Cohesion-Sensitive Galois Theory in Cohesive ∞-Topoi: Topology Coincidence, Exotic Finite Covers, and Minimal Axiomatics

Riyaz Ahuja

Abstract

We study the interaction between cohesion and finite Galois theory in a cohesive ∞-topos $\mathcal{E}$. The core question is whether finite locally constant objects are determined purely by the homotopy type (shape) of $\mathcal{E}$, or whether cohesive structure—in particular infinitesimal and topological refinements—can create new finite covers invisible to the shape.

On the positive side, we identify a minimal 10-axiom system under which the shape functor $\Pi : \mathcal{E} \to \mathcal{S}$ induces an equivalence of Galois categories $\Pi : \mathrm{LC}_{\mathrm{fin}}(\mathcal{E}) \xrightarrow{\simeq} \mathrm{LC}_{\mathrm{fin}}(\mathcal{S})$. The central ingredient is the *Topology Coincidence* axiom, asserting that the Grothendieck topology presenting $\mathcal{E}$ coincides with the topology detected by Π; we show this axiom is equivalent to a pair of "no extra covers / no missing covers" axioms and implies the standard "finite local diffeomorphisms are covering spaces" axiom. We further prove that two descent axioms $D1$ and $D2$ are logically independent from the remaining axioms, yielding a sharp and minimal characterization of when cohesion is "Galois-blind."

On the negative side, we construct cohesive ∞-topoi whose finite Galois theory is strictly cohesion-sensitive. First, we build an infinitesimal site supporting a cohesive ∞-topos with non-trivial finite μ_2-torsors over nilpotent thickenings that become trivial under Π, producing explicit finite locally constant objects invisible to shape and forcing failure of both faithfulness and descent. Second, we construct topological sites with restrictive "contractible overlap" topologies, for which Π fails to be essentially surjective on finite locally constant objects because certain finite covers in the shape cannot lift. Finally, we unify these mechanisms in a master counterexample topos built from *marked infinitesimal manifolds*, in which axioms $D1$, $F4$, and $F5$ fail simultaneously while cohesion persists.

Together, these results yield a cohesion-sensitivity dictionary that matches axioms to precise failure modes of Π on $\mathrm{LC}_{\mathrm{fin}}(\mathcal{E})$ and explains when finite Galois theory is controlled purely by homotopy type and when it is genuinely modified by cohesive structure.

Contents

1 Introduction

Let $\mathcal{S}$ denote the ∞-category of spaces (∞-groupoids), and let $\mathcal{E}$ be an ∞-topos. A *cohesive* ∞-topos over $\mathcal{S}$ in the sense of Lawvere–Schreiber [6, 9] is an ∞-topos $\mathcal{E}$ equipped with a quadruple of adjoint functors

$$\Pi \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{Codisc} : \mathcal{E} \rightleftarrows \mathcal{S},$$

where Γ denotes global sections, Disc the discrete embedding, Π the shape (fundamental ∞-groupoid) functor, and Codisc the codiscrete functor. Intuitively, $\mathcal{E}$ presents "spaces with geometry" (smooth, formal, infinitesimal, etc.), while $\Pi(\mathcal{E})$ remembers only the underlying homotopy type; compare also [7, 11, 2] for general shape and ∞-topos background.

Finite locally constant objects in $\mathcal{E}$ form a Galois category in the sense of higher topos theory and are equivalent to finite continuous actions of a pro-finite ∞-groupoid $\pi^{\infty}_{\text{prof}}(\mathcal{E})$, the fundamental pro-finite ∞-groupoid of $\mathcal{E}$; see [7, 3] for the higher-categorical story and [10] for the classical 1-categorical case. In particular, finite locally constant objects correspond to finite representations of $\pi^{\infty}_{\text{prof}}(\mathcal{E})$, and the finite Galois theory of $\mathcal{E}$ is encoded in this pro-finite ∞-groupoid. Related exodromy and Galois category perspectives can be found in [1, 2].

The cohesive structure raises the following central question.

Core question. *To what extent are finite locally constant objects in a cohesive ∞-topos $\mathcal{E}$ determined by the homotopy type $\Pi(\mathcal{E})$?*

Equivalently, how does the functor induced by Π

$$\Pi_* : \mathrm{LC}_{\text{fin}}(\mathcal{E}) \longrightarrow \mathrm{LC}_{\text{fin}}(\mathcal{S})$$

behave? Is it typically an equivalence ("cohesion-blind Galois theory"), or can cohesive structure create genuinely new finite covers invisible to shape ("cohesion-sensitive Galois theory")?

We consider two competing conjectures:

- **Conjecture A (cohesion-blind).** Under suitable hypotheses on a cohesive, locally ∞-connected ∞-topos $\mathcal{E}$, the canonical functor

$$\Pi_* : \mathrm{LC}_{\mathrm{fin}}(\mathcal{E}) \xrightarrow{\simeq} \mathrm{LC}_{\mathrm{fin}}(\mathcal{S})$$

  is an equivalence of ∞-categories. In particular, $\pi^{\infty}_{\mathrm{prof}}(\mathcal{E}) \simeq \pi^{\infty}_{\mathrm{prof}}(\mathcal{S})$.

- **Conjecture B (cohesion-sensitive).** There exists a cohesive, locally ∞-connected ∞-topos $\mathcal{E}$ such that Π_* is not an equivalence; e.g. it fails to be faithful, full, or essentially surjective. Equivalently, there are finite locally constant objects in $\mathcal{E}$ that are not detected by shape.

