Assignment 1 – Discrete Review

Answer the questions in the boxes provided on the question sheets. If you run out of room for an answer, add a page to the end of the document.

Related Readings: http://pages.cs.wisc.edu/~hasti/cs240/readings/

Name: Riya Kore Wisc id: rykore

Logic

1. Using a truth table, show the equivalence of the following statements.

(a) $P \vee (\neg P \wedge Q) \equiv P \vee Q$

Solution:

P	Q	$\neg P$	$\neg P \wedge Q$	$P \vee (\neg P \wedge Q)$	$P \vee Q$
T	T	F	F	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	F	T	F	F	F

column 5 $\equiv$ column 6

Hence,

$P \vee (\neg P \wedge Q) \equiv P \vee Q$

(b) $\neg P \vee \neg Q \equiv \neg(P \wedge Q)$

Solution:

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$
T	T	F	F	F	T	F
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

column 5 $\equiv$ column 7

Hence,

$\neg P \vee \neg Q \equiv \neg(P \wedge Q)$

(c) $\neg P \vee P \equiv \text{true}$

Solution:

P	$\neg P$	$\neg P \vee P$
T	F	T
F	T	T

The final column is all truth values.

Hence,

$\neg P \vee P \equiv \text{true}$.

(d) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

Solution:

P	Q	R	$P \vee Q$	$P \vee R$	$Q \wedge R$	$P \vee (Q \wedge R)$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F
F	F	T	F	T	F	F	F
F	F	F	F	F	F	F	F

column 7 $\equiv$ column 8

$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

Sets

2. Based on the definitions of the sets A and B, calculate the following: $|A|$, $|B|$, $A \cup B$, $A \cap B$, $A \setminus B$, $B \setminus A$.

   (a) $A = \{1, 2, 6, 10\}$ and $B = \{2, 4, 9, 10\}$

   Solution:
   $|A| = 4$
   $|B| = 4$
   $A \cup B = \{1, 2, 4, 6, 9, 10\}$
   $A \cap B = \{2, 10\}$
   $A \setminus B = \{1, 6\}$
   $B \setminus A = \{4, 9\}$

   (b) $A = \{x \mid x \in \mathbb{N}\}$ and $B = \{x \in \mathbb{N} \mid x \text{ is even}\}$

   Solution:
   $|A| =$ infinite
   $|B| =$ infinite
   $A \cup B = \{x \mid x \in \mathbb{N}\} \approx A$
   $A \cap B = \{x \in \mathbb{N} \mid x \text{ is even}\} \approx B$
   $A \setminus B = \{x \in \mathbb{N} \mid x \text{ is odd}\}$
   $B \setminus A = \{\phi\}$

Relations and Functions

3. For each of the following relations, indicate if it is reflexive, antireflexive, symmetric, antisymmetric, or transitive.

   (a) $\{(x, y) : x \leq y\}$

   Solution:
   reflexive, antisymmetric, transitive.

   (b) $\{(x, y) : x > y\}$

   Solution:
   antireflexive, anticsymmetric, transitive

(c) $\{(x, y) : x < y\}$

Solution:
transitive, antireflexive, antisymmetric

(d) $\{(x, y) : x = y\}$

Solution:
reflexive, symmetric, transitive

4. For each of the following functions (assume that they are all $f : \mathbb{Z} \to \mathbb{Z}$), indicate if it is surjective (onto), injective (one-to-one), or bijective.

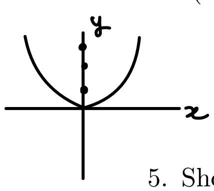
(a) $f(x) = x$

Solution:
bijective

(b) $f(x) = 2x - 3$

Solution:
injective

(c) $f(x) = x^2$



Solution:
none.

5. Show that $h(x) = g(f(x))$ is a bijection if $g(x)$ and $f(x)$ are bijections.

Solution:
To prove $h(x)$ is a bijection, you need $h(x)$ to be injective and surjective.
for proving injection: we know $f(x)$ is bijective, which means $f(x)$ is injective. If $f(a) = f(b)$ then $a = b$. So, every value corresponds to a unique value.
$g(f(a)) = g(f(b))$, so $h(a) = h(b)$, so $h(x)$ is injective.

for proving surjection: since $f(x)$ and $g(x)$ are bijections, the size of the domains and ranges are the same. This implies that their composition $h(x) = g(f(x))$ is also surjective.

$h(x)$ has been proved injective and surjective. Hence, we have proved that $h(x) = g(f(x))$ is bijective.

