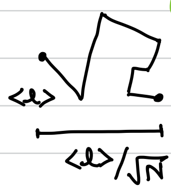
If $\tau = 1$: you see the surface of the sun, this surface has $\tau = 1$.

$$t_{esc} = \frac{\Delta}{c} = \frac{R-r}{c}$$

(time for photon to espace the sun's surface)

↳ speed of light

(a photon being scattered and absorped)



$$N = \frac{(R-r)^2}{\langle l \rangle^2} \approx \tau^2$$

$$\Delta = N\langle l \rangle \approx (R-r)\,\tau$$

(total distance travelled)

time taken.

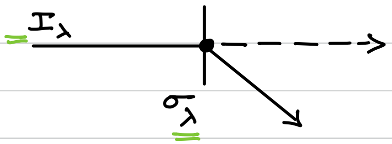
$$t_{esc} = \left(\frac{R-r}{c}\right)\tau$$

$N = 10^{25}$ } each photon takes about 10^{25} jumps before it gets out.

$\Delta = 70$ kpc $= 70 \times 10^3$ pc – distance travelleled by one photon before it escapes the sun

$t_{esc} = 200{,}000$ yrs.

time taken for one photon to escape.



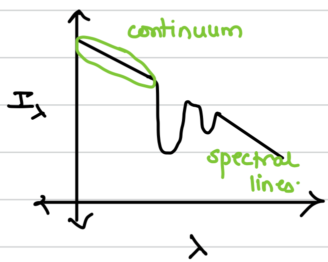
how do we figure out optical depth?

$$\tau = K\rho \qquad K = \frac{\sigma}{\mu m_H}$$

wavelength dependent

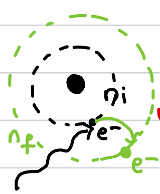
σ_λ varies slowly wrt λ. , continuum opacity (in an emission spectrum).

σ_λ varies quickly wrt λ, you get spectral lines.



sources of opacity:

① Bound-bound transitions:



jumps to a new orbit e^- while photon gets absurped.

$$E_n = Z^2 \frac{-R_H hc}{n^2} \quad — ①$$

no. of protons.

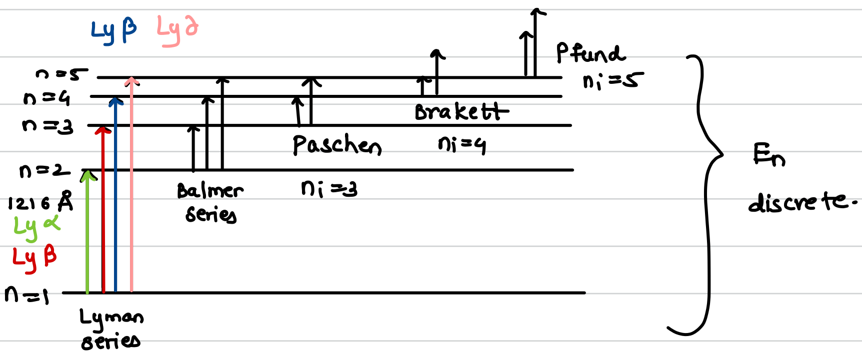
$$E_r = \frac{hc}{\lambda} \quad — ②$$

combining ①, ②

$$\frac{1}{\lambda} = Z^2 R_H \left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right)$$

also how you measure temp in stellar atmosphere

hydrogen spectral series:



Balmer series: in the optical spectrum

2-3 : Hα

2-4 : Hβ

Lecture 8:

Midterm 1: open book, open notes, calculator.
HW 1 + 2, might have HW 3 as well.
till week 4 tuesday content.

a) star in thermal equilibrium, uniform thermal energy generation.

$\varepsilon_{nuc} = 2 \text{ erg g}^{-1} \text{s}^{-1}$ } efficiency.

$M = 3 M_\odot$

$R = 5 R_\odot$

(effective temperature)

what is the luminosity of the star in solar units? T_{eff} in kelvin?

($M_\odot = 1.989 \times 10^{33}$ g, $L_\odot = 3.83 \times 10^{33}$ erg s^{-1})

$R_\odot = 6.96 \times 10^{10}$ cm.

$$\frac{\partial l}{\partial r} = 4\pi r^2 \rho (\varepsilon_{nuc}) \qquad \underset{\text{(density)}}{\rho} = \frac{3M_\odot}{\frac{4}{3}\pi R^3} = \frac{9M_\odot}{4\pi R^3}$$

$$dl = 4\pi r^2 \times \frac{9M_\odot}{4\pi R^3} \times \varepsilon_{nuc}\, dr$$

integrating both sides

$$\int_0^R \partial l = \int_0^R \cancel{4\pi} r^2 \frac{9M_\odot}{\cancel{4\pi} R^3} \times \varepsilon_{nuc}\, \partial r$$

$$* \quad \frac{1.1934 \times 10^{34}}{3.83 \times 10^{33}} = 3.116\, L_\odot$$

$$\underset{\text{(luminosity)}}{L} = \frac{9M_\odot \varepsilon_{nuc}}{R^3} \int_0^R r^2\, dr$$

$$L = \frac{9M_\odot \varepsilon_{nuc}}{R^3} \times \left[\frac{r^3}{3}\right]_0^R = \frac{\overset{3}{\cancel{9}} M_\odot \times \varepsilon_{nuc}}{\cancel{R^3}} \times \frac{\cancel{R^3}}{\cancel{3}}$$

$$= 3 \times 1.989 \times 10^{33} \times 2 = 1.1934 \times 10^{34} \; *$$

Answer:

$$\frac{dl}{dr} = \underbrace{4\pi r^2 \rho}_{dm/dr \text{ (mass equation)}} \varepsilon_{nuc}$$

integrating both sides.

$$L = M \varepsilon_{nuc} \approx 3.12 L_\odot$$

for effective temp: $L = 4\pi R^2 \sigma (T_{eff})^4$

$$3.116 \times 3.83 \times 10^{33} = 4 \times \pi \times (5R_\odot)^2 \times 5.67 \times 10^{-5} \times (T_{eff})^4$$

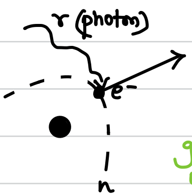
$$(T_{eff})^4 = \frac{L}{4\pi(5R_\odot)^2 \sigma}$$

$$T_{eff} = \sqrt[4]{\frac{3.116 \times 3.83 \times 10^{33}}{4\pi \times (5 \times 6.96 \times 10^{10})^2 \times 5.67 \times 10^{-5}}}$$

$$T_{eff} = \sqrt[4]{\frac{L}{4\pi(5R_\odot)^2 \sigma}} = 3429.345 \text{ K}$$

(somewhere in the middle of the star)

bound free transition (photo-ionization)



$$\frac{hc}{\lambda} > \frac{z^2 R_H hc}{n^2}$$ } energy comparision

R_H → Rydberg's constant

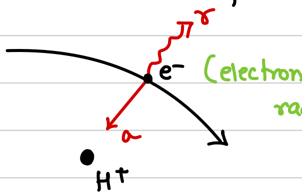
n → orbit no. of electron (K, L, M, N)

gives you the upper bound of energy needed

$$\sigma_{bf} = 1.31 \times 10^{-15} \times \frac{1}{n^5} \left(\frac{\lambda}{5000 \text{Å}}\right)^3 \text{cm}^2$$

(bound free)

free - free absorption



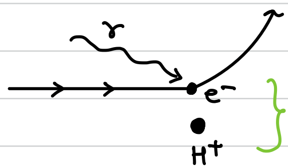
(electron gets deflected, it experiences some acceleration. Meaning, it radiates. It releases a photon and this radiation is called bremsstrahlung radiation (breaking radiation).)

emission process.

(ion at the center of a star)

(κ (kappa) = opacity)

$$\kappa = \frac{\sigma}{\mu m_H}$$



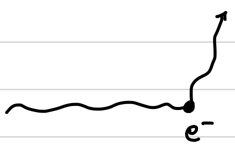
(ion)

conservation of momentum and energy.

it deflects away

$$\kappa_{ff} \propto \lambda^3 \times n_e \times T^{1/2}$$

e^- scattering (Thomson scattering).



cross section of Thomson scattering

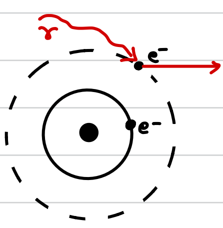
$$\sigma_T = \frac{1}{6\pi \varepsilon_0^2}\left(\frac{e^2}{m_e c^2}\right)^2 = 6.65 \times 10^{25}\ \text{cm}^{-2}$$

gaunt factor = the quantum mechanical fixations you need to apply.

for this to happen, you need the material to be hot and dense (mostly found in the core of stars).

H^- opacity

(happens when temp is not too high)



0.754 eV $\Rightarrow \lambda_c < 16{,}400$ Å

(takes place very far away from the star.)

$$\kappa \propto \rho^{1/2}\, T^{15/2}$$

(opacity)

heavily dependent of temp and not much on density.

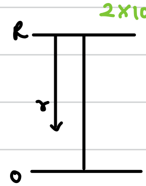
Lecture 9:

Q) Stellar atmosphere with constant opacity $\kappa = 1 \text{ cm}^2 \text{ g}^{-1}$, but the atmosphere is linearly varying with the radius, given by, $\rho(r) = 1\times10^{-5} \text{ g cm}^{-3} \left(\frac{R-r}{2\times10^5 \text{ cm}}\right)$

$\rho(z) = 1\times10^{-5} \times \frac{(z)}{2\times10^5}$

What is the optical depth τ at $R-r = 4\times10^5$ cm ?

$z = 4\times10^5$ cm



$$\tau = \int_r^R \kappa \rho \, dr$$

$$\tau = K\int_r^R 1\times10^{-5} \times \frac{(\overset{z}{R-r})}{2\times10^5} dr = K \int_r^R z\, dr = K \times \left[\frac{1}{2} z^2\right]_0^z$$

* Radiative diffusion: $F_{rad} = -K_{rad} \frac{\partial U_{rad}}{\partial r}$

diffusivity constant

$U_{rad} = aT^4$ ← erg $\text{cm}^{-3}\,T^{-4}$

here, $a = \frac{4\sigma}{c}$ } very similar to Stefon-Boltzmon const

$K_{rad} = \frac{1}{3} \times c \times \langle l \rangle = \frac{c}{3\kappa\rho}$

This one is how a star transferrs energy through radiative transfer / transport. The other way is convective energy transfer.

Lets look at the radiative transfer first, to determine which one is more efficient

$$l_{rad} = 4\pi r^2 F_{rad}$$

$$= 4\pi r^2 \times \underbrace{K_{rad}}_{\frac{c}{3\kappa\rho}} \times \frac{\partial U_{rad}}{\partial r}$$

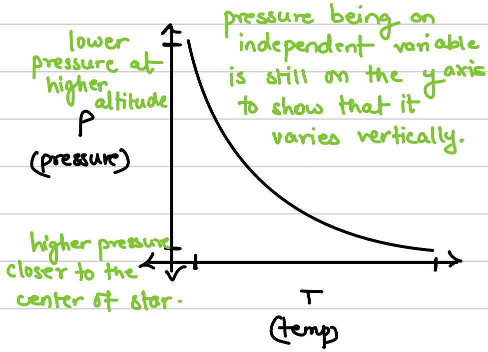
$$U_{rad} = aT^4$$

$$\therefore \frac{\partial U_{rad}}{\partial r} = 4aT^3 \frac{\partial T}{\partial r}$$

$$l_{rad} = \frac{-16\pi r^2 acT^3}{3\kappa\rho}\frac{\partial T}{\partial r}$$

$$\therefore \frac{\partial T}{\partial r} = \frac{-3\kappa\rho\, l_{rad}}{16\pi r^2 ac\, T^3}$$

$$\therefore \frac{\partial \ln T}{\partial \ln P} \equiv \nabla_T = \frac{P}{T}\frac{\partial T/\partial r}{\partial P/\partial r} \quad —— ①$$



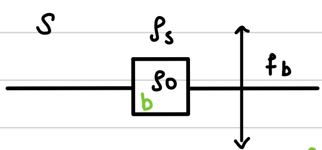
$$\frac{\partial P}{\partial r} = -\rho g = -\rho\frac{Gm}{r^2} \quad —— ②$$

combining ① and ②, we get: $\nabla_T = \dfrac{3}{16\pi ac G}\dfrac{\kappa\, l_{rad}\, P}{mT^4}$

The radiative energy emission is given by: $\underbrace{\nabla_{rad} = \dfrac{3}{16\pi ac G}\dfrac{\kappa\, l P}{mT^4}}_{}$

this is all the energy that is transferred from the core of the star to the surface only because of radiation.

