Kripke-Joyal Semantics

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2025-11-30

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1. Introduction

1.1 External vs Internal

Throughout this class, we have attempted to find different models of logical theories and analyze their properties as categories. We've also shown dualities between syntax and semantics. However, our underlying languages are always fully syntactic ($\vdash$).

Here, we hope to be able to understand the underlying logic of a theory completely within a category, only speaking w.r.t its objects and arrows.

1.2 Subobjects and membership

We have often discussed the concept of generalized elements in a category $\mathbb{C}$, such that such an “element” of $C \in \mathbb{C}$ is given by some $z : Z \to C$.

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We have often discussed the concept of generalized elements in a category $\mathbb{C}$, such that such an “element” of $C \in \mathbb{C}$ is given by some $z : Z \to C$.

If we consider a subobject $S \rightarrowtail C$, we can say that such a generalized element $z : Z \to C$ is a *member* of this subobject – $z \underset{C}{\in} S$ – iff z factors through the subobject:

$$\begin{tikzcd} & S \arrow[d, tail] \\ Z \arrow[ur, dashed] \arrow[r, "z"] & C \end{tikzcd}$$

1.3 Formulas in Context

For $[x : A \mid \varphi]$, for $M \in \mathrm{Mod}(\mathbb{T}, \mathbb{C})$, we interpret it as $[x : A \mid \varphi]^M \rightarrowtail [A]^M$ – in $\mathbf{Sub}([A]^M)$.

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Then, for first order $\mathbb{T}$ and $M \in \mathrm{Mod}(\mathbb{T}, \mathbb{C})$, M satisfies $[x : A|\varphi]$ iff its interpretation is the maximal subobject. This means $[x : A \mid \varphi]^M \rightarrowtail [A]^M$ is the identity (so $[x : A \mid \varphi] = [A]^M$).

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$$M \vDash [x : A \, | \varphi] \Longleftrightarrow \left([x : A | \varphi]^M \rightarrowtail [A]^M\right) = \mathrm{id}_{[A]^M}$$

$$\Longleftrightarrow M \vDash [x : A \mid \top \vdash \varphi]$$

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Then, for first order $\mathbb{T}$ and $M \in \mathrm{Mod}(\mathbb{T}, \mathbb{C})$, M satisfies $[x : A|\varphi]$ iff its interpretation is the maximal subobject. This means $[x : A \mid \varphi]^M \rightarrowtail [A]^M$ is the identity (so $[x : A \mid \varphi] = [A]^M$).

$$M \vDash [x : A \mid\varphi] \Longleftrightarrow \left([x : A|\varphi]^M \rightarrowtail [A]^M\right) = \mathrm{id}_{[A]^M}$$

$$\Longleftrightarrow M \vDash [x : A \mid \top \vdash \varphi]$$

(write : $\left([x : A|\varphi]^M \rightarrowtail [A]^M\right) = \mathrm{id}_{[A]^M} \Longleftrightarrow [x : A|\varphi]^M = \mathrm{id}$)

1.4 Factoring

Claim: All generalized elements $z : Z \to [A]^M$ factor through $[x : A \ |\varphi]^M \in \mathbf{Sub}([A]^M)$ iff $[x : A|\varphi]^M = \mathrm{id}$.

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Claim: All generalized elements $z : Z \to [A]^M$ factor through $[x : A \mid \varphi]^M \in \mathbf{Sub}([A]^M)$ iff $[x : A|\varphi]^M = \mathrm{id}$.

Proof:

- $(\Longrightarrow)$

$$\begin{tikzcd}
 & & {[x : A \mid \phi]^M} \arrow[d, tail, "v"'] \\
{[A]^M} \arrow[urr, dashed, "u"'] \arrow[rr, "\mathrm{id}"] & & {[A]^M}
\end{tikzcd}$$

- $(\Longleftarrow)$

$$\begin{tikzcd}
 & & {[x : A \mid \phi]^M} \arrow[d, equal] \\
Z \arrow[urr, dashed, "z"'] \arrow[rr, "z"] & & {[A]^M}
\end{tikzcd}$$

1.5 The Forcing Relation

So, for a Heyting category $\mathbb{C}$, we define the forcing relation "$\Vdash$" such that $Z \Vdash \varphi(z) \iff z \underset{[A]^M}{\in} [x : A \mid \varphi]^M$.

