Theoretical Computer Science 609 (2016) 143–161

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Extended color visual cryptography for black and white secret image

Ching-Nung Yang *, Li-Zhe Sun, Song-Ruei Cai

Department of Computer Science and Information Engineering, National Dong Hwa University, #1, Sec. 2, Da Hsueh Rd., Hualien, Taiwan

ARTICLE INFO

Article history:
Received 19 November 2014
Received in revised form 12 March 2015
Accepted 17 September 2015
Available online 26 September 2015
Communicated by G. Persiano

Keywords:
Visual secret sharing
Visual cryptography
Extended visual cryptography
Black and white secret image
Color shadow image

ABSTRACT

Recently, De Prisco and De Santis introduced a (k, n) colored-black-and-white visual cryptography scheme (CBW-VCS), which adopts colored pixels in shadow images to share a black and white secret image. Only k or more participants can reconstruct the secret image by combining their shadow images. It seems that no one study (k, n)-CBW-VCS with the extended ability revealing meaningful shadow images (denoted as CBW-extended VCS (CBW-EVCS)). In this paper, we extend the conventional EVCS (i.e., BW-EVCS) to the CBW-EVCS. This paper has two main contributions: (i) we construct (k, n)-CWB-EVCSs, (ii) all constructions are proven to satisfy *security*, *contrast*, and *cover image* conditions.

1. Introduction

The cryptographic technique for the visual sharing of secret images, known as a visual cryptographic scheme (VCS) or a Visual Secret Sharing scheme (VSSS) was first proposed by Naor and Shamir [16]. VCS is a method to decompose a secret image into shadow images (called shadows) and distribute them to a number of participants. Only legitimate subsets of shadows can be used to reconstruct the original image. VCS is usually implemented as a threshold k-out-of-n scheme. A (k, n)-VCS, where $k \leq n$, divides a secret image into n shadows by expanding a pixel in the secret image into m (the pixel expansion) pixels in shadows. In a (k, n)-VCS, we can reconstruct the secret image by simply stacking together the shadows they own; but $(k-1)$ or fewer shadows cannot recover the secret image. VCS has a novel "stacking-to-see" property by which participants may photocopy their shadows onto transparencies and stack them to visually decode the secret. Its decoding requires neither knowledge of cryptography nor a computer.

Naor and Shamir's (k, n)-VCS shares a black-and-white secret image into black-and-white shadows. Because sizes of the pixel in secret image and the pixel in shadow are equal, the shadow size is expanded m times. Therefore, most studies attempted to reduce pixel expansion and enhance contrast [11,13,24,3,22], and some of them even had no pixel expansion. Various VCSs with specific functions, such as sharing multiple secrets, cheating prevention, achieving the ideal contrast, sharing gray image, retaining invariant aspect ratio, providing region incrementing property, providing progressive recovery, and adopting XOR operation were subsequently proposed [18,15,4,20,25,28,29,10,30].

Ito et al. [11] first proposed a VCS with no pixel expansion (i.e., $m = 1$) via randomly choosing a column from the black and white matrices of VCS. Yang [25] formally defined the so-called probabilistic VCS (PVCS) and provided various construction methods. The conventional VCS with the fixed m (>1) is referred to as the deterministic VCS (DVCS). The

* Corresponding author. Tel.: +886 3 8634025; fax: +886 3 8634010.
E-mail address: cnyang@mail.ndhu.edu.tw (C.-N. Yang).

http://dx.doi.org/10.1016/j.tcs.2015.09.016

major advantage of PVCS over DVCS is its non-expandable shadow size, while DVCS has the clear reconstructed image. Afterwards, a generalized PVCS with the pixel expansion $1 \le m_P \le m$ was further extended by Cimato et al. [3]. For $m_P = 1$, there is no pixel expansion and Cimato et al.'s VCS is the PVCS. On the other hand, Cimato et al.'s VCS becomes DVCS for $m_P = m$.

A very similar secret imaging technology like PVCS, the random grid (RG), was introduced by Kafri and Keren [12]. Kafri and Keren's RG shares a binary secret image into two noise-like random grids. When superimposing two random grids, only two stacked white pixels will let the light through it, while other stacked results yielding a black pixel stops the light. Finally, RG also has both the stacking-to-see property and the un-expandable shadows. In fact, RG and VCS have no difference other than the terminology. They are only different ways of looking at the same problem. Recently, the authors in [8,31] have shown that these models (PVCS, DVCS, and RG) are related to each other.

VCS can also be applied on colored secret images. A trivial solution is to convert the colored image into a binary image by the halftoning technique and then processes it by black-and-white VCS [9]. Various colored VCS (CVCS) were chronologically proposed [19,23,2,17,5,27,22,6]. Recently, De Prisco and De Santis [7] introduced a new notion of colored-black-and-white VCS (known as CBW-VCS). Same as the CVCS, the CBW-VCS has colored pixels in shadows but it shares the black-and-white secret image not the colored image. When compared with the black-and-white VCS, the CBW-VCS can reduce the pixel expansion and enhance the contrast due to using colored pixels in shadows.

In implementing VCS it would be useful to have a meaningful image on shadow. Any user will recognize the cover image on his shadow, so that the involved participants can easily identify and manage their shadows. Also, the meaningful shadows can conceal the existence of the secret image [1]. However, De Prisco and De Santis's CWB-VCS has noise-like shadows. In this paper, we study the CWB-VCS with the extended capability revealing a meaningful image on shadow (CWB-EVCS for short). There were already some (k, n)-EVCSs [1,26,21,14,32]. However, it seems that no one study CBW-EVCS. In this paper, via the idea of Ateniese et al.'s EVCS in [1], we give the conditions (*security*, *contrast*, and *cover image* conditions) of (k, n)-CWB-EVCS and extend the EVCS from BW model to CBW model.

The rest of this paper is organized as follows. In Section 2, (k, n)-VCS and (k, n)-CBW-VCS are briefly reviewed. In Section 3, we give the conditions of (k, n)-CBW-EVCS and show different constructions. Additionally, we prove that our (k, n)-CBW-EVCSs satisfy these conditions. Experiments and comparisons are included in Section 4. Examples and experimental results demonstrate that our (k, n)-CBW-EVCSs at least have the same pixel expansion and contrast when compared with the (k, n)-VCS. Meanwhile, our scheme provides the extended capability that shadows reveal the cover image. Finally, the conclusion is provided in Section 5.

2. Preliminary

2.1. (k, n)-VCS

A conventional (k, n)-VCS (a black-and-white VCS) is defined as the black and white $(n \times m)$ Boolean base matrices $B_\bullet$ and $B_\circ$, on which we can determine the color. The collection $C_\bullet$ (respectively, $C_\circ$) is obtained by permuting the columns of the corresponding matrix $B_\bullet$ (respectively, $B_\circ$) in all possible ways. When sharing a black (respectively, white) secret pixel, the dealer randomly chooses one row of the matrix in $C_\bullet$ (respectively, $C_\circ$) to a relative shadow. In (k, n)-VCS, every pixel is subdivided into m black-and-white pixels in each of n shadows. A (k, n)-VCS uses h black pixels and $(m - h)$ white pixels (denoted as $(h \bullet (m - h)\circ)$), and $(l \bullet (m - l)\circ)$, where $0 \le l < h \le m$, to represent black and white secret pixels, respectively. The values of h and l are the blackness of black color and white color. A VCS is the perfect black VCS (PB-VCS) for $h = m$. Let X be a set on involved participants. We denote $w_\bullet(v)$ as the number of elements of v equal to $\bullet$. Suppose that M is an $n \times m$ matrix. Let $(M|X)$ be a $|X| \times m$ matrix selecting the rows of the corresponding participants in X from M. Then, $\text{add}(M|X)$ denotes the OR-ed vector of all rows in $(M|X)$ and $D(M|X)$ denotes a set including all distribution matrices obtained by permuting all the columns in $(M|X)$. The formal definition of a VCS is given as follows [16,19].

Definition 1. A (k, n)-VCS is given by $(n \times m)$ black and white base matrices $B_\bullet$ and $B_\circ$ satisfying the following two conditions.

(V-1) (Contrast condition): Given any qualified set X, where $|X| = k$, we have that $w_\bullet(\text{add}(B_\bullet \mid X)) \ge h$ (respectively, $w_\bullet(\text{add}(B_\circ \mid X)) \le l$), where $0 \le l < h \le m$.

(V-2) (Security condition): Given any forbidden set X, where $|X| < k$, we have $D_\bullet = D(B_\bullet \mid X)$ and $D_\circ = D(B_\circ \mid X)$. Then, the collections $D_\bullet$ and $D_\circ$ are indistinguishable in the sense that they contain the same matrices with the same frequencies. For simplicity, we say that both collections $D_\bullet$ and $D_\circ$ are equivalent.

The first condition (V-1) is the contrast condition and the second condition (V-2) is the security condition. In [16], the contrast is defined as the difference in blackness between a black pixel and a white pixel in the reconstructed image, i.e., $\alpha = \frac{(h-l)}{m}$. Obviously, our aim in designing a VCS is to achieve the contrast that is as large as possible (or to minimize the pixel expansion as small as possible) for reconstructing a clearer image.

Table 1.1
Full intensity colors.

Color	(R, G, B)
R	$(\circ\bullet\bullet)$
G	$(\bullet\circ\bullet)$
B	$(\bullet\bullet\circ)$
C	$(\bullet\circ\circ)$
M	$(\circ\bullet\circ)$
Y	$(\circ\circ\bullet)$
$\bullet$	$(\bullet\bullet\bullet)$
$\circ$	$(\circ\circ\circ)$

Table 1.2
Stacked results of full intensity colors.

$+$	R	G	B	C	M	Y	$\bullet$	$\circ$
R	R	$\bullet$	$\bullet$	$\bullet$	R	R	$\bullet$	R
G	$\bullet$	G	$\bullet$	G	$\bullet$	G	$\bullet$	G
B	$\bullet$	$\bullet$	B	B	B	$\bullet$	$\bullet$	B
C	$\bullet$	G	B	C	B	G	$\bullet$	C
M	R	$\bullet$	B	B	M	R	$\bullet$	M
Y	R	G	$\bullet$	G	R	Y	$\bullet$	Y
$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
$\circ$	R	G	B	C	M	Y	$\bullet$	$\circ$

2.2. De Prisco and De Santis's (k, n)-CBW-VCS

In a color mixing formula, there are three primary color lights with equal intensities – red (R), green (G) and blue (B). Two or more colors come together can form a different color. For example, R mixed with G (respectively, B) yields yellow (Y) (respectively, magenta (M)), and B mixed with G produces cyan (C). Additionally, a combination of all R, G, and B creates white ($\circ$), whereas black ($\bullet$) color is shown when there is an absence of all three primary colors. These eight colors are full intensity colors, as shown in Table 1.1, and their stacked results are given in Table 1.2.

In [7], De Prisco and De Santis showed that using only full intensity colors can construct the CBW-VCS with the optimal contrast. Without loss of generality, they only use full intensity colors to construct $(2, 2)$-CBW-VCS, $(2, 3)$-CBW-VCS, $(2, n)$-CBW-VCS, $(2, n)$-PB-CBW-VCS, and (k, n)-PB-CBW-VCS. In these CBW-VCSs, the black pixel in the reconstructed image is used as the ($\bullet$) color and any non-black pixel ($R, G, B, C, M, Y, \circ$) are regarded as white pixel like the ($\circ$) in the conventional VCS. Here, we denote these non-black pixels as ($*$) in CBW-VCS. A (k, n)-CBW-VCS uses $(h \bullet (m - h)*))$ and $(l \bullet (m - l)*)$, as black and white secret pixels, respectively. Conditions (V-1) and (V-2) in VCS are also satisfied for the CBW-VCS. We use the following simple example, a $(2, 2)$-CWB-VCS, to show the design concept of De Prisco and De Santis's (k, n)-CBW-VCS.

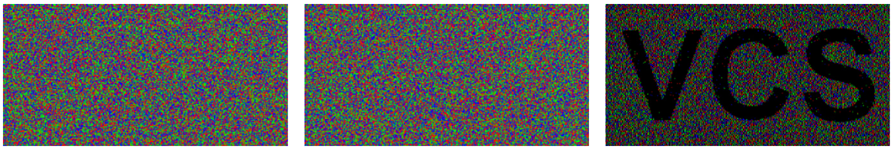
Example 1. Construct De Prisco and De Santis's $(2, 2)$-CBW-EVCS with background colors $\{RGB\}$ and $\{RGBYMC\}$, respectively.

The collections $C_\bullet$ and $C_\circ$ of $(2, 2)$-CBW-EVCS with $\{RGB\}$ and $(2, 2)$-CBW-EVCS with $\{RGBYMC\}$ are shown in Eq. (1.1) and Eq. (1.2), respectively.

$$C_\bullet = \left\{ \begin{bmatrix} R \\ G \end{bmatrix} \begin{bmatrix} G \\ R \end{bmatrix} \begin{bmatrix} R \\ B \end{bmatrix} \begin{bmatrix} B \\ R \end{bmatrix} \begin{bmatrix} G \\ B \end{bmatrix} \begin{bmatrix} B \\ G \end{bmatrix} \right\}, \quad C_\circ = \left\{ \begin{bmatrix} R \\ R \end{bmatrix} \begin{bmatrix} G \\ G \end{bmatrix} \begin{bmatrix} B \\ B \end{bmatrix} \right\}. \tag{1.1}$$

$$C_\bullet = \left\{ \begin{bmatrix} R \\ C \end{bmatrix} \begin{bmatrix} C \\ R \end{bmatrix} \begin{bmatrix} G \\ M \end{bmatrix} \begin{bmatrix} M \\ G \end{bmatrix} \begin{bmatrix} B \\ Y \end{bmatrix} \begin{bmatrix} Y \\ B \end{bmatrix} \right\}, \quad C_\circ = \left\{ \begin{bmatrix} R \\ R \end{bmatrix} \begin{bmatrix} G \\ G \end{bmatrix} \begin{bmatrix} B \\ B \end{bmatrix} \begin{bmatrix} C \\ C \end{bmatrix} \begin{bmatrix} M \\ M \end{bmatrix} \begin{bmatrix} Y \\ Y \end{bmatrix} \right\}. \tag{1.2}$$

In Eq. (1.1), it is observed that $w_\bullet(M \mid X_2) = 1$ for $M \in C_\bullet$, $w_\bullet(M \mid X_2) = 0$ for $M \in C_\circ$, and $D_\bullet = D_\circ = \{R, G, B\}$ satisfy contrast condition (V-1) and security condition (V-2). Here we use $(x \bullet y^*)$ to represent $(\overbrace{\bullet \cdots \bullet}^{x}\overbrace{* \cdots *}^{y})$ and its permutations, where $*$ denotes a color in $\{R, G, B\}$. In a reconstructed image, the black color is $(1\bullet)$ and the white color is $(1*)$. Because every pixel in two shadows S_1 and S_2 is $(1*)$, shadows are noise-like (see Fig. 1(a)). As shown in Fig. 1(b), we can visually decode the secret image, a printed-text $\boxed{\text{VCS}}$. It can be verified that Eq. (1.2) also satisfies conditions (V-1) and (V-2). The $(2, 2)$-CBW-EVCS with $\{RGBYMC\}$ also has $(1\bullet)$ and $(1*)$ as black color and white color in a reconstructed image, where $* \in \{R, G, B, Y, M, C\}$. Fig. 2 shows two shadows and the reconstructed image. For this case, the background color is a mixture of the colored pixels in $\{R, G, B, C, M, Y\}$. The average light transmission of background λ can be enhanced from



(a-1) S_1 (a-2) S_2 (b) (S_1+S_2)

Fig. 1. $(2,2)$-CBW-VCS with $\{RGB\}$: (a) shadows, (b) the reconstructed image.

(a-1) S_1 (a-2) S_2 (b) (S_1+S_2)

Fig. 2. $(2,2)$-CBW-VCS with $\{RGBCMY\}$: (a) shadows, (b) the reconstructed image.

Table 2
Description and example of notations.

Notation	Description/Example
$S^{(1)}$	A set of all the strings of length m' over the alphabet Σ. For example, in Construction 1, $m'=1$ and $\Sigma=\{R,G,B\}$ and thus $S^{(1)}=\{R,G,B\}$ has 3 elements. On the other hand, in Construction 4, we have $S^{(1)}=\{RR,RG,RB,GR,GG,GB,BR,BG,BB\}$ with 9 elements for $m'=2$ and $\Sigma=\{R,G,B\}$.
$S^{(i)}$	$S^{(i)}$, $2\le i\le n$, is a set by removing the chosen element from the set $S^{(i-1)}$, above which is the set $S^{(i)}$ located in the same column of a matrix. For example, in Construction 1, if R is chosen from $S^{(1)}=\{R,G,B\}$, then $S^{(2)}$ is $\{G,B\}$.
$\bar{s}$	$\bar{s}$ is complementary of s. For example, $\bar{s}=(\bullet\circ\circ)=C$ for $s=R=(\circ\bullet\bullet)$ and $\bar{s}=(\bullet\bullet\circ)=B$ for $s=Y=(\circ\circ\bullet)$.