Induction

6. Prove the following by induction.

(a) $\sum_{i=1}^{n} i = n(n+1)/2$

Solution:

Base case: $n=1$

$\therefore \frac{n(n+1)}{2} = \frac{1\times 2}{2} = 1$

Inductive hypothesis: Assume $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

Inductive step: for $k+1$,

$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)$

$\frac{k(k+1)}{2} + k+1 = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$

(b) $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$

Solution:

Base case: $n=1$ $\therefore \frac{n(n+1)(2n+1)}{6} = \frac{1\times 2\times 3}{6} = 1$

Inductive hypothesis:

Assume $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$

Inductive step: for $k+1$ $6k^2+12k+6$

$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$

$\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = \frac{(k+1)(2k^2+k+6k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$

(c) $\sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$

Solution:

Base case: $n=1$ $\therefore \frac{n^2(n+1)^2}{4} = \frac{1\times 4}{4} = 1$

Inductive hypothesis: Lets assume, $\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$

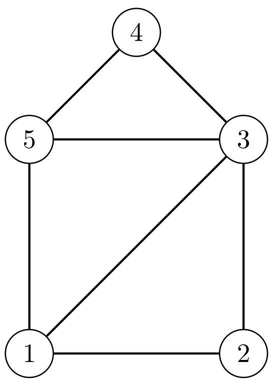
Inductive step: For $k+1$,

$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3$

$\frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2 + 4(k+1)^3}{4} = \frac{(k+1)^2(k^2+4k+4)}{4} = \frac{(k+1)^2(k+2)^2}{4}$

Graphs and Trees

7. Give the adjacency matrix, adjacency list, edge list, and incidence matrix for the following graph.



Solution: Adjacency Matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Adjacency List:

$$\begin{pmatrix} (2,3,5) \\ (1,3) \\ (1,2,4,5) \\ (3,5) \\ (1,3,4) \end{pmatrix}$$

Edge List:

$((1,2), (1,3), (1,5), (2,3), (3,4), (3,5), (4,5))$

Incidence Matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

8. How many edges are there is a complete graph of size n? Prove by induction.

Solution:

In a complete graph, there are $h(n) = \frac{n(n-1)}{2}$ edges.

you perform induction on the number of nodes.

Base case:

$n = 1 \quad \therefore h(1) = 0.$

Inductive hypothesis:

Lets assume, K_i has $\frac{i(i-1)}{2}$ edges.

Inductive step:

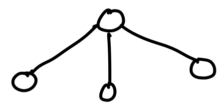
when you add one more node to the complete graph, the edges become K_{i+1} adds i edges.

$$h(i+1) = \frac{i(i-1)}{2} + i = \frac{i(i-1)+2i}{2} = \frac{i((i-1)+2)}{2} = \frac{(i+1) \times i}{2}$$

9. Draw all possible (unlabelled) trees with 4 nodes.

Solution:

① 

② 

10. Show by induction that, for all trees, $|E| = |V| - 1$.

edges = vertices − 1.

Solution: To show $|E| = |V| - 1$ for all trees.

Base step:

for 1 vertex: no. of edges = 0 $|E| = 1 - 1 = 0$ ∴ proved.

Inductive hypothesis:

assuming true for $|V| = k$, so $|E| = k - 1$

Inductive step:

for $|V| = k+1$. , the formula says that $|E| = k$

From the inductive hypothesis : $|E| = k - 1$ for k vertices but when you add another vertex, the edges should increase by 1. A tree is a connected and an acyclic graph. So, when you add a leaf node, it can have atmost one edge.

$|E| = (k+1) - 1 = k$, which is in correspondence to our formula.

Hence, proved. $= |V| - 1$

Counting

11. How many 3 digit pin codes are there?

Solution:
10 10 10 total number of pin codes $= 10 \times 10 \times 10 = 10^3$

12. What is the expression for the sum of the ith line (indexing starts at 1) of the following:

1
2 3
4 5 6
7 8 9 10
⋮

Solution:
Let the last value in line $(i-1)$ be x.

$$x = \sum_{n=1}^{i-1} n = \frac{i(i+1)}{2} - i = \frac{i(i-1)}{2}$$

The sum of line i:

$$\sum_{K=1}^{i} (x+K) = ix + \frac{i(i+1)}{2} = \frac{i^2(i-1)}{2} + \frac{i(i+1)}{2} = \frac{i^3+i}{2}$$

13. A standard deck of 52 cards has 4 suits, and each suit has card number 1 (ace) to 10, a jack, a queen, and a king. A standard poker hand has 5 cards. For the following, how many ways can the described hand be drawn from a standard deck.

(a) A royal flush: all 5 cards have the same suit and are 10, jack, queen, king, ace.

Solution: diamond, hearts, clubs, spades
$1 + 1 + 1 + 1 = \boxed{4}$ (one for each suit, and the deck has 4 suits)

(b) A straight flush: all 5 cards have the same suit and are in sequence, but not a royal flush.

Solution:
$\underbrace{4}_{\text{suits}} \times \underbrace{10}_{\text{possibilities}} - \underbrace{4}_{\text{royal flush}} = 40 - 4 = 36$ hands of the straight flush.