1.5 The Forcing Relation

So, for a Heyting category $\mathbb{C}$, we define the forcing relation "$\Vdash$" such that $Z \Vdash \varphi(z) \iff z \underset{[A]^M}{\in} [x : A \mid \varphi]^M$.

Theorem: for $M \in \mathrm{Mod}(\mathbb{T}, \mathbb{C})$:

$$M \vDash [x : A \mid \varphi] \iff \forall z : Z \to [A]^M, Z \Vdash \varphi(z)$$

Proof: Follows from previous claim.

1.6 The Forcing Relation (general!)

For propriety, note:

- $[\cdot \mid \varphi]^M \in \mathbf{Sub}(\mathbf{1})$, and thus $M \vDash [\cdot \mid \varphi] \iff \forall z : Z \to \mathbf{1}, Z \Vdash \varphi \iff !_Z \in_{\mathbf{1}} [\cdot \mid \varphi]^M$.
- $M \vDash [x_1 : A_1, ..., x_n : A_n \mid \varphi] \iff \forall z : Z \to \Pi_{i=1}^n [A_i]^M, Z \Vdash \varphi(z)$.
  - Note that here, we are writing $\varphi(z) = \varphi(z_1, ..., z_n)$ for $z_i = \pi_i \circ z : Z \to \Pi_{i=1}^n [A_i]^M \to [A_i]^M$.

1.7 Presheaf Categories

Consider $\hat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{\mathsf{op}}}$ and restrict to considering generalized elements of the form $z : yZ \to [A]^M$ (works as well with $\Gamma = [x_i : A_i, ...]$).

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Claim: The previous result holds if we only consider the representable generalized elements $z : yZ \to [A]^M$.

Proof: Sufficient to show that:

$$(\forall z : yZ \to [A]^M, yZ \Vdash \varphi(c)) \Longrightarrow (\forall w : W \to [A]^M), W \Vdash \varphi(C)$$

Fix $w : W \to [A]^M$. By *Co-Yoneda Lemma*, we know that W is the colimit of some $\{yR_i\}_{i \in I}$, so $w : W \to [A]^M$, so there exists $\alpha_i : yR_i \to W$ and $yR_i \Vdash \varphi(w \circ \alpha_i)$ for $i \in I$.

This allows us to lift factorings from the $\{yR_i\}$'s to the colimit.

1.8 Co-Yoneda Lemma

Lemma: For all $P \in \hat{\mathbb{C}}$, there exists index set I and diagram $R : I \to \mathbb{C}$ such that $P \cong \lim_{\to I}(yR)$.

Proof: Let $I = \int P$ i.e. the category of elements of P – objects (C, p) for $C \in \mathbb{C}, p \in P(C)$ and morphisms $(C', p') \to (C, p)$ are morphisms $u : C' \to C$ with $p \circ u = p'$. Moreover, consider the projection function $\pi_P : \int P \to \mathbb{C}$.

For a functor $A : \mathbb{C} \to \mathbb{E}$ ($\mathbb{E}$ cocomplete), define $R(E) : C \mapsto \mathrm{Hom}(A(C), E)$ and $L(P) = \mathrm{Colim}(\{A \circ \pi_P\})$ and by [Mac Lane I.5.2], $L : \hat{\mathbb{C}} \leftrightarrows \mathbb{E} : R$ are adjoint.

Now let $\mathbb{E} = \hat{\mathbb{C}}$ and $A = y$, so $R(E)(C) = \mathrm{Hom}(y(C), E) \cong E(C)$. Thus by adjoint isomorphism, we get that:

$$P \cong \lim_{\to I}(y \circ \pi_P)$$

2. Kripke-Joyal Semantics

2.1 Kripke-Joyal Semantics

Fix a presheaf category $\hat{\mathbb{C}}$ and $M \in \mathrm{Mod}(\mathbb{T}, \mathbb{C})$ in FOL, formulas $[x : A \mid \varphi], [x : A \mid \psi], [x : A, y : B \mid \chi]$ in $\mathcal{L}(\mathbb{T})$, $C \in \mathbb{C}$ and morphisms $c, c_1, c_2 : yC \to [A]^M$.