* Convection:



$f_b = (\delta\rho_s - \delta\rho_0)\, g\, \Delta V$ } weight of the displaced water

(buoyancy force)

if mass of block is less than the mass of the surrounding material, then its going to go up.

$$\delta\rho = \underbrace{\left(\frac{\partial\rho}{\partial T}\right)_P}_{\text{pressure is constant}} \delta T + \underbrace{\left(\frac{\partial\rho}{\partial P}\right)_T}_{\text{temperature is constant.}} \delta P \qquad \left(\rho = \frac{P}{T}\right)$$

$$\delta\rho = \frac{-P}{T^2}\delta T + \frac{1}{T}\delta P = \frac{-\rho}{T}\delta T + \frac{\rho}{P}\delta P$$

$$\delta\rho = \rho\left(\frac{-\delta T}{T} + \frac{\delta P}{P}\right)$$ } small change in density.

$$\frac{\delta T_s}{T} = \nabla_T \cdot \frac{\delta P_s}{P}, \quad \nabla_T = \frac{\partial \ln T}{\partial \ln P} = \frac{P}{T}\frac{\partial T/\partial r}{\partial P/\partial r}$$

$$\nabla_{adiabatic} \equiv \left(\frac{\partial \ln T}{\partial \ln P}\right)_{adiabatic} = \frac{\gamma - 1}{\gamma}$$

→ depends on the adiabatic const for the gas
for ideal gas $\gamma = 5/3$

taking the small change in density equation:

$$\delta\rho = \rho\left(\frac{-\delta T}{T} + \frac{\delta P}{P}\right) = \rho\left(\frac{\delta P}{P} - \nabla\frac{\delta P}{P}\right)$$

stability condition for convection:

$$\boxed{\nabla_T < \nabla_{adiabatic}}$$

buoyancy force: $f_b = (\nabla_{adi} - \nabla_T)\dfrac{\partial P_s}{P}\rho g \Delta V$

Schwarzchild criterion. for convective stability.

If the inequality is met, then the star is stable on no convection occurs

Lecture 10:

$\nabla_{radiative} < \nabla_{adiabatic} \longrightarrow$ radiative energy transfer.

$\nabla_{radiative} > \nabla_{adiabatic} \longrightarrow$ convective energy transfer.

Q) Star : $\nabla_T = 0.4$, $\nabla_{rad} = 1.2$ } this means that the star has radiative transfer of energy over convective.

luminosity of a layer (l) = 1.1×10^{34} erg s^{-1}

luminosity of radiative energy (l_{rad}) = ?

luminosity of convective energy (l_{conv}) = ?

$$\nabla_T = \frac{3}{16\pi ac G} \frac{K l_{rad} P}{m T^4}$$

$$\nabla_{rad} = \frac{3}{16\pi ac G} \frac{K l P}{m T^4}$$

$$\frac{\nabla_{rad}}{\nabla_T} = \frac{\frac{\cancel{3}}{\cancel{16\pi ac G}} \frac{\cancel{K} l \cancel{P}}{\cancel{mT^4}}}{\frac{\cancel{3}}{\cancel{16\pi ac G}} \frac{\cancel{K} l_{rad} \cancel{P}}{\cancel{mT^4}}} = \frac{l}{l_{rad}}$$

$$l_{rad} = \frac{l \times \nabla_T}{\nabla_{rad}} = \frac{1.1 \times 10^{34} \times 0.4}{1.2}$$

$$l = l_{rad} + l_{conv}$$

$$l_{conv} = l - l_{rad}.$$

for $\nabla_T = \nabla_{rad}$,

$$\nabla_T = \phi \nabla_{ad} + (1-\phi)\nabla_{rad}$$

convective efficiency $= \phi$

$0 \le \phi \le 1$ } limit for convective efficiency.

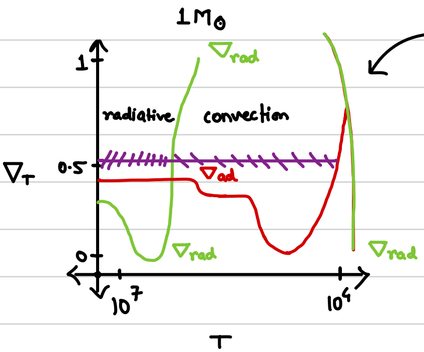
$$\nabla_T = \nabla_{rad} - \phi(\nabla_{rad} - \nabla_{ad})$$

$$\frac{\nabla_T}{\nabla_{rad}} = 1 - \phi\left(\frac{\nabla_{rad} - \nabla_{ad}}{\nabla_{rad}}\right)$$

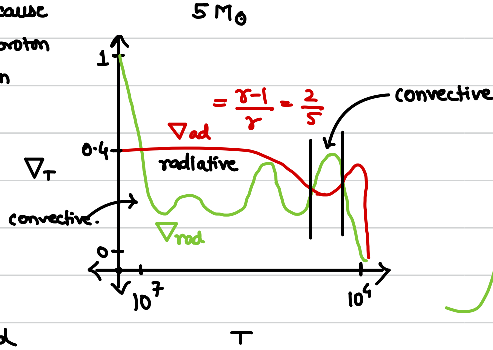
$$\therefore \; l_{rad} = l\frac{\nabla_T}{\nabla_{rad}} = l\left(1 - \phi\left(\frac{\nabla_{rad} - \nabla_{ad}}{\nabla_{rad}}\right)\right)$$

$$\therefore \; l_{conv} = l\left(\phi\left(\frac{\nabla_{rad} - \nabla_{ad}}{\nabla_{rad}}\right)\right)$$

} another way of finding luminosity because of radiative transport and convective transport



burns energy because of proton-proton p-p chain reaction



} for stars greater than 1.25M⊙

HCNO process burns energy.

convective energy envolopes ← mag field breaks early in life.

- sun spots.
- sun rotates slowly 28 – 29 days. / rotation

  4 km/s.

  ↳ smaller L

↳ 200 km/s. rotation for bigger stars.

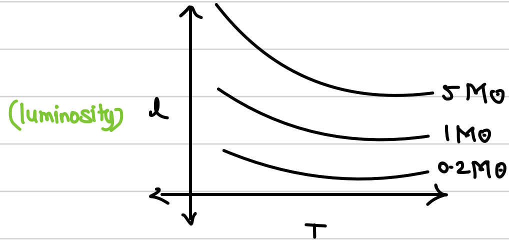
↳ bigger L.

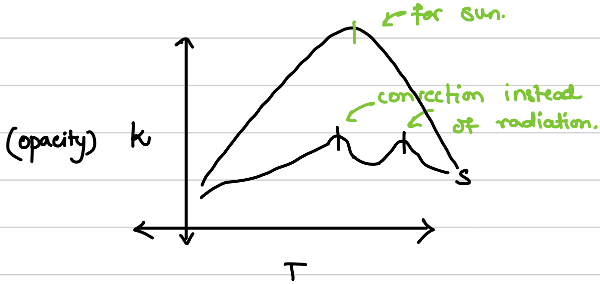
- these graphs tells us rotational history of star.
- sun like stars are well aligned solar system.
- stars rotating very fast dont have a well aligned system.

$$\nabla_{rad} = \frac{3}{16\pi a c G} \times \frac{\kappa l P}{m T^4}$$

for ideal gas: $\gamma = 5/3$.

$$\left.\frac{\gamma - 1}{\gamma} = \frac{2}{5}\right\} = 0.4 = \nabla_T.$$





Structure equations:

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}$$

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4}$$

$$\frac{\partial l}{\partial m} = \epsilon_{nuc} - \epsilon_{\nu} - \frac{\partial u}{\partial t} + \frac{P}{\rho}\frac{\partial \rho}{\partial t}$$

$$\frac{\partial T}{\partial m} = \frac{-GmT}{4\pi r^2 P}\nabla_T$$

solving all these equations together will give you the stellar structure of the star.

equation of state for pressure inside a star (not assuming star is ideal gas)

derive this !!!

$$P = \underbrace{P_{ion}}_{\text{pressure from ions repelling each other}} + \underbrace{P_{e^-}}_{\text{electron pressure}} + \underbrace{P_{rad}}_{\text{pressure because of photons}}$$

ans should be: $P_{rad} = \frac{aT^4}{3}$

for stars in the range:
$M < 0.1 M_\odot$
$M > 10 M_\odot$
and for later evolutionary stages

$P_{ion} = n^j_{ion} k_B T$

j = isotope referencing

n_{ion} = number density of the isotope of that ion.

$$P_{ion} = \left(\sum \frac{X_j}{A_j}\right) \frac{\rho kT}{m_H} = \frac{\rho k_B T}{\mu_{ion} m_H} \quad \text{— } 1$$

$X_j \rightarrow$ mass fraction of that isotope

$A_j \rightarrow$ mass number of the same isotope.

$$\mu_{ion} = \underbrace{\frac{1}{\left(\sum \frac{X_j}{A_j}\right)}}_{\text{mean molecular mass/weight for the ion}}$$

$P_{electron} = P_{e^-}$ — quantum mechanical pressure

but for now, it is approximately equal to the ideal gas pressure

$$P_e^j = N_j P_{ion}^j$$

$$P_e = \left(\sum_j \frac{N_j X_j}{A_j}\right) \frac{\rho k_B T}{m_H}$$

$$= \frac{\rho k_B T}{\mu_e m_H} \quad \text{— } 2$$

$$\mu_e = \frac{1}{\left(\sum \frac{N_j X_j}{A_j}\right)}$$

mean molecular mass

fully ionized

$P = P_{ion} + P_e$.

$$P = \left[\frac{\sum (1+N_j) X_j}{A_j}\right]^{-1}$$

$$\mu = \left[\frac{2x}{1} + \frac{3y}{4} + \frac{z}{2}\right]^{-1}$$

non-ionized

$P_e = 0$.

(no free electrons)

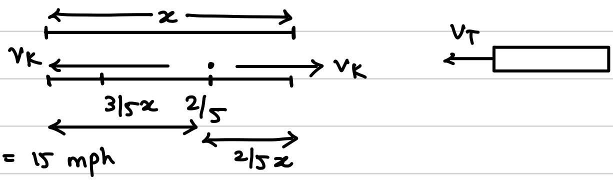
$P = P_{ion}$

$$\mu = \left[\sum \frac{X_j}{A_j}\right]^{-1}$$

$$\mu = \left[\underset{H}{\frac{x}{1}} + \underset{He}{\frac{y}{4}} + \underset{C}{\frac{z}{12}}\right]^{-1}$$

Lecture 11:

a)



$V_K = 15$ mph

$V_T = ?$

two kids run in opposite directions of the tunnel and both of them barely escape the train. Find the speed of the train.