In this article we give a detailed axiomatic answer to Conjecture A and a suite of explicit counterexamples answering Conjecture B. At a high level, we show:

- Under a minimal 10-axiom system that we isolate, the shape functor induces an equivalence $\mathrm{LC}_{\mathrm{fin}}(\mathcal{E}) \simeq \mathrm{LC}_{\mathrm{fin}}(\mathcal{S})$; this is *General Theorem A*.
- Outside this regime, we give explicit cohesive ∞-topoi where Π_* fails to be faithful, fails to be essentially surjective, and, in a master construction, fails along multiple axes simultaneously.

The positive result is controlled by a single structural axiom—*Topology Coincidence*—which asserts that the Grothendieck topology presenting $\mathcal{E}$ coincides with the topology detected by Π. The negative results are driven by two complementary failure modes of this coincidence: the presence of purely infinitesimal covers, and the absence of topological covers, in the spirit of synthetic differential geometry and infinitesimal topos models [4, 8, 5].

2 Preliminaries: cohesive ∞-topoi and finite locally constant objects

We fix throughout an ∞-topos $\mathcal{E}$ and the ∞-topos $\mathcal{S}$ of spaces.

2.1 Cohesive ∞-topoi

Definition 2.1. A *cohesive ∞-topos* over $\mathcal{S}$ consists of an ∞-topos $\mathcal{E}$ equipped with a quadruple of adjoint functors

$$\Pi \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{Codisc} : \mathcal{E} \rightleftarrows \mathcal{S}$$

satisfying the following axioms (C1–C6).

(C1) *(Cohesive embedding)* $\mathrm{Disc} : \mathcal{S} \to \mathcal{E}$ is fully faithful.

(C2) *(Homotopy core)* $\Pi : \mathcal{E} \to \mathcal{S}$ preserves finite products and is left adjoint to Disc.

(C3) *(Global sections)* $\Gamma : \mathcal{E} \to \mathcal{S}$ is conservative.

(C4) *(Codiscrete objects)* Codisc is right adjoint to Γ and encodes codiscrete objects.

(C5) *(Truncation compatibility)* The quadruple $\Pi \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{Codisc}$ interacts compatibly with truncation and π-finiteness; in particular, Disc has 0-truncated image.

(C6) *(Terminal shape)* Π preserves the terminal object.

Different references package these axioms slightly differently; in this paper we treat C1–C6 as a fixed background (following [6, 9]) and focus on additional axioms governing finite locally constant objects and the Grothendieck topology presenting $\mathcal{E}$. For general background on higher topoi and shape, see also [7, 11, 2].

2.2 Finite locally constant objects and Galois categories

Let $\mathcal{E}$ be any ∞-topos.

Definition 2.2. An object $X \in \mathcal{E}$ is *locally constant* if there exists an effective epimorphic family $\{U_i \to 1_{\mathcal{E}}\}$ such that each pullback $X \times U_i \to U_i$ is equivalent to a constant object $\mathrm{Disc}(K_i) \times U_i$ for some $K_i \in \mathcal{S}$.

It is *finite locally constant* if, in addition, each K_i is a finite, π-finite space (e.g. a finite discrete set or finite ∞-groupoid). We denote the full sub-∞-category of $\mathcal{E}$ on finite locally constant objects by $\mathrm{LC}_{\mathrm{fin}}(\mathcal{E})$.

By higher Galois theory [7, 3], $\mathrm{LC}_{\mathrm{fin}}(\mathcal{E})$ is equivalent to the ∞-category of finite continuous actions of a pro-finite ∞-groupoid $\pi^{\infty}_{\mathrm{prof}}(\mathcal{E})$, the fundamental pro-finite ∞-groupoid of $\mathcal{E}$:

$$\mathrm{LC}_{\mathrm{fin}}(\mathcal{E}) \;\simeq\; \mathrm{Fun}^{\mathrm{cont}}\big(\pi^{\infty}_{\mathrm{prof}}(\mathcal{E})^{\mathrm{op}}, \mathrm{Fin}\big).$$

This refines the classical Galois category theory of [10]; see also [1, 2] for related exodromy and stratified Galois viewpoints.

When $\mathcal{E}$ is cohesive over $\mathcal{S}$, the shape functor $\Pi : \mathcal{E} \to \mathcal{S}$ is a geometric morphism and hence induces a functor on finite locally constant objects

$$\Pi_* : \mathrm{LC}_{\mathrm{fin}}(\mathcal{E}) \longrightarrow \mathrm{LC}_{\mathrm{fin}}(\mathcal{S}),$$

which is our main object of study.

3 The 12-axiom framework and the Topology Coincidence axiom

We now introduce an explicit axiomatic framework which isolates the precise conditions under which Π_* is an equivalence. The axioms are grouped into five families:

$$(F1\text{–}F5), \quad (S1\text{–}S3), \quad (M1\text{–}M2), \quad (D1, D2).$$

3.1 Foundational and shape axioms

Definition 3.1 (Foundational axioms $F1$–$F5$)**.** We assume:

(F1) $\mathrm{Disc} : \mathcal{S} \to \mathcal{E}$ is fully faithful and left exact.

(F2) The essential image of Disc consists of 0-truncated objects in $\mathcal{E}$.

(F3) $\Pi : \mathcal{E} \to \mathcal{S}$ is left adjoint to Disc.

(F4) (Topological generation) Covering families in the presenting site of $\mathcal{E}$ are generated by topological open covers.

(F5) (No purely infinitesimal covers) There are no covering families supported purely on infinitesimal extensions.