$\lambda=33\%$ $(=(\frac{1}{3}+\frac{1}{3}+\frac{1}{3})/3)$ in Fig. 1(b) to $\lambda=50\%$ $(=(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3})/6)$ in Fig. 2(b), so that we can more clearly decode the secret. We use the definition of $\alpha=\frac{(h-l)}{m}$ to calculate the contrasts. All the contrasts of these two $(2,2)$-CBW-VCS are $\alpha=\frac{(1-0)}{1}=1$. Notice that a contrast of 1 cannot be achieved in the conventional VCS. Because any non-black pixel is regarded as a "white" pixel, it is possible for the CBW-VCS.

3. The proposed CBW-EVCS

Let the black and white matrices be $A_s^{c_1c_2\cdots c_n}$, where $c_i\in\{\bullet,\circ\}$ and $1\le i\le n$, denotes the color on the i-th cover image C_i, and the secret pixel is $s\in\{\bullet,\circ\}$. For example, the $A_\circ^{\bullet\bullet}$ of a $(2,2)$-CBW-EVCS is a base matrix that the pixels of both cover images c_1 and c_2 are all black colors, and their stacked result is a white color. We use the notations $(h_s\bullet(m-h_s)*))$ and $(l_s\bullet(m-l_s)*)$ to represent black and white pixels on shadows, where h_s and l_s are the blackness of black and white colors in cover images. Analogous to Ateniese et al.'s EVCS [1], the formal definition of the proposed (k,n)-CBW-EVCS is given as follows.

Definition 2. A (k,n)-CBW-EVCS is given by $(n\times m)$ black and white base matrices $A_\bullet^{c_1c_2\cdots c_n}$ and $A_\circ^{c_1c_2\cdots c_n}$ satisfying the following three conditions.

(C-1) (Contrast condition): Given any $c_1,c_2,\ldots,c_n$, and any qualified set X with $|X|\ge k$, we have that $w_\bullet(\mathrm{add}(A_\bullet^{c_1c_2\cdots c_n}\mid X))\ge h$ and $w_\bullet(\mathrm{add}(A_\circ^{c_1c_2\cdots c_n}\mid X))\le l$). Note: h and l are the blackness of black and white colors in the reconstructed image and $0\le l<h\le m$.

(C-2) (Security condition): Given any $c_1,c_2,\ldots,c_n$, and any forbidden set X with $|X|<k$, we have $D_\bullet=D(A_\bullet^{c_1c_2\cdots c_n}\mid X)$ and $D_\circ=D(A_\circ^{c_1c_2\cdots c_n}\mid X)$). Then, the collections $D_\bullet$ and $D_\circ$ are equivalent.

(C-3) (Cover image condition): Given any $c_1,\ldots,c_{i-1}$, $c_{i+1},\ldots,c_n$, $i\in\{1,n\}$ and the secret pixel $s\in\{\bullet,\circ\}$, we have that $w_\bullet(\mathrm{add}(A_s^{c_1,\cdots,c_i=\bullet,\cdots,c_n}\mid i))\ge h_s$ and $w_\bullet(\mathrm{add}(A_s^{c_1,\cdots,c_i=\circ,\cdots,c_n}\mid i))\le l_s$, where $0\le l_s<h_s\le m$. Note: the notation $(A_s^{c_1,c_2,\cdots,c_n}\mid i)$ denotes the i-th row in $A_s^{c_1,c_2,\cdots,c_n}$.

In subsequent sub sections, we describe $(2,n)$-CBW-EVCS and (k,n)-PB-CBW-EVCS. Some notations in our constructions are first described in Table 2.

The 2-out-f-2 CBW-EVCS is a special case for $k = n = 2$. We first introduce $(2,2)$-CBW-EVCS, and then describe the general $(2,n)$-CBW-EVCS and (k,n)-CBW-EVCS. In all constructions, we show the generation of $A_s^{c_1c_2\cdots c_n}$. The collection $C_s^{c_1c_2\cdots c_n}$ is composed of all possible distribution matrices $A_s^{c_1c_2\cdots c_n}$.

3.1. $(2,n)$-CBW-EVCS

3.1.1. $(2,2)$-CBW-EVCS

There are three different ways to implement $(2,2)$-CBW-EVCS. In Construction 1, Construction 2, and Construction 3, we show how to construct $(2,2)$-CBW-EVCS with background colors $\{RGB\}$, background colors $\{RGBCMY\}$, and two background colors $\{b_1b_2\}$, respectively.

Construction 1. Let $A_\bullet^{c_1c_2}$ and $A_\circ^{c_1c_2}$ be $2 \times m$ black and white base matrices. Then, $(2,2)$-CBW-EVCS with background colors $\{RGB\}$, where $c_i \in \{\bullet, \circ\}$, $i = 1, 2$, has $A_\bullet^{c_1c_2} = \begin{bmatrix} a_\bullet^{11} & a_\bullet^{12} \\ a_\bullet^{21} & a_\bullet^{22} \end{bmatrix}$ and $A_\circ^{c_1c_2} = \begin{bmatrix} a_\circ^{11} & a_\circ^{12} \\ a_\circ^{21} & a_\circ^{22} \end{bmatrix}$. Let $S^{(1)}$ have $m' = 1$ and $\Sigma = \{R, G, B\}$, i.e., $S^{(1)} = \{R, G, B\}$. Elements of $A_\bullet^{c_1c_2}$ and $A_\circ^{c_1c_2}$ are obtained from Eq. (2).

$$\begin{cases} \text{Case 1: } c_1 = \circ \text{ and } c_2 = \circ\text{:} \\ a_\bullet^{11} \leftarrow S^{(1)}, a_\bullet^{12} \leftarrow S^{(1)}, a_\bullet^{21} \leftarrow S^{(2)}, a_\bullet^{22} \leftarrow S^{(2)};\, a_\circ^{11} \leftarrow S^{(1)}, a_\circ^{12} \leftarrow S^{(1)}, a_\circ^{21} = a_\circ^{11}, a_\circ^{22} \leftarrow S^{(2)} \\ \text{Case 2: } c_1 = \circ \text{ and } c_2 = \bullet\text{:} \\ a_\bullet^{11} \leftarrow S^{(1)}, a_\bullet^{12} \leftarrow S^{(1)}, a_\bullet^{21} \leftarrow S^{(2)}, a_\bullet^{22} = \bullet;\, a_\circ^{11} \leftarrow S^{(1)}, a_\circ^{12} \leftarrow S^{(1)}, a_\circ^{21} = a_\circ^{11}, a_\circ^{22} = \bullet \\ \text{Case 3: } c_1 = \bullet \text{ and } c_2 = \circ\text{:} \\ a_\bullet^{11} \leftarrow S^{(1)}, a_\bullet^{12} = \bullet, a_\bullet^{21} \leftarrow S^{(2)}, a_\bullet^{22} \leftarrow S^{(2)};\, a_\circ^{11} \leftarrow S^{(1)}, a_\circ^{12} = \bullet, a_\circ^{21} = a_\circ^{11}, a_\circ^{22} \leftarrow S^{(2)} \\ \text{Case 4: } c_1 = \bullet \text{ and } c_2 = \bullet\text{:} \\ a_\bullet^{11} \leftarrow S^{(1)}, a_\bullet^{12} = \bullet, a_\bullet^{21} \leftarrow S^{(2)}, a_\bullet^{22} = \bullet;\, a_\circ^{11} \leftarrow S^{(1)}, a_\circ^{12} = \bullet;\, a_\circ^{21} = a_\circ^{11}, a_\circ^{22} = \bullet \end{cases} \tag{2}$$

According to the definition of $S^{(i)}$, we can obtain the sets $S^{(2)}$ in Eq. (2). For example, in Case 1 of Eq. (2), the sets $S^{(2)}$ in $(a_\bullet^{21} \leftarrow S^{(2)})$ and $(a_\bullet^{22} \leftarrow S^{(2)})$ are $\{S^{(1)} - \{a_\bullet^{11}\}\}$ and $\{S^{(1)} - \{a_\bullet^{12}\}\}$, respectively. In Case 3 of Eq. (2), the set $S^{(2)}$ in $(a_\bullet^{22} \leftarrow S^{(2)})$ is $\{S^{(1)} - \{a_\bullet^{12}\}\} = S^{(1)} - \{\bullet\} = S^{(1)}$. By the same argument, other sets $S^{(2)}$ can be obtained.

Considering the example of generating all possible $A_\bullet^{\bullet\bullet}$, we always have $a_\bullet^{12} = a_\bullet^{22} = \bullet$. Because $a_\bullet^{11} \leftarrow S^{(1)}$ and $a_\bullet^{21} \leftarrow S^{(2)}$, we have 6 matrices of $A_\bullet^{\bullet\bullet} = \begin{bmatrix} a_\bullet^{11} & a_\bullet^{12} \\ a_\bullet^{21} & a_\bullet^{22} \end{bmatrix}$: $\begin{bmatrix} R\bullet \\ G\bullet \end{bmatrix}$, $\begin{bmatrix} R\bullet \\ B\bullet \end{bmatrix}$, $\begin{bmatrix} G\bullet \\ R\bullet \end{bmatrix}$, $\begin{bmatrix} G\bullet \\ B\bullet \end{bmatrix}$, $\begin{bmatrix} B\bullet \\ R\bullet \end{bmatrix}$, and $\begin{bmatrix} B\bullet \\ G\bullet \end{bmatrix}$. After column permutation, we have the collection $C_\bullet^{\bullet\bullet} = \left\{ \begin{bmatrix} R\bullet \\ G\bullet \end{bmatrix} \begin{bmatrix} R\bullet \\ B\bullet \end{bmatrix} \begin{bmatrix} \bullet R \\ \bullet G \end{bmatrix} \begin{bmatrix} \bullet R \\ \bullet B \end{bmatrix} \begin{bmatrix} G\bullet \\ R\bullet \end{bmatrix} \begin{bmatrix} G\bullet \\ B\bullet \end{bmatrix} \begin{bmatrix} \bullet G \\ \bullet R \end{bmatrix} \begin{bmatrix} \bullet G \\ \bullet B \end{bmatrix} \begin{bmatrix} B\bullet \\ R\bullet \end{bmatrix} \begin{bmatrix} B\bullet \\ G\bullet \end{bmatrix} \begin{bmatrix} \bullet B \\ \bullet R \end{bmatrix} \begin{bmatrix} \bullet B \\ \bullet G \end{bmatrix} \right\}$. The collections $C_\bullet^{c_1c_2}$ and $C_\circ^{c_1c_2}$ of all distribution matrices are shown in Table A1.

Theorem 1. *The scheme from Construction 1 is a $(2,2)$-CBW-EVCS with the pixel expansion $m = 2$. The blackness is $h = 2$, $l = 1$, $h_s = 1$, and $l_s = 0$.*

Proof. There are 2 columns in $A_\bullet^{c_1c_2}$ and $A_\circ^{c_1c_2}$, and thus the pixel expansion is $m = 2$. We first prove that this $(2,2)$-CBW-EVCS satisfies contrast condition (C-1). Consider the case $c_1 = \circ, c_2 = \circ$. Because $a_\bullet^{11} \in S^{(1)}$, $a_\bullet^{21} \in \{S^{(1)} - \{a_\bullet^{11}\}\}$, $a_\bullet^{12} \in S^{(1)}$ and $a_\bullet^{22} \in \{S^{(1)} - \{a_\bullet^{11}\}\}$ in $A_\bullet^{c_1c_2}$, we have $a_\bullet^{11} + a_\bullet^{21} = \bullet$ and $a_\bullet^{12} + a_\bullet^{22} = \bullet$, respectively. In fact, it can be verified that the stacked results of two rows in $A_\bullet^{c_1c_2}$ and $A_\circ^{c_1c_2}$ are $(\bullet\bullet)$ and $(\bullet *)$, respectively, where $* \in \{R, G, B\}$. Then, we have $w_\bullet(\text{add}(A_\bullet^{c_1c_2} \mid X_2)) = 2$ and $w_\bullet(\text{add}(A_\circ^{c_1c_2} \mid X_2)) = 1$, and thus $h = 2$ and $l = 1$, where the notation X_k denotes a set with any k participants.

For a proof of Condition (C-2), given any c_1 and c_2, and a set X_1 (note: the threshold of $(2,2)$-CBW-EVCS is 2), we should prove that the collections $D_\bullet$ and $D_\circ$ are equivalent. Suppose $c_1 = \bullet$ and $c_2 = \bullet$. For the first row, because $a_\bullet^{11} \leftarrow S^{(1)}$ and $a_\bullet^{12} = \bullet$, we have $D_\bullet = \{[R\bullet][G\bullet][B\bullet][\bullet R][\bullet G][\bullet B]\}$, and because $a_\circ^{11} \leftarrow S^{(1)}$ and $a_\circ^{12} = \bullet$, we have $D_\circ = \{[R\bullet][G\bullet][B\bullet][\bullet R][\bullet G][\bullet B]\}$. For the second row, since $a_\bullet^{21} \leftarrow S^{(2)} = \{S^{(1)} - a_\bullet^{11}\}$ and after randomly choosing $a_\bullet^{11}$ from $S^{(1)} = \{R, G, B\}$, the value of $a_\bullet^{21}$ could be R, G, and B, and meantime $a_\bullet^{22}$ is always $\bullet$. Thus, $D_\bullet = \{[R\bullet][G\bullet][B\bullet][\bullet R][\bullet G][\bullet B]\}$. On the other hand, because $a_\circ^{21} = a_\circ^{11}$ and $a_\circ^{11}$ is chosen from $S^{(1)} = \{R, G, B\}$, the value of $a_\circ^{21}$ could be R, G, and B, and meantime $a_\circ^{22}$ is always $\bullet$. Thus, we have $D_\circ = \{[R\bullet][G\bullet][B\bullet][\bullet R][\bullet G][\bullet B]\}$. The above implies that $D_\bullet$ and $D_\circ$ are equivalent (i.e., contain the same matrices with the same frequencies) for $c_1 = \bullet$ and $c_2 = \bullet$. By the same argument, it can be verified that $D_\bullet$ and $D_\circ$ are also equivalent for other c_1 and c_2. Next, we prove the condition (C-3). From Eq. (2), we have $w_\bullet(\text{add}(A_s^{c_1=\bullet,c_2} \mid 1)) = 1$, $w_\bullet(\text{add}(C_s^{c_1=\circ,c_2} \mid 1)) = 0$, $w_\bullet(\text{add}(A_s^{c_1,c_2=\bullet} \mid 2)) = 1$, and $w_\bullet(\text{add}(C_s^{c_1,c_2=\circ} \mid 2)) = 0$. Thus, we have $h_s = 1$ and $l_s = 0$. □

In Construction 1, the background colors of shadows are colored pixels in $\{R, G, B\}$. We can construct another $(2,2)$-CBW-EVCS with $\{R, G, B, C, M, Y\}$, as shown in Construction 2. Same as the $(2,2)$-CBW-VCS (see Example 1), λ in

(1) choose any n ($\le 3m'$) elements from $S^{(1)}$ to form a new set $S^{(1)}$
(2) **for** $i=1$ to n do $\{a^i \leftarrow S^{(i)}\}$;
(3) **for** $i=1$ to n do $\{$**if** $c_i=\bullet$ **then** $a^i=(\overbrace{\bullet\cdots\bullet}^{m'})\}$;

Fig. 3. Generation of $A^{c_1c_2\cdots c_n}$.

Construction 1 can be enhanced from 33% to 50% in Construction 2, so that we can decode the clearer image. Meanwhile, the clearness of shadow image is enhanced.