$\text{time} = \frac{\text{dist}}{\text{speed}}$

↓

same

$$\frac{\frac{1}{5}\cancel{x}}{15} = \frac{\cancel{x}}{V_T}$$

$$\frac{1}{75} = \frac{1}{V_T}$$

$$V_T = 75$$

electron degeneracy pressure

- quantum mechanical pressure – electrons cannot be in the same quantum state (Pauli exclusion principle).
  - no same spin in the same orbital.

momentum distribution :

↳ for particles around us.

$f(p)$ – momentum distribution.

$$n = \int_0^\infty f(p)\, dp.$$

n: total no. of particles in the system

we can calculate pressure from this.

$$P = \frac{1}{3}\int_0^{\infty} f(p)\, p\, v(p)\, dp$$

velocity = $\frac{\text{momentum}}{\text{mass}}$ } assuming the velocity is non-relativistic.

Fermi mechanics.

ψ : degeneracy

$$f_e(p) = \frac{8\pi p^2}{h^3} \times \frac{1}{e^{(p^2/2m_e kT - \psi)} + 1}$$

there are two limits for ψ :

① classical : $\psi << 0$ (very negative).

$$f_e(p) = \frac{8\pi p^2}{h^3}\, e^{(-p^2/2m_e kT + \psi)}$$

could be written as $e^{-p^2/2m_e kT} \times e^{\psi}$

$$n = \int_0^{\infty} \frac{8\pi p^2}{h^3}\, e^{-p^2/2m_e kT} \cdot e^{\psi}\, dp$$

$$n = \frac{8\pi}{h^3}\, e^{\psi} \int_0^{\infty} p^2\, e^{-p^2/2m_e kT}\, dp \quad = \frac{2(2m_e kT)^{3/2}}{h^3} \times e^{\psi} = n_e$$

$$e^{\psi} = \frac{n_e h^3}{2(2\pi m_e kT)^{3/2}}$$

plugging all these values in the (P) equation, you get:

$$P = \frac{1}{3}\int_0^{\infty} \frac{8\pi p^2}{h^3} \times \frac{n_e h^3}{2(2\pi m_e kT)^{3/2}} \times e^{\frac{-p^2}{2m_e kT}} \times p \times \frac{p}{m_e}\, dp.$$

$$P = \frac{1}{3} \frac{4\pi n_e}{(2\pi m_e K T)^{3/2}} \times \frac{1}{m_e} \int_0^{\infty} p^4 e^{-p^2/2m_e KT} \, dp$$

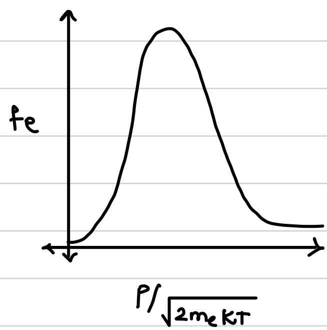
$$P = \frac{1}{3} \frac{4\pi n_e}{(2\pi m_e K T)^{3/2}} \times \frac{1}{m_e} \left(\frac{3}{8} \sqrt{\pi} \, (2 m K T)^{5/2} \right)$$

After solving the equation, and cancelling out the constants, you get:

$P = n_e K_B T$ } ideal gas law.

(most of the electrons behave as particles in ideal gas.

Maxwell - Boltzmann graph!



② degenerate : $\psi >> 0$

pressure momentum.

$$f_e(p) = \begin{cases} \frac{8\pi p^2}{h^3} & p < p_f \\ 0 & p > p_f \end{cases}$$

$$\psi = \frac{(P_f)^2}{2 m K T}$$

critical momentum.

$$n = \int_0^{P_f} \frac{8\pi p^2}{h^3}\,dp \quad , \quad n_e = \frac{8\pi}{3h^3} P_f \longrightarrow P_f = \left(\frac{3h^3 n_e}{8\pi}\right)^{1/3}$$

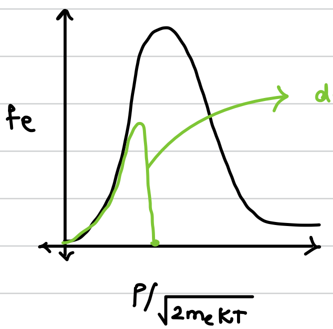
$$P = \frac{1}{3}\int_0^{P_f} \frac{8\pi p^2}{h^3} \times p \times \frac{p}{m_e}\,dp = \frac{1}{3} \times \frac{8\pi}{h^3 m_e} \times \frac{1}{5} \times (P_f)^5$$

$$P = \frac{8\pi}{15 h^3 m_e}\left(\frac{3h^3 n_e}{8\pi}\right)^{5/3}$$

how many electrons are there and how close they are ? (number density)

$$P = \left(\frac{3}{\pi}\right)^{3/2} \frac{h^2}{20\, m_e} (n_e)^{5/3}$$

this pressure does not depend on the temperature.



degenerate pressure is a part of the Maxwell-Boltzmann graph.

Fermi energy : $\varepsilon_F = \dfrac{P_f^2}{2\, m_e}$

if $\varepsilon_F \ll kT \longrightarrow$ classical.

if $\varepsilon_F \gg kT \longrightarrow$ degenerate.

for stellar interior, $\overset{H,0}{H} \leftrightarrow \overset{H,1}{H^+} + e^-$ } ionization pressure (P_{ion})

(momenta distibution for neutral hydrogen)

$$n = \int_0^\infty f(p)\, dp$$

$$f_{H,0}(p) = \frac{8\pi p^2}{h^3} \frac{1}{e^{p^2/2m_p kT - \underbrace{\chi_H}_{\text{binding energy}}/kT - \psi_{H,0}} + 1}$$

$\chi_H \rightarrow$ binding energy

$= 13.6 \text{ eV}$

$= 2.18 \times 10^{-11} \text{ erg}$

$$f_{H,1}(p) = \frac{1}{2} \frac{8\pi p^2}{h^3} \frac{1}{e^{p^2/2m_p kT - \psi_{H,1}} + 1}$$

$$f_e(p) = \frac{8\pi p^2}{h^3} \frac{1}{e^{p^2/2m_e kT - \psi_e} + 1}$$

$$n_{H,0} = \int_0^\infty \frac{8\pi p^2}{h^3} e^{-p^2/2m_H kT} \times e^{\chi_H/kT} \times e^{\psi_{H,0}}\, dp$$

$$\therefore n_{H,0} = \frac{2(2\pi m_H kT)^{3/2}}{h^3} e^{\chi_H/kT + \psi_{H,0}}$$

$$n_{H,1} = \frac{(2\pi m_H kT)^{3/2}}{h^3} \times e^{\psi_{H,1}}$$

$$n_e = \frac{2(2\pi m_e kT)^{3/2}}{h^3} e^{\psi_e}$$

from what we know

$H \leftrightarrow H^+ + e^-$

$\psi_{H,0} = \psi_{H,1} + \psi_e$

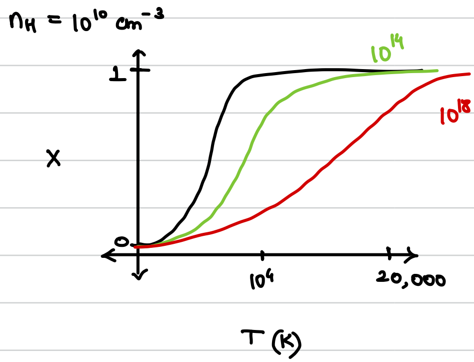
Saha's equation:

$$\frac{n_{H,1} \times n_e}{n_{H,0}} = \frac{(2\pi m_e kT)^{3/2}}{h^3} \times e^{-\chi_H/kT}$$

$$n_H = n_{H,0} + n_{H,1} \quad , \text{ and } \quad n_{H,1} = n_e$$

$$X = \frac{n_{H,1}}{n_H}$$ } no. of protons over total number of hydrogen atoms.

$$\frac{X^2}{1-X} = \frac{(2\pi m_e kT)^{3/2}}{n_H h^2} \times e^{-\chi_H/kT}$$



Lecture 12:

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}$$

$$\frac{\partial P}{\partial m} = \frac{-Gm}{4\pi r^4}$$

$$\frac{\partial l}{\partial m} = \epsilon_{nuc} - \epsilon_{\nu} - \frac{du}{dt} + \frac{P}{\rho}\frac{\partial \rho}{\partial t}$$

$$\frac{\partial T}{\partial m} = \frac{-GmT}{4\pi r^2 P}\nabla_T$$

$$P = \frac{\rho k_B T}{\mu m_H}$$

if your star is puffing up in the red giant phase.

here, the independent variables are mass, time.

dependent variables.

$r, l, \rho, T, \{x_k\}$ → initial composition.

$$\frac{\partial x_j}{\partial t} = \frac{A_t m_H}{\rho}\sum_{k=j}\left(R_{kj} - R_{jk}\right)$$

Boundary conditions:

- At the center of the star, all the quantities go to zero.

  $l \to 0$ as $m \to 0$

  $r \to 0$

- Eddington approximation: $T^4(\tau) = \frac{3}{4}(T_{eff})^4\left(\tau + \frac{2}{3}\right)$ Here τ = optical depth of the star.

  temperature of the stellar surface.

  $P(\tau) = \frac{GM}{R^2 K}\tau$ optical depth is zero at the surface and goes up as you go down.

$$P \to \frac{2}{3}\frac{GM}{RK}$$

as $m \to M$

$$T \to T_{eff}$$

usually we can ask a computer to solve these equations. But in the past, when computers did not exist, astronomers made some simple assumptions.

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \quad , \quad \frac{\partial P}{\partial m} = \frac{-Gm}{4\pi r^4}$$ mechanical equations

$$\frac{\partial l}{\partial m} = \epsilon_{nuc} - \epsilon_\nu - \frac{du}{dt} + \frac{P}{\rho}\frac{\partial \rho}{\partial t}$$

$$\frac{\partial T}{\partial m} = \frac{-GmT}{4\pi r^2 P} \nabla_T$$

thermo energetic equations.

- Polytropes

$$P = K \rho^{\frac{1+n}{n}}$$

$n \rightarrow$ polytropic index

$$n = \frac{1}{\gamma - 1} \qquad \gamma = 5/3 \qquad n = 3/2$$

$$P = \frac{\rho k_B T}{\mu m_H}$$

$$P = K \left(\frac{P \mu m_H}{kT} \right)^{\frac{1+n}{n}}$$

$$T = K^{\frac{1+n}{n}} \left(\frac{\mu m_H}{k} \right) P^{1 - \frac{1+n}{n}}$$

LE equations.

$$\frac{1}{z^2} \frac{d}{dz} \left(z^2 \frac{dw}{dz} \right) = -w^n$$

$$z = \left[\frac{4\pi G}{K \rho_c^{(1-n)/n} (n+1)} \right]^{1/2} \times r$$

Boundary conditions.

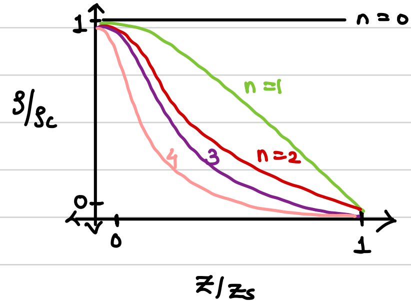
at the center of the star,

$$w = 1 \, , \quad \frac{dw}{dz} = 0$$ at $z = 0$ (center)

$$w = 0 \quad \text{at} \quad z = z_s$$

w = rescaling of the density.

you need to figure out what z_s is.



for the sun, it is very close to $n=3$.