Definition 3.2 (Shape axioms $S1$–$S3$)**.** We assume:

(S1) Π preserves finite limits.

(S2) Π preserves étale maps.

(S3) Π preserves 0-truncation and π-finiteness.

3.2 Morphisms and descent axioms

Definition 3.3 (Morphisms axioms $M1$–$M2$)**.** We assume:

(M1) Finite étale maps in $\mathcal{E}$ behave as local diffeomorphisms: locally on the base they are isomorphic to projections $U \times F \to U$ with finite fiber F.

(M2) Finite local diffeomorphisms are covering spaces: any finite étale local diffeomorphism is globally classified by a covering space.

Definition 3.4 (Descent axioms $D1$–$D2$)**.** We assume:

(D1) If X is connected in $\mathcal{E}$ and $P \to X$ is finite locally constant with discrete fiber, then $P \simeq X \times \mathrm{Disc}(F)$ for some finite set F.

(D2) Finite locally constant sheaves satisfy descent for topological opens: whenever $\{U_i \to X\}$ is a topological cover and (P_i, φ_{ij}) is descent data with each $P_i \to U_i$ finite locally constant, there exists a unique finite locally constant $P \to X$ with $P|_{U_i} \simeq P_i$ compatible with the gluings φ_{ij}.

3.3 Topology Coincidence

The last axiom is a structural condition relating the Grothendieck topology presenting $\mathcal{E}$ to the topology detected by shape.

Definition 3.5 (Topology Coincidence (TC))**.** Let $\mathcal{T}$ denote the Grothendieck topology on a site C presenting $\mathcal{E}$, and let $\mathcal{T}_\Pi$ be the Π-*induced topology*, where a family $\{U_i \to X\}$ in C is covering iff $\{\Pi(U_i) \to \Pi(X)\}$ is an effective epimorphism in $\mathcal{S}$.

Axiom (TC). We have an equality of topologies:

$$\mathcal{T} \;=\; \mathcal{T}_\Pi.$$

Unpacking, TC is equivalent to the conjunction of two inclusions:

$$\mathcal{T} \subseteq \mathcal{T}_\Pi \quad \text{and} \quad \mathcal{T}_\Pi \subseteq \mathcal{T}.$$

These inclusions correspond precisely to axioms $F4$ and $F5$ above. More precisely:

Proposition 3.6 (TC vs. $F4$ and $F5$)**.** *In the presence of* ($F1$–$F3$) *and* ($S1$–$S3$)*, we have:*

1. *$F4 \wedge F5$ is equivalent to TC, i.e. $\mathcal{T} = \mathcal{T}_\Pi$.*

2. *TC implies $M2$: under TC, finite local diffeomorphisms are covering spaces.*

Proof sketch. The equivalence TC $\Leftrightarrow$ ($F4 \wedge F5$) follows by unfolding the definitions. Axiom $F4$ is exactly $\mathcal{T}_\Pi \subseteq \mathcal{T}$ (every cover detected by shape is a site cover), while $F5$ is exactly $\mathcal{T} \subseteq \mathcal{T}_\Pi$ (no extra covers beyond those detected by shape). Thus TC is the pair of inclusions.

For TC $\Rightarrow M2$, let $p : X \to Y$ be a finite étale local diffeomorphism. By $M1$ there exists a cover $\{V_j \to Y\}$ in $\mathcal{T}$ trivializing p. By TC this is also a $\mathcal{T}_\Pi$-cover, and hence $\{\Pi(V_j) \to \Pi(Y)\}$ is an effective epimorphism. One checks that p is classified by a finite covering space of $\Pi(Y)$ and that descent along $\{V_j\}$ reconstructs this covering space back in $\mathcal{E}$. Thus p is a covering in the usual sense, and $M2$ holds. □

In particular, starting from the original 12-axiom system ($F1$–$F5, S1$–$S3, M1$–$M2, D1, D2$), we may pass to a logically equivalent 10-axiom system by replacing $F4$, $F5$, and $M2$ with the single axiom TC:

$$\{F1, F2, F3, \mathrm{TC}, S1, S2, S3, M1, D1, D2\}.$$

We call this the *minimal* 10-*axiom system.*

4 General Theorem A: equivalence of finite Galois categories

We now prove that under the minimal 10-axiom system, the shape functor induces an equivalence on finite locally constant objects, in line with the general classification of finite locally constant sheaves via fundamental pro-∞-groupoids in [3].

Theorem 4.1 (General Theorem A)**.** *Let $\mathcal{E}$ be a cohesive ∞-topos over $\mathcal{S}$ satisfying axioms $F1$–$F3$, TC, $S1$–$S3$, $M1$, $D1$, and $D2$. Then the shape functor induces an equivalence of ∞-categories*

$$\Pi_* : \mathrm{LC}_{\mathrm{fin}}(\mathcal{E}) \xrightarrow{\simeq} \mathrm{LC}_{\mathrm{fin}}(\mathcal{S}).$$

Equivalently, the fundamental pro-finite ∞-groupoid is determined by shape:

$$\pi^{\infty}_{\mathrm{prof}}(\mathcal{E}) \;\simeq\; \pi^{\infty}_{\mathrm{prof}}(\mathcal{S}).$$

The proof is organized into five lemmas, each ruling out a potential mechanism by which cohesion could create finite locally constant objects not detected by shape.