Then we have that:

1) $yC \Vdash \top(c)$ always
2) $yC \Vdash \bot(c)$ never
3) $yC \Vdash c_1 = c_2$ iff $c_1 = c_2 : yC \to [A]^M$
4) $yC \Vdash (\varphi \land \psi)(c)$ iff $yC \Vdash \varphi(c)$ and $yC \Vdash \psi(c)$
5) $yC \Vdash (\varphi \lor \psi)(c)$ iff $yC \Vdash \varphi(c)$ or $yC \Vdash \psi(c)$
6) $yC \Vdash (\varphi \implies \psi)(c)$ iff $\forall d : yD \to yC, yD \Vdash \varphi(c \circ d)$ implies $D \Vdash \psi(c \circ d)$.
7) $yC \Vdash (\neg \varphi)(c)$ iff for no $d : yD \to yC, yD \Vdash \varphi(c \circ d)$
8) $yC \Vdash (\exists y : B, \chi(\cdot, y))(c)$ iff $\exists c' : yC \to [B]^M, yD \Vdash \chi(c, c')$.
9) $yC \Vdash (\forall y : B, \chi(\cdot, y))(c)$ iff $\forall d : yD \to yC, \forall d' : yD \to [B]^M, yD \Vdash \chi(c \circ d, d')$.

2.2 Base case ($\top$, $\bot$)

- $yC \Vdash \top(c) \iff c : yC \to [A]^M$ factors through $[x : A \mid \top]^M \in \mathbf{Sub}([A]^M)$. But $[x : A \mid \top]^M$ is always the maximal subobject, and thus id. So c must factor through $[x : A \mid \top]^M$ always.
- $yC \Vdash \bot(c) \iff c$ factors through $[x : A \mid \bot]^M$, which is the (canonical morphism of) initial object $\mathbf{0}$. Thus, $yC \Vdash \bot(c)$ iff there exists $u : yC \to \mathbf{0}$ s.t.

$$\begin{tikzcd} yC \arrow[r, dashed, "u"] \arrow[dr, "c"'] & \mathbf{0} \arrow[d, "!_{[A]^M}"] \\ & {[A]^M} \end{tikzcd}$$

Note there exists $!_{yC} : \mathbf{0} \to yC$, so if such u exists, yC is initial. As $\hat{\mathbb{C}}$ is nontrivial, exists C noninitial, so yC noninitial, so $yC \Vdash \bot(c)$ never (for general yC).

2.3 Equality

Recall $yC \Vdash \left[x \underset{A}{=} y\right](\langle c_1, c_2 \rangle)$ iff $\langle c_1, c_2 \rangle : yC \to [A]^M \times [A]^M$ factors through $\left[x : A, y : A \mid x \underset{A}{=} y\right]^M \in \mathbf{Sub}([A]^M \times [A]^M)$.

Recall that $\left[x : A, y : A \mid x \underset{A}{=} y\right]^M$ is the equalizer of $[x : A, y : A \mid x]^M = p_1$ and $[x : A, y : A \mid y]^M = p_2$. This is exactly the diagonal $\Delta_{[A]^M}$.

So $yC \Vdash c_1 = c_2$ iff exists u s.t.

$$
\begin{tikzcd}
{[x : A, y : A \mid x = y]^M} \arrow[r, "\Delta_{[A]^M}"] & {[A]^M \times [A]^M} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & {[A]^M} \\
yC \arrow[u, dashed, "u"] \arrow[ur, "\langle c_1, c_2 \rangle"'] & &
\end{tikzcd}
$$

As $\langle c_1, c_2 \rangle$ factors through the diagonal, we get that $\langle c_1, c_2 \rangle = \langle u, u \rangle$, so $c_1 = c_2$.

