Homework 3

July 5, 2024

1 Homework #3 (4 points)

1.1 Submission requirements

Upload a **single PDF or HTML file** of your IJulia notebook for this entire assigment. Do not submit an .ipynb file. Clearly denote which question each section of your file corresponds to.

1.2 Problem 1 - Duality Practice

Consider the following primal linear program:

$$\begin{array}{lrrrrr} \max z = & 7x_1 - & 3x_2 + & 2x_3 + & x_4 & \\ \text{s.t.} & x_1 - & x_2 & & & = 5 \\ & & 4x_2 + & 6x_3 & & \leq 9 \\ & x_1 & - & x_3 - & 2x_4 & \leq 4 \\ & x_1 \geq 0, & x_2 \text{ Free }, & x_3 \geq 0, & x_4 \leq 0 & \end{array}$$

Write the dual of this linear program.

1.3 Problem 2 - Max Flow Formulation

Delta Airlines has four different types of planes in its fleet labeled A, B, C, and D. There are 4 different types of routes these planes can take, labeled $R1, R2, R3, R4$. Each type of plane can be assigned to one route type per day. Type A planes can fly any of the routes. Type B planes can fly on routes $R1$, $R3$, and $R4$. Type C planes can fly on routes $R2$ and $R3$. Finally, type D planes can only fly route $R3$. The required total numbers of each route to be flown per day are given in the table below:

Route Category (j)	Required # routes (D_j)
$R1$	9
$R2$	12
$R3$	10
$R4$	15

Delta isn't worried about cost, only about the total number of routes it is feasible to fly under this setup. The available number of each type of plane are listed in the table below.

Plane Type (i)	Number available (M_i)
A	25
B	17
C	12
D	9

Formulate a **maximum flow problem** that will help determine which types planes should fly which routes in order to cover all the required routes, if possible. Draw the network and label the capacities of the arcs. Implement this model in Julia/JuMP and determine if it is possible for Delta to cover the routes with the current fleet.

1.4 Problem 3 - Craft Brewing Duality

Our ever-present local craft brewery is conntinuing to optimize their production processes. Currently, they are producing 4 beers for summer. The profit contribution (per keg), labor hours, and vat usage (in hours) per keg for each type of beer are given below:

Beer	Profit (\$)	Labor (Hr.)	Vat (Hr.)
1	60	2	5
2	120	3	5
3	200	3	10
4	300	5	15

A maximum of 12,000 labor hours and 32,000 vat hours are available this summer. Each keg of beer 1 spends an average of 0.25 months in inventory; beer 2, an average of 1 month; beer 3, an average of 2 months; beer 4, an average of 3.5 months. The brewery's warehouse can handle an average inventory level of 5,000 kegs. Determine how much of each type of beer should be produced this year to maximize profit.

A linear program model for this problem is:

$$
\begin{aligned}
\max z = \quad & 60x_1 + & 120x_2 + & & 200x_3 + & & 300x_4 & \\
\text{s.t.} \quad & 2x_1 + & 3x_2 + & & 3x_3 + & & 5x_4 & \leq 12000 \\
& 5x_1 + & 5x_2 + & & 10x_3 + & & 15x_4 & \leq 32000 \\
& 0.25x_1 + & x_2 + & & 2x_3 + & & 3.5x_4 & \leq 5000 \\
& x_1 \geq 0, & x_2 \geq 0, & & x_3 \geq 0, & & x_4 \geq 0 &
\end{aligned}
$$

where x_j is the kegs of beer j to produce this year.

In the sensitivity analysis questions below, answer each question independnetly – e.g., when answering part (c), consider only the changes suggested in part (c), not those in addition to the ones considered in part (b).

(a) Solve this model in Julia/JuMP and write the optimal primal and dual solutions, and the optimal objective function value. (To obtain the dual solution associated with a constraint called "c1" in JuMP, use the command "dual(c1)")

(b) What is the maximum amount the brewery should be willing to pay for an additional hour of labor?

(c) The marketing department is considering running an advertising campaign that would increase the profit per keg of beer 2 by \$10 and beer 3 by \$15. Provide an estimate of the new optimal profit (z^*_{NEW}) after this change and indicate if your estimate is a lower or upper bound on the new optimal profit. Your answer should be of the form $z^*_{NEW} \geq$ ____ or $z^*_{NEW} \leq$ ____, where you fill in a number in the blank.

(d) The brewery is considering a facility redesign that would decrease the labor availability by 1000 hours, but increase the vat availabity by 4000 hours. Provide an estimate of the new optimal profit (z^*_{NEW}) after this change and indicate if your estimate is a lower or upper bound on the new optimal profit. Your answer should be of the form $z^*_{NEW} \geq$ ____ or $z^*_{NEW} \leq$ ____, where you fill in a number in the blank.

(e) The brewery is considering a new beer that they would sell to Der Rathskeller for profit per keg of \$400. This beer would require 10 labor hours per keg and on average each keg would use 4 vat hours. Let T denote the average time a keg of the new beer spends in inventory. Provide a bound on the value of T in order for this new beer to potentially be profitable. Your answer should be of the form $T \geq$ ____ or $T \leq$ ____, where you fill in a number in the blank. (You probably won't be able to find a closed-form solution to this question; it's okay to "guess and check" to estimate bounds on T.)

1.5 Problem 4 - A graphical approach to duality

Consider the following linear program:

$$\begin{aligned} \max\ & x_1 - x_2 && (1)\\ s.t.\ & x_1 + 3x_2 \leq 6 && (2)\\ & 2x_1 + 4x_2 \leq 5 && (3)\\ & x_1, x_2 \geq 0 && (4) \end{aligned}$$

(a) Write the dual of this linear program.

(b) Is an objective value of -1/2, $x_1 = 1/2$, and $x_2 = 1$ an optimal solution to this problem? Justify your answer with appropriate theory.

(c) Use the graphical method of solving LPs to determine the range of values of the right-hand side of constraint 1 for which the optimal **dual** solution remains optimal.

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