Construction 2. Let $A_\bullet^{c_1c_2}$ and $A_\circ^{c_1c_2}$ be $2\times m$ black and white matrices. Then, (2, 2)-CBW-EVCS with background colors $\{RGBCMY\}$, where $c_i\in\{\bullet,\circ\}$, $i=1,\ 2$, has $A_\bullet^{c_1c_2}=\begin{bmatrix} a_\bullet^{11} & a_\bullet^{12} \\ a_\bullet^{21} & a_\bullet^{22}\end{bmatrix}$ and $A_\circ^{c_1c_2}=\begin{bmatrix} a_\circ^{11} a_\circ^{12} \\ a_\circ^{21} a_\circ^{22}\end{bmatrix}$. Let $S^{(1)}$ have $m'=1$ and $\Sigma=\{R,G,B,C,M,Y\}$, i.e., $S^{(1)}=\{R,G,B,C,M,Y\}$. Elements of $A_\bullet^{c_1c_2}$ and $A_\circ^{c_1c_2}$ are obtained from Eq. (3).

$$\begin{cases} \text{Case 1: } c_1=\circ \text{ and } c_2=\circ: \\ a_\bullet^{11}\leftarrow S^{(1)}, a_\bullet^{12}\leftarrow S^{(1)}, a_\bullet^{21}=\overline{a_\bullet^{11}}, a_\bullet^{22}=\overline{a_\bullet^{12}};\ a_\circ^{11}\leftarrow S^{(1)}, a_\circ^{12}\leftarrow S^{(1)};\ a_\circ^{21}=a_\circ^{11}, a_\circ^{22}=\overline{a_\circ^{12}} \\ \text{Case 2: } c_1=\circ \text{ and } c_2=\bullet: \\ a_\bullet^{11}\leftarrow S^{(1)}, a_\bullet^{12}\leftarrow S^{(1)}, a_\bullet^{21}=\overline{a_\bullet^{11}}, a_\bullet^{22}=\bullet;\ a_\circ^{11}\leftarrow S^{(1)}, a_\circ^{12}\leftarrow S^{(1)};\ a_\circ^{21}=a_\circ^{11}, a_\circ^{22}=\bullet \\ \text{Case 3: } c_1=\bullet \text{ and } c_2=\circ: \\ a_\bullet^{11}\leftarrow S^{(1)}, a_\bullet^{12}=\bullet, a_\bullet^{21}=\overline{a_\bullet^{11}}, a_\bullet^{22}\leftarrow S^{(2)};\ a_\circ^{11}\leftarrow S^{(1)}, a_\circ^{12}=\bullet;\ a_\circ^{21}=a_\circ^{11}, a_\circ^{22}\leftarrow S^{(2)} \\ \text{Case 4: } c_1=\bullet \text{ and } c_2=\bullet: \\ a_\bullet^{11}\leftarrow S^{(1)}, a_\bullet^{12}=\bullet, a_\bullet^{21}=\overline{a_\bullet^{11}}, a_\bullet^{22}=\bullet;\ a_\circ^{11}\leftarrow S^{(1)}, a_\circ^{12}=\bullet;\ a_\circ^{21}=a_\circ^{11}, a_\circ^{22}=\bullet \end{cases} \tag{3}$$

According to the definition of $S^{(i)}$, in Case 3 of Eq. (3), the sets $S^{(2)}$ in $(a_\bullet^{22}\leftarrow S^{(2)})$ and $(a_\circ^{22}\leftarrow S^{(2)})$ are $\{S^{(1)}-\{a_\bullet^{12}\}\}=S^{(1)}-\{\bullet\}=S^{(1)}$ and $\{S^{(1)}-\{a_\circ^{12}\}\}=S^{(1)}-\{\bullet\}=S^{(1)}$.

Obviously, it is possible to construct (2, 2)-CBW-EVCS with only two colors, as shown in Construction 3.

Construction 3. Let $A_\bullet^{c_1c_2}$ and $A_\circ^{c_1c_2}$ be $2\times m$ black and white matrices. Then, (2, 2)-CBW-EVCS with two background colors $\{b_1b_2\}=\{RG\},\{RB\},\{GB\},\{CR\},\{MG\}$, or $\{YB\}$, where $c_i\in\{\bullet,\circ\}$, $i=1,2$, has $A_\bullet^{c_1c_2}=\begin{bmatrix} a_\bullet^{11} & a_\bullet^{12} \\ a_\bullet^{21} & a_\bullet^{22}\end{bmatrix}$ and $A_\circ^{c_1c_2}=\begin{bmatrix} a_\circ^{11} & a_\circ^{12} \\ a_\circ^{21} & a_\circ^{22}\end{bmatrix}$. Let $S^{(1)}$ have $m'=1$ and $\Sigma=\{b_1,\ b_2\}$, i.e., $S^{(1)}=\{b_1,\ b_2\}$. Elements of $A_\bullet^{c_1c_2}$ and $A_\circ^{c_1c_2}$ can be obtained by the same approach in Eq. (2).

The proofs of Construction 2 and Construction 3 being (2, 2)-CBW-EVCS are straight forward, and exactly the same as the proof of Theorem 1.

3.1.2. (2, n)-CBW-EVCS

Construction 4. Let $A_\bullet^{c_1c_2\cdots c_n}$ and $A_\circ^{c_1c_2\cdots c_n}$ be $n\times m$ black and white matrices. Then, $(2,n)$-CBW-EVCS with background colors $\{RGB\}$, where $c_i\in\{\bullet,\circ\}$, $i=1,2,\ldots,n$, has $A_\bullet^{c_1c_2\cdots c_n}=(A_\bullet \parallel A^{c_1c_2\cdots c_n})$ and $A_\circ^{c_1c_2\cdots c_n}=(A_\circ \parallel A^{c_1c_2\cdots c_n})$, where $A_\bullet$, $A_\circ$, and $A^{c_1c_2\cdots c_n}$ are obtained from the following. Let $S^{(1)}$ have $m'=\lceil \log_3 n\rceil$ characters over the alphabet $\Sigma=\{R,G,B\}$, i.e., $S^{(1)}=\{\overbrace{R\cdots RR}^{m'},\overbrace{R\cdots RG}^{m'},\overbrace{R\cdots RB}^{m'},\cdots,\overbrace{B\cdots BB}^{m'}\}$ with $3^{m'}$ elements. We first choose any n ($\le 3^{m'}$) elements from $S^{(1)}$ to form a new set $S^{(1)}$. Black and white matrices $A_\bullet=\begin{bmatrix} a_\bullet^1 \\ a_\bullet^2 \\ \vdots \\ a_\bullet^n\end{bmatrix}$ and $A_\circ=\begin{bmatrix} a_\circ^1 \\ a_\circ^2 \\ \vdots \\ a_\circ^n\end{bmatrix}$ are obtained from De Prisco and De Santis's $(2,n)$-CBW-VCS (Construction 5.5 in [7]) with the pixel expansion m', which are constructed from Eq. (4).

$$\begin{cases} (1)\ a_\bullet^1\leftarrow S^{(1)}, a_\bullet^2\leftarrow S^{(2)},\cdots,a_\bullet^n\leftarrow S^{(n)} \\ (2)\ a_\circ^1\leftarrow S^{(1)}, a_\circ^2=a_\circ^1,\cdots,a_\circ^n=a_\circ^1. \end{cases} \tag{4}$$

For the matrices of cover images $A^{c_1c_2\cdots c_n}$, we use another $S^{(1)}$ of $m'=\lceil n/3\rceil$ characters with $(m'-1)\bullet$ and 1 alphabet in $\Sigma=\{R,G,B\}$, i.e., $S^{(1)}=\{\overbrace{\bullet\cdots\bullet}^{m'-1}R,\overbrace{\bullet\cdots\bullet}^{m'-1}G,\overbrace{\bullet\cdots\bullet}^{m'-1}B,\cdots,R\overbrace{\bullet\cdots\bullet}^{m'-1},G\overbrace{\bullet\cdots\bullet}^{m'-1},B\overbrace{\bullet\cdots\bullet}^{m'-1}\}$ with $3m'$ elements. Then, $A^{c_1c_2\cdots c_n}=\begin{bmatrix} a^1 \\ a^2 \\ \vdots \\ a^n\end{bmatrix}$, where $n\le 3m'$, with the pixel expansion m' can be constructed from the algorithm in Fig. 3.

Lemma 1. *The matrix of cover image $A^{c_1c_2\cdots c_n}$ has $w_\bullet(\mathrm{add}(A^{c_1,\cdots c_i=\bullet,\cdots c_n} \mid i)) = \lceil n/3\rceil$ and $w_\bullet(\mathrm{add}(A^{c_1,\cdots c_i=\circ,\cdots c_n} \mid i)) = (\lceil n/3\rceil - 1)$ where $i = 1, 2, \ldots, n$. Also, $w_\bullet(\mathrm{add}(A^{c_1c_2\cdots c_n} \mid X_{k\geq 2})) = \lceil n/3\rceil$.*

Proof. According to the generation of $A^{c_1c_2\cdots c_n}$ in Fig. 3, we have the element a^i chosen from $S^{(i)}$ for $c_i = \circ$. From the definition of $S^{(1)} = \{\overbrace{\bullet\cdots\bullet}^{\lceil n/3\rceil-1} R, \overbrace{\bullet\cdots\bullet}^{\lceil n/3\rceil-1} G, \overbrace{\bullet\cdots\bullet}^{\lceil n/3\rceil-1} B, \cdots, R\overbrace{\bullet\cdots\bullet}^{\lceil n/3\rceil-1}, G\overbrace{\bullet\cdots\bullet}^{\lceil n/3\rceil-1}, B\overbrace{\bullet\cdots\bullet}^{\lceil n/3\rceil-1}\}$, we have $w_\bullet(\mathrm{add}(A^{c_1,\cdots c_i=\circ,\cdots c_n} \mid i)) = w_\bullet(a^i) = (\lceil n/3\rceil - 1)$. Because the element a^i is $(\overbrace{\bullet\cdots\bullet}^{\lceil n/3\rceil})$ for $c_i = \bullet$, we have $w_\bullet(add(A^{c_1,\cdots c_i=\bullet,\cdots c_n} \mid i)) = w_\bullet(a^i) = \lceil n/3\rceil$. Obviously, it can be easily verified that $\mathrm{add}(A^{c_1c_2\cdots c_n} \mid X_{k\geq 2}) = (\overbrace{\bullet\cdots\bullet}^{\lceil n/3\rceil})$, and thus we have $w_\bullet(\mathrm{add}(A^{c_1c_2\cdots c_n} \mid X_{k\geq 2})) = \lceil n/3\rceil$. □

Theorem 2. *The scheme from Construction 4 is a $(2, n)$-CBW-EVCS with the pixel expansion $m = \lceil\log_3 n\rceil + \lceil n/3\rceil$. The blackness is $h \geq (\lceil n/3\rceil + 1)$, $l = \lceil n/3\rceil$, $h_s = \lceil n/3\rceil$, and $l_s = (\lceil n/3\rceil - 1)$.*

Proof. Because $A_\bullet^{c_1c_2\cdots c_n} = (A_\bullet \parallel A^{c_1c_2\cdots c_n})$ and $A_\circ^{c_1c_2\cdots c_n} = (A_\circ \parallel A^{c_1c_2\cdots c_n})$, we have $|A_\bullet^{c_1c_2\cdots c_n}| = |A_\bullet| + |A^{c_1c_2\cdots c_n}| = \lceil\log_3 n\rceil + \lceil n/3\rceil$ and $|A_\circ^{c_1c_2\cdots c_n}| = |A_\circ| + |A^{c_1c_2\cdots c_n}| = \lceil\log_3 n\rceil + \lceil n/3\rceil$. The pixel expansion is $m = \lceil\log_3 n\rceil + \lceil n/3\rceil$. We first prove that $(2, n)$-CBW-EVCS satisfies contrast condition (C-1). Because $A_\bullet$ and $A_\circ$ are base matrices of De Prisco and De Santis's $(2, n)$-CBW-VCS, we have $w_\bullet(\mathrm{add}(A_\bullet \mid X_2)) \geq 1$ and $w_\bullet(\mathrm{add}(A_\circ \mid X_2)) = 0$. From Lemma 1, we have $w_\bullet(\mathrm{add}(A^{c_1c_2\cdots c_n} \mid X_2)) = \lceil n/3\rceil$. The values of $w_\bullet(\mathrm{add}(A_\bullet^{c_1c_2\cdots c_n} \mid X_2))$ and $w_\bullet(\mathrm{add}(A_\circ^{c_1c_2\cdots c_n} \mid X_2))$ are determined from Eq. (5.1) and Eq. (5.2), respectively. Finally, we have the blackness $h = w_\bullet(\mathrm{add}(A_\bullet^{c_1c_2\cdots c_n} \mid X_2)) \geq (\lceil n/3\rceil + 1)$ and $l = w_\bullet(\mathrm{add}(A_\circ^{c_1c_2\cdots c_n} \mid X_2)) = \lceil n/3\rceil$.

$$\begin{cases} w_\bullet(\mathrm{add}(A_\bullet^{c_1c_2\cdots c_n} \mid X_2)) = w_\bullet(\mathrm{add}((A_\bullet \parallel A^{c_1c_2\cdots c_n}) \mid X_2)) = w_\bullet(\mathrm{add}(A_\bullet \mid X_2)) \\ \qquad\qquad + w_\bullet(\mathrm{add}(A^{c_1c_2\cdots c_n} \mid X_2)) \geq (\lceil n/3\rceil + 1)(5-1) \qquad (5.1) \\ w_\bullet(\mathrm{add}(A_\circ^{c_1c_2\cdots c_n} \mid X_2)) = w_\bullet(\mathrm{add}((A_\circ \parallel A^{c_1c_2\cdots c_n}) \mid X_2)) \\ \qquad\qquad = w_\bullet(\mathrm{add}(A_\circ \mid X_2)) + w_\bullet(\mathrm{add}(A^{c_1c_2\cdots c_n} \mid X_2)) = \lceil n/3\rceil \qquad (5.2) \end{cases} \tag{5}$$

For a proof of Condition (C-2), by definition, we can derive $D_\bullet = D(A_\bullet^{c_1c_2\cdots c_n} \mid X)$ and $D_\circ = D(A_\circ^{c_1c_2\cdots c_n} \mid X)$, where $|X| < 2$, as follows.

$$\begin{cases} D(A_\bullet^{c_1c_2\cdots c_n} \mid X) = D((A_\bullet \parallel A^{c_1c_2\cdots c_n}) \mid X) = D(A_\bullet \mid X) \cup D(A^{c_1c_2\cdots c_n} \mid X), \\ D(A_\circ^{c_1c_2\cdots c_n} \mid X) = D((A_\circ \parallel A^{c_1c_2\cdots c_n}) \mid X) = D(A_\circ \mid X) \cup D(A^{c_1c_2\cdots c_n} \mid X). \end{cases} \tag{6}$$

Collections $D(A_\bullet \mid X)$ and $D(A_\circ \mid X)$ are equivalent since they comes from De Prisco and De Santis's $(2, n)$-CBW-VCS. Therefore, $D_\bullet = D(A_\bullet^{c_1c_2c_3} \mid X)$ and $D_\circ = D(A_\circ^{c_1c_2c_3} \mid X)$ are equivalent. Next, we prove the condition (C-3). Because $A_\bullet$ and $A_\circ$ are the base matrices of De Prisco and De Santis's $(2, n)$-CBW-VCS, we have $w_\bullet(\mathrm{add}(A_\bullet \mid i)) = w_\bullet(\mathrm{add}(A_\circ \mid i)) = 0$. We can derive $w_\bullet(\mathrm{add}(A_s^{c_1c_2\cdots c_n} \mid i)) = w_\bullet(\mathrm{add}(A_s \parallel A^{c_1c_2\cdots c_n}) \mid i)) = w_\bullet(\mathrm{add}(A_s \mid i)) + w_\bullet(\mathrm{add}(A^{c_1c_2\cdots c_n} \mid i)) = w_\bullet(\mathrm{add}\, A^{c_1c_2\cdots c_n} \mid i))$. By Lemma 1, we have $w_\bullet(\mathrm{add}(A^{c_1,\cdots c_i=\bullet,\cdots c_n} \mid i)) = \lceil n/3\rceil$ and $w_\bullet(\mathrm{add}(A^{c_1,\cdots c_i=\circ,\cdots c_n} \mid i)) = (\lceil n/3\rceil - 1)$, i.e., $h_s = \lceil n/3\rceil$, and $l_s = (\lceil n/3\rceil - 1)$. □

Construction 4 shows a $(2, n)$-CBW-EVCS. However, a PB-VCS has the clearer reconstructed image via perfect reconstruction of black pixels. Therefore, we propose a $(2, n)$-PB-CBW-EVCS.

Construction 5. Let $A_\bullet^{c_1c_2\cdots c_n}$ and $A_\circ^{c_1c_2\cdots c_n}$ be $n \times m$ black and white matrices. Then, $(2, n)$-PB-CBW-EVCS with background colors $\{RGB\}$, where $c_i \in \{\bullet, \circ\}$, $i = 1, 2, \ldots, n$, has $A_\bullet^{c_1c_2\cdots c_n} = (A_\bullet \parallel A^{c_1c_2\cdots c_n})$ and $A_\circ^{c_1c_2\cdots c_n} = (A_\circ \parallel A^{c_1c_2\cdots c_n})$, where $A_\bullet$ and $A_\circ$ are obtained from the following. The matrix $A^{c_1c_2\cdots c_n}$ is the same as that in Construction 4.