(c) A flush: all 5 cards have the same suit, but not a royal or straight flush.

Solution:
$\underbrace{4}_{\text{suits}} \times \underbrace{{}^{13}C_5}_{\text{choosing 5 from 13}} - \underbrace{40}_{\text{straight and royal flushes}} = 4 \times \frac{13!}{8!\,5!} - 40 = 5108$

(d) Only one pair (2 of the 5 cards have the same number/rank, while the remaining 3 cards all have different numbers/ranks):

Solution:
$13 \times {}^4C_2 \times {}^{12}C_3 \times 4^3 = 13 \times \frac{4 \times 3}{2 \times 1} \times \frac{12 \times 11 \times 10}{3 \times 2 \times 1} \times 4 \times 4 \times 4 = 2{,}196{,}480$

Proofs

14. Show that $2x$ is even for all $x \in \mathbb{N}$.

(a) By direct proof.

Solution:
For direct proof, $p \to q$.
An even number is divisible by 2, and $2x$ is divisible by 2.

(b) By contradiction.

Solution:
Lets assume that $2x$ is odd.
This means that $2x$ cannot be perfectly divided by 2.
However, $\frac{2x}{2} = x$, and $x \in \mathbb{N}$.

So, our assumption was false.
Hence, $2x$ is even.

15. For all $x, y \in \mathbb{R}$, show that $|x + y| \leq |x| + |y|$. (Hint: use proof by cases.)

Solution:
Case 1: If $x \geq 0, y \geq 0$, then $|x+y| = x+y$ and $|x| + |y| = x+y$. Since, $x+y = x+y$, it follows that $|x+y| \leq |x| + |y|$

Case 2: If $x<0$ and $y<0$, then $|x+y| = -(x+y)$ and $|x|+|y| = -x-y$
Since, $-(x+y) \leq -x-y$, it follows that $|x+y| \leq |x| + |y|$

Case 3: If $x \geq 0$ and $y<0$, then $|x+y| = x-y$ and $|x| + |y| = x + (-y)$
Since $x-y \leq x+(-y)$, it follows that $|x+y| \leq |x| + |y|$

Case 4: If $x<0$ and $y \geq 0$, then $|x+y| = y-x$ and $|x|+|y| = (-x)+y$
Since $y-x \leq (-x)+y$, it follows that $|x+y| \leq |x| + |y|$

In all the cases, we have shown that $|x+y| \leq |x| + |y|$. Thus, the statement is true for all $x, y \in \mathbb{R}$.

Program Correctness (and Invariants)

sound, complete,

16. For the following algorithms, describe the loop invariant(s) and prove that they are sound and complete.

(a)

Algorithm 1: findMin

Input: a: A non-empty array of integers (indexed starting at 1)
Output: The smallest element in the array
begin
    $min \leftarrow \infty$
    for $i \leftarrow 1$ **to** $len(a)$ **do**
        if $a[i] < min$ **then**
            $min \leftarrow a[i]$
        end
    end
    return min
end

Solution:

Invariant:
when step i of the loop ends, "min" contains the minimum value from index 1 to i of a. You can prove this by performing induction on the iterations.

First iteration becomes the base case: min = a[1]

Inductive hypothesis: Assume that min is the minimum value of a[1, ..., k] after the Kth iteration.

Inductive step: After the (k+1)st iteration, min will be the minimum of min and a[k+1]. Hence, min becomes the minimum of a[1, ..., k+1].

Soundness: From the loop invariant, after completing the step equal to the length of the input array, min will contain the minimum element.

Completeness: With each iteration of the loop, the counter i moves closer to the end of the input array, and the loop ends when it reaches the array's last element.

(b)

Algorithm 2: InsertionSort

Input: a: A non-empty array of integers (indexed starting at 1)
Output: a sorted from largest to smallest
begin
  for $i \leftarrow 2$ **to** $len(a)$ **do**
    $val \leftarrow a[i]$
    for $j \leftarrow 1$ **to** $i-1$ **do**
      if $val > a[j]$ **then**
        shift $a[j..i-1]$ to $a[j+1..i]$
        $a[j] \leftarrow val$
        break
      end
    end
  end
  return a
end

Solution:

Completeness:
For each iteration of the loop, the outer loop counter 'i' moves one step closer to the end of the input array. After reaching the last element in the array, the outer loop ends. When it comes to the inner loop, it goes from 1 to i and will terminate when the outer loop does.

Invariant: At the end of step i of the outer loop, the items in the array are arranged from largest to smallest.
You can prove this using induction:
Base case: a[1] is sorted after the first iteration of the loop.
Inductive hypothesis: Let's assume that a[1,...,K] become sorted after the Kth iteration.
Inductive step: when the (K+1)st iteration happens, a[K+1] will be placed at a[n] after sorting a[1,.. ,n-1] elements greater than a[K+1]. The sorted a[n,K] elements less than a[K+1] are shifted to a[n+1,K+1].
Therefore, a[1,. . .,K+1] is sorted.