$n=0$

$$w(z) = 1 - \frac{z^2}{6}$$

$$z_s = +\sqrt{6}$$

$n=1$

$$w(z) = \frac{\sin(z)}{z}$$

$$z_s = \pi$$

for $n=5$,

$$w(z) = \frac{1}{(1+z^2/3)^{1/2}} \Rightarrow z_s \to \infty$$

- Lookup table :

n	z_s	$\left(-z^2 \frac{dw}{dz}\right)_{z_s}$
3	6.8968	2.018

$$m(r) = 4\pi \rho_c r^3 \left(\frac{-1}{z} \frac{dw}{dz}\right)$$

mass equation in terms of polytropes.

$$M = 4\pi \rho_c R^3 \left(\frac{-1}{z} \frac{dw}{dz}\right)_{z=z_s}$$

(total mass)

$$\rho_c = \frac{1}{4\pi} \frac{M}{R^3} \left(\frac{-1}{z} \frac{dw}{dz}\right)^{-1}_{z=z_s}$$

Substituting this in pressure expression.

$$P_c = \frac{G}{4\pi(n+1)} \frac{M^2}{R^4} \left(\frac{-dw}{dz}\right)^{-2}_{z=z_s}$$

Lecture 18:

Q) What is the central density ρ_c for an $n=1$ polytrope?

$$w(z) = \frac{\sin z}{z}$$

$$\frac{dw}{dz} = \frac{\cos z}{z} - \frac{\sin z}{z^2}$$

$$\rho_c = \frac{1}{4\pi}\frac{M}{R^3}\left(\frac{-1}{z}\frac{dw}{dz}\right)^{-1}_{z=z_s}$$

$$\rho_c = \frac{1}{4\pi}\frac{M}{R^3}\left(\frac{-1}{\pi}\left(\frac{\cos\pi}{\pi} - \frac{\sin\pi}{\pi^2}\right)\right)^{-1}_{z=z_s}$$

$$\rho_c = \frac{1}{4\pi}\frac{M}{R^3}\left(\frac{+1}{\pi^2}\right)^{-1}$$

$$\rho_c = \frac{1}{4\cancel{\pi}}\frac{M}{R^3}\,{+\pi^{\cancel{2}}} = \frac{+M\pi}{4R^3}$$

For a convective star, $\nabla_T = \frac{\partial \ln T}{\partial \ln P} = \nabla_{ad}$, $\gamma = 5/3$, $\nabla_{ad} = 2/5$

$$\therefore \frac{\partial \ln T}{\partial \ln P} = 2/5$$

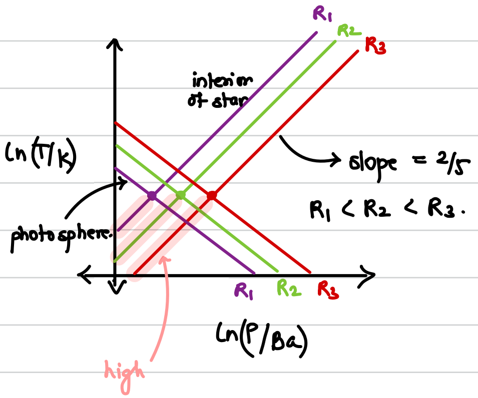
$$\therefore \ln\left(\frac{T}{T_c}\right) = \frac{2}{5}\ln\left(\frac{P}{P_c}\right)$$

you get this relation between central temperature and central pressure.

$$\rho_c \propto \frac{M}{R^3}, \quad P_c \propto \frac{M^2}{R^4}, \quad T_c \propto \frac{M}{R}$$

plug these relations in the above eq.

$$\ln\left(\frac{T}{K}\right) = \frac{2}{5}\ln\left(\frac{P}{Ba}\right) + \frac{1}{5}\ln\left(\frac{M}{M_\odot}\right) + \frac{3}{5}\ln\left(\frac{R}{R_\odot}\right) + C_{int}$$



- Modelling the photosphere of the star

$$P_{photosphere} = \frac{2}{3}\frac{GM}{R^2\kappa}, \quad T_{photosphere} = T_{eff}$$

$$\kappa = \kappa_0 P_{photo}^{a} T_{photo}^{b}$$ plug this in the above equation to find the opacity.

$$\ln\left(\frac{T_{photo}}{K}\right) = \frac{1+a}{b}\ln\left(\frac{P_{photo}}{Ba}\right) + \frac{1}{b}\ln\left(\frac{M}{M_\odot}\right) - \frac{2}{b}\ln\left(\frac{R}{R_\odot}\right) + C_{photo}$$

$$\ln\left(\frac{P_{photo}}{Ba}\right) = \frac{(10-3b)\ln(R/R_\odot) + (5-b)\ln(M/M_\odot) + C_P}{5+5a+2b}$$

$$\ln\left(\frac{T_{photo}}{K}\right) = \frac{(3a-1)\ln(R/R_\odot) + (3+a)\ln(M/M_\odot) + C_T}{5+5a+2b}$$

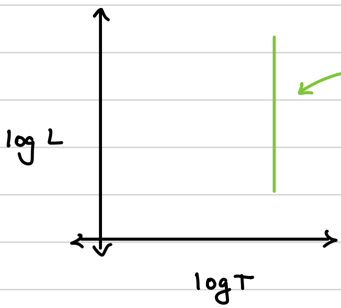
$$\ln\left(\frac{L}{L_\odot}\right) = \frac{(6+22a+4b)\ln(T/K) - (6+2a)\ln(M/M_\odot) + C_L}{3a-1}$$

$$\frac{d\ln L}{d\ln T} = \frac{6+22a+4b}{3a-1}$$

for H^- opacity,

$a=1$ $\quad\therefore\quad \frac{d\ln L}{d\ln T} \approx 20$

$b=3$



for a given star with mass M, where radius changes, the graph is a straight vertical line (hayashi track

Lecture 14:

Q) 100 people , $100

$5/man , $2/woman , 10¢ or $0.1/child

$m + w + c = 100 \Rightarrow c = 100 - m - w$

$5m + 2w + 0.1c = 100$

$50m + 20w + 1c = 1000$

$c = 60$

$5m + 2w = 94$

$m + w = 40$ } m, w are not integers.

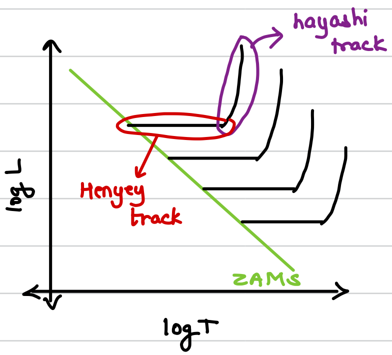
$$
\begin{array}{r}
50m + 20w + 1\cancel{c} = 1000 \\
-\quad m + w + \cancel{c} = 100 \\
\hline
49m + 19w = 900
\end{array}
$$

$49m = 900 - 19w$ — ①

$c = 70$

$5m + 2w = 93$

$m + w = 30$ } $m = 11$, $w = 19$



$\rho_c \propto \frac{M}{R^3}$, $P_c \propto \frac{M^2}{R^4}$, $T_c \propto \frac{M}{R}$

$$l_{rad} = \frac{-16\pi r^2 a c T^3}{3\kappa \rho_c} \times \frac{\partial T}{\partial r}$$

$r \rightarrow R$, $T \rightarrow T_c$, $\frac{\partial T}{\partial r} \rightarrow \frac{T_c}{R}$, $l \rightarrow L$

$L \propto \frac{RT^4}{\rho_c}$ } assuming κ (opacity) remains constant.

So, for radiative , $L \propto M^3$ } represents the luminosity when the star is on the henyey track.

For a main-sequence star:

$$\frac{\partial L}{\partial r} = \underset{R}{4\pi r^2}\, \underset{\rho_c}{\rho} \left(\epsilon_{nuc} - \cancel{\epsilon_\nu}^{\,0} \right)$$ we can ignore the neutrino fusion losses.

R (radius of the star)

ρ_c (central density)

$$\frac{L}{R} \propto R^2 \rho_c \epsilon_{nuc} \longrightarrow \epsilon_{nuc} = \epsilon_{nuc,0}\, \rho_c^{\alpha} T_c^{\beta}$$

$$\therefore \frac{L}{R} \propto R^2 \rho_c^{1+\alpha} T_c^{\beta}$$

from $L \propto M^3$, we get

$$M^3 \propto R^3 \frac{M^{1+\alpha}}{R^{3+3\alpha}} \frac{M^{\beta}}{R^{\beta}}$$

$$R \propto M^{\frac{\alpha+\beta-3}{3\alpha+\beta}}$$

for $\alpha = 1$, $\underset{\text{proton-proton fusion}}{\beta = 5}$, $\underset{\text{CNO}}{\beta = 17}$ $\Rightarrow$ $\underbrace{\beta \sim 11}_{\text{average.}}$,

$$R \propto M^{0.64}$$

fraction of Mass of the star converted from Hydrogen to Helium.

$$M f_e = L \tau_{MS}$$

$$\tau_{MS} = \frac{f M_e}{L} \propto M^{-2}$$

- proton-proton fusion (hydrogen burning) chain reaction : $\underbrace{4\left({}^{1}_{1}H + e^-\right)}_{\text{H-atom}} \rightarrow \underbrace{\left({}^{4}He + 2e^-\right)}_{\text{He-atom}} + 2\nu_e + N\gamma$

out of 4 protons, two of them undergo beta-decay.

step 1: $2({}^{1}H + e^-) \longrightarrow \underbrace{({}^{2}D + e^-)} + \underline{e^- + e^+} + \nu_e$

energy release because of mass difference between deuterium and hydrogen. $E = mc^2$

γ

electrons and positrons annhilate each other to get a gamma ray photon

annhilation releases energy.

$\Delta E_1 = 1.44$ MeV

step 2: $({}^{2}D + e^-) + ({}^{1}H + e^-) \longrightarrow ({}^{3}He + 2e^-) + \gamma$ } energy released because of mass difference again.

$\Delta E_2 = 5.49$ MeV

Chain I: 69% } happens 69% of the time.

$2({}^{3}He + 2e^-) \longrightarrow ({}^{4}He + 2e^-) + 2({}^{1}H + e^-)$ $\qquad \Delta E_3 = 12.86$ MeV

$2\nu \qquad 6\gamma \qquad \Delta E = 26.72$ MeV

energy for all of them is the same

Chain II/III: (step 3) $({}^{3}He + 2e^-) + ({}^{4}He + 2e^-) \longrightarrow ({}^{7}Be + 4e^-) + \gamma$

p-p III: $\left.\begin{aligned} {}^{7}Be + {}^{1}H &\longrightarrow {}^{8}B + \gamma \\ {}^{8}B &\longrightarrow {}^{8}Be + e^+ + \nu_e \\ {}^{8}Be &\longrightarrow 2\,{}^{4}He \end{aligned}\right\}$ not including electrons in the equations.

p-p II: $\left.\begin{aligned} {}^{7}Be + e^- &\longrightarrow {}^{7}Li + \nu_e \\ {}^{7}Li + {}^{1}H &\longrightarrow 2\,{}^{4}He \end{aligned}\right\}$ again, not including electrons in the equations.