Let $S^{(1)}$ be the set $S^{(1)}$ used to generate $A^{c_1c_2\cdots c_n}$ in Fig. 3. We first choose any n $(\leq 3^{m'})$ elements from $S^{(1)}$ to form a new set $S^{(1)}$. Black and white matrices $A_\bullet = \begin{bmatrix} a_\bullet^1 \\ a_\bullet^2 \\ \vdots \\ a_\bullet^n \end{bmatrix}$ and $A_\circ = \begin{bmatrix} a_\circ^1 \\ a_\circ^2 \\ \vdots \\ a_\circ^n \end{bmatrix}$ are base matrices of De Prisco and De Santis's $(2, n)$-PB-CBW-VCS (Construction 7.1 in [7]), which can also be constructed by the same approach in Eq. (4).

The proof of Construction 5 being a $(2, n)$-PB-CBW-VCS is similar to the proof of Theorem 2. This $(2, n)$-PB-CBW-VCS has $m = 2 \times \lceil n/3\rceil$, $h = 2 \times \lceil n/3\rceil$, $l = (2 \times \lceil n/3\rceil - 1)$, $h_s = (2 \times \lceil n/3\rceil - 1)$ and $l_s = (2 \times \lceil n/3\rceil - 2)$.

3.2. (k, n)-PB-CBW-EVCS

This section shows a general (k, n)-PB-CBW-EVCS for any k and n, where $k \leq n$. This scheme is constructed from the (k, n)-PB-VCS.

Construction 6. Let $A_{\bullet}^{c_1c_2\cdots c_n}$ and $A_{\circ}^{c_1c_2\cdots c_n}$ be $n \times m$ black and white matrices. Then, (k,n)-PB-CBW-EVCS with $2 < k \le n$ and background colors $\{RGBCMY \bullet \circ\}$, where $c_i \in \{\bullet, \circ\}$, $i = 1, 2, \ldots, n$, has $A_{\bullet}^{c_1c_2\cdots c_n} = (A_{\bullet} \parallel \overbrace{(A^{c_1c_2\cdots c_n} \| \cdots \| A^{c_1c_2\cdots c_n})}^{d})$ and $A_{\circ}^{c_1c_2\cdots c_n} = (A_{\circ} \parallel \overbrace{(A^{c_1c_2\cdots c_n} \| \cdots \| A^{c_1c_2\cdots c_n})}^{d})$, where $A_{\bullet}$ and $A_{\circ}$ are obtained from the following. The matrix $A^{c_1c_2\cdots c_n}$ is the same as that in Construction 4.

Matrices $A_{\bullet}$ and $A_{\circ}$ are the black and white matrices of De Prisco and De Santis's (k,n)-PB-CBW-VCS (Construction 7.2 in [7]), which are transformed from the (k, n)-PB-VCS. This transformation is briefly described as follows. By adding $(\lceil m_B/3 \rceil \times 3) - m_B$ all-$\bullet$ columns into base matrices of (k,n)-PB-VCS where m_B is the pixel expansion of this (k,n)-PB-VCS, we can keep the pixel expansion a multiple of 3. When generating shadows, we randomly permute the columns in matrices and group every triple to form a color into 8 full intensity colors. Finally De Prisco and De Santis's (k,n)-PB-CBW-VCS has the pixel expansion $\lceil m_B/3 \rceil$. Let the maximum number and minimum number of $\bullet$ of a row in this (k,n)-PB-CBW-VCS be $n_{\max}$ and $n_{\min}$. Then, we concatenate $A^{c_1c_2\cdots c_n} d$ times to $A_{\bullet}$ and $A_{\circ}$ to form $A_{\bullet}^{c_1c_2\cdots c_n}$ and $A_{\circ}^{c_1c_2\cdots c_n}$, respectively, where $d = (n_{\max} - n_{\min} + 1)$.

Lemma 2. *The matrix* $\overbrace{(A^{c_1c_2\cdots c_n} \| \cdots \| A^{c_1c_2\cdots c_n})}^{d}$ *has* $w_{\bullet}(\mathrm{add}((\overbrace{A^{c_1,\cdots,c_i=\bullet,\cdots c_n} \| \cdots \| A^{c_1,\cdots,c_i=\bullet,\cdots c_n}}^{d}) \mid i)) = (\lceil n/3 \rceil \times d)$ *and* $w_{\bullet}(\mathrm{add}((\overbrace{A^{c_1,\cdots,c_i=\circ,\cdots c_n} \| \cdots \| A^{c_1,\cdots,c_i=\circ,\cdots c_n}}^{d}) \mid i)) = ((\lceil n/3 \rceil - 1) \times d)$ *for* $1 \le i \le n$. *Also,* $w_{\bullet}(\mathrm{add}(\overbrace{A^{c_1c_2\cdots c_n} \| \cdots \| A^{c_1c_2\cdots c_n}}^{d} \mid X_{k\ge 2})) = (\lceil n/3 \rceil \times d)$.

Proof. Obviously, we have $w_{\bullet}(\mathrm{add}(\overbrace{A^{c_1c_2\cdots c_n} \| \cdots \| A^{c_1c_2\cdots c_n}}^{d} \mid i)) = w_{\bullet}(\overbrace{a^i \cdots a^i}^{d}) = \overbrace{w_{\bullet}(a^i) + \cdots + w_{\bullet}(a^i)}^{d} = w_{\bullet}(a^i) \times d$. Because $w_{\bullet}(a^i) = \lceil n/3 \rceil$ and $w_{\bullet}(a^i) = (\lceil n/3 \rceil - 1)$ for $c_i = \bullet$ and $c_i = \circ$ from Lemma 1, we have $w_{\bullet}(\mathrm{add}((\overbrace{A^{c_1,\cdots,c_i=\bullet,\cdots c_n} \| \cdots \| A^{c_1,\cdots,c_i=\bullet,\cdots c_n}}^{d}) \mid i)) = (\lceil n/3 \rceil \times d)$ and $w_{\bullet}(\mathrm{add}((\overbrace{A^{c_1,\cdots,c_i=\circ,\cdots c_n} \| \cdots \| A^{c_1,\cdots,c_i=\circ,\cdots c_n}}^{d}) \mid i)) = ((\lceil n/3 \rceil - 1) \times d)$. It can be verified that $w_{\bullet}(\mathrm{add}(M \mid X_{k\ge 2})) = w_{\bullet}(\overbrace{(\underbrace{\bullet \cdots \bullet}_{\lceil n/3 \rceil})(\underbrace{\bullet \cdots \bullet}_{\lceil n/3 \rceil}) \cdots (\underbrace{\bullet \cdots \bullet}_{\lceil n/3 \rceil})}^{d}) = (\lceil n/3 \rceil \times d)$. $\square$

Theorem 3. *The scheme from Construction 6 is a (k,n)-PB-CBW-EVCS with $m = (\lceil m_B/3 \rceil + (\lceil n/3 \rceil \times d))$, $h = (\lceil m_B/3 \rceil + (\lceil n/3 \rceil \times d))$, $l \le (\lceil m_B/3 \rceil + (\lceil n/3 \rceil \times d)) - 1$, $h_s \ge ((\lceil n/3 \rceil \times d) + n_{\min})$, and $l_s \le ((\lceil n/3 \rceil \times d) + n_{\min} - 1)$.*

Proof. Because $A_{\bullet}^{c_1c_2\cdots c_n} = (A_{\bullet} \parallel \overbrace{(A^{c_1c_2\cdots c_n} \| \cdots \| A^{c_1c_2\cdots c_n})}^{d})$ and $A_{\circ}^{c_1c_2\cdots c_n} = (A_{\circ} \parallel \overbrace{(A^{c_1c_2\cdots c_n} \| \cdots \| A^{c_1c_2\cdots c_n})}^{d})$, we have $|A_{\bullet}^{c_1c_2\cdots c_n}| = |A_{\bullet}| + \overbrace{|A^{c_1c_2\cdots c_n}| + \cdots |A^{c_1c_2\cdots c_n}|}^{d} = (\lceil m_c/3 \rceil + (\lceil n/3 \rceil \times d))$, and $|A_{\circ}^{c_1c_2\cdots c_n}| = |A_{\circ}| + \overbrace{|A^{c_1c_2\cdots c_n}| + \cdots |A^{c_1c_2\cdots c_n}|}^{d} = (\lceil m_B/3 \rceil + (\lceil n/3 \rceil \times d))$. The pixel expansion is $m = (\lceil m_B/3 \rceil + (\lceil n/3 \rceil \times d))$. We first prove that (k,n)-PB-CBW-EVCS satisfies contrast condition (C-1). Because $A_{\bullet}$ and $A_{\circ}$ are base matrices of De Prisco and De Santis's (k,n)-PB-CBW-VCS, we have $w_{\bullet}(\mathrm{add}(A_{\bullet} \mid X_k)) = \lceil m_B/3 \rceil$ and $w_{\bullet}(\mathrm{add}(A_{\circ} \mid X_k)) \le (\lceil m_B/3 \rceil - 1)$. Additionally, from Lemma 2, we have $w_{\bullet}(\mathrm{add}(A^{c_1c_2\cdots c_n} \mid X_{k\ge 2})) = (\lceil n/3 \rceil \times d)$. The values of $h = w_{\bullet}(\mathrm{add}(A_{\bullet}^{c_1c_2\cdots c_n} \mid X_k))$ and $l = w_{\bullet}(\mathrm{add}(A_{\circ}^{c_1c_2\cdots c_n} \mid X_k))$ are derived from Eq. (7.1) and Eq. (7.2), respectively.

$$\begin{cases} h = w_{\bullet}(\mathrm{add}(A_{\bullet}^{c_1c_2\cdots c_n} \mid X_k)) = w_{\bullet}(\mathrm{add}((A_{\bullet} \parallel \overbrace{A^{c_1c_2\cdots c_n} \| \cdots \| A^{c_1c_2\cdots c_n}}^{d}) \mid X_k))) \\ \quad = w_{\bullet}(\mathrm{add}(A_{\bullet} \mid X_k)) + w_{\bullet}(\mathrm{add}((\overbrace{A^{c_1c_2\cdots c_n} \| \cdots \| A^{c_1c_2\cdots c_n}}^{d}) \mid X_k)) = (\lceil m_c/3 \rceil + (\lceil n/3 \rceil \times d)) \end{cases} \tag{7.1}$$

$$\begin{cases} l = w_{\bullet}(\mathrm{add}(A_{\circ}^{c_1c_2\cdots c_n} \mid X_k)) = w_{\bullet}(\mathrm{add}((A_{\circ} \parallel \overbrace{A^{c_1c_2\cdots c_n} \| \cdots \| A^{c_1c_2\cdots c_n}}^{d}) \mid X_k))) \\ \quad = w_{\bullet}(\mathrm{add}(A_{\circ} \mid X_k)) + w_{\bullet}(\mathrm{add}((\overbrace{A^{c_1c_2\cdots c_n} \| \cdots \| A^{c_1c_2\cdots c_n}}^{d}) \mid X_k)) \le (\lceil m_c/3 \rceil + (\lceil n/3 \rceil \times d)) - 1 \end{cases} \tag{7.2}$$

By the same argument in the proof of Theorem 2, $D_{\bullet} = D(A_{\bullet}^{c_1c_2\cdots c_n} \mid X)$ and $D_{\circ} = D(A_{\circ}^{c_1c_2\cdots c_n} \mid X)$, where $|X| < k$, are equivalent. Thus condition (C-2) is satisfied. Next, we prove the condition (C-3). Since $A_{\bullet}$ and $A_{\circ}$ are the base matrices of De Prisco and De Santis's (k,n)-PB-CBW-VCS, we have $n_{\min} \le w_{\bullet}(A_s \mid i) \le n_{\max}$. From Lemma 2,

$w_\bullet(\mathrm{add}((\overbrace{A^{c_1,\cdots,c_i=\bullet,\cdots c_n}\|\cdots\|A^{c_1,\cdots,c_i=\bullet,\cdots c_n}}^{d})\mid i))=(\lceil n/3\rceil\times d)$ and $w_\bullet(\mathrm{add}((\overbrace{A^{c_1,\cdots,c_i=\circ,\cdots c_n}\|\cdots\|A^{c_1,\cdots,c_i=\circ,\cdots c_n}}^{d})\mid i))=((\lceil n/3\rceil-1)\times d)$, we can determine the blackness of shadows $h_s=w_\bullet(\mathrm{add}(A_s^{c_1,\cdots c_i=\bullet,\cdots c_n}\mid i))$ and $l_s=w_\bullet(\mathrm{add}(A_s^{c_1,\cdots c_i=\circ,\cdots c_n}\mid i))$ from the following equation.

$$\begin{cases} h_s=w_\bullet(\mathrm{add}(A_s^{c_1,\cdots c_i=\bullet,\cdots c_n}\mid i))=w_\bullet(\mathrm{add}(A_s\parallel A^{c_1,\cdots c_i=\bullet,\cdots c_n})\mid i)) \\ \quad =w_\bullet(\mathrm{add}(A_s\mid i))+w_\bullet(add(A^{c_1,\cdots c_i=\bullet,\cdots c_n}\mid i)) \\ \quad \geq(\lceil n/3\rceil\times d)+n_{\min} \end{cases} \tag{8.1}$$

$$\begin{cases} l_s=w_\bullet(\mathrm{ad}(A_s^{c_1,\cdots c_i=\circ,\cdots c_n}\mid i))=w_\bullet(\mathrm{add}((A_s\parallel A^{c_1,\cdots c_i=\circ,\cdots c_n})\mid i)) \\ \quad =w_\bullet(\mathrm{add}(A_s\mid i))+w_\bullet(\mathrm{add}(A^{c_1,\cdots c_i=\circ,\cdots c_n}\mid i)) \\ \quad \leq n_{\max}+(\lceil n/3\rceil-1)\times d)=(\lceil n/3\rceil\times d)+n_{\max}-d=(\lceil n/3\rceil\times d)+n_{\min}-1 \end{cases} \qquad \square \tag{8.2}$$

3.3. Examples

Five examples are given to easily understand the proposed constructions. For $(2,n)$-CWB-EVCS, we shows the $(2,2)$-CWB-EVCSs from Constructions 1, 2, and 3, the $(2,3)$-CWB-EVCS and the $(2,4)$-CWB-EVCS from Construction 4, and the $(2,4)$-PB-CWB-EVCS from Construction 5. For (k,n)-PB-CWB-EVCS, we show a $(3,3)$-PB-CWB-EVCS from Construction 6.

Example 2. Construct $(2,2)$-CBW-EVCSs by Construction 1, Construction 2, and Construction 3.

From Table A1, it is observed that this $(2,2)$-CBW-EVCS has $(1\bullet 1*)$ and $(2*)$, where $*\in\{R,G,B\}$, in shadows to represent the black and white color, i.e., $h_s=1$ and $l_s=0$, s. The black and white colors in reconstructed image are $(2\bullet)$ and $(1\bullet 1*)$, i.e., $h=2$ and $l=1$. By the definition of Naor and Sham'r's contrast, all the contrasts of these three constructions are $\alpha=\frac{(h-l)}{m}=\frac{(2-1)}{2}=\frac{1}{2}$ and $\alpha_s=\frac{(h_s-l_s)}{m}=\frac{(1-0)}{2}=\frac{1}{2}$ for the reconstructed image and the shadows. The collections $C_\bullet^{c_1c_2}$ and $C_\circ^{c_1c_2}$ of all distribution matrices using Construction 2 and Construction 3 with $\{RG\}$ are shown in Table A2 and Table A3, respectively. Construction 2 has $\lambda=50\%$, while Construction 3 may have $\lambda=50\%$ or $\lambda=33\%$. Construction 3 with $\{RG\}$, $\{RB\}$, and $\{GB\}$ has $\lambda=33\%$, while Construction 3 with $\{CR\}$, $\{MG\}$, and $\{YB\}$ has the high $\lambda=(\frac{1}{3}+\frac{2}{3})/2=50\%$.

Example 3. Construct a $(2,3)$-CBW-EVCS by Construction 4.