2.4 Conjunction (Disjunction)

$yC \Vdash (\varphi \wedge \psi)(c)$ iff c factors through $[x : A \mid \varphi \wedge \psi]^M = [x : A \mid \varphi]^M \wedge [x : A \mid \psi]^M \in \mathbf{Sub}([A]^M)$.

- ($\Longrightarrow$) Suppose $yC \Vdash (\varphi \wedge \psi)(c)$, so then there exists $u : yC \to \operatorname{dom}([x : A \mid \varphi \wedge \psi]^M)$ such that the following commutes:

$$\begin{tikzcd}
& {[x : A \mid \phi \wedge \psi]^M} \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & {[x : A \mid \psi]^M} \arrow[d, tail] \\
& {[x : A \mid \phi]^M} \arrow[r, tail] & {[A]^M} \\
yC \arrow[uur, dotted, "u"] \arrow[urr, "c"'] & &
\end{tikzcd}$$

- ($\Longleftarrow$) Suppose $yC \Vdash \varphi(c)$ and $yC \Vdash \psi(c)$. Then by the pullback UMP, there exists u s.t. the following commutes

$$\begin{tikzcd}
yC \arrow[dr, dashed, "u"] \arrow[drr] \arrow[ddr] \arrow[ddrr, bend left, "c"] & & \\
& {[x : A \mid \phi \wedge \psi]^M} \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & {[x : A \mid \psi]^M} \arrow[d, tail] \\
& {[x : A \mid \phi]^M} \arrow[r, tail] & {[A]^M}
\end{tikzcd}$$

2.5 Implication

$yC \Vdash (\varphi \implies \psi)(c)$ iff c factors through $[x : A \mid \varphi \implies \psi]^M = \left([x : A \mid \psi]^M\right)^{[x:A \mid \psi]^M} = \Psi^\Phi \in \mathbf{Sub}([A]^M)$.

($\Longrightarrow$) Suppose c factors through Ψ^Φ via $u \in \mathrm{Hom}(yC, \Psi^\Phi)$, so there exists $\overline{u} : \varepsilon \circ (u \times \mathrm{id}) : yC \times \Phi \to \Psi$. Fix $d : yD \to yC$ and suppose $D \Vdash \varphi(c \circ d)$ (via factor v). Then we get that by the product UMP:

$$\begin{tikzcd}
 & yD \arrow[dl, "d"'] \arrow[d, dashed, "w"] \arrow[dr, "v"] & \\
yC \arrow[ddr, "c"] & yC \times \Phi \arrow[l, "p_1"] \arrow[r, "p_2"'] \arrow[d, "\bar{u}"] & \Phi \arrow[ddl, tail] \\
 & \Psi \arrow[d, tail] & \\
 & {[A]^M} &
\end{tikzcd}$$

($\Longleftarrow$) Suppose $\forall d : yD \to yC, D \Vdash \varphi(c \circ d) \implies D \Vdash \psi(c \circ d)$, so $c \circ d$ factors through Φ. Take pullback of Φ, c to get $(V, d : V \to yC, p : V \to \Phi)$, and using hypothesis (+co-yoneda) get $q : V \to \Psi$. Apply product UMP + exponential transposition and observe:

$$\begin{tikzcd}
 & & \Psi^\Phi \times \Phi \arrow[drr, "\varepsilon"] & & \\
V \times \Phi \arrow[urr, "\widetilde{(q \circ pi_1)}"'] \arrow[rrrr, "q \circ \pi_1"] \arrow[dr, "\pi_1"] \arrow[drrr, "\pi_2"] & & & & \Psi \arrow[ddl, tail] \\
 & V \arrow[ul, dotted, "u"] \arrow[urrr, dotted, "q"] \arrow[rr, dashed, "p"'] \arrow[d, dashed, "d"] \arrow[dr, phantom, "\lrcorner", very near start] & & \Phi \arrow[d, tail] & \\
 & yC \arrow[rr, "c"] & & {[A]^M} &
\end{tikzcd}$$

2.6 Negation

Recall that we define $\neg\varphi \equiv (\varphi \implies \bot)$. Thus, $C \Vdash \neg\varphi(c)$ iff $C \Vdash (\varphi \implies \bot)(c)$.

