Lecture @ 4 ①

Example 3! For the given table construct
(i) CDF, graph (plot)
(ii) PDF, graph. (plot)

$X=\{x_i\}$	0	1	2	3	4
$P(x_i)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$

Solution!

ⓐ $F_X(0) = \frac{1}{8}$

$F_X(0.6) = P(x \le 0.6) = \frac{1}{8}$

$F_X(0.9) = P(x \le 0.9) = \frac{1}{8}$

$F_X(1) = P(x \le 1) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

$F_X(1.5) = \frac{1}{4}$

$F_X(2) = P(x \le 2) = \frac{1}{8} + \frac{1}{8} + \frac{1}{2} = \frac{1}{4} + \frac{1}{2}$
$= 3/4$

$F_X(2.5) = 3/4$

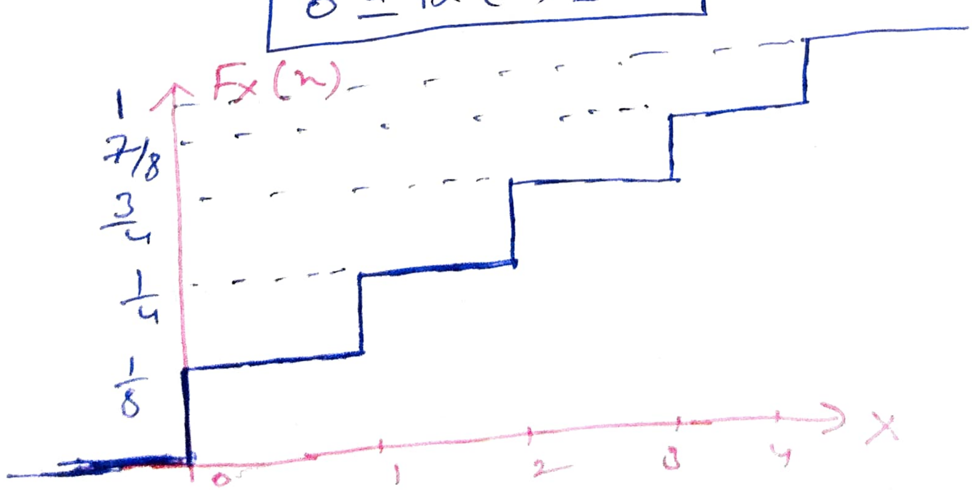
$F_X(3) = P(x \le 3) = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$

$F_X(4) = P(x \le 4) = \frac{7}{8} + \frac{1}{8} = \frac{8}{8} = 1$

⋮

$F_X(\infty) = P(x \le \infty) = 1$

$$\boxed{0 \le F_X(x) \le 1}$$

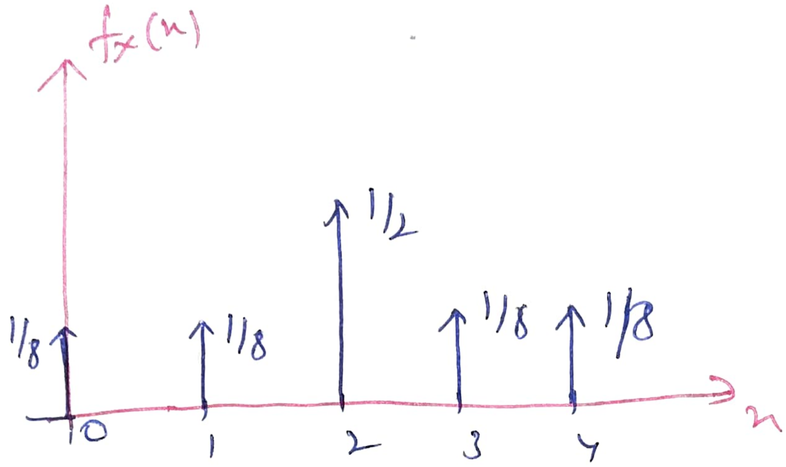


we know that

$$f_X(x) = \frac{d(F_X(x))}{dx}$$

$$\Rightarrow f_X(x) = \frac{1}{8}\delta(x) + \frac{1}{8}\delta(x-1) + \frac{1}{2}\delta(x-2) + \frac{1}{8}\delta(x-3) + \frac{1}{8}\delta(x-4)$$

$\Rightarrow$



Probability Mass Function (PMF)

$\Rightarrow$ Probability Mass function (PMF) specifies probability of Random Variable 'X' taking each of its possible values.

Mathematically:

$$\boxed{P_X(x_i) = P(x = x_i)}$$

Example: Construct PMF for a descrete Random Variable 'X', which specifies number of possible heads in the experiment of tossing a coin twice.