For $n=3$, Construction 4 has the pixel expansion $m=(\lceil(\log_3 3)\rceil+\lceil 3/3\rceil)=2$. The matrices $A_\bullet$ and $A_\circ$ of $(2,3)$-CBW-VCS are $A_\bullet=\left\{\begin{bmatrix}R\\G\\B\end{bmatrix}\begin{bmatrix}B\\R\\G\end{bmatrix}\begin{bmatrix}G\\B\\R\end{bmatrix}\right\}$ and $A_\circ=\left\{\begin{bmatrix}R\\R\\R\end{bmatrix}\begin{bmatrix}G\\G\\G\end{bmatrix}\begin{bmatrix}B\\B\\B\end{bmatrix}\right\}$. The matrices of cover mage $A^{c_1c_2c_3}$ are given in Table A4. Consider the example $c_1=\circ$, $c_2=\circ$, $c_3=\bullet$. Matrices $A_\bullet^{\circ\circ\bullet}=(A_\bullet\parallel A^{\circ\circ\bullet})$ and $A_\circ^{\circ\circ\bullet}=(A_\circ\parallel A^{\circ\circ\bullet})$ are shown in Eq. (9).

$$\begin{cases} A_\circ^{\circ\circ\bullet}=\left\{\begin{bmatrix}RR\\RG\\R\bullet\end{bmatrix}\begin{bmatrix}RB\\RR\\R\bullet\end{bmatrix}\begin{bmatrix}RG\\RB\\R\bullet\end{bmatrix}\begin{bmatrix}GR\\GG\\G\bullet\end{bmatrix}\begin{bmatrix}GB\\GR\\G\bullet\end{bmatrix}\begin{bmatrix}GG\\GB\\G\bullet\end{bmatrix}\begin{bmatrix}BR\\BG\\B\bullet\end{bmatrix}\begin{bmatrix}BB\\BR\\B\bullet\end{bmatrix}\begin{bmatrix}BG\\BB\\B\bullet\end{bmatrix}\right\}, \\ A_\bullet^{\circ\circ\bullet}=\left\{\begin{bmatrix}RR\\GG\\B\bullet\end{bmatrix}\begin{bmatrix}RB\\GR\\B\bullet\end{bmatrix}\begin{bmatrix}RG\\GB\\B\bullet\end{bmatrix}\begin{bmatrix}BR\\RG\\G\bullet\end{bmatrix}\begin{bmatrix}BB\\RR\\G\bullet\end{bmatrix}\begin{bmatrix}BG\\RB\\G\bullet\end{bmatrix}\begin{bmatrix}GR\\BG\\R\bullet\end{bmatrix}\begin{bmatrix}GB\\BR\\R\bullet\end{bmatrix}\begin{bmatrix}GG\\BB\\R\bullet\end{bmatrix}\right\}. \end{cases} \tag{9}$$

This $(2,3)$-CBW-EVCS has $(1\bullet 1*)$ and $(2*)$ in shadows to represent the black and white colors, i.e., $h_s=1$ and $l_s=0$. When stacking any two shadows, $A_\bullet^{\circ\circ\bullet}$ has $(2\bullet)$, while $A_\circ^{\circ\circ\bullet}$ only has $(1\bullet 1*)$, i.e., $h=2$ and $l=1$. Finally, we have the contrasts $\alpha=\frac{1}{2}$ and $\alpha_s=\frac{1}{2}$.

Example 4. Construct a $(2,4)$-CBW-EVCS by Construction 4.

For $n=4$, the matrices $A_\bullet$ and $A_\circ$ of $(2,4)$-CBW-VCS with the pixel expansion $\lceil\log_3 4\rceil=2$ are shown in Eq. (10).

$$A_\bullet=\left\{\begin{bmatrix}RR\\GG\\BB\\RG\end{bmatrix}\begin{bmatrix}RG\\RR\\GG\\BB\end{bmatrix}\begin{bmatrix}BB\\RG\\RR\\GG\end{bmatrix}\begin{bmatrix}GG\\BB\\RG\\RR\end{bmatrix}\right\},\qquad A_\circ=\left\{\begin{bmatrix}RR\\RR\\RR\\RR\end{bmatrix}\begin{bmatrix}GG\\GG\\GG\\GG\end{bmatrix}\begin{bmatrix}BB\\BB\\BB\\BB\end{bmatrix}\begin{bmatrix}RG\\RG\\RG\\RG\end{bmatrix}\right\}. \tag{10}$$

The matrices of cover image $A^{c_1c_2c_3c_4}$ of $\lceil 4/3 \rceil = 2$ characters with 4 elements $\{(R\bullet), (G\bullet), (B\bullet), (\bullet R)\}$ are given in Table A5. Consider the example $c_1 = \circ$, $c_2 = \circ$, $c_3 = \circ$, $c_4 = \bullet$. Matrices $A_\bullet^{\circ\circ\circ\bullet} = (A_\bullet \parallel A^{\circ\circ\circ\bullet})$ and $A_\circ^{\circ\circ\circ\bullet} = (A_\circ \parallel A^{\circ\circ\circ\bullet})$ are shown in Eq. (11).

$$\left\{\begin{array}{l}
A_\bullet^{\circ\circ\circ\bullet} = \left\{\begin{array}{l}
\begin{bmatrix} RRR\bullet \\ GGG\bullet \\ BBB\bullet \\ RG\bullet\bullet \end{bmatrix}
\begin{bmatrix} RR\bullet R \\ GGR\bullet \\ BBG\bullet \\ RG\bullet\bullet \end{bmatrix}
\begin{bmatrix} RRB\bullet \\ GG\bullet R \\ BBR\bullet \\ RG\bullet\bullet \end{bmatrix}
\begin{bmatrix} RRG\bullet \\ GGB\bullet \\ BB\bullet R \\ RG\bullet\bullet \end{bmatrix}
\begin{bmatrix} RGR\bullet \\ RRG\bullet \\ GGB\bullet \\ BB\bullet\bullet \end{bmatrix}
\begin{bmatrix} RG\bullet R \\ RRR\bullet \\ GGG\bullet \\ BB\bullet\bullet \end{bmatrix}
\begin{bmatrix} RGB\bullet \\ RR\bullet R \\ GGR\bullet \\ BB\bullet\bullet \end{bmatrix}
\begin{bmatrix} RGG\bullet \\ RRB\bullet \\ GG\bullet R \\ BB\bullet\bullet \end{bmatrix} \\
\begin{bmatrix} BBR\bullet \\ RGG\bullet \\ RRB\bullet \\ GG\bullet\bullet \end{bmatrix}
\begin{bmatrix} BB\bullet R \\ RGR\bullet \\ RRG\bullet \\ GG\bullet\bullet \end{bmatrix}
\begin{bmatrix} BBB\bullet \\ RG\bullet R \\ RRR\bullet \\ GG\bullet\bullet \end{bmatrix}
\begin{bmatrix} BBG\bullet \\ RGB\bullet \\ RR\bullet R \\ GG\bullet\bullet \end{bmatrix}
\begin{bmatrix} GGR\bullet \\ BBG\bullet \\ RGB\bullet \\ RR\bullet\bullet \end{bmatrix}
\begin{bmatrix} GG\bullet R \\ BBR\bullet \\ RGG\bullet \\ RR\bullet\bullet \end{bmatrix}
\begin{bmatrix} GGB\bullet \\ BB\bullet R \\ RGR\bullet \\ RR\bullet\bullet \end{bmatrix}
\begin{bmatrix} GGG\bullet \\ BBB\bullet \\ RG\bullet R \\ RR\bullet\bullet \end{bmatrix}
\end{array}\right\}, \\
A_\circ^{\circ\circ\circ\bullet} = \left\{\begin{array}{l}
\begin{bmatrix} RRR\bullet \\ RRG\bullet \\ RRB\bullet \\ RR\bullet\bullet \end{bmatrix}
\begin{bmatrix} RR\bullet R \\ RRR\bullet \\ RRG\bullet \\ RR\bullet\bullet \end{bmatrix}
\begin{bmatrix} RRB\bullet \\ RR\bullet R \\ RRR\bullet \\ RR\bullet\bullet \end{bmatrix}
\begin{bmatrix} RRG\bullet \\ RRB\bullet \\ RR\bullet R \\ RR\bullet\bullet \end{bmatrix}
\begin{bmatrix} GGR\bullet \\ GGG\bullet \\ GGB\bullet \\ GG\bullet\bullet \end{bmatrix}
\begin{bmatrix} GG\bullet R \\ GGR\bullet \\ GGG\bullet \\ GG\bullet\bullet \end{bmatrix}
\begin{bmatrix} GGB\bullet \\ GG\bullet R \\ GGR\bullet \\ GG\bullet\bullet \end{bmatrix}
\begin{bmatrix} GGG\bullet \\ GGB\bullet \\ GG\bullet R \\ GG\bullet\bullet \end{bmatrix} \\
\begin{bmatrix} BBR\bullet \\ BBG\bullet \\ BBB\bullet \\ BB\bullet\bullet \end{bmatrix}
\begin{bmatrix} BB\bullet R \\ BBR\bullet \\ BBG\bullet \\ BB\bullet\bullet \end{bmatrix}
\begin{bmatrix} BBB\bullet \\ BB\bullet R \\ BBR\bullet \\ BB\bullet\bullet \end{bmatrix}
\begin{bmatrix} BBG\bullet \\ BBB\bullet \\ BB\bullet R \\ BB\bullet\bullet \end{bmatrix}
\begin{bmatrix} RGR\bullet \\ RGG\bullet \\ RGB\bullet \\ RG\bullet\bullet \end{bmatrix}
\begin{bmatrix} RG\bullet R \\ RGR\bullet \\ RGG\bullet \\ RG\bullet\bullet \end{bmatrix}
\begin{bmatrix} RGB\bullet \\ RG\bullet R \\ RGR\bullet \\ RG\bullet\bullet \end{bmatrix}
\begin{bmatrix} RGG\bullet \\ RGB\bullet \\ RG\bullet R \\ RG\bullet\bullet \end{bmatrix}
\end{array}\right\}.
\end{array}\right. \tag{11}$$

As shown in Eq. (11), in shadows, we have $(2\bullet 2*)$ for black color (see S_4) and $(1\bullet 3*)$ for white color (see S_1, S_2, and S_3), i.e., $h_s = 2$ and $l_s = 1$. Thus, we have we have $\alpha_s = \frac{(h_s - l_s)}{4} = \frac{(2-1)}{4} = \frac{1}{4}$. When stacking any two shadows, $A_\bullet^{\circ\circ\circ\bullet}$ has $(4\bullet)$ or $(3\bullet 1*)$. However, $A_\circ^{\circ\circ\circ\bullet}$ only has $(2\bullet 2*)$, i.e., $h \geq 3$ and $l = 2$. Any two stacked shadows has $(4\bullet)$ with $\frac{5}{6}$ probability and $(3\bullet 1*)$ with $\frac{1}{6}$ probability. We have the average contrast $\alpha = (\frac{(4-2)}{4} \times \frac{5}{6}) + (\frac{(3-2)}{4} \times \frac{1}{6}) = \frac{11}{24}$.

Example 5. Construct a $(2,4)$-PB-CBW-EVCS by Construction 5.

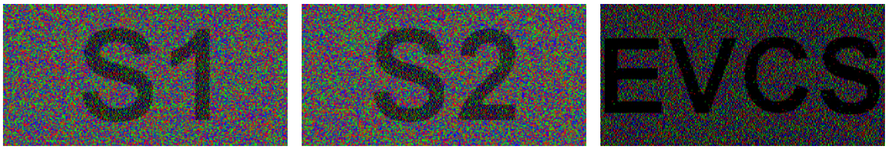
For $n = 4$, the matrices of cover images $A^{c_1c_2c_3c_4}$ are the same as those in Table A5. Consider the example $c_1 = \circ$, $c_2 = \circ$, $c_3 = \circ$, $c_4 = \bullet$. Matrices $A_\bullet^{\circ\circ\circ\bullet} = (A_\bullet \parallel A^{\circ\circ\circ\bullet})$ and $A_\circ^{\circ\circ\circ\bullet} = (A_\circ \parallel A^{\circ\circ\circ\bullet})$ are similar to those in Eq. (11), but using $A_\bullet = \left\{ \begin{bmatrix} \bullet R \\ \bullet G \\ \bullet B \\ R\bullet \end{bmatrix} \begin{bmatrix} R\bullet \\ \bullet R \\ \bullet G \\ \bullet B \end{bmatrix} \begin{bmatrix} \bullet B \\ R\bullet \\ \bullet R \\ \bullet G \end{bmatrix} \begin{bmatrix} \bullet G \\ \bullet B \\ R\bullet \\ \bullet R \end{bmatrix} \right\}$ and $A_\circ = \left\{ \begin{bmatrix} \bullet R \\ \bullet R \\ \bullet R \\ \bullet R \end{bmatrix} \begin{bmatrix} \bullet G \\ \bullet G \\ \bullet G \\ \bullet G \end{bmatrix} \begin{bmatrix} \bullet B \\ \bullet B \\ \bullet B \\ \bullet B \end{bmatrix} \begin{bmatrix} R\bullet \\ R\bullet \\ R\bullet \\ R\bullet \end{bmatrix} \right\}$ instead. This $(2,4)$-PB-CBW-EVCS has $(3\bullet 1*)$ and $(2\bullet 2*)$ in shadows to represent the black and white colors, i.e., $h_s = 3$ and $l_s = 2$. The black and white colors in reconstructed image are $(4\bullet)$ and $(3\bullet 1*)$, i.e., $h = 4$ (note: $h = m = 4$, this scheme is perfect black). We have the contrasts $\alpha = \frac{(4-3)}{4} = \frac{1}{4}$ and $\alpha_s = \frac{(3-2)}{4} = \frac{1}{4}$.

Example 6. Construct a $(3,3)$-PB-CBW-EVCS by Construction 6.

By transforming Naor and Shamir's base matrices $B_\bullet = \begin{bmatrix} \bullet\circ\circ\bullet \\ \circ\bullet\circ\bullet \\ \circ\circ\bullet\bullet \end{bmatrix}$ and $B_\circ = \begin{bmatrix} \circ\bullet\circ\circ \\ \bullet\circ\bullet\circ \\ \bullet\bullet\circ\circ \end{bmatrix}$, we obtain the collections $C_\circ$ and $C_\bullet$ of $(3,3)$-PB-CBW-VCS (see Table 7 and Table 8 in [7]). After adding 2 all-$\bullet$ columns into $B_\bullet$ and $B_\circ$, we have $(4\bullet 2\circ)$ in each shadow. We have $n_{max} = 1$ and $n_{min} = 0$ for this case, and thus $d = (n_{max} - n_{min} + 1) = 2$. The matrix of cover image $A^{c_1c_2c_3}$ is the concatenation of two $A^{c_1c_2c_3}$ in Table A4. Consider the example $c_1 = \circ$, $c_2 = \circ$, $c_3 = \bullet$, and then we have the matrix $(A^{\circ\circ\bullet} \parallel A^{\circ\circ\bullet}) = \left\{ \begin{bmatrix} R \\ G \\ \bullet \end{bmatrix} \begin{bmatrix} B \\ R \\ \bullet \end{bmatrix} \begin{bmatrix} G \\ B \\ \bullet \end{bmatrix} \right\} \parallel \left\{ \begin{bmatrix} R \\ G \\ \bullet \end{bmatrix} \begin{bmatrix} B \\ R \\ \bullet \end{bmatrix} \begin{bmatrix} G \\ B \\ \bullet \end{bmatrix} \right\} = \left\{ \begin{bmatrix} RR \\ GG \\ \bullet\bullet \end{bmatrix} \begin{bmatrix} RB \\ GR \\ \bullet\bullet \end{bmatrix} \begin{bmatrix} RG \\ GB \\ \bullet\bullet \end{bmatrix} \begin{bmatrix} BR \\ RG \\ \bullet\bullet \end{bmatrix} \begin{bmatrix} BB \\ RR \\ \bullet\bullet \end{bmatrix} \begin{bmatrix} BG \\ RB \\ \bullet\bullet \end{bmatrix} \begin{bmatrix} GR \\ BG \\ \bullet\bullet \end{bmatrix} \begin{bmatrix} GB \\ BR \\ \bullet\bullet \end{bmatrix} \begin{bmatrix} GG \\ BB \\ \bullet\bullet \end{bmatrix} \right\}$.