ν_e → neutrino

Flow-chart format:

$$\begin{tikzcd}[column sep=small]
 & \text{Step 1} \arrow[d] & & \\
 & \text{Step 2} \arrow[dl, "69\%"'] \arrow[dr, "31\%"] & & \\
\text{Chain I} & & \text{Step 3} \arrow[dl, "99.7\%"'] \arrow[dr, "0.3\%"] & \\
 & \text{Chain 2} & & \text{Chain 3}
\end{tikzcd}$$

reaction writing notation: $a\,(b, c)\,d$ → final.

target incoming outgoing

$$p(p, e^+)\,d\,(p, \gamma)\,{}^3He\,({}^3He, 2p)\,\alpha$$

→ α particle is the ^{4}Helium particle.

$${}^8B\,(\cdot\,, e^+)\,{}^8Be$$

- $\cdot$ when there is nothing incoming

Lecture 15:

CNO cycle } instead of proton-proton fusion for stars having $M > 1.4 M_\odot$

→ essentially the same as p-p fusion but occurs in carbon, nitrogen, oxygen.

$$^{12}C\,(p,\gamma)\,^{13}N\,(\cdot, e^+)\,^{13}C\,(p,\gamma)\,^{14}N\,(p,\gamma)\,^{15}O\,(\cdot, e^+)\,^{15}N\,(p,\alpha)\,^{12}C$$

Beta decay

positron annhilates

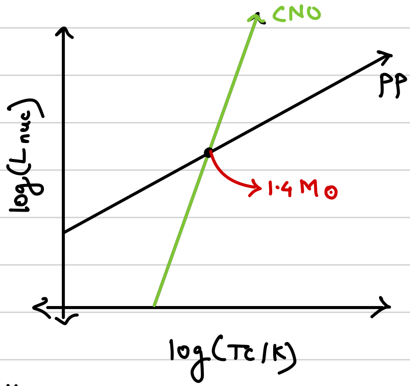
decays as ^{15}O is unstable

^{4}He

The ^{4}He comes from the 4 protons added in the reaction. This process releases the same energy as p-p fusion does. The above reaction is Chain I, and it occurs 99.7 % of the time. In some rare instances, Chain II occurs and ^{4}He is created.

$$\underbrace{\epsilon_{nuc} \approx \epsilon_{nuc,0,p\text{-}p}\, \rho\, T^{4.9}}_{\text{proton-proton}}$$

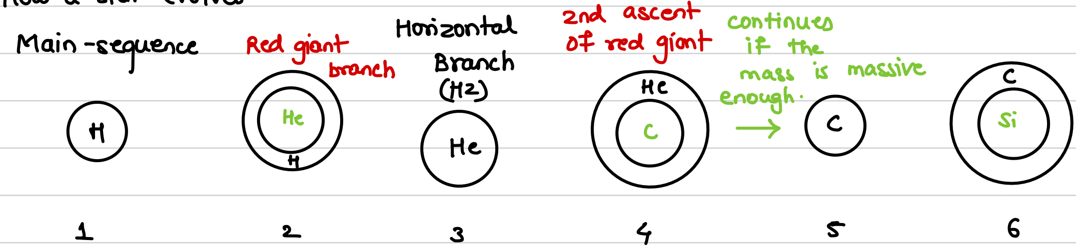
$$\underbrace{\epsilon_{nuc} \approx \epsilon_{nuc,0,CNO}\, \rho\, T^{16}}_{\text{CNO}}$$

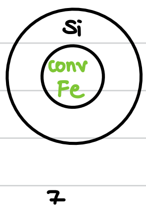


At $1.4 M_\odot$, the interior of the star changes as mass increases. Higher mass stars tend to rotate more rapidly than low mass stars (like the sun).

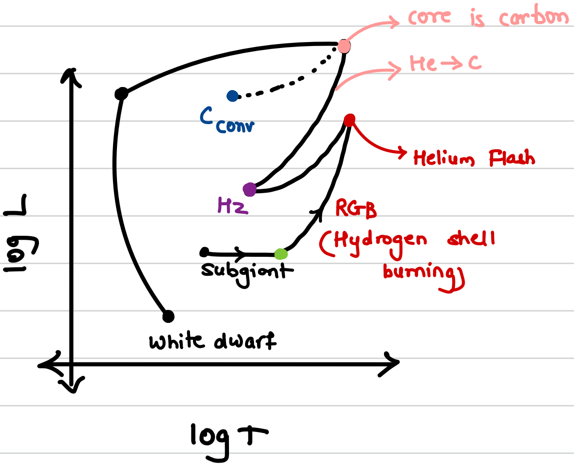
Also called as the Chandrashekar Limit.

How a star evolves:

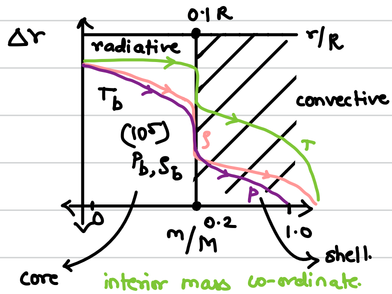


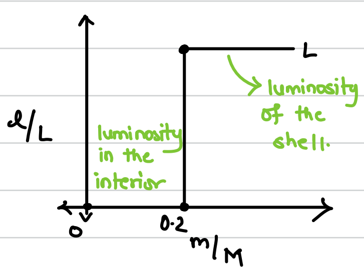


as you go from hydrogen burning to silicon and iron, the time it takes to burn it stars decreasing (the burning gets faster).



- Subgiant: fusion in the core stops the temperature starts falling, but luminosity remains the same approximately.
- RGB (Red giant branch):





Also, assume neutrino losses are negligible $\epsilon_\nu = 0$.

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \Rightarrow \frac{\Delta r}{\Delta m} = \frac{1}{4\pi R_c^2 \rho_b} \propto \frac{1}{R_c^2 \rho_b}$$

$$\frac{P_n}{\Delta m} \propto \frac{M_c}{R_c^4}, \quad \frac{l_t}{\Delta m} \propto \rho_b^{\alpha} T_b^{\beta}, \quad \frac{T_b}{\Delta m} \propto \frac{l_t}{R_c^4 T_b^3}$$

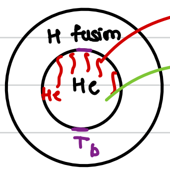
$$l_t \propto M_c^{\left(\frac{4\alpha+\beta+4}{\alpha+2}\right)} R_c^{\left(\frac{-3\alpha-\beta}{\alpha+2}\right)}, \quad T_b \propto M_c R_c^{-1}$$

$$\Delta m \propto M_c^{\left(\frac{-\beta+4}{\alpha+2}\right)} R_c^{\left(\frac{3\alpha+\beta}{\alpha+2}\right)}, \quad \left. l_t \propto M_c^{7} R_c^{-16/3} \right\} \text{for } \alpha \cong 1, \ \beta \cong 13$$

$$\Delta m \propto M_c^{-3} R_c^{16/3}$$

Lecture 16:

RGB



Helium is raining down in the core.

start getting electron degeneracy pressure.

$$P_e = \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{20 m_e} n_e^{5/3} \qquad n_e = \frac{\rho}{\mu_e m_H}$$

$$P \propto \rho^{5/3} \equiv n = 3/2 \text{ polytrope}$$

(for a polytrope $P = K \rho^{1+n/n}$)

$$T_b \propto \frac{M_c}{R_c}$$ temperature at the bottom of the shell.

$$\rho_c \propto \frac{M_c}{R_c^3}, \quad P_c \propto \frac{M_c^2}{R_c^4}$$

using these

$$R_c \propto M_c^{-1/3} \; *$$

As mass increases, radius decreases. (of the core)

using *

$$T_b \propto M_c^{4/3}$$

this becomes the temperature for the whole shell (constant).

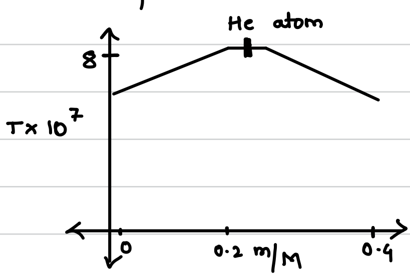
Triple-α process (net energy release : 7.275 MeV)

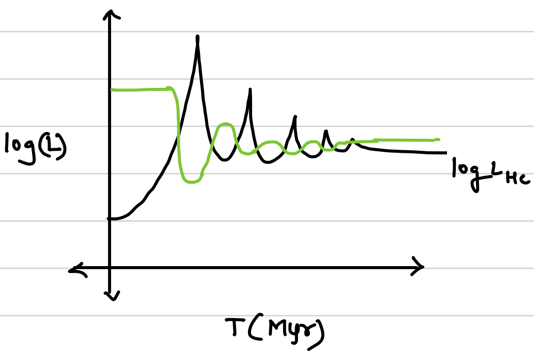
$$3(\underbrace{{}^4He + 2e^-}_{\text{He atom}}) \longrightarrow \underbrace{{}^{12}C + 6e^-}_{C} + N\gamma$$

this process is so rare, that you can assume it to be effectively zero.

You get a lot of neutrino losses in the center/core.

If you see the temperature distribution in the core, the center of the core is the coolest.

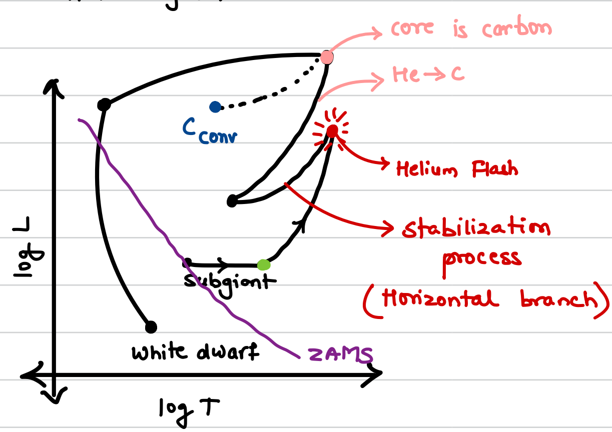




$l_t \propto M_c^7 R_c^{-16/3}$

shell increases in size as it puffs up.

H-R diagram



Triple α process (detailed) :

$$2(^4He + 2e^-) \longrightarrow \underbrace{^8Be}_{\text{extremely unstable.}} + 4e^- \qquad \Delta\varepsilon = -92\,keV$$

$$\underbrace{^8Be + 4e^- + {^4He} + 2e^-}_{\text{(He atom)}} \longrightarrow \underbrace{^{12}C^*}_{\substack{\text{Hoil state} \\ \text{(quantum mechanical state)} \\ n=0}} + 6e^-$$

$\Delta\varepsilon = -287$ MeV

triple α cannot happen if this quantum mechanical state didn't exist.

$$^{12}C^* + 6e^- \longrightarrow {^{12}C} + 6e^- + 2\gamma$$

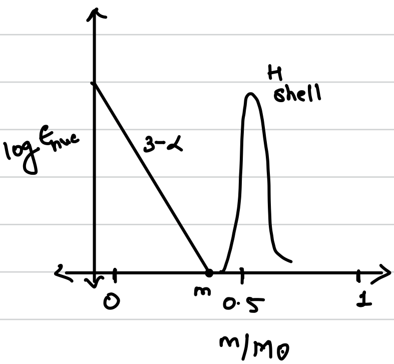
$\Delta\varepsilon = 7.654$ MeV

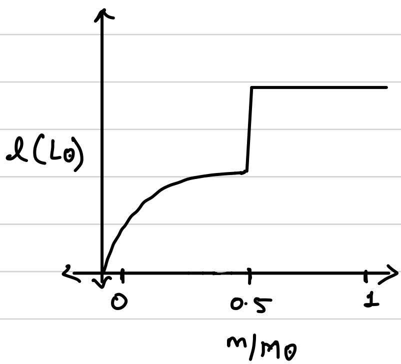
Beta decay

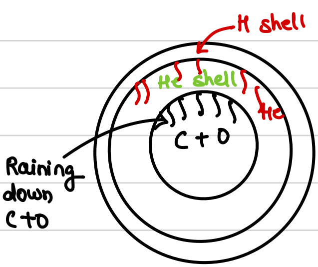
Horizontal branch: Star's core runs out of He → C faster than H → He, as the energy release because of triple α process is lesser.