Soundness:
From the loop invariant, after finishing the step equal to the length of the input array, a will be sorted from largest to smallest.

Recurrences

17. Solve the following recurrences.
    (a) $c_0 = 1; c_n = c_{n-1} + 4$

Solution:

$c_0 = 1$
$c_1 = c_0 + 4 = 1+4 = 5 = 4(1)+1$
$c_2 = c_1 + 4 = 5+4 = 9 = 4(2)+1$
$c_3 = c_2 + 4 = 9+4 = 13 = 4(3)+1$

$\therefore c_n = 4(n)+1$

$\boxed{c_n = 4n+1}$

    (b) $d_0 = 4; d_n = 3 \cdot d_{n-1}$

Solution:

$d_0 = 4$
$d_1 = 3 \times d_0 = 3\times 4 = 12 = (3)^1 \times 4$
$d_2 = 3 \times d_1 = 3 \times 12 = 36 = (3)^2 \times 4$
$d_3 = 3 \times d_2 = 3 \times 36 = 108 = (3)^3 \times 4$

$\boxed{d_n = (3)^n \times 4}$

(c) $T(1) = 1; T(n) = 2T(n/2) + n$ — eq ① (An upper bound is sufficient.)

Solution:

Lets say we have:

$T(n/2) = 2T(n/4) + n/2$ - eq 2

$T(n/4) = 2T(n/8) + n/4$ - eq 3

$$T(n) = 2[2T(n/4) + n/2] + n$$
$$= 2^2 T(n/2^2) + 2n$$
$$\hookrightarrow 2^2 [2T(n/8) + n/4] + 2n$$
$$= 2^3 T(n/2^3) + 3n$$
$$\vdots \text{ k times}$$
$$2^k T(n/2^k) + kn$$

Here, you assume $\frac{n}{2^k} = 1 \Rightarrow n = 2^k$

$\log n = \log 2^k$

$\log_2 n = k$

$\therefore 2^k T(1) + kn = n \times 1 + \log_2 n \times n = n + n \times \log_2 n \approx \boxed{O(n \log_2 n)}$ upper bound.

since $n \log_2 n$ is dominating and not n.

(d) $f(1) = 1; f(n) = \sum_1^{n-1} (i \cdot f(i))$
(Hint: compute $f(n+1) - f(n)$ for $n > 1$)

Solution:

$$f(n+1) - f(n) = \sum_1^{n} (i \cdot f(i)) - \sum_1^{n-1} (i \cdot f(i))$$
$$= (n) \cdot f(n)$$
$$f(n+1) = (n+1) f(n)$$

$n+1 > 1$, Hence,

$$f(n) = n\, f(n-1), \quad \text{for } n > 2$$
$$= n.(n-1) \ldots 3 \times f(2)$$
$$= \frac{n!}{2} \times 1 \times f(1)$$
$$= \frac{n!}{2}$$

$f(2) = 1 \times f(1) = 1 = \frac{2!}{2}$. Therefore,

$$f(n) = \begin{cases} 1, & \text{when } n = 1 \\ \frac{n!}{2}, & \text{otherwise} \end{cases}$$

Coding Question: Hello World

Most assignments will have a coding question. You can code in C, C++, C#, Java, Python, or Rust. You will submit a Makefile and a source code file.

Makefile: In the Makefile, there needs to be a build command and a run command. Below is a sample Makefile for a C++ program. You will find this Makefile in assignment details. Download the sample Makefile and edit it for your chosen programming language and code.

```
#Build commands to copy:
#Replace g++ -o HelloWorld HelloWord.cpp below with the appropriate command.
#Java:
#       javac source_file.java
#Python:
#       echo "Nothing to compile."
#C#:
#       mcs -out:exec_name source_file.cs
#C:
#       gcc -o exec_name source_file.c
#C++:
#       g++ -o exec_name source_file.cpp
#Rust:
#       rustc source_file.rs

build:
        g++ -o HelloWorld HelloWord.cpp

#Run commands to copy:
#Replace ./HelloWorld below with the appropriate command.
#Java:
#       java source_file
#Python 3:
#       python3 source_file.py
#C#:
#       mono exec_name
#C/C++:
#       ./exec_name
#Rust:
#       ./source_file

run:
        ./HelloWorld
```

18. **HelloWorld Program Details**

The input will start with a positive integer, giving the number of instances that follow. For each instance, there will be a string. For each string s, the program should output Hello, s! on its own line.

A sample input is the following:

```
3
World
Marc
Owen
```

The output for the sample input should be the following:

```
Hello, World!
Hello, Marc!
Hello, Owen!
```