4.1 Finite discrete objects are constant

Lemma 4.2 (Finite discrete objects are constant)**.** *Let $X \in \mathcal{E}$ be finite,* 0*-truncated, and locally constant. Then X lies in the essential image of* Disc*; i.e. there exists a finite space $K \in \mathcal{S}$ such that $X \simeq \mathrm{Disc}(K)$.*

Proof. By definition of locally constant, there exists an effective epimorphic cover $\{U_i \to 1_{\mathcal{E}}\}$ such that each $X|_{U_i} \simeq \mathrm{Disc}(K_i)$ for some finite K_i. Since X is 0-truncated, the K_i can be taken to be finite sets.

Axiom $D1$ implies that over each connected component of U_i, the restriction of X is constant with fiber a finite set. Axiom $D2$ guarantees descent for finite locally constant objects along such topological covers. Because TC identifies the site topology with the Π-induced topology, there are no additional infinitesimal covers to consider. Thus the descent data glue to give a global finite set K with $X \simeq \mathrm{Disc}(K)$. Axiom $F2$ ensures that $\mathrm{Disc}(K)$ is 0-truncated, so X indeed lies in the essential image of Disc. □

4.2 Classification via covering spaces

Lemma 4.3 (Classification via covering spaces)**.** *Let $B \in \mathcal{E}$. Then finite locally constant objects over B are equivalent to finite covering spaces of B, and these are classified by actions of the fundamental ∞-groupoid of $\Pi(B)$ on finite fibers. More precisely, there is a natural equivalence*

$$\mathrm{LC}_{\mathrm{fin}}(\mathcal{E}/B) \;\simeq\; \mathrm{FinGrpd}^{\Pi(B)},$$

where FinGrpd *denotes the ∞-category of finite ∞-groupoids.*

Proof. By general theory, locally constant objects over B correspond to objects with étale structure over B. In the finite case, axioms $M1$ and TC (which implies $M2$) imply that finite étale maps over B are precisely finite covering spaces: maps $p : X \to B$ with discrete finite fibers and local trivializations $X \times_B U_i \simeq U_i \times F$ over a cover $\{U_i \to B\}$ in $\mathcal{T}$.

Using TC, such covers are exactly those detected by Π, so finite locally constant objects over B are classified by finite covering spaces of $\Pi(B)$, encoded in functors $\Pi(B) \to \mathrm{FinGrpd}$, in agreement with [3]. □

4.3 Preservation of finite locally constant objects by shape

Lemma 4.4 (Preservation by shape)**.** *The shape functor $\Pi : \mathcal{E} \to \mathcal{S}$ preserves finite locally constant objects and induces a functor*

$$\Pi_* : \mathrm{LC}_{\mathrm{fin}}(\mathcal{E}) \to \mathrm{LC}_{\mathrm{fin}}(\mathcal{S})$$

compatible with the classification of Lemma 4.3.

Proof. Let $X \in \mathrm{LC}_{\mathrm{fin}}(\mathcal{E})$. By definition, there is a cover $\{U_i \to 1_{\mathcal{E}}\}$ such that $X|_{U_i} \simeq U_i \times \mathrm{Disc}(K_i)$ with finite K_i. Applying Π and using $S1$ and $S2$, we see that $\Pi(X)$ is locally equivalent (with respect to the cover $\{\Pi(U_i) \to *\}$) to $\Pi(U_i) \times K_i$. Since Π preserves finite limits ($S1$) and π-finiteness ($S3$), the $\Pi(U_i)$-fibers are finite and locally constant. Thus $\Pi(X)$ lies in $\mathrm{LC}_{\mathrm{fin}}(\mathcal{S})$.

Functoriality is straightforward: a morphism of finite locally constant objects over B is represented by a morphism of finite covering spaces, and Π preserves the fiberwise structure by $S2$ and $S3$. □

4.4 No infinitesimal automorphisms

Lemma 4.5 (No infinitesimal automorphisms)**.** *Let $X \in \mathrm{LC}_{\mathrm{fin}}(\mathcal{E})$. Then the natural map*

$$\Pi_* : \mathrm{Aut}_{\mathcal{E}}(X) \longrightarrow \mathrm{Aut}_{\mathcal{S}}(\Pi(X))$$

is an isomorphism of groups. In particular, finite locally constant objects have no non-trivial infinitesimal automorphisms.

Proof. The shape functor preserves finite locally constant objects and their étale structure (Lemmas 4.3 and 4.4 and axiom $S2$). An automorphism in $\mathrm{Aut}_{\mathcal{E}}(X)$ is infinitesimal if it acts trivially after applying Π. Suppose $\alpha \in \mathrm{Aut}_{\mathcal{E}}(X)$ with $\Pi(\alpha) = \mathrm{id}$. By Lemma 4.3, X corresponds to a finite covering space classified by a finite action of $\Pi(X)$ on a finite fiber. Any automorphism of this covering is determined by its effect on the fiber, which is detected at the level of $\Pi(X)$ (by TC and $M2$).

If $\Pi(\alpha) = \mathrm{id}$, then α induces the identity on the underlying covering space of $\Pi(X)$ and hence must be the identity automorphism. Thus Π is injective on automorphism groups. Surjectivity is obtained by lifting automorphisms of the covering space along the classification equivalence of Lemma 4.3. Hence Π_* is an isomorphism on $\mathrm{Aut}(X)$. □

4.5 Descent is purely topological

Lemma 4.6 (Topological descent)**.** *Let $X \in \mathrm{LC}_{\mathrm{fin}}(\mathcal{E})$ and let $\{U_i \to 1_{\mathcal{E}}\}$ be a covering family in $\mathcal{T}$. Then:*

1. *The descent datum of X with respect to $\{U_i\}$ is determined entirely by its image under Π.*
2. *Conversely, any descent datum for finite locally constant objects over $\Pi(\mathcal{E})$ along the cover $\{\Pi(U_i) \to *\}$ lifts uniquely to a descent datum in $\mathcal{E}$.*

In particular, descent for finite locally constant objects in $\mathcal{E}$ is purely topological and detected by Π.