Solution: Since $N = 2$

$\Rightarrow S = 2^2 = 4$

$\Rightarrow S = [HH, HT, TH, TT]$

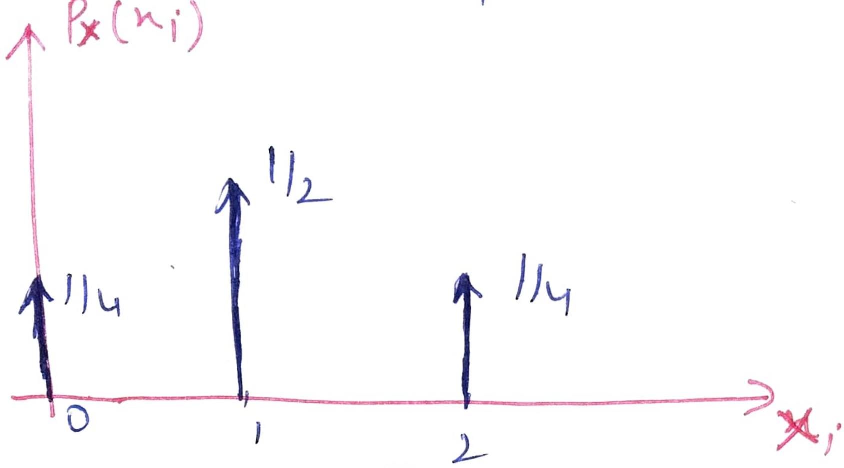
$S = [2, 1, 1, 0]$

$\Rightarrow$ $P_X(0) = P(X=0) = \frac{1}{4}$

$P_X(1) = P(X=1) = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$

$P_X(2) = P(X=2) = \frac{1}{4}$

$\Rightarrow$



Note! $\boxed{\sum_i P_X(x_i) = 1}$

$\Rightarrow$ Statistical averages of Random Variable :-

(i). Expectation / Mean / First Moments.

(ii) Mean square value / Second Moments.

(iii) Variance / Second Central moment

(i) Expectation / Mean / First Moments !

$\Rightarrow$. Expectation is a mathematical operator which is used to calculate ~~the~~ mean value of a random variable 'X'. It is denoted by capital 'E'.

i.e. $E[X]$ = Expectation of 'X'

or

Mean of 'X'

or

Statistical average of 'X'

(5) square value

i.e.

(4)

$$\boxed{\text{Mean}[X] = E[X] = \overline{X} = m_1} = \mu_1$$

average of 'x' First moment

⇒. In general, the expected value of any random variable 'x' is defined by:

$$\boxed{E[X] = \overline{X} = \int_{-\infty}^{\infty} x\, f_X(x)\, dx}$$ --- For Continuous Random Variable.

$$\boxed{E[X] = \sum_i x_i\, P_X(x_i)}$$ ---- For Discrete Random variable.

(iii). Note! Mean gives D.C. or Average Value of Random Variable.

m_1 = D.C. value.

m_1^2 = D.C. power.

(ii). Mean square value:

$$MSQ[X] = E[X^2] = \overline{X^2} = m_2 = \mu_2$$

⇒ $$\boxed{E[X^2] = \sum_i x_i^2\, P_X(x_i)}$$ ---- For DRV

$$\boxed{E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx}$$ ---- For CRV

Note! Mean square value gives Total power of Random Variable 'X'

(iii) Variance (σ_x^2) :- Mean square value centered around mean value ($\bar{X}$) is known as variance of Random variable $\bar{X}$, it is denoted by σ_x^2 :

i.e. $\sigma_x^2 = MS[X - \bar{X}]$

$= E[(X - M_1)^2]$

$= E[(X - M_1)^2]$

$$\boxed{\sigma_x^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 f_X(x)\, dx}$$ --- For CRV

$$\boxed{\sigma_x^2 = \sum_i (x_i - \bar{x})^2 P_i(x)}$$ --- For DRV

$\Rightarrow$ since $\sigma_x^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 f_X(x)\, dx$

$$= \int_{-\infty}^{\infty} (x - M_1)^2 f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x^2 + M_1^2 - 2xM_1) f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx + M_1^2 \int_{-\infty}^{\infty} f_X(x)\, dx - 2M_1 \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$\Rightarrow \sigma_x^2 = E[X^2] + M_1^2 - 2M_1^2$

$\Rightarrow$ $$\boxed{\sigma_x^2 = E[x^2] - M_1^2}$$ --- V.Imp.