Suppose that the secret s is $\bullet$, and that two matrices $A_\bullet = \begin{bmatrix} BR \\ Y\bullet \\ GR \end{bmatrix}$ and $A_\bullet = \begin{bmatrix} C\bullet \\ Y\bullet \\ M\bullet \end{bmatrix}$ are chosen (note: because $(B + Y + G) = \bullet$ and $(C + Y + M) = \bullet$ we have $(\bullet\bullet)$ in these two matrices for staking 3 shadows). The matrices $A_\bullet^{\circ\circ\bullet} = (A_\bullet \parallel A^{\circ\circ\bullet} \parallel A^{\circ\circ\bullet})$ using $A_\bullet = \begin{bmatrix} BR \\ Y\bullet \\ GR \end{bmatrix}$ and $\begin{bmatrix} C\bullet \\ Y\bullet \\ M\bullet \end{bmatrix}$ are given in Eq. (12).

$$\left\{\begin{array}{l}
A_\bullet^{\circ\circ\bullet} = \left\{
\begin{bmatrix} BRRR \\ Y\bullet GG \\ GR\bullet\bullet \end{bmatrix}
\begin{bmatrix} BRRB \\ Y\bullet GR \\ GR\bullet\bullet \end{bmatrix}
\begin{bmatrix} BRRG \\ Y\bullet GB \\ GR\bullet\bullet \end{bmatrix}
\begin{bmatrix} BRBR \\ Y\bullet RG \\ GR\bullet\bullet \end{bmatrix}
\begin{bmatrix} BRBB \\ Y\bullet RR \\ GR\bullet\bullet \end{bmatrix}
\begin{bmatrix} BRBG \\ Y\bullet RB \\ GR\bullet\bullet \end{bmatrix}
\begin{bmatrix} BRGR \\ Y\bullet BG \\ GR\bullet\bullet \end{bmatrix}
\begin{bmatrix} BRGB \\ Y\bullet BR \\ GR\bullet\bullet \end{bmatrix}
\begin{bmatrix} BRGG \\ Y\bullet BB \\ GR\bullet\bullet \end{bmatrix}
\right\}, \\
A_\bullet^{\circ\circ\bullet} = \left\{
\begin{bmatrix} C\bullet RR \\ Y\bullet GG \\ M\bullet\bullet\bullet \end{bmatrix}
\begin{bmatrix} C\bullet RB \\ Y\bullet GR \\ M\bullet\bullet\bullet \end{bmatrix}
\begin{bmatrix} C\bullet RG \\ Y\bullet GB \\ M\bullet\bullet\bullet \end{bmatrix}
\begin{bmatrix} C\bullet BR \\ Y\bullet RG \\ M\bullet\bullet\bullet \end{bmatrix}
\begin{bmatrix} C\bullet BB \\ Y\bullet RR \\ M\bullet\bullet\bullet \end{bmatrix}
\begin{bmatrix} C\bullet BG \\ Y\bullet RB \\ M\bullet\bullet\bullet \end{bmatrix}
\begin{bmatrix} C\bullet GR \\ Y\bullet BG \\ M\bullet\bullet\bullet \end{bmatrix}
\begin{bmatrix} C\bullet GB \\ Y\bullet BR \\ M\bullet\bullet\bullet \end{bmatrix}
\begin{bmatrix} C\bullet GG \\ Y\bullet BB \\ M\bullet\bullet\bullet \end{bmatrix}
\right\}.
\end{array}\right. \tag{12}$$



(a-1) S_1 (a-2) S_2 (b) $(S_1{+}S_2)$

Fig. 4. $(2,2)$-CBW-EVCS with $\{RGB\}$: (a) shadows, (b) the reconstructed image.

Suppose that the secret s is $\circ$, and that two matrices $A_\circ = \begin{bmatrix} BR \\ RR \\ GR \end{bmatrix}$ and $A_\circ = \begin{bmatrix} C\bullet \\ Y\bullet \\ GR \end{bmatrix}$ are chosen (note: because $(R+R+R)=R$ and $(C+Y+G)=G$ we have $(\bullet *)$ in these two matrices for staking 3 shadows). Then the matrices $A_\circ^{\circ\circ\bullet} = (A_\circ \parallel A^{\circ\circ\bullet} \parallel A^{\circ\circ\bullet})$ using $A_\circ = \begin{bmatrix} BR \\ RR \\ GR \end{bmatrix}$ and $\begin{bmatrix} C\bullet \\ Y\bullet \\ GR \end{bmatrix}$ are given in Eq. (13).

$$\begin{cases} A_\circ^{\circ\circ\bullet} = \left\{ \begin{bmatrix} BRRR \\ RRGG \\ GR\bullet\bullet \end{bmatrix} \begin{bmatrix} BRRB \\ RRGR \\ GR\bullet\bullet \end{bmatrix} \begin{bmatrix} BRRG \\ RRGB \\ GR\bullet\bullet \end{bmatrix} \begin{bmatrix} BRBR \\ RRRG \\ GR\bullet\bullet \end{bmatrix} \begin{bmatrix} BRBB \\ RRRR \\ GR\bullet\bullet \end{bmatrix} \begin{bmatrix} BRBG \\ RRRB \\ GR\bullet\bullet \end{bmatrix} \begin{bmatrix} BRGR \\ RRBG \\ GR\bullet\bullet \end{bmatrix} \begin{bmatrix} BRGB \\ RRBR \\ GR\bullet\bullet \end{bmatrix} \begin{bmatrix} BRGG \\ RRBB \\ GR\bullet\bullet \end{bmatrix} \right\}, \\ A_\circ^{\circ\circ\bullet} = \left\{ \begin{bmatrix} C\bullet RR \\ Y\bullet GG \\ GR\bullet\bullet \end{bmatrix} \begin{bmatrix} C\bullet RB \\ Y\bullet GR \\ GR\bullet\bullet \end{bmatrix} \begin{bmatrix} C\bullet RG \\ Y\bullet GB \\ GR\bullet\bullet \end{bmatrix} \begin{bmatrix} C\bullet BR \\ Y\bullet RG \\ GR\bullet\bullet \end{bmatrix} \begin{bmatrix} C\bullet BB \\ Y\bullet RR \\ GR\bullet\bullet \end{bmatrix} \begin{bmatrix} C\bullet BG \\ Y\bullet RB \\ GR\bullet\bullet \end{bmatrix} \begin{bmatrix} C\bullet GR \\ Y\bullet BG \\ GR\bullet\bullet \end{bmatrix} \begin{bmatrix} C\bullet GB \\ Y\bullet BR \\ GR\bullet\bullet \end{bmatrix} \begin{bmatrix} C\bullet GG \\ Y\bullet BB \\ GR\bullet\bullet \end{bmatrix} \right\}. \end{cases} \tag{13}$$

In the reconstructed image, we have $(4\bullet)$ and $(3\bullet 1*)$ for black and white colors. Thus, the contrast is $\alpha = \frac{(4-3)}{4} = \frac{1}{4}$. The contrast of shadow is calculated as follows. From Eqs. (12) and (13), for S_3, we have $(3\bullet 1*)$ or $(2\bullet 2*)$ for black color in shadows. Additionally, we have $(1\bullet 3*)$ or $(4*)$ for white color in shadows (see S_1 and S_2). From all the distribution matrices in $(3,3)$-CBW-VCS (see Table 7 and Table 8 in [7]), we have $h_s = 2$ and 3 with probability 1/2 for each, an $l_s = 0$ and 1 with probability 1/2 for each. Thus, we have the average $h_s = \frac{(3+2)}{2} = 2.5$ and the average $l_s = \frac{(1+0)}{2} = \frac{1}{2}$. Then, the average contrast is $\alpha_s = \frac{(2.5-0.5)}{4} = \frac{1}{2}$. This average contrast can be represented as a general term derived as follows. Because the maximum number and minimum number of $\bullet$ for each row in this (k,n)-PB-CBW-VCS is $n_{\max}$ and $n_{\min}$, we have the following average blackness of h_s and l_s when concatenating the matrix $(\overbrace{A^{c_1c_2\cdots c_n} \parallel \cdots \parallel A^{c_1c_2\cdots c_n}}^{d})$.

$$\begin{cases} h_s = \dfrac{(\lceil n/3 \rceil + n_{\min}) + (\lceil n/3 \rceil + n_{\min} + 1) + \cdots + (\lceil n/3 \rceil + n_{\max})}{(n_{\max} - n_{\min} + 1)} = \lceil n/3 \rceil + \dfrac{(n_{\min} + n_{\max})}{2} \\ l_s = \dfrac{(\lceil n/3 \rceil - d + n_{\min}) + (\lceil n/3 \rceil - d + n_{\min} + 1) + \cdots + (\lceil n/3 \rceil - d + n_{\max})}{(n_{\max} - n_{\min} + 1)} = (\lceil n/3 \rceil - d) + \dfrac{(n_{\min} + n_{\max})}{2} \end{cases} \tag{14}$$

We then have the average contrast $\alpha_s = \frac{(h_s - l_s)}{m} = ((\lceil n/3 \rceil + \frac{(n_{\min}+n_{\max})}{2}) - (\lceil n/3 \rceil - d + \frac{(n_{\min}+n_{\max})}{2}))/m = \frac{d}{m}$. By this general term, we have the average contrast of $(3,3)$-PB-CBW-VCS $\alpha_s = \frac{d}{m} = \frac{2}{4} = \frac{1}{2}$.

4. Experiments and comparisons

4.1. Experimental results

Four experiments are conducted to evaluate the performance of the proposed CBW-EVCSs. There are 7 schemes (denoted as Schemes A–G) in these four experiments. Experiment 1 tests the clearness of shadow and reconstructed image among three $(2,2)$-CWB-EVCSs: $(2,2)$-CWB-EVCS with $\{RGB\}$ from Construction 1 (Scheme A), $(2,2)$-CWB-EVCS with $\{RGBCMY\}$ from Construction 2 (Scheme B), and $(2,2)$-CWB-EVCS with $\{b_1b_2\}$ background colors from Construction 3 (Scheme C). Experiment 1 only demonstrates the simple 2-out-of-2 schemes. In Experiment 2, we show the $(2,3)$-CWB-EVCS with $\{RGB\}$ from Construction 4 (Scheme D). Experiment 3 shows the $(2,4)$-CWB-EVCS with $\{RGB\}$ from Construction 4 (Scheme E) and the $(2,4)$-PB-CWB-EVCS with $\{RGB\}$ from Construction 5 (Scheme F). Both Scheme E and Scheme F are $(2,4)$ schemes, but Scheme F has the perfect reconstruction of black pixels. In Experiment 4, we gives the proposed $(3,3)$-PB-CBW-EVCS from Construction 6 (Scheme G), which is an example with $k > 2$.

Experiment 1. Consider Scheme A, Scheme B and Scheme C in Example 2. All tested images are printed-text images. The secret is $\boxed{\text{EVCS}}$, and the cover images on shadows are $\boxed{\text{S1}}$ and $\boxed{\text{S2}}$.

All the three schemes have $m = 1$. Fig. 4 and Fig. 5 illustrate 2 shadows and the reconstructed images of Scheme A and Scheme B, respectively. It is observed that Scheme B has $\lambda = 50\%$, while Scheme A only has $\lambda = 33\%$. Fig. 6 shows the stacked results of Scheme C with $\{RG\}$, $\{RB\}$, $\{GB\}$, $\{CR\}$, $\{MG\}$, and $\{YB\}$. The average light transmission of Figs. 6(a–c) are $\lambda = 33\%$, which are the same as that of Fig. 4, while its background color only has a mixture of 2 colors. Figs. 6(d–f)

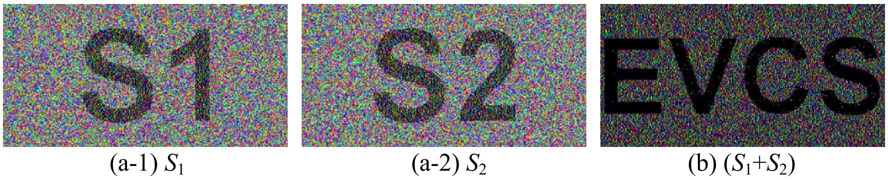


Fig. 5. (2, 2)-CBW-EVCS with $\{RGBCMY\}$: (a) shadows (b) the reconstructed image.

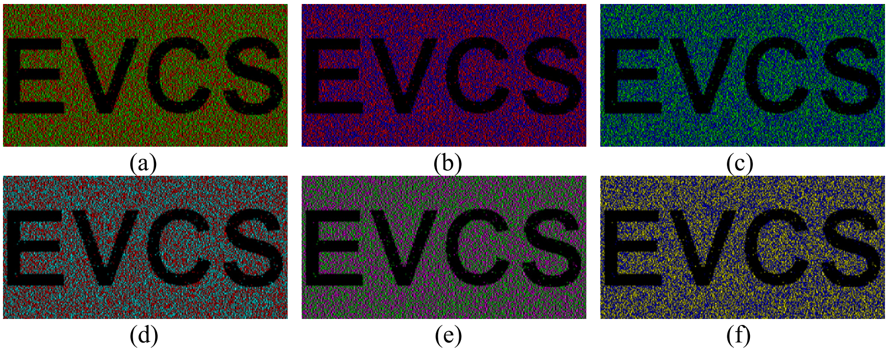


Fig. 6. The stacked results of (2, 2)-CBW-EVCS with $\{b_1b_2\}$: (a) $\{RG\}$, (b) $\{RB\}$, (c) $\{GB\}$, (d) $\{CR\}$, (e) $\{MG\}$, (f) $\{YB\}$.

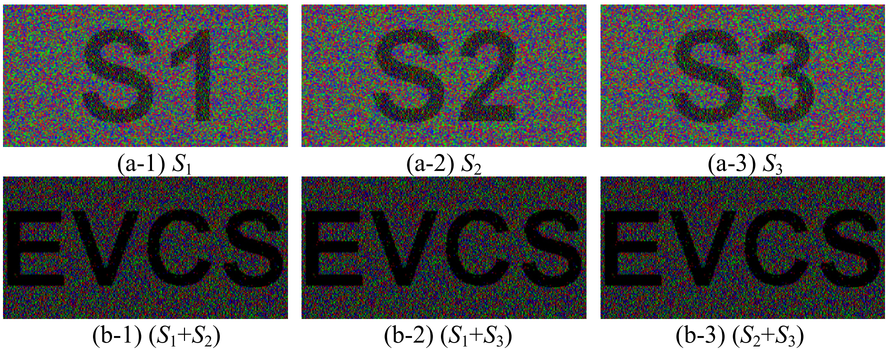


Fig. 7. (2, 3)-CBW-EVCS with $\{RGB\}$: (a) shadows, (b) the reconstructed images.

have $\lambda = (\frac{1}{3} + \frac{2}{3})/2 = 50\%$. All the contrasts of Schemes A, B, and C are $\alpha_s = \frac{1}{2}$ and $\alpha = \frac{1}{2}$. So, we can see $\boxed{\text{S1}}$ and $\boxed{\text{S2}}$ on shadows, and visually decode the secret image $\boxed{\text{EVCS}}$.

Experiment 2. Consider Scheme D in Example 3. The secret is $\boxed{\text{EVCS}}$, and the cover images on shadows are $\boxed{\text{S1}}$, $\boxed{\text{S2}}$, and $\boxed{\text{S3}}$.

From Example 3, Scheme D has $m = 2$, $\alpha_s = \frac{1}{2}$ and $\alpha = \frac{1}{2}$. Three shadows (S_1, S_1 and S_3) with cover images $\boxed{\text{S1}}$, $\boxed{\text{S2}}$, and $\boxed{\text{S3}}$ are shown in Fig. 7(a). The reconstructed images $\boxed{\text{EVCS}}$ for $(S_1 + S_2)$, $(S_1 + S_3)$, and $(S_2 + S_3)$ are illustrated in Fig. 7(b). Since the background color is a mixture of $\{RGB\}$, thus $\lambda = 33\%$.

Experiment 3. Consider Scheme E in Example 4 and Scheme F in Example 5. The secret is $\boxed{\begin{array}{c}\text{EV}\\ \text{CS}\end{array}}$, and the cover images on shadows are $\boxed{\text{S1}}$, $\boxed{\text{S2}}$, $\boxed{\text{S3}}$, and $\boxed{\text{S4}}$.

Both schemes are $(2, 4)$ schemes and have the pixel expansion $m = 4$. Actually, by using $m = 4$, the values of n for $(2, n)$ schemes from Construction 4 and Construction 5 can be, respectively, at most up to 9 and 6. Scheme E has $\alpha_s = \frac{1}{4}$ and the average contrast $\alpha = \frac{11}{24}$, while Scheme F has $\alpha_s = \frac{1}{4}$ and $\alpha = \frac{1}{4}$. The blackness h of Scheme E is 3 or 4, but the blackness of Scheme F is $h = m = 4$. Thus, Scheme F is a perfect black scheme. Figs. 8 and 9 show 4 shadows and two reconstructed images $(S_1 + S_2)$ and $(S_3 + S_4)$ for Scheme E and Scheme F, respectively. It is observed Fig. 9(b) has the perfect blackness.

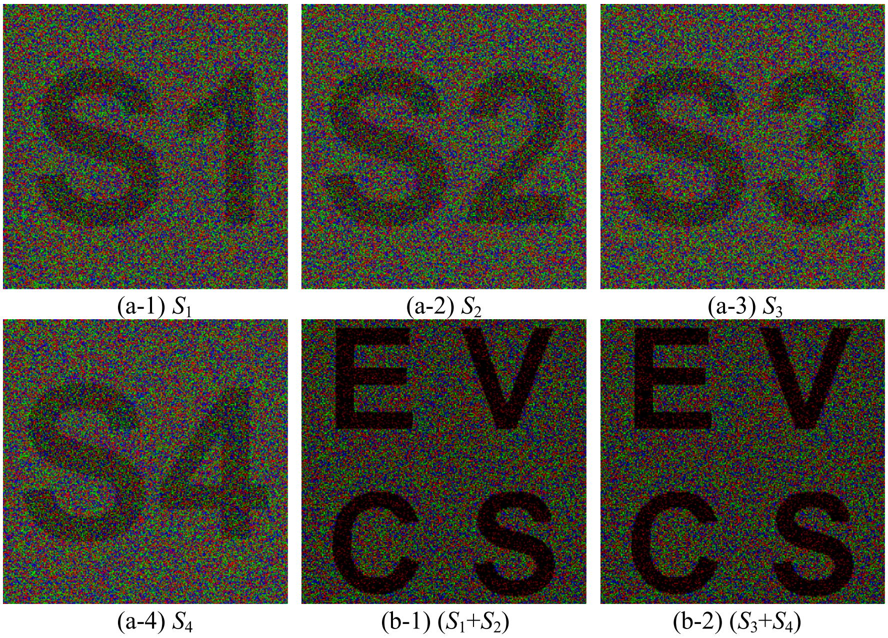


Fig. 8. $(2,4)$-CBW-EVCS with $\{RGB\}$: (a) shadows, (b) the reconstructed images.