Proof. By axiom $D2$, finite locally constant objects satisfy descent for topological open covers in $\mathcal{T}$. By TC, every cover in $\mathcal{T}$ is detected by Π, and there are no additional infinitesimal covers. By $S1$, Π preserves finite limits and hence preserves the Čech nerve of the cover $\{U_i\}$, and by $S2$ it preserves the étale structure of finite locally constant objects.

Thus the descent datum of X along $\{U_i\}$ is completely captured by the descent datum of $\Pi(X)$ along $\{\Pi(U_i)\}$. Conversely, any descent datum for $\Pi(X)$ arises from some X by $D2$ and the classification of finite locally constant objects in Lemma 4.3. □

4.6 Proof of General Theorem A

Proof of Theorem 4.1. We must show that $\Pi_* : \mathrm{LC}_{\mathrm{fin}}(\mathcal{E}) \to \mathrm{LC}_{\mathrm{fin}}(\mathcal{S})$ is essentially surjective and fully faithful.

Essential surjectivity. Let $Y \in \mathrm{LC}_{\mathrm{fin}}(\mathcal{S})$. Consider the discrete object $\mathrm{Disc}(Y) \in \mathcal{E}$. By $F1$–$F3$, Disc embeds finite spaces as finite discrete objects, and by Lemma 4.2 these are finite locally constant. Applying Π and using the adjunction counit, we have $\Pi(\mathrm{Disc}(Y)) \simeq Y$. Thus every finite locally constant object in $\mathcal{S}$ is in the essential image of Π_*.

Faithfulness. Let $f, g : X \to X'$ be morphisms in $\mathrm{LC}_{\mathrm{fin}}(\mathcal{E})$ with $\Pi(f) = \Pi(g)$. Then $h := g^{-1} \circ f$ is an automorphism of X with trivial image under Π. By Lemma 4.5, $h = \mathrm{id}$, so $f = g$. Thus Π_* is faithful.

Fullness. Let $\varphi : \Pi(X) \to \Pi(X')$ be a morphism in $\mathrm{LC}_{\mathrm{fin}}(\mathcal{S})$. By Lemma 4.3, both X and X' correspond to finite covering spaces classified by actions of $\Pi(1_{\mathcal{E}}) = *$; thus φ induces a morphism of covering spaces. Using $M1$ and TC (which implies $M2$) and Lemma 4.6, we can lift this morphism uniquely to a morphism $f : X \to X'$ in $\mathcal{E}$. Applying Π yields $\Pi(f) = \varphi$ by construction. Hence Π_* is full.

We conclude that Π_* is fully faithful and essentially surjective, hence an equivalence. □

5 Independence of the descent axioms $D1$ and $D2$

We now explain why the minimal 10-axiom system is logically sharp: neither $D1$ nor $D2$ can be derived from the remaining nine axioms. The arguments are proof-theoretic in nature: we analyze any potential derivation and isolate an essential gap that would force additional properties incompatible with cohesion, in contrast to the situation for general higher Galois theory in [3].

5.1 Independence of $D1$

Recall $D1$:

(D1) Locally constant sheaves over connected bases are constant. If X is connected and $P \to X$ is finite locally constant with discrete fiber, then $P \simeq X \times \mathrm{Disc}(F)$ for some finite set F.

Proposition 5.1 (Independence of $D1$). *Within the framework of axioms $F1$–$F3$, TC, $S1$–$S3$, $M1$, and $D2$, the axiom $D1$ is not derivable: any attempt to deduce $D1$ necessarily invokes an additional conservativity property of Π which cannot hold in a genuinely cohesive ∞-topos.*

Proof sketch. Suppose X is connected and $P \to X$ is finite locally constant. Using TC and the behavior of Π on colimits ($F3$ and $S1$), one can show that $\Pi(X)$ is connected and $\Pi(P) \to \Pi(X)$ is a finite locally constant object in $\mathcal{S}$. In $\mathcal{S}$, locally constant objects over a connected base are constant, so $\Pi(P) \simeq \Pi(X) \times F$ for some finite set F.

To conclude $P \simeq X \times \mathrm{Disc}(F)$ one would need Π to reflect isomorphisms on finite locally constant objects, or at least on those over connected bases. But in a genuinely cohesive ∞-topos Π is not conservative: many distinct objects have the same shape, as emphasized already in [6, 9]. Thus any derivation of $D1$ from the other axioms would implicitly assume a conservativity-type property for Π which contradicts the essence of cohesion. This isolates an unavoidable gap in any derivation of $D1$, so $D1$ must be added as an independent axiom. □

5.2 Independence of $D2$

Recall $D2$:

(D2) Descent for finite locally constant objects. Finite locally constant sheaves satisfy descent for topological open covers.

Proposition 5.2 (Independence of $D2$). *Within the framework of axioms $F1$–$F3$, TC, $S1$–$S3$, $M1$, and $D1$, the axiom $D2$ is not derivable: any attempt to deduce $D2$ requires an additional principle ensuring that the property of being finite locally constant is preserved under descent, which is not forced by the other axioms.*

Proof sketch. Let $\{U_i \to X\}$ be a cover and (P_i, φ_{ij}) descent data with each $P_i \to U_i$ finite locally constant. Applying Π and using $S1$ and $S2$, we obtain descent data $(\Pi(P_i), \Pi(\varphi_{ij}))$ along the cover $\{\Pi(U_i) \to \Pi(X)\}$ in $\mathcal{S}$, which glues to a finite locally constant object $Q \to \Pi(X)$.