$\Rightarrow \sigma^2 = E[x^2] - \{E[x]\}^2$

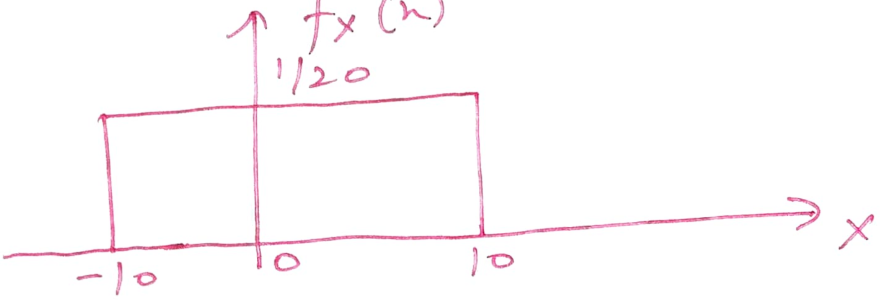
$\Rightarrow \sigma^2 =$ Total power $-$ D.C. power

$\Rightarrow \boxed{\sigma^2 = \text{AC power}}$

Note: Standard deviation (σ_x) (S.D.)
square root of Variance is known as
standard deviation.

$\Rightarrow \boxed{\sigma_x = S.D = \sqrt{\text{Variance}} = \sqrt{\text{A.C. power}}}$

Example: A R.V. is uniformly distributed in $(-10, 10)$. Find all stastical averages of Random variable:



Solution: (i) Mean (M_1 or $\overline{X}$ or $E[X]$) :
or m_1

we know that
$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$\Rightarrow E[X] = \int_{-10}^{10} x f_X(x)\, dx$$

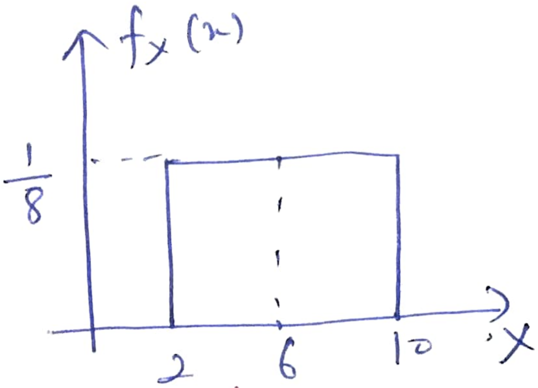
$$= \int_{-10}^{10} \frac{1}{20} \cdot x\, dx = \frac{1}{20}\left[\frac{x^2}{2}\right]_{-10}^{10}$$

$$= \frac{1}{40}[100-100]$$

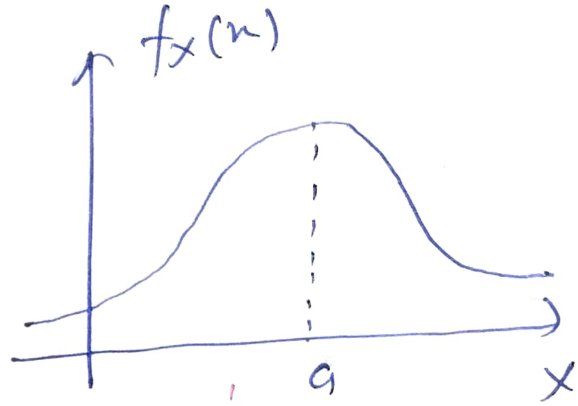
$=$

$\Rightarrow$ ~~E[X]~~ $\boxed{E[\bar{X}] = 0}$ Ans

Note: If the density function is symmetrical about x-axis, then mean will be equal to symmetrical points:



$E[\bar{X}] = 6$



$E[\bar{X}] = 9$

(ii) MSQ or second moment :-

we know that

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

$$= \int_{-10}^{10} x^2 \cdot \frac{1}{20}\,dx$$

$$E[X^2] = \frac{1}{20}\left[\frac{x^3}{3}\right]_{-10}^{10} = \frac{100}{3} \quad \text{Ans}$$

(iii). Variance (σ_x^2)

we know that

$$\sigma_x^2 = E[X^2] - E[X]$$

$$\Rightarrow \sigma_x^2 = \frac{100}{3} - 0 = \frac{100}{3} \quad \text{Ans}$$

(iv) S.D :- $\sigma_x = \sqrt{\text{Variance}} = \sqrt{\frac{100}{3}} = \frac{10}{\sqrt{3}}$ Ans

Home work!

~~Ex.~~ Q. 1! The probability Density function of a R.V. 'X' is given by!

$$f_X(x) = 5e^{-kx}u(x)$$

then calculate following!

(i) D.C. power
(ii) A.C. power.
(iii) Total power
(iv) standard deviation.

Q.2! For the table given below. calculate following

(i) M_1 or μ_1 (ii) M_2 or $E[X^2]$
(iii) σ_x^2 (iv) σ_x

$X = \{x_i\}$	0	1	2
$P_X(x_i)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$