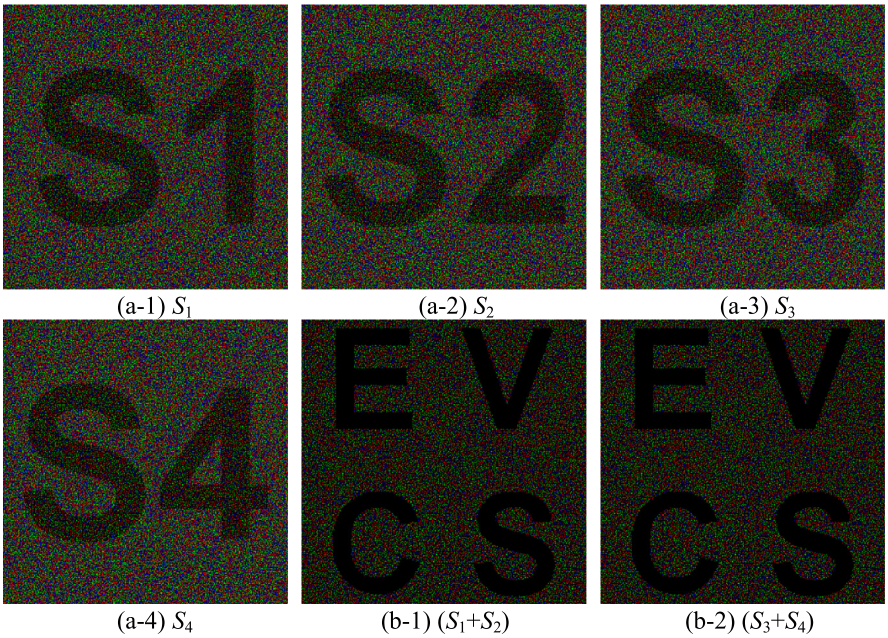


Fig. 9. $(2,4)$-PB-CBW-EVCS with $\{RGB\}$: (a) shadows, (b) the reconstructed images.

Experiment 4. Consider Scheme G in Example 6. The secret is $\boxed{\begin{matrix}\text{EV}\\ \text{CS}\end{matrix}}$, and the cover images on shadows are $\boxed{\text{S1}}$, $\boxed{\text{S2}}$, and $\boxed{\text{S3}}$.

Scheme G is a scheme with $k > 2$. From Example 6, we have $m = 4$, $\alpha_s = \frac{1}{2}$, and $\alpha = \frac{1}{4}$. Additionally, the blackness is $h = m = 4$, and thus Scheme G has the perfect blackness. Fig. 10 shows 3 shadows, the stacked results of any 2 stacked shadows, and the reconstructed image of stacking all 3 shadows. Since the threshold is 3, so we cannot see anything from any 2 stacked shadows. In Fig. 10(c), we can visually decode the secret when stacking 3 shadows.

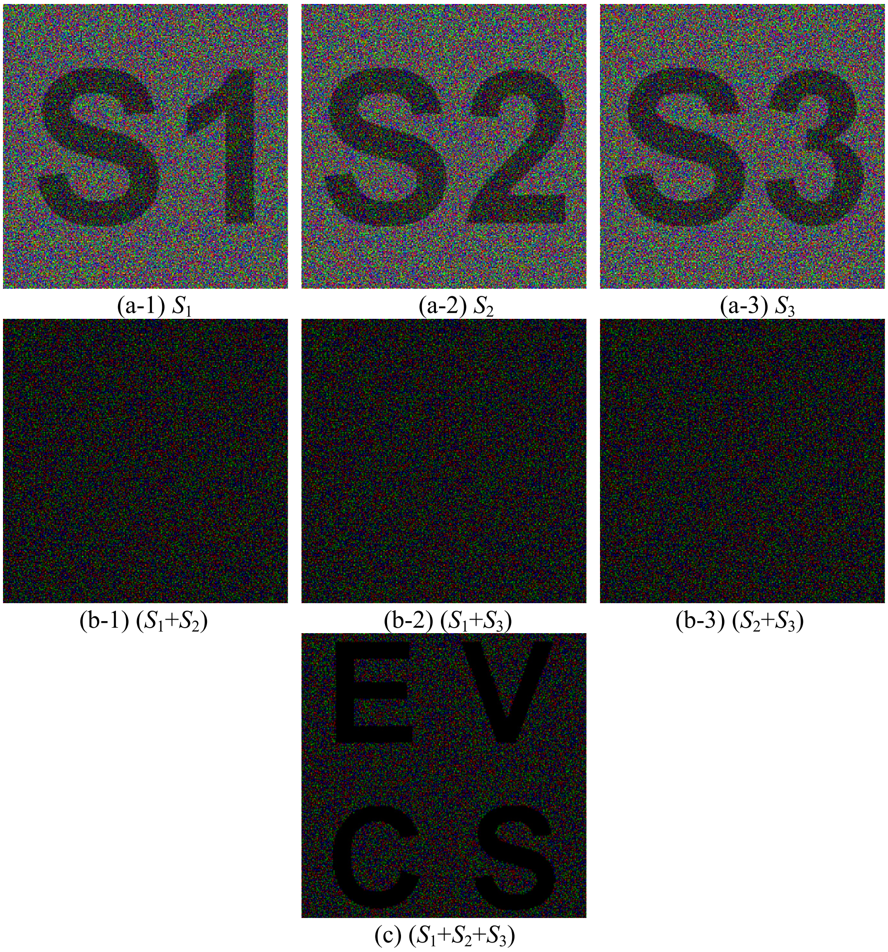


Fig. 10. $(3, 3)$-PB-CBW-EVCS with $\{RGB\}$: (a) shadows, (b) the results of stacking any 2 shadows, (c) the reconstructed image of stacking all 3 shadows.

4.2. Comparison

Generally, the CBW-VCS has the less pixel expansion when compared with the conventional VCS. Even our pixel expansion is not larger than the conventional scheme. The reason is that CBW-VCS and our CBW-EVCS use the color pixels, while the conventional VCS uses the black and white pixels.

In comparing with the black and white (k, n)-EVCS, our (k, n)-CBW-EVCS has the less pixel expansion. For example, Naor and Shamir's (2, 2)-EVCS has $m = 4$, where using $(\bullet\bullet\bullet\circ)/(\bullet\bullet\circ\circ)$ to represent black/white colors in shadows and $(\bullet\bullet\bullet\bullet)/(\bullet\bullet\bullet\circ)$ to represent black/white colors in the reconstructed image. The proposed (2, 2)-CBW-EVCS only has $m = 2$. Although the proposed (k, n)-CBW-EVCS has the larger pixel expansion than De Prisco and De Santis's (k, n)-CBW-VCS, our scheme has the extended capability revealing cover images on shadows. The pixel expansions of (k, n)-VCS, (k, n)-CBW-VCS, and our (k, n)-CBW-EVCS, where $2 \le k \le n \le 9$, are illustrated in Table 3. Consider the example $(3, 5)$ scheme. For $(3, 5)$-CBW-VCS, we should add 2 all-$\bullet$ columns into a $(3, 5)$-PB-VCS with $m_B = 16$ to get 18 pixels (a multiple of 3) and we can implement a $(3, 5)$-CBW-VCS with $m = 6$. There are $(14\bullet 4\circ)$ in these 18 pixels, and thus $d = (n_{\max} - n_{\min}) + 1 = (4 - 2) + 1 = 3$ (note: Table 4 shows the values of d for the schemes in Table 3). So, we append $\lceil n/3 \rceil \times d = \lceil 5/3 \rceil \times 3 = 6$ colored pixels to construct our $(3, 5)$-CBW-EVCS with $m = 6 + 6 = 12$.

Contrast comparison for some small values of k and n is shown in Table 5. All these schemes are perfect black schemes. It is observed that our scheme at least has the same contrast of the reconstructed image when compared with VCS. However, shadows of VCS and CBW-VCS are noise-like, but our CBW-EVCS has the extended capability. In Table 5, our $(2, n)$-PB-CBW-EVCS has $\alpha = \frac{1}{m}$ (see Construction 5). About the contrasts of $(3, n)$-PB-CBW-EVCS, and (n, n) -PB-CBW-EVCS, since they are transformed from the conventional (k, n)-PB-VCS with the pixel expansion m_B and the blackness $h_B = m_B$ and $l_B = (m_B - 1)$. After adding $(\lceil m_B/3 \rceil \times 3) - m_B$ all-$\bullet$ columns, this (k, n)-PB-VCS has new $h_B = (\lceil m_B/3 \rceil \times 3)$ and $l_B = (\lceil m_B/3 \rceil \times 3) - 1$. By grouping every 3 black/white pixels to $\lceil m_B/3 \rceil$ color pixels, we have $w_\bullet(\mathrm{add}(A_\bullet \mid X_k)) = \lceil m_B/3 \rceil$ (since $h_B = (\lceil m_B/3 \rceil \times 3)$) and $w_\bullet(\mathrm{add}(A_\circ \mid X_k)) = (\lceil m_c/3 \rceil - 1)$ (since $l_B = (\lceil m_B/3 \rceil \times 3) - 1$). Therefore, the contrasts of both $(3, n)$-PB-CBW-EVCS

Table 3
Comparison of pixel expansion.

$(2,n)$				$(3,n)^{\blacklozenge}$				$(n-1,n)^{\blacklozenge}$				$(n,n)^{\blacklozenge}$			
n	#1	#2	#3	n	#1	#2	#3	n	#1	#2	#3	n	#1	#2	#3
2	2	1 (1♦)	2 (2♦)	2	–	–	–	2	–	–	–	2	2	1	2
3	3	1 (1♦)	2 (2♦)	3	4	2	4	3	3	1	2	3	4	2	4
4	4	2 (2♦)	4 (4♦)	4	9	3	9	4	9	3	9	4	8	3	7
5	5	2 (2♦)	4 (4♦)	5	16	6	12	5	25	9	23	5	16	6	14
6	6	2 (2♦)	4 (4♦)	6	25	9	17	6	65	22	50	6	32	12	24
7	7	2 (3♦)	5 (6♦)	7	36	12	27	7	161	54	153	7	64	22	58
8	8	2 (3♦)	5 (6♦)	8	49	17	32	8	385	129	357	8	128	43	109
9	9	2 (3♦)	5 (6♦)	9	64	22	40	9	897	299	812	9	256	86	218

#1: (k,n)-VCS; #2: (k,n)-CBW-VCS; #3: (k,n)-CBW-EVCS; ♦: Perfect Black.

Table 4
The values of d for constructing (k,n)-PB-CBW-EVCS.

$(3,n)$			$(n-1,n)$			(n,n)		
n	$(x\bullet y\circ)$	d	n	$(x\bullet y\circ)$	d	n	$(x\bullet y\circ)$	d
2	–	–	2	–	–	2	$(2\bullet 1\circ)$	1
3	$(4\bullet 2\circ)$	2	3	$(2\bullet 1\circ)$	1	3	$(4\bullet 2\circ)$	2
4	$(6\bullet 3\circ)$	3	4	$(6\bullet 3\circ)$	3	4	$(5\bullet 4\circ)$	2
5	$(14\bullet 4\circ)$	3	5	$(18\bullet 9\circ)$	7	5	$(10\bullet 8\circ)$	4
6	$(22\bullet 5\circ)$	4	6	$(41\bullet 25\circ)$	14	6	$(17\bullet 16\circ)$	6
7	$(33\bullet 6\circ)$	5	7	$(97\bullet 65\circ)$	33	7	$(34\bullet 32\circ)$	12
8	$(44\bullet 7\circ)$	5	8	$(226\bullet 161\circ)$	76	8	$(65\bullet 64\circ)$	22
9	$(58\bullet 8\circ)$	6	9	$(512\bullet 385\circ)$	171	9	$(130\bullet 128\circ)$	44

Table 5
Comparison of contrast for (k,n) schemes for some small values of k and n.

(k,n)		The reconstructed image (α)			Shadow (α_s)	
		#1	#2	#3	#1, #2	#3
$k=2$	$n=2$	1/2	1	1/2	0	1/2
	$n=3$	1/3	1	1/2	0	1/2
	$n=4$	1/4	1/2	1/4	0	1/4
	$n=5$	1/5	1/2	1/4	0	1/4
	$n=6$	1/6	1/2	1/4	0	1/4
$k=3$	$n=3$	1/4	1/2	1/4	0	1/2
	$n=4$	1/9	1/3	1/9	0	1/3
	$n=5$	1/16	1/6	1/12	0	1/4
$k=4$	$n=4$	1/8	1/3	1/7	0	2/7

#1: (k,n)-PB-VCS (note: for $k=2$ we use Naor and Shamir's reversed $(2,n)$-PB-VCS).
#2: De Prisco and De Santis's $(2,n)$-PB-CBW-VCS.
#3: the proposed (k,n)-PB-CBW-EVCS.

and (n,n)-PB-CBW-EVCS are all $\alpha = \frac{1}{m}$. For example, in Table 5, our $(3,n)$-PB-CBW-EVCS has $\alpha = \frac{1}{4}$, $\frac{1}{9}$ and $\frac{1}{12}$ for $n = 3, 4$ and 5, and the $(4,4)$-PB-CBW-EVCS has $\alpha = \frac{1}{7}$.

5. Conclusion

In this paper, we take the lead to study CBW-EVCS. Our paper presented $(2,2)$-CBW-EVCS, $(2,n)$-CBW-EVCS, $(2,n)$-PB-CBW-EVCS, and (k,n)-PB-CBW-EVCS. Via the idea of Ateniese et al.'s EVCS, we give the conditions (*security*, *contrast*, and *cover image* conditions) of (k,n)-CWB-EVCS, and prove that the proposed CBW-EVCSs satisfy these conditions. Our scheme at least has the same pixel expansion and contrast when compared with (k,n)-VCS. Meanwhile, our scheme provides the extended capability. For arbitrary k and n, De Prisco and De Santis's (k,n)-PB-CBW-VCS and our (k,n)-PB-CBW-EVCS are all perfect black. How to construct the non-perfect black (k,n)-CBW-VCS and (k,n)-CBW-EVCS is interesting and deserve further studying.

Acknowledgements

This research was supported in part by Ministry of Science and Technology, Taiwan, under Grant MOST 102-2221-E-259-009-MY2 and MOST 103-2221-E-259-015.