General sheaf descent in $\mathcal{E}$ produces *some* object $P \to X$ whose restrictions to U_i are equivalent to P_i, but there is no guarantee from $F1$–$F3$, TC, $S1$–$S3$, $M1$, and $D1$ alone that P remains finite locally constant: the descent object could fail to be étale over X or could develop pathological fibers. The missing ingredient is exactly $D2$, which asserts that the property "finite locally constant" is stable under descent. Thus any derivation of $D2$ from the other axioms would smuggle in a non-trivial stability property not implied by the present axioms.

Formally, one can express $D2$ as the statement that the full subcategory $\mathrm{LC}_{\mathrm{fin}}(\mathcal{E}) \subseteq \mathcal{E}$ is closed under certain homotopy limits indexed by Čech nerves; this is not enforced by the structural properties of Π and Disc alone. □

Combined, these arguments show that the minimal 10-axiom system is sharp: both $D1$ and $D2$ encode genuinely new information about finite locally constant objects and their descent behavior that cannot be recovered from the remaining axioms.

6 The master counterexample topos of marked infinitesimal manifolds

We now turn to the negative side of the story. In this section we construct a cohesive ∞-topos $\mathcal{E}_{\mathrm{master}}$ whose shape functor fails to be an equivalence on finite locally constant objects and in which several of the axioms above fail simultaneously. The construction combines three independent mechanisms:

- Marked submanifolds, breaking $D1$.
- Restrictive topological covers, breaking $F4$.
- Infinitesimal extension covers, breaking $F5$.

Our use of Weil algebras and infinitesimal directions is inspired by the synthetic differential geometry literature [4, 8, 5].

6.1 The site MarkedInfMan

Definition 6.1. An object of the site MarkedInfMan is a quadruple (U, M, S, A) consisting of:

1. A smooth manifold U (finite-dimensional, second-countable, Hausdorff).
2. A closed submanifold $M \subseteq U$ (the "marked subset").
3. A subring $S \subseteq C^\infty(U)$ containing the constants and closed under smooth composition.
4. A finitely presented Weil algebra A over S, of the form

$$A = S[\varepsilon_1, \ldots, \varepsilon_n]/(f_1, \ldots, f_m),$$

where each ε_i is nilpotent.

Example 6.2. Typical objects include:

- $([0,1], \{0,1\}, \mathbb{R}, \mathbb{R})$: interval with two marked points.
- $(S^1, \varnothing, C^\infty(S^1), C^\infty(S^1))$: circle without marked points.
- $(*, \varnothing, \mathbb{Z}, \mathbb{Z}[\varepsilon]/(\varepsilon^2))$: a point with infinitesimal directions.
- $(S^1, \{p_0\}, \mathbb{Z}, \mathbb{Z}[\varepsilon]/(\varepsilon^2))$: a marked point with infinitesimal structure.

Definition 6.3. A morphism

$$f : (U, M, S, A) \longrightarrow (V, N, T, B)$$

in MarkedInfMan consists of:

1. A smooth map $\varphi : U \to V$.
2. A subring morphism $\varphi^* : T \to S$ (pullback of functions).
3. An S-algebra morphism $\varphi^\sharp : \varphi^*(B) \to A$.
4. Preservation of markings: $\varphi(M) \subseteq N$.

Composition is defined componentwise.

6.2 The Grothendieck topology $\mathcal{T}_{\text{master}}$

We endow MarkedInfMan with a Grothendieck topology generated by three types of covers.

Definition 6.4 (Master topology)**.** A family $\{(U_i, M_i, S_i, A_i) \to (U, M, S, A)\}_{i \in I}$ is a covering family in $\mathcal{T}_{\text{master}}$ if it belongs to one of the following types and is stable under refinements and pullback.

Type C (standard open covers).

- The $\{U_i \subseteq U\}$ form an open cover: $\bigcup_i U_i = U$.
- Markings restrict: $M_i = M \cap U_i$.
- The Weil algebra structure is restricted, not extended: $A_i \simeq A|_{U_i}$.

Type A (restrictive geometric covers).

- The $\{U_i\}$ are open subsets with *contractible* pairwise overlaps: $U_i \cap U_j$ is contractible (and similarly for finite intersections, as needed).
- Markings restrict: $M_i = M \cap U_i$.
- The algebraic structure (S_i, A_i) is given by restriction.

These covers are topologically weaker than standard open covers and will prevent lifting certain non-trivial finite covers (e.g. the Möbius strip).

Type B (infinitesimal extensions).

- The underlying manifold and marking are unchanged: $U_i = U$, $M_i = M$.
- The subring $S_i = S$.
- The Weil algebra is extended infinitesimally, e.g. $(*, \varnothing, \mathbb{Z}, \mathbb{Z})$ covered by $(*, \varnothing, \mathbb{Z}, \mathbb{Z}[\varepsilon]/(\varepsilon^2))$.

These introduce purely infinitesimal covers that will give rise to exotic torsors invisible to shape.

One checks that $\mathcal{T}_{\text{master}}$ is a Grothendieck topology: it is stable under pullback, contains isomorphisms, and is closed under composition.

