Question ③ :

(a) we can write this system of equations as an agmented matrix and row reduce it to see if we find a solution or not.

$$\begin{array}{ccc} x_1 & x_2 & \\ \end{array}$$
$$\left[\begin{array}{cc|c} 3 & 1 & 7 \\ 2 & 5 & 5 \\ 1 & -1 & 3 \end{array}\right] \xrightarrow{3R_3 - R_1 = R_3} \left[\begin{array}{cc|c} 3 & 1 & 7 \\ 2 & 5 & 5 \\ 0 & -4 & 2 \end{array}\right] \xrightarrow{\frac{R_3}{3} = R_3}$$

$$\left[\begin{array}{cc|c} 3 & 1 & 7 \\ 2 & 5 & 5 \\ 0 & -4/3 & 2/3 \end{array}\right] \xrightarrow{\cancel{R_2 - 2R_1 = R_2}} \left[\begin{array}{ccc} \cancel{3} & \cancel{1} & \cancel{7} \\ & & \\ & & \end{array}\right] \xrightarrow{\frac{1}{3} \times R_1 = R_1}$$

$$\left[\begin{array}{cc|c} 1 & 1/3 & 7/3 \\ 2 & 5 & 5 \\ 0 & -4/3 & 2/3 \end{array}\right] \xrightarrow{R_2 - 2R_1 = R_2} \left[\begin{array}{cc|c} 1 & 1/3 & 7/3 \\ 0 & 13/3 & 1/3 \\ 0 & -4/3 & 2/3 \end{array}\right] \xrightarrow{R_2 \times \frac{3}{13} = R_2}$$

$$\left[\begin{array}{cc|c} 1 & 1/3 & 7/3 \\ 0 & 1 & 1/13 \\ 0 & -4/3 & 2/3 \end{array}\right] \xrightarrow{R_3 + \frac{4}{3} R_2 \Rightarrow R_3} \left[\begin{array}{cc|c} 0 & 1/3 & 7/3 \\ 0 & 1 & 1/13 \\ 0 & 0 & 10/13 \end{array}\right]$$

From the third row, we can see that there is a contradiction $10/13 \neq 0$. Thus, this system of equations has no solution. This situation is called as inconsistent system.

(b) Let's say "z" is the total absolute deviation:

objective function: minimize (z)

s.t.

$$3x_1 + x_2 - 7 \leq z$$
$$3x_1 + x_2 - 7 \geq -z$$
$$2x_1 + 5x_2 - 5 \leq z$$
$$2x_1 + 5x_2 - 5 \geq -z$$
$$x_1 - x_2 - 3 \leq z$$
$$x_1 - x_2 - 3 \geq -z$$

This will be the linear program we can use

(c) Let's introduce deviation variables: e_1, e_2, e_3 for each equation respectively.

objective: minimize $(e_1 + e_2 + e_3)$

s.t.

$$3x_1 + x_2 - 7 \leq e_1$$
$$3x_1 + x_2 - 7 \geq -e_1$$
$$2x_1 + 5x_2 - 5 \leq e_2$$
$$2x_1 + 5x_2 - 5 \geq -e_2$$
$$x_1 - x_2 - 3 \leq e_3$$
$$x_1 - x_2 - 3 \geq -e_3$$
$$e_1, e_2, e_3 \leq 5$$

(d) Minimize the 2-norm of the violation:

objective: minimize $(e_1^2 + e_2^2 + e_3^2)$

s.t.

$$3x_1 + x_2 - 7 \le e_1$$
$$3x_1 + x_2 - 7 \ge -e_1$$
$$2x_1 + 5x_2 - 5 \le e_2$$
$$2x_1 + 5x_2 - 5 \ge -e_2$$
$$x_1 - x_2 - 3 \le e_3$$
$$x_1 - x_2 - 3 \ge -e_3$$

This type of problem is a quadratic problem because the objective function is quadratic.

(e) To convert this to linear equalities, we introduce auxiliary variables:

objective: minimize $(u_1 + u_2 + u_3)$

s.t.

$$u_1 \ge e_1^2, \quad u_2 \ge e_2^2, \quad u_3 \ge e_3^2$$
$$3x_1 + x_2 - 7 \le e_1$$
$$3x_1 + x_2 - 7 \ge -e_1$$
$$2x_1 + 5x_2 - 5 \le e_2$$
$$2x_1 + 5x_2 - 5 \ge -e_2$$
$$x_1 - x_2 - 3 \le e_3$$
$$x_1 - x_2 - 3 \ge -e_3$$

This will give us the optimal solution to part (d).