$M_c \sim 0.47\, M_\odot$ } Amount of mass that should be in the core for the burning.

as metallicity ↑ , opacity ↑.
as you go to triple α process, you burn out of He faster. (you are on the HZ branch for lesser time than the main sequence).







Asymptotic giant branch (AGB) } very bright (nuclear reactions)

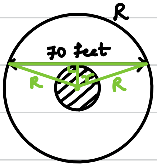
↳ first ascent AGB star. (has more oxygen).

α - capture:

$$^{12}C + {}^{4}He \rightarrow \underbrace{{}^{16}O}_{\text{oxygen content.}} \qquad \Delta E = 7.161 \text{ MeV.}$$

Lecture 17 :

Q) Some painters trying to paint a merry-go round.



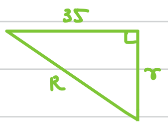
Area of bigger circle $= \pi R^2$

$\pi R^2 = x + \pi r^2$

$\pi R^2 = x + \pi (R^2 - 35^2)$

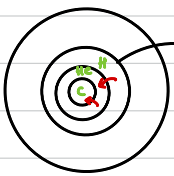
$\cancel{\pi R^2} = x + \cancel{\pi R^2} - \pi \times 35^2$

$x = \pi \times 35^2 = 3846.5 \text{ ft}^2$



$35^2 + r^2 = R^2$

$r^2 = R^2 - 35^2$



Asymptotic giant branch.

hydrogen is fusing into helium because of the CNO cycle.

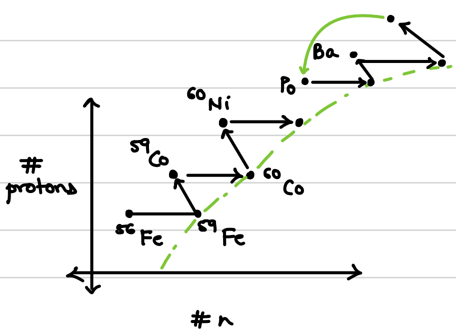
$$^{13}C + {}^{4}He \longrightarrow {}^{16}O + n$$

(neutron)

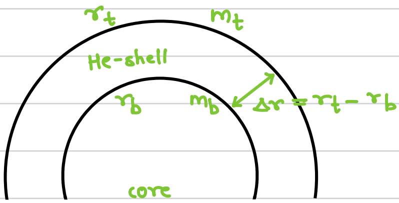
s-process, where all the heavy elements (atomic mass greater than iron) are formed.

$$^{56}Fe\,(n,\gamma)\,^{57}Fe\,(n,\gamma)\,^{58}Fe\,(n,\gamma)\,^{59}Fe\,(\cdot\,, e^-)\,^{59}Co$$

unstable



Thin helium layer in the AGB star is where this occurs



$$\rho \approx \frac{\Delta m}{4\pi r_b^2 \Delta r} \quad , \quad m_m = \frac{m_t + m_b}{2} \Big\} \text{ avg mass.}$$

$$P \approx \int_{m_m}^{M} \frac{Gm}{4\pi r^4}\, dm \Big\} \text{ Hydrostatic equilibrium}$$

$$\delta\rho = \frac{\partial \rho}{\partial r_t} \times \delta r_t \approx -\rho \frac{\delta r_t}{\Delta r} \quad , \quad \delta P = \frac{\partial P}{\partial r_t}\, \delta r_t \approx -4P \frac{\delta r_t}{r_t}$$

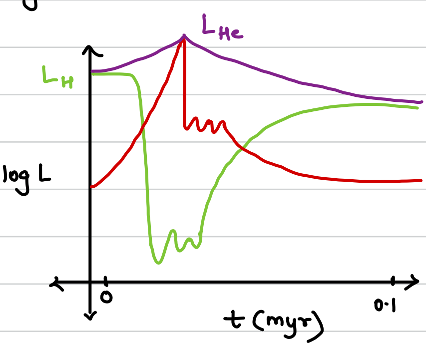
(small change in density)

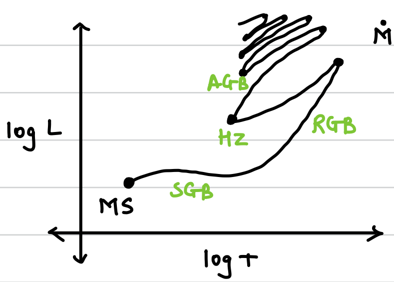
Helium fusion process is very temperature dependent, so using ideal gas law (ideal gas equation of states).

$$\frac{\delta T}{T} = \frac{\delta P}{P} - \frac{\delta \rho}{\rho} \approx \left(1 - 4\cancel{\frac{\Delta r}{r_t}}^{\,0}\right) \frac{\delta r_t}{\Delta r}$$

Δr is very small compared to r_t or r_b.

This makes the pulses that we see in the star as the He-shell becomes unstable. The puffing and shrinking of He-shell shuts off the fusing of H to He in the outer shell of He. The hydrogen burning luminosity (or hydrogen luminosity) goes to zero.



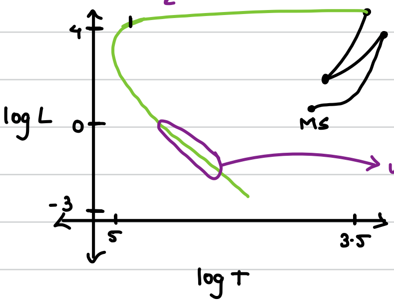


$\dot{M} \sim 10^{-5}$ Mo/yr log L

super wind.

$$\dot{M}_{AGB} \sim 5 \times 10^{-9}\ M_\odot/yr \left(\frac{M}{M_\odot}\right)^{-2.1} \left(\frac{L}{L_\odot}\right)^{2.7}$$

luminosity stays pretty much the same.



where most of the WD lie

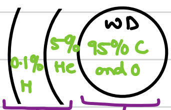
All stars $< 9 M_\odot$ will end their life as a white dwarf.

For a white dwarf: $M = 0.54 M_\odot$

$R \sim 1 R_E$ (radius of earth)

$\rho \sim 10^6 g\ cm^{-3}$

$T \lesssim 10^7 K$



classical.

degenerate

(Electron degeneracy pressure depends on the number density).

this is like having an atmosphere of H and He around C and O.

Most of the temperature is concentrated for C and O.

$$K \propto \rho_b T_b^{-7/2} \quad \text{(Kramer's Law)}$$

opacity of the outer atmosphere.

$$L = L_t \propto M T_b^{7/2}$$ total luminosity is basically luminosity of the surface.

$$L = -\frac{du}{dt}, \text{ where } u = M T_b$$

$$\therefore \frac{dL}{dt} \propto M^{-5/7} L^{12/7}$$

$$\therefore L \propto T^{-7/5}$$

D-A white dwarf 80%

enough H in the atmosphere.

D-B white dwarfs 20%

don't have any H in the atmosphere.

extremely low mass stars don't go through thermal pulses and just become white dwarfs.

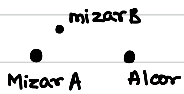
High mass stars have Ne in their cores.

Lecture 18:

Binary systems:

- Visual Binary

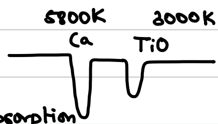
Eg:



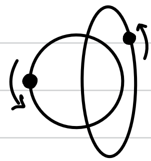
we use spectroscopy to detect such systems.

you can see them with the naked eye.

- Spectrum Binary:



spectrum that combines absorption lines from stars at different temperatures.



doppler shifted.

- Spectroscopic Binaries:

you can see the stars moving on top of each other (double-lined).

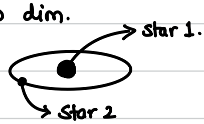
you can see only one star and the influence from the other star (single-lined).

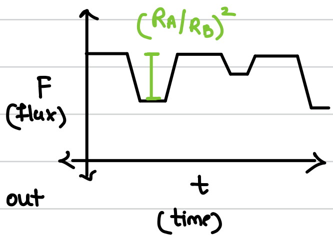
doppler shifted.

- Astrometric Binaries:

periodic changes in the position of one star on the sky, as it orbits around the system center of mass; other star too dim.

- Eclipsing Binary:





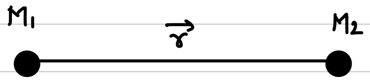
periodic changes in the overall brightness of the system, occuring as one star passes in front of the other and blocks out its light. Inclination angle = 90°.

Kepler's 3rd Law: $$p^2 = \frac{4\pi}{G(M_A+M_B)} \times a^3$$

time period

semi-major axis of the ellipse.

when you have mass in $M_\odot$, and a in AU, $G = 4\pi$



$\left(F = ma = \frac{G m_1 m_2}{r^2}\right)$

$$M_1 \ddot{r}_1 = \frac{G M_1 M_2}{r^3}\vec{r} \qquad M_2 \ddot{r}_2 = -\frac{G M_1 M_2}{r^3}\vec{r}$$

reduced mass : $\mu = \frac{M_1 M_2}{M_1 + M_2}$ $\qquad \mu \ddot{r} = -\frac{G M_1 M_2}{r^3}\vec{r}$

$$(x, y, z) \rightarrow (r, \theta)$$

$$\mu(\ddot{r} - r\dot{\theta}^2) = \frac{-G M_1 M_2}{r^2}, \qquad \mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$$

integrates to

your orbital angular momentum must be conserved. (Kepler's 2nd Law).

$$\underbrace{\mu r^2}_{I\ (MR^2)} \underbrace{\dot{\theta}}_{\omega} = L \rightarrow \text{angular momentum (orbital)}$$

$$\mu r^2 \dot{\theta} = L$$

$$r^2 \Delta + \dot{\theta} = \left(\frac{L}{\mu}\right)\Delta t$$

*

* Substitutions you need to use:

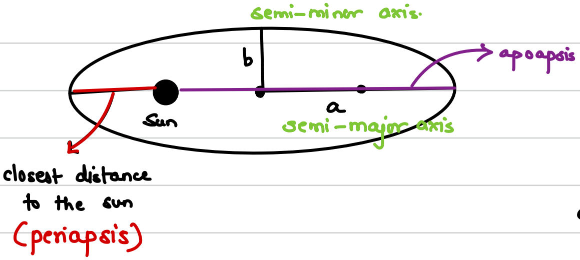
$$u = \frac{1}{r} \qquad \frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta} = \frac{L}{\mu r^2}\frac{d}{d\theta}$$

$$\frac{d^2u}{d\theta} + u = \frac{G M_1 M_2 \mu}{L^2}$$

$$\therefore \frac{1}{r(\theta)} = u = \frac{G M_1 M_2 \mu}{L^2}(1 + e\cos\theta)$$

$$\therefore \underbrace{r(\theta) = \frac{a(1-e^2)}{1 + e\cos\theta}} \qquad \therefore a = \frac{L^2}{G M_1 M_2 \mu} \times \frac{1}{1-e^2}$$

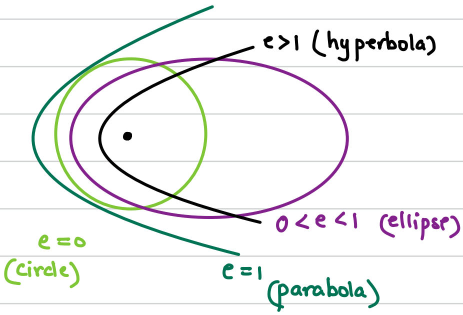
Kepler's 1st law as the equation represents an ellipse.



$$r_{per} = a(1-e) \quad \text{(periastron)}$$
$$r_{ap} = a(1+e) \quad \text{(apoastron)}$$

for stars.