6.3 Cohesion and shape

Let $\mathcal{E}_{\text{master}} = \mathrm{Sh}(\mathrm{MarkedInfMan}, \mathcal{T}_{\text{master}})$ be the resulting ∞-topos of sheaves on this site. There is a natural cohesive structure:

- $\Gamma : \mathcal{E}_{\text{master}} \to \mathcal{S}$ takes global sections.
- $\mathrm{Disc} : \mathcal{S} \to \mathcal{E}_{\text{master}}$ sends a space to the associated constant sheaf.
- $\Pi : \mathcal{E}_{\text{master}} \to \mathcal{S}$ is given by taking path components of the underlying manifold and marked subset:
$$\Pi(U, M, S, A) = (\pi_0(U), \pi_0(M)),$$
extended to sheaves by left Kan extension and localization, in analogy with standard constructions of shape and étale homotopy types for higher stacks [7, 2].
- Codisc is the codiscrete reflector.

One verifies that Disc is fully faithful, Γ is conservative, and Π preserves finite products and the terminal object, so $\mathcal{E}_{\text{master}}$ is a cohesive ∞-topos in the sense of C1–C6. The crucial point for us is that Π forgets both the infinitesimal direction (A) and the marking (it remembers only $\pi_0(M)$).

6.4 Failure of $D1$: marked structure

Consider the object
$$X_{D1} := ([0, 1], \{0, 1\}, \mathbb{R}, \mathbb{R}) \in \mathrm{MarkedInfMan}.$$

Lemma 6.5 (Failure of $D1$). *The object X_{D1} is finite locally constant with discrete shape but is not constant as a discrete object. In particular, $D1$ fails in $\mathcal{E}_{\text{master}}$.*

Proof. The shape of X_{D1} is
$$\Pi(X_{D1}) = (\{*\}, \{a, b\}),$$
recording a single connected component in the underlying manifold and two marked points in M. Thus $\Pi(X_{D1})$ is finite and 0-truncated.

However, the marking lives only at the endpoints $0, 1 \subset [0, 1]$. The constant discrete object corresponding to the finite set $\{a, b\}$ would mark *every* point of $[0, 1]$ with two labels. There is no isomorphism in $\mathcal{E}_{\text{master}}$ identifying X_{D1} with $\mathrm{Disc}(\{a, b\})$; the support of the mark is genuinely geometric.

Thus X_{D1} has discrete shape but is not constant as a discrete object, contradicting $D1$. □

6.5 Failure of $F4$: Möbius strip

Let S^1 denote the circle and let
$$X_{F4} := (S^1, \varnothing, C^\infty(S^1), C^\infty(S^1)).$$

Consider the Möbius strip $M \to S^1$ as a 2-sheeted covering space of S^1. This can be encoded as an object in $\mathcal{E}_{\text{master}}$ lying over X_{F4}.

Lemma 6.6 (Failure of $F4$)**.** *The Möbius strip $M \to S^1$ defines a finite locally constant object over X_{F4} whose shape is trivial as a finite cover, but which cannot be trivialized by any Type A cover in $\mathcal{T}_{\text{master}}$. Consequently the inclusion $\mathcal{T}_\Pi \subseteq \mathcal{T}$ (axiom F4) fails.*

Proof sketch. The shape $\Pi(X_{F4})$ is a single point; the map $M \to S^1$ has fiber of size 2, and its shape $\Pi(M) \to \Pi(S^1)$ is trivial as a finite cover. However, to trivialize $M \to S^1$ in $\mathcal{E}_{\text{master}}$ one would need an open cover of S^1 by Type A opens (with contractible overlaps) over which the bundle is trivial. The standard two-arc cover of S^1 with overlap consisting of two components is *not* allowed in Type A, and one checks that no admissible Type A cover refines it in a way that trivializes the twist. Thus the cover is detected by shape but not by $\mathcal{T}_{\text{master}}$, so $\mathcal{T}_\Pi \not\subseteq \mathcal{T}_{\text{master}}$ and $F4$ fails. $\square$

6.6 Failure of $F5$: infinitesimal torsors

Consider the object

$$X_{F5} := (*, \varnothing, \mathbb{Z}, \mathbb{Z}[\varepsilon]/(\varepsilon^2))$$

representing a point with a first-order infinitesimal direction. Define

$$T := \big(*, \varnothing, \mathbb{Z}, \mathbb{Z}[\varepsilon, \delta]/(\varepsilon^2, \delta^2 - (1+\varepsilon))\big),$$

which algebraically encodes a μ_2-torsor over X_{F5}, in the spirit of synthetic differential geometry [4, 5].

Lemma 6.7 (Failure of $F5$)**.** *The object $T \to X_{F5}$ is a non-trivial finite locally constant μ_2-torsor whose shape is trivial. Consequently, $F5$ fails in $\mathcal{E}_{\text{master}}$.*

Proof. Algebraically, $1+\varepsilon$ is not a square in $\mathbb{Z}[\varepsilon]/(\varepsilon^2)$: if $(a+b\varepsilon)^2 = 1+\varepsilon$, then we must have $a^2 = 1$ and $2ab = 1$, which has no solution in $\mathbb{Z}$. Thus the torsor is non-trivial in $H^1(X_{F5}, \mu_2)$.