This happens iff for all $d : yD \to yC, yD \Vdash \varphi(c \circ d)$ implies $D \Vdash \bot(c \circ d)$. But we know this never happens, so that implication is true iff for all $d : yD \to yC, \neg(D \Vdash \varphi(c \circ d))$.

2.7 Quantifiers Background

Note by [Awodey 3.3.29], for $U \in \mathbf{Sub}([A]^M \times [B]^M)$, $V \in \mathbf{Sub}([A]^M)$, we can define quantifiers pointwise as:

- $\exists_B(U) : C \mapsto \{a \in [A]^M(C) \mid \exists y \in [B]^M(C), (a, y) \in U(C)\}$
- $\forall_B(U) : C \mapsto \{a \in [A]^M(C) \mid \forall h : D \to C, \forall (x, y) \in ([A]^M \times [B]^M)(D), x = [A]^M(h)(a) \implies (x, y) \in U(D)\}$

2.8 Existential Quantifier

$yC \Vdash \exists y : B, \chi(c, y)$ iff c factors through $[x : A \mid \exists y : B, \chi(\cdot, y)]^M = \exists_B [x : A, y : B \mid \chi]^M$. Taking $U = [x : A, y : B \mid \chi]^M$, we can apply the previous to get that this happens iff c factors through $\exists_B U$

$\iff$ exists $c' : yC \to [B]^M$ such that for all D, $\langle c, c' \rangle(D) \in U(D) = [x : A, y : B \mid \chi]^M(D)$.

$\iff$ there exists c' such that $\langle c, c' \rangle$ factors through $[x : A, y : B \mid \chi]^M$

$\iff$ there exists c' such that $yC \Vdash \chi(c, c')$.

2.9 Universal Quantifier

$yC \Vdash \forall y : B, \chi(c, y)$ iff c factors through $[x : A \mid \forall y : B, \chi(\cdot, y)]^M = \forall_B [x : A, y : B \mid \chi]^M$. Taking $U = [x : A, y : B \mid \chi]^M$, we can apply the previous to get that this happens iff c factors through $\forall_B U$

$\Longleftrightarrow$ for all $h : yD \to yC, (a, b) : yD \to [A]^M \times [B]^M$, if $a = c \circ h$ then for all K, $\langle (c \circ h)(K), b(K) \rangle \in U(K)$.

$\Longleftrightarrow$ for all $d : yD \to yC, d' : yD \to [B]^M$, $\langle c \circ h, b \rangle \in U$.

$\Longleftrightarrow \forall d : yD \to yC, \forall d' : yD \to [B]^M, \langle c \circ d, d' \rangle \in [x : A, y : A \mid \chi]^M$.

$\Longleftrightarrow \forall d : yD \to yC, \forall d' : yD \to [B]^M, yD \Vdash \chi(c \circ d, d')$

3. Analysis

3.1 IFOL vs FOL

We claim that KJ semantics models internal IFOL, and not full FOL. It is sufficient to show that it is possible in some presheaves for LEM/DN to fail.

Consider $\mathbb{C} = \mathbb{2}$ w/ objects $0, 1$ and morphism $0 \to 1$. Then $\hat{\mathbb{C}}$ is given by sets $P(0) = \{*\}, P(1) = \emptyset$ with a trivial function $f : \emptyset \to \{*\}$. Let $U \in \mathbf{Sub}(P)$ be maximal, and then consider the formula $\varphi = [U = \emptyset]$.

Then for an element $z : Z \to \mathbf{1}$ (WLOG consider $Z = y(1), z = \mathrm{id}_{y(1)}$), WTS both φ and $\neg\varphi$ are not forced.

- z forces φ iff at $U(1)$ is empty *and* $U(f)(1) = U(0)$ is empty, which is false.
- Similarly, z forces $\neg\varphi$ iff both $U(0), U(1)$ are nonempty, which is false.

3.2 Extensions

KJ semantics can be extended to many other settings, such as:

- General Heyting categories
- Grothendieck topoi
- Elementary topoi
- G-sets

etc.

4. Q+A