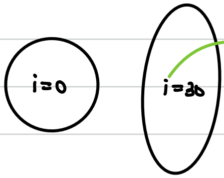
$$a_1 = \frac{M_2}{M_1+M_2}\,a \qquad a_2 = \frac{M_1}{M_1+M_2}\,a$$



when your orbit is completely circular,

$$\dot{\theta} = \frac{2\pi}{P}, \quad M_1 \gg M_2$$

Lecture 19:

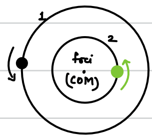


inclination (when you look at a circular orbit at an inclination, it looks like an ellipse).

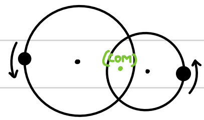
$$\frac{a_1}{a_2} = \frac{\alpha_1}{\alpha_2} = \frac{M_2}{M_1} = q$$ } you take the ratio of the semi-major axis.

$$\frac{a_1}{1\,AU} = \frac{\alpha_1}{1''} \times \frac{d}{1\,pc} \qquad \therefore\ a_2 = \alpha_2 d$$

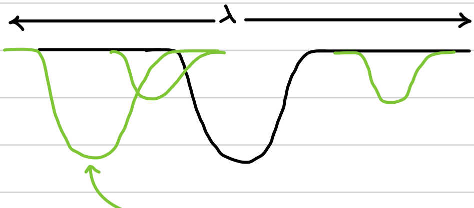
eg①:



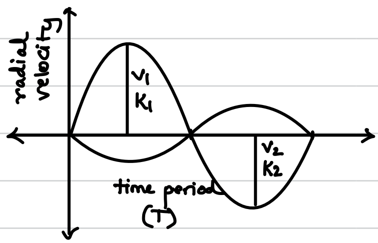
Eg②:



$$a = a_1 + a_2 \qquad P^2 = \frac{4\pi^2}{G(M_1+M_2)} a^3 \qquad G = 4\pi^2$$



Blue shifted.



double lined binary.

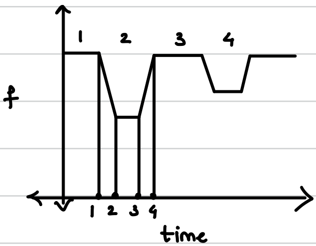
$$\frac{M_2}{M_1} = \frac{K_1}{K_2} \qquad 2\pi a_1 = \frac{K_1}{\sin(i)} P \ , \quad 2\pi a_2 = \frac{K_2}{\sin(i)} P$$

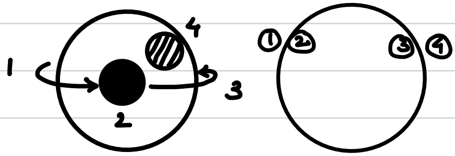
$$M_1 + M_2 = \frac{(K_1+K_2)^3 P}{2\pi G \times \sin^3(i)}$$

mass function: $\frac{M_2 \sin i}{(M_1+M_2)^{2/3}}$

for a singly-lined binary : $2\pi a_1 = \frac{K_1}{\sin(i)} P$

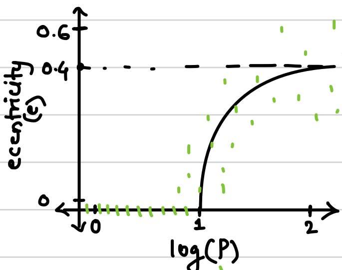
$$\frac{2\pi}{P}\left(\frac{M_2}{M_1+M_2}\right)^{a} \sin(i) = K_1 \Rightarrow 2\pi\left(\frac{G}{4\pi}\right)^{1/3} P^{-1/3} (M_1+M_2)^{-2/3} M_2 \sin(i) = K_1$$





$$R_1 = k_1 \frac{t_2 - t_1}{2}$$

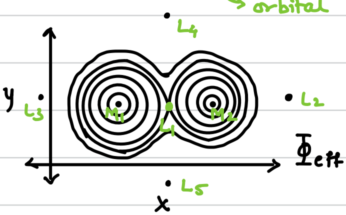
$$R_2 = k_2 \frac{t_3 - t_1}{2}$$



tidal evolution in binary stars.

Stars distort each other, thus tidal effects occur.

orbital period



$$\Phi_{eff}(\vec{r}) = \frac{-GM_1}{|\vec{r} - \vec{r}_1|} - \frac{GM_2}{|\vec{r} - \vec{r}_2|} - \frac{1}{2}|\vec{r} \times \Omega|^2$$

$$\vec{\alpha} = -\nabla \Phi_{eff}$$

Roche Lobe — As stars get closer, Roche Lobe starts to shrink

$$\frac{\dot{a}}{a} = -2 \frac{\dot{M}_2}{M_1}\left(1 - \frac{M_2}{M_1}\right)$$ change in mass because of mass transfer.

$$\frac{2\dot{P}}{P} = 3\frac{\dot{a}}{a}$$ change in time period.

$$R_{L,1} \approx 0.49\, a \left(\frac{M_1}{M_1 + M_2}\right)^{1/3}$$

$$\frac{\dot{R}_{L,1}}{R_{L,1}} = -2 \frac{\dot{M}_2}{M_2}\left(1 - \frac{5M_2}{6M_1}\right)$$

$\dot{R}_{L,1} < 0$ change is -ve.

$$\frac{M_2}{M_1} < \frac{6}{5}$$

Lecture 20:

very observationally driven categories

- Classical Nova

  $\Delta L \sim 10^4 - 10^5 L_\odot$

- Recurrent Nova (classical nova seen again and again).

  classical x 2

cataclysmic variables*

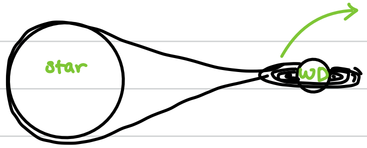
- Dwarf nova (mechanism for dwarf nova is not well understood as compared to classical / recurrent nova)

  $\Delta L \sim L_\odot$

- Super Nova

  $\Delta L = 10^{11} L_\odot$

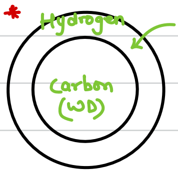
* Cataclysmic variables.



* During this process, the white dwarfs luminosity becomes the eddington luminosity.

$$L_{edd} = \frac{4\pi G M c}{\kappa}$$

opacity



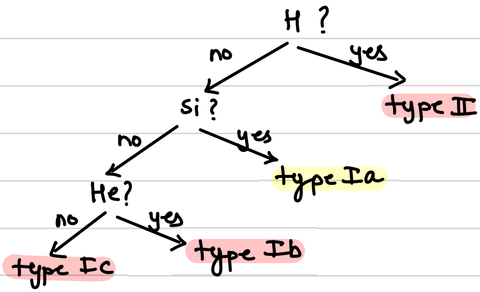
This process keeps happening till the star mass is over.

- Supernova: $L \sim 10^{11} L_\odot$, $F \sim 10^{51}$ ergs.

(one of the most luminous events that occur in the universe).

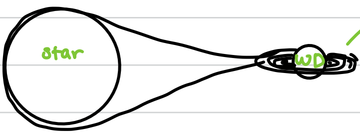
- Classify: based on spectral lines.

supernovae.



— high mass star core collapse.

— white dwarf explosion.



if the accretion disk/ mass is buiding up very slowly.

while working on the degenerate core, (Handout 24) eq 24.3.

$R \propto M^{-1/3}$
$\rho_c \propto M^2$ } core decreases as mass increases.

$\varepsilon_F \propto \rho_c^{2/3} \propto M^{4/3}$ → fermi energy increases as mass increases.

$\varepsilon_F >> k_B T$ (classical / non-relativistic limit). → $v(p) = \frac{p}{m_e}$

$\varepsilon_F >> m_e c^2$ (relativistic). → $v(p) = c$

electrons travelling have velocities closer to speed of light

for non-relativistic: $P_e = n_e k_B T$

for relativistic : $P_e = \left(\frac{3}{\pi}\right)^{1/3} \frac{hc}{8} n_e^{4/3}$

$\therefore \; P \propto \rho^{4/3}$ } electron pressure becomes total pressure.

also equation of state for $n=3$ polytrope.

in the relativistic case, electrons are degenerate.
(relativistic white dwarfs can only have one mass.)

$$M \propto R^0$$

$M_{chandrashekar} = 1.44 M_\odot$ } If mass more than this gets added, white dwarf collapses. (electron degeneracy pressure losses).

Heavier elements are formed when a white dwarf collapses:

$^{12}C(^{12}C, \alpha)^{20}Ne$
$^{16}O(^{16}O, \alpha)^{28}Si$ } fusion process is degenerate.

α - capture process:

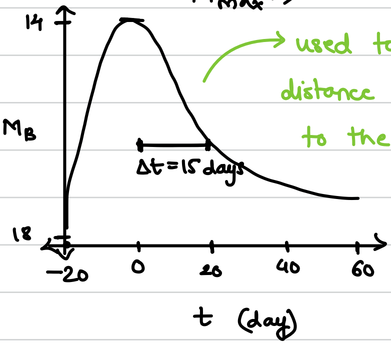
$^{26}Si \longrightarrow {}^{56}Ni$

dark energy theory comes from observing type Ia supernova.

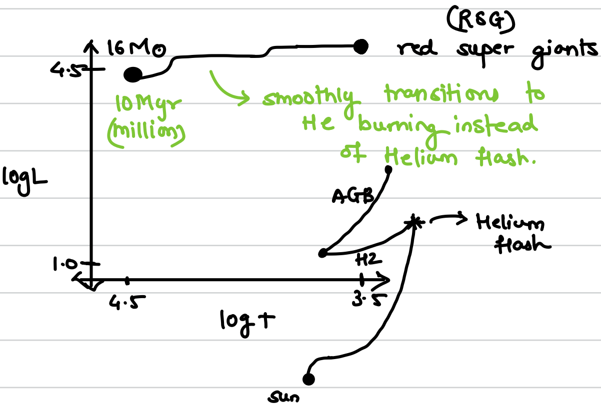
maximum absolute magnitude

$$M_{max}(B) = -21.726 + 2.698\,\Delta m_{15}(B)$$

Δm_{15}: change in the apparent magnitude.



used to measure distance from earth to the supernovae.



$H \to He \to C \to Ne \to O \to Si \to Fe$

11 Myr, 1.3 Myr, 2000 yrs, 3 yrs, 9 months, 1 week

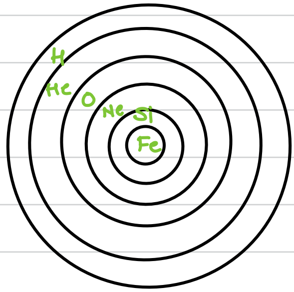
$^{20}Ne(\gamma, \alpha)^{16}O$

$^{20}Ne(\alpha, \gamma)^{24}Mg$

$^{16}O(^{16}O, \alpha)^{28}Si$

α-capture fusion: $^{28}Si \to ^{56}Ni$

$\alpha + ^{56}Ni \to ^{56}Fe$



Lecture 21:

As fusion is happening, the temperature in the core exceeds 10 billion K, and the iron in the core starts disintegrating.