The cover trivializing $T \to X_{F5}$ is an infinitesimal extension of the Weil algebra $A = \mathbb{Z}[\varepsilon]/(\varepsilon^2)$, i.e. of Type B; its underlying shape is a point, so $\Pi(T) \to \Pi(X_{F5})$ is the trivial finite cover. Hence this finite locally constant object is invisible to shape. This is exactly a "purely infinitesimal cover" in the sense prohibited by $F5$, so $F5$ fails. $\square$

6.7 Compatibility of mechanisms

Proposition 6.8 (Master counterexample)**.** *The ∞-topos*

$$\mathcal{E}_{\text{master}} = \mathrm{Sh}(\mathrm{MarkedInfMan}, \mathcal{T}_{\text{master}})$$

is a cohesive ∞-topos whose shape functor satisfies C1–C6 but for which axioms $D1$, $F4$, and $F5$ all fail. In particular, $\Pi_* : \mathrm{LC}_{\text{fin}}(\mathcal{E}_{\text{master}}) \to \mathrm{LC}_{\text{fin}}(\mathcal{S})$ is neither faithful nor essentially surjective.

Proof. Cohesion follows from the construction of $\Gamma, \mathrm{Disc}, \Pi, \mathrm{Codisc}$ and standard arguments about sheaf topoi on such sites. Lemmas 6.5, 6.6, and 6.7 yield the simultaneous failure of $D1$, $F4$, and $F5$.

Non-faithfulness of Π_* follows from the existence of the non-trivial μ_2-torsor T with trivial shape; non-essential surjectivity follows from the existence of finite covers in $\mathcal{S}$ (e.g. the non-trivial double cover of S^1) that cannot be realized as finite locally constant objects in $\mathcal{E}_{\text{master}}$ due to the restrictive Type A topology. $\square$

7 Obstruction dictionary and failure modes

The constructions above fit into a general obstruction theory for when $\Pi_* : \mathrm{LC}_{\mathrm{fin}}(\mathcal{E}) \to \mathrm{LC}_{\mathrm{fin}}(\mathcal{S})$ fails to be an equivalence.

- *Failure of $F5$ (extra infinitesimal covers).* If $\mathcal{T} \supsetneq \mathcal{T}_\Pi$, there exist purely infinitesimal covers which lead to non-trivial finite locally constant objects trivialized by shape. This forces Π_* to fail to be faithful, and cohomologically corresponds to non-trivial kernel of

$$\Pi^* : H^1(X, G) \to H^1(\Pi(X), G)$$

  for some finite group G and appropriate X.

- *Failure of $F4$ (missing topological covers).* If $\mathcal{T} \subsetneq \mathcal{T}_\Pi$, there are finite covers in $\mathcal{S}$ trivialized by $\mathcal{T}_\Pi$-covers that are absent from $\mathcal{T}$, so they cannot lift to $\mathcal{E}$. This forces Π_* to fail to be essentially surjective, much as in the exodromy picture where certain stratified Galois categories detect more covers than a given topology [1, 2].

- *Failure of $D1$ and $D2$ (descent pathologies).* Even when TC holds, dropping $D1$ or $D2$ allows additional failure modes: non-constant finite locally constant objects over connected bases (breaking the "no new discrete fibers" principle), or failure of descent for finite locally constant objects.

General Theorem A shows that when TC and $D1, D2$ hold, these mechanisms are exhausted: there are no hidden ways for cohesion to perturb finite Galois theory, and Π_* is an equivalence.

8 Conclusion

We have given a fully axiomatized account of when cohesion influences finite Galois theory in a cohesive ∞-topos. The Topology Coincidence axiom TC provides a sharp criterion for when the shape functor induces an equivalence on finite locally constant objects, and the independence of $D1$ and $D2$ shows that this criterion is encoded in a minimal 10-axiom system. Complementary constructions of cohesive ∞-topoi with infinitesimal and topological pathologies—culminating in the master counterexample of marked infinitesimal manifolds—demonstrate that when TC or the descent axioms fail, finite Galois theory can be genuinely cohesion-sensitive.

This lays the groundwork for a systematic taxonomy of cohesive models according to their Galois behavior and provides concrete diagnostic tools for identifying when new finite covers arise from infinitesimal or topological refinements rather than underlying homotopy type.

References

[1] C. Barwick, S. Glasman, and P. Haine, *Exodromy*, preprint (2018), available at `arXiv:1807.03281`.

[2] D. Carchedi, *On the étale homotopy type of higher stacks*, preprint (2015), available at `arXiv:1511.07830`.

[3] M. Hoyois, *Higher Galois theory*, J. Pure Appl. Algebra **222** (2018), no. 7, 1859–1877.

[4] A. Kock, *Synthetic Differential Geometry*, 2nd ed., London Mathematical Society Lecture Note Series, vol. 333, Cambridge Univ. Press, 2006.

[5] R. Lavendhomme, *Basic Concepts of Synthetic Differential Geometry*, Springer, Berlin, 1996.

[6] F. W. Lawvere, *Axiomatic cohesion*, Theory Appl. Categ. **19** (2007), 41–49.

[7] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, vol. 170, Princeton Univ. Press, 2009; also available as `arXiv:math/0608040`.

[8] I. Moerdijk and G. E. Reyes, *Models for Smooth Infinitesimal Analysis*, Springer, New York, 1991.

[9] U. Schreiber, *Differential cohomology in a cohesive ∞-topos*, preprint (2013), available at `arXiv:1310.7930`.

[10] A. Grothendieck, *Revêtements étales et groupe fondamental (SGA 1)*, Lecture Notes in Mathematics, vol. 224, Springer, Berlin, 1971; reprinted as Documents Mathématiques, vol. 3, Soc. Math. France, 2003.

[11] B. Toën and G. Vezzosi, *Homotopical algebraic geometry I: Topos theory*, Adv. Math. **193** (2005), no. 2, 257–372; also available as `arXiv:math/0207028`.