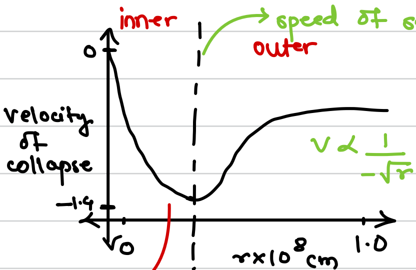
$^{56}Fe\,(\gamma, 4n)\,\beta\alpha$
$\alpha\,(\gamma, 2n)\,2p$
} matter of seconds.

$p\,(e^-, \nu_e)\,n$

core is mainly supported by radiation pressure, but that starts reducing as protons and electrons get fused to neutrons. Pressure that's supporting the core reduces.

Inner core collapses.

speed of sound (how long it takes for an astronomical object to realise what happened).

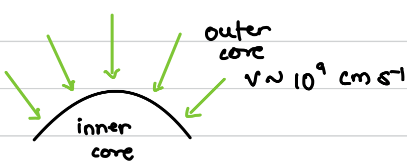


shrinks much faster than outer core.

inner part of core is kinda free falling with the speed of sound.

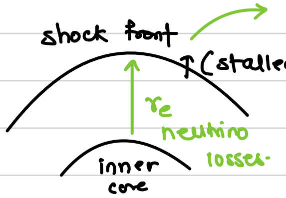
it is free falling as it collapses.

The inner core ends up becoming a neutron star. (becomes incompressible – neutron degeneracy pressure).



The outer core bounces off the inner core at relativistic velocities $v \sim 10^9$ cm s^{-1} and this causes the supernova explosion. There is a shock wave propogating outwards.

When you have a shock wave, the ideal gas equation of state doesn't work.
Shock wave stalling.
As outer core bounces off inner core, it starts compressing it more and inner core loses neutrinos. (neutrino losses.)



neutrinos are stopped and stored by the shock front. When this collapses, the neutrinos get released.

Energy release in this process:

$$\Delta\Omega = \frac{GM_c^2}{R_{c,i}} - \frac{GM_c^2}{R_{c,f}} \approx 10^{53}\ \text{erg}$$

(total)

$R_{c,i}$: 10^6 cm; $R_{c,f}$: 10^9 cm

energy given to neutrinos. (most of it). 1% energy approx given to energy of shock front and EM radiation.

$\approx 10^{51}$ erg (kinetic energy)

$\approx 10^{49}$ erg (electromagnetic radiation)

Type Ib → no hydrogen, but helium spectral lines.
Ic → no H and He, but
II → when you see hydrogen spectral lines.

Interior of a neutron star:

Neutron stars :

$M = 1.4 M_\odot$

$T = 10^{10}$ K

$R = 10$ km

depth :

ocean : 10 m H, He.

outer crust : 30 m crystallized protons and neutrons in a lattice
+ bunch of free electrons floating around.
ratio of p, n, e^- (started arranging themselves in a quantum mechanical state

inner crust : 60 m neutron drip density

outer core : 4 km lattice spacing $\sim R_{nuclei}$

inner core : 6 km everything starts breaking down quantum mechanically.
the ratio of neutron : proton : electron = 8 : 1 : 1
neutron start becoming relativistic.

center : 0 km neutrons moving at the speed of light.

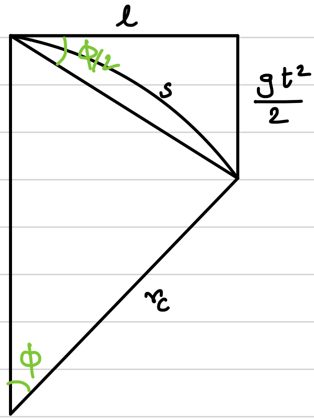
hydrostatic equilibrium, (magnetic effects, gravitational effects.
→ and rotating very fast. (as angular momentum (L) dramatically increases).

pulsars : rapidly rotating neutron stars.
(perfectly repeating radiations).
perfect atomic clocks, when callibrating clocks you compare it to pulsar frequencies.

X-ray binaries : neutron stars are usually found in binaries.
you always have an accretion disk in a binary system.

Lecture 22:

gravity alters the relative distance between two different points.



$$\frac{\phi}{2} = \frac{gt^2}{2\ell} \quad , \quad t = \frac{\ell}{c}$$

$$\phi = \frac{g\ell}{c^2}$$

$$\phi r_c = s = \ell$$

$$r_c = \frac{c^2}{g} = 9.17 \times 10^{17} \text{ cm}$$

generically, space-time distance $= (ds)$

$$ds^2 = \Bigg(c\,dt \underbrace{\sqrt{1 - \frac{2GM}{r_c^2}}}_{0 \text{ for } R_s}\Bigg)^2 - \Bigg(\frac{dr}{\underbrace{\sqrt{1 - \frac{2GM}{r_c^2}}}_{0 \text{ for } R_s}}\Bigg)^2 - (r\,d\theta)^2 - (r\sin\theta\, d\phi)^2$$

$d\theta = 0$ $\quad$ 90° $\quad$ $\omega\, dt$

$$\frac{v^2}{\cancel{r}} = \frac{GM}{r^{\cancel{2}}} \Rightarrow v = \sqrt{\frac{GM}{r}}$$

$$\therefore\ ds^2 = \left(c^2 - \frac{2GM}{r} - r^2\omega^2\right) dt^2$$

$$\Delta s = \int_0^{2\pi/\omega} \left(c^2 - \frac{2GM}{r} - r^2\omega^2\right)^{1/2} dt$$

$$\frac{d}{dr}(\Delta s) = \frac{d}{dr}\left(c^2 - \frac{2GM}{v} - r^2\omega^2\right)^{1/2} = 0$$

$$\frac{2GM}{r^2} - 2r\omega^2 = 0 \Rightarrow v = \sqrt{\frac{GM}{r}} = r\omega$$

$$d\tau = \frac{ds}{c} = dt\sqrt{1 - \frac{2GM}{rc^2}}$$

$$R_s = \frac{2GM}{c^2}$$ Schwarzschild radius.
that is the event horizon.

After you cross event horizon, $d\tau = 0$. Time on the clock stops.
For a beam of light, $ds = 0$.

$$\underbrace{\frac{dr}{dt}}_{\text{for a ray of light}} = c\left(1 - \frac{2GM}{rc^2}\right) = c\left(1 - \frac{R_s}{r}\right)$$

$\rightarrow v_{esc} = c$ (escape velocity)

$r < R_s$ } inside the event horizon
$ds^2 < 0$ $\rightarrow$ not supposed to happen in the normal universe.
"spacelike intervals". — means time-travel

Review Lecture:

Structure equations.

① Mechanical equations.

$$\frac{\partial r}{\partial m} = \frac{1}{\underbrace{4\pi r^2}_{\text{sphere}}\,\underbrace{\rho}_{\text{density}}}$$

how the mass goes with the radius of the star.

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4}$$

hydrostatic equilibrium equation

pressure ↑ , gravity ↓

$$\frac{\partial P}{\partial r} = \rho g$$

interior luminosity — how much energy in that radius shell.

② $$\frac{\partial l}{\partial m} = \epsilon_{nuc} - \epsilon_{\nu} - \underbrace{\frac{\partial u}{\partial t} + \frac{P}{\rho}\frac{\partial \rho}{\partial t}}$$

time-evolution of the star

per unit mass
how much efficiency
energy is released
by nuclear reaction

ϵ_{ν} : how much energy
is lost because of
neutrino losses.

$$\frac{\partial T}{\partial m} = \underbrace{-\frac{GmT}{4\pi r^4 P}}_{\text{bunch of constants.}} \nabla_T$$

how temperature changes with mass.

∇_T → how energy is transferred in a star. (derivative of temp)

too steep : convection

too shallow/ flat : radiation.

$$P = nk_BT = \frac{\rho k_B T}{\mu m_H}$$

pressure as a function of density and temperature and composition of a star.

μ = mean molecular weight

Till $10 M_\odot$, you can use the equation of state., but for higher mass stars, electrons get degenerate

$$P = P_{ion} + P_e + P_{rad} \qquad P_{rad} = \frac{aT^4}{3}$$

P_e → classical, degenerate.

$P_{e,\,degenerate} \propto \underbrace{n_e}_{\text{number density}}{}^{5/3}$ } Helium flash, white dwarfs explode

opacity, optical depth : per unit mass, how much scattering is there

$$\underset{\text{optical depth}}{\tau} = \int_{r_1}^{r_2} \kappa \rho \, dr \qquad \kappa = \frac{\sigma}{\mu m_H}$$

σ → cross sectional area

$$I_f = I_0 e^{-\tau}$$

I_f → intensity

} as you go down in the star, intensity decreases by a factor $e^{-\tau}$.

Blackbody radiation!

L - luminosity (erg s^{-1}) } total energy output of an object per unit time

f - flux (erg s^{-1} cm^{-2}) } observed flux, depends on the distance you are from the object

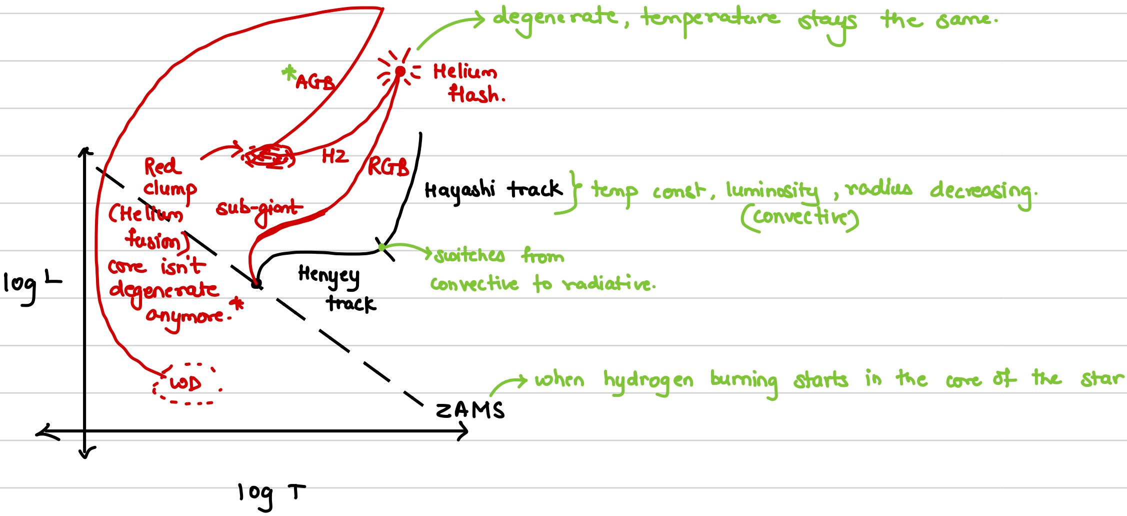
F - surface flux (erg s^{-1} cm^{-2}) } energy released per unit area at the surface of the star.

$$L = 4\pi r^2 F = 4\pi r^2 \sigma T^4$$

$$F = \sigma T^4$$ } Stefan - Boltzmann Law.

Wein's Law. } max wavelength emitted.

Polytropes : equations of state.
(pressure varying with density)
n = 0 } uniform density



Main sequence stops when core hydrogen runs out., you only get shell hydrogen burning.

* Helium fusing in the core, hydrogen fusing in the shell.

* AGB - first ascent - shell helium, shell hydrogen (they move out but they don't turn off). Core can't fuse carbon / oxygen anymore. The outer layers leave (planetary nebula).

For 30 $M_\odot$ stars, you have carbon fusion to oxygen, then more elements fusing, until iron (in the core). } going up and down the AGB.
Then you have the core bounce. : core collapse supernova.

White dwarf - goes above the Chandrashekar mass : another supernova.