Appendix A

Table A1
The collections of $C_\circ^{c_1c_2}$ and $C_\bullet^{c_1c_2}$ in Construction 1.

$$C_\circ^{\circ\circ} = \left\{ \begin{array}{l} \begin{bmatrix} RR \\ RG \end{bmatrix} \begin{bmatrix} RR \\ RB \end{bmatrix} \begin{bmatrix} RG \\ RR \end{bmatrix} \begin{bmatrix} RG \\ RB \end{bmatrix} \begin{bmatrix} RB \\ RR \end{bmatrix} \begin{bmatrix} RB \\ RG \end{bmatrix} \begin{bmatrix} RR \\ GR \end{bmatrix} \begin{bmatrix} RR \\ BR \end{bmatrix} \begin{bmatrix} GR \\ RR \end{bmatrix} \begin{bmatrix} GR \\ BR \end{bmatrix} \begin{bmatrix} BR \\ RR \end{bmatrix} \begin{bmatrix} BR \\ GR \end{bmatrix} \begin{bmatrix} GR \\ GG \end{bmatrix} \begin{bmatrix} GR \\ GB \end{bmatrix} \begin{bmatrix} GG \\ GR \end{bmatrix} \begin{bmatrix} GG \\ GB \end{bmatrix} \begin{bmatrix} GB \\ GR \end{bmatrix} \begin{bmatrix} GB \\ GG \end{bmatrix} \\ \begin{bmatrix} RG \\ GG \end{bmatrix} \begin{bmatrix} RG \\ BG \end{bmatrix} \begin{bmatrix} GG \\ RG \end{bmatrix} \begin{bmatrix} GG \\ BG \end{bmatrix} \begin{bmatrix} BG \\ RG \end{bmatrix} \begin{bmatrix} BG \\ GG \end{bmatrix} \begin{bmatrix} BR \\ BG \end{bmatrix} \begin{bmatrix} BR \\ BB \end{bmatrix} \begin{bmatrix} BG \\ BR \end{bmatrix} \begin{bmatrix} BG \\ BB \end{bmatrix} \begin{bmatrix} BB \\ BR \end{bmatrix} \begin{bmatrix} BB \\ BG \end{bmatrix} \begin{bmatrix} RB \\ GB \end{bmatrix} \begin{bmatrix} RB \\ BB \end{bmatrix} \begin{bmatrix} GB \\ RB \end{bmatrix} \begin{bmatrix} GB \\ BB \end{bmatrix} \begin{bmatrix} BB \\ RB \end{bmatrix} \begin{bmatrix} BB \\ GB \end{bmatrix} \end{array} \right\}$$

$$C_\circ^{\circ\bullet} = \left\{ \begin{bmatrix} RR \\ R\bullet \end{bmatrix} \begin{bmatrix} RG \\ R\bullet \end{bmatrix} \begin{bmatrix} RB \\ R\bullet \end{bmatrix} \begin{bmatrix} RR \\ \bullet R \end{bmatrix} \begin{bmatrix} GR \\ \bullet R \end{bmatrix} \begin{bmatrix} BR \\ \bullet R \end{bmatrix} \begin{bmatrix} GR \\ G\bullet \end{bmatrix} \begin{bmatrix} GG \\ G\bullet \end{bmatrix} \begin{bmatrix} GB \\ G\bullet \end{bmatrix} \begin{bmatrix} RG \\ \bullet G \end{bmatrix} \begin{bmatrix} GG \\ \bullet G \end{bmatrix} \begin{bmatrix} BG \\ \bullet G \end{bmatrix} \begin{bmatrix} BR \\ B\bullet \end{bmatrix} \begin{bmatrix} BG \\ B\bullet \end{bmatrix} \begin{bmatrix} BB \\ B\bullet \end{bmatrix} \begin{bmatrix} RB \\ \bullet B \end{bmatrix} \begin{bmatrix} GB \\ \bullet B \end{bmatrix} \begin{bmatrix} BB \\ \bullet B \end{bmatrix} \right\}$$

$$C_\circ^{\bullet\circ} = \left\{ \begin{bmatrix} R\bullet \\ RR \end{bmatrix} \begin{bmatrix} R\bullet \\ RG \end{bmatrix} \begin{bmatrix} R\bullet \\ RB \end{bmatrix} \begin{bmatrix} \bullet R \\ RR \end{bmatrix} \begin{bmatrix} \bullet R \\ GR \end{bmatrix} \begin{bmatrix} \bullet R \\ BR \end{bmatrix} \begin{bmatrix} G\bullet \\ GR \end{bmatrix} \begin{bmatrix} G\bullet \\ GG \end{bmatrix} \begin{bmatrix} G\bullet \\ GB \end{bmatrix} \begin{bmatrix} \bullet G \\ RG \end{bmatrix} \begin{bmatrix} \bullet G \\ GG \end{bmatrix} \begin{bmatrix} \bullet G \\ BG \end{bmatrix} \begin{bmatrix} B\bullet \\ BR \end{bmatrix} \begin{bmatrix} B\bullet \\ BG \end{bmatrix} \begin{bmatrix} B\bullet \\ BB \end{bmatrix} \begin{bmatrix} \bullet B \\ RB \end{bmatrix} \begin{bmatrix} \bullet B \\ GB \end{bmatrix} \begin{bmatrix} \bullet B \\ BB \end{bmatrix} \right\}$$

$$C_\circ^{\bullet\bullet} = \left\{ \begin{bmatrix} R\bullet \\ R\bullet \end{bmatrix} \begin{bmatrix} G\bullet \\ G\bullet \end{bmatrix} \begin{bmatrix} B\bullet \\ B\bullet \end{bmatrix} \begin{bmatrix} \bullet R \\ \bullet R \end{bmatrix} \begin{bmatrix} \bullet G \\ \bullet G \end{bmatrix} \begin{bmatrix} \bullet B \\ \bullet B \end{bmatrix} \right\}$$

$$C_\bullet^{\circ\circ} = \left\{ \begin{array}{l} \begin{bmatrix} RR \\ GG \end{bmatrix} \begin{bmatrix} RR \\ GB \end{bmatrix} \begin{bmatrix} RG \\ GR \end{bmatrix} \begin{bmatrix} RG \\ GB \end{bmatrix} \begin{bmatrix} RB \\ GR \end{bmatrix} \begin{bmatrix} RB \\ GG \end{bmatrix} \begin{bmatrix} RR \\ BG \end{bmatrix} \begin{bmatrix} RR \\ BB \end{bmatrix} \begin{bmatrix} RG \\ BR \end{bmatrix} \begin{bmatrix} RG \\ BB \end{bmatrix} \begin{bmatrix} RB \\ BR \end{bmatrix} \begin{bmatrix} RB \\ BG \end{bmatrix} \begin{bmatrix} GR \\ RG \end{bmatrix} \begin{bmatrix} GR \\ RB \end{bmatrix} \begin{bmatrix} GG \\ RR \end{bmatrix} \begin{bmatrix} GG \\ RB \end{bmatrix} \begin{bmatrix} GB \\ RR \end{bmatrix} \begin{bmatrix} GB \\ RG \end{bmatrix} \\ \begin{bmatrix} GR \\ BG \end{bmatrix} \begin{bmatrix} GR \\ BB \end{bmatrix} \begin{bmatrix} GG \\ BR \end{bmatrix} \begin{bmatrix} GG \\ BB \end{bmatrix} \begin{bmatrix} GB \\ BR \end{bmatrix} \begin{bmatrix} GB \\ BG \end{bmatrix} \begin{bmatrix} BR \\ RG \end{bmatrix} \begin{bmatrix} BR \\ RB \end{bmatrix} \begin{bmatrix} BG \\ RR \end{bmatrix} \begin{bmatrix} BG \\ RB \end{bmatrix} \begin{bmatrix} BB \\ RR \end{bmatrix} \begin{bmatrix} BB \\ RG \end{bmatrix} \begin{bmatrix} BR \\ GG \end{bmatrix} \begin{bmatrix} BR \\ GB \end{bmatrix} \begin{bmatrix} BG \\ GR \end{bmatrix} \begin{bmatrix} BG \\ GB \end{bmatrix} \begin{bmatrix} BB \\ GR \end{bmatrix} \begin{bmatrix} BB \\ GG \end{bmatrix} \end{array} \right\}$$

$$C_\bullet^{\circ\bullet} = \left\{ \begin{array}{l} \begin{bmatrix} RR \\ G\bullet \end{bmatrix} \begin{bmatrix} RG \\ G\bullet \end{bmatrix} \begin{bmatrix} RB \\ G\bullet \end{bmatrix} \begin{bmatrix} RR \\ B\bullet \end{bmatrix} \begin{bmatrix} RG \\ B\bullet \end{bmatrix} \begin{bmatrix} RB \\ B\bullet \end{bmatrix} \begin{bmatrix} RR \\ \bullet G \end{bmatrix} \begin{bmatrix} GR \\ \bullet G \end{bmatrix} \begin{bmatrix} BR \\ \bullet G \end{bmatrix} \begin{bmatrix} RR \\ \bullet B \end{bmatrix} \begin{bmatrix} GR \\ \bullet B \end{bmatrix} \begin{bmatrix} BR \\ \bullet B \end{bmatrix} \begin{bmatrix} GR \\ R\bullet \end{bmatrix} \begin{bmatrix} GG \\ R\bullet \end{bmatrix} \begin{bmatrix} GB \\ R\bullet \end{bmatrix} \begin{bmatrix} GR \\ B\bullet \end{bmatrix} \begin{bmatrix} GG \\ B\bullet \end{bmatrix} \begin{bmatrix} GB \\ B\bullet \end{bmatrix} \\ \begin{bmatrix} RG \\ \bullet R \end{bmatrix} \begin{bmatrix} GG \\ \bullet R \end{bmatrix} \begin{bmatrix} BG \\ \bullet R \end{bmatrix} \begin{bmatrix} RG \\ \bullet B \end{bmatrix} \begin{bmatrix} GG \\ \bullet B \end{bmatrix} \begin{bmatrix} BG \\ \bullet B \end{bmatrix} \begin{bmatrix} BR \\ R\bullet \end{bmatrix} \begin{bmatrix} BG \\ R\bullet \end{bmatrix} \begin{bmatrix} BB \\ R\bullet \end{bmatrix} \begin{bmatrix} BR \\ G\bullet \end{bmatrix} \begin{bmatrix} BG \\ G\bullet \end{bmatrix} \begin{bmatrix} BB \\ G\bullet \end{bmatrix} \begin{bmatrix} RB \\ \bullet R \end{bmatrix} \begin{bmatrix} GB \\ \bullet R \end{bmatrix} \begin{bmatrix} BB \\ \bullet R \end{bmatrix} \begin{bmatrix} RB \\ \bullet G \end{bmatrix} \begin{bmatrix} GB \\ \bullet G \end{bmatrix} \begin{bmatrix} BB \\ \bullet G \end{bmatrix} \end{array} \right\}$$

$$C_\bullet^{\bullet\circ} = \left\{ \begin{array}{l} \begin{bmatrix} R\bullet \\ GR \end{bmatrix} \begin{bmatrix} R\bullet \\ GG \end{bmatrix} \begin{bmatrix} R\bullet \\ GB \end{bmatrix} \begin{bmatrix} R\bullet \\ BR \end{bmatrix} \begin{bmatrix} R\bullet \\ BG \end{bmatrix} \begin{bmatrix} R\bullet \\ BB \end{bmatrix} \begin{bmatrix} \bullet R \\ RG \end{bmatrix} \begin{bmatrix} \bullet R \\ GG \end{bmatrix} \begin{bmatrix} \bullet R \\ BG \end{bmatrix} \begin{bmatrix} \bullet R \\ RB \end{bmatrix} \begin{bmatrix} \bullet R \\ GB \end{bmatrix} \begin{bmatrix} \bullet R \\ BB \end{bmatrix} \begin{bmatrix} G\bullet \\ RR \end{bmatrix} \begin{bmatrix} G\bullet \\ RG \end{bmatrix} \begin{bmatrix} G\bullet \\ RB \end{bmatrix} \begin{bmatrix} G\bullet \\ BR \end{bmatrix} \begin{bmatrix} G\bullet \\ BG \end{bmatrix} \begin{bmatrix} G\bullet \\ BB \end{bmatrix} \\ \begin{bmatrix} \bullet G \\ RR \end{bmatrix} \begin{bmatrix} \bullet G \\ GR \end{bmatrix} \begin{bmatrix} \bullet G \\ BR \end{bmatrix} \begin{bmatrix} \bullet G \\ RB \end{bmatrix} \begin{bmatrix} \bullet G \\ GB \end{bmatrix} \begin{bmatrix} \bullet G \\ BB \end{bmatrix} \begin{bmatrix} B\bullet \\ RR \end{bmatrix} \begin{bmatrix} B\bullet \\ RG \end{bmatrix} \begin{bmatrix} B\bullet \\ RB \end{bmatrix} \begin{bmatrix} B\bullet \\ GR \end{bmatrix} \begin{bmatrix} B\bullet \\ GG \end{bmatrix} \begin{bmatrix} B\bullet \\ GB \end{bmatrix} \begin{bmatrix} \bullet B \\ RR \end{bmatrix} \begin{bmatrix} \bullet B \\ GR \end{bmatrix} \begin{bmatrix} \bullet B \\ BR \end{bmatrix} \begin{bmatrix} \bullet B \\ RG \end{bmatrix} \begin{bmatrix} \bullet B \\ GG \end{bmatrix} \begin{bmatrix} \bullet B \\ BG \end{bmatrix} \end{array} \right\}$$

$$C_\bullet^{\bullet\bullet} = \left\{ \begin{bmatrix} R\bullet \\ G\bullet \end{bmatrix} \begin{bmatrix} R\bullet \\ B\bullet \end{bmatrix} \begin{bmatrix} \bullet R \\ \bullet G \end{bmatrix} \begin{bmatrix} \bullet R \\ \bullet B \end{bmatrix} \begin{bmatrix} G\bullet \\ R\bullet \end{bmatrix} \begin{bmatrix} G\bullet \\ B\bullet \end{bmatrix} \begin{bmatrix} \bullet G \\ \bullet R \end{bmatrix} \begin{bmatrix} \bullet G \\ \bullet B \end{bmatrix} \begin{bmatrix} B\bullet \\ R\bullet \end{bmatrix} \begin{bmatrix} B\bullet \\ G\bullet \end{bmatrix} \begin{bmatrix} \bullet B \\ \bullet R \end{bmatrix} \begin{bmatrix} \bullet B \\ \bullet G \end{bmatrix} \right\}$$

Table A2
The collections of $C_\circ^{c_1c_2}$ and $C_\bullet^{c_1c_2}$ in Construction 2.

$$C_\circ^{\circ\circ} = \left\{ \begin{array}{l} \begin{bmatrix} RR \\ RC \end{bmatrix} \begin{bmatrix} RG \\ RM \end{bmatrix} \begin{bmatrix} RB \\ RY \end{bmatrix} \begin{bmatrix} RC \\ RR \end{bmatrix} \begin{bmatrix} RM \\ RG \end{bmatrix} \begin{bmatrix} RY \\ RB \end{bmatrix} \begin{bmatrix} RR \\ CR \end{bmatrix} \begin{bmatrix} GR \\ MR \end{bmatrix} \begin{bmatrix} BR \\ YR \end{bmatrix} \begin{bmatrix} CR \\ RR \end{bmatrix} \begin{bmatrix} MR \\ GR \end{bmatrix} \begin{bmatrix} YR \\ BR \end{bmatrix} \begin{bmatrix} GR \\ GC \end{bmatrix} \begin{bmatrix} GG \\ GM \end{bmatrix} \begin{bmatrix} GB \\ GY \end{bmatrix} \begin{bmatrix} GC \\ GR \end{bmatrix} \begin{bmatrix} GM \\ GG \end{bmatrix} \begin{bmatrix} GY \\ GB \end{bmatrix} \\ \begin{bmatrix} RG \\ CG \end{bmatrix} \begin{bmatrix} GG \\ MG \end{bmatrix} \begin{bmatrix} BG \\ YG \end{bmatrix} \begin{bmatrix} CG \\ RG \end{bmatrix} \begin{bmatrix} MG \\ GG \end{bmatrix} \begin{bmatrix} YG \\ BG \end{bmatrix} \begin{bmatrix} BR \\ BC \end{bmatrix} \begin{bmatrix} BG \\ BM \end{bmatrix} \begin{bmatrix} BB \\ BY \end{bmatrix} \begin{bmatrix} BC \\ BR \end{bmatrix} \begin{bmatrix} BM \\ BG \end{bmatrix} \begin{bmatrix} BY \\ BB \end{bmatrix} \begin{bmatrix} RB \\ CB \end{bmatrix} \begin{bmatrix} GB \\ MB \end{bmatrix} \begin{bmatrix} BB \\ YB \end{bmatrix} \begin{bmatrix} CB \\ RB \end{bmatrix} \begin{bmatrix} MB \\ GB \end{bmatrix} \begin{bmatrix} YB \\ BB \end{bmatrix} \\ \begin{bmatrix} CR \\ CC \end{bmatrix} \begin{bmatrix} CG \\ CM \end{bmatrix} \begin{bmatrix} CB \\ CY \end{bmatrix} \begin{bmatrix} CC \\ CR \end{bmatrix} \begin{bmatrix} CM \\ CG \end{bmatrix} \begin{bmatrix} CY \\ CB \end{bmatrix} \begin{bmatrix} RC \\ CC \end{bmatrix} \begin{bmatrix} GC \\ MC \end{bmatrix} \begin{bmatrix} BC \\ YC \end{bmatrix} \begin{bmatrix} CC \\ RC \end{bmatrix} \begin{bmatrix} MC \\ GC \end{bmatrix} \begin{bmatrix} YC \\ BC \end{bmatrix} \begin{bmatrix} MR \\ MC \end{bmatrix} \begin{bmatrix} MG \\ MM \end{bmatrix} \begin{bmatrix} MB \\ MY \end{bmatrix} \begin{bmatrix} MC \\ MR \end{bmatrix} \begin{bmatrix} MM \\ MG \end{bmatrix} \begin{bmatrix} MY \\ MB \end{bmatrix} \\ \begin{bmatrix} RM \\ CM \end{bmatrix} \begin{bmatrix} GM \\ MM \end{bmatrix} \begin{bmatrix} BM \\ YM \end{bmatrix} \begin{bmatrix} CM \\ RM \end{bmatrix} \begin{bmatrix} MM \\ GM \end{bmatrix} \begin{bmatrix} YM \\ BM \end{bmatrix} \begin{bmatrix} YR \\ YC \end{bmatrix} \begin{bmatrix} YG \\ YM \end{bmatrix} \begin{bmatrix} YB \\ YY \end{bmatrix} \begin{bmatrix} YC \\ YR \end{bmatrix} \begin{bmatrix} YM \\ YG \end{bmatrix} \begin{bmatrix} YY \\ YB \end{bmatrix} \begin{bmatrix} RY \\ CY \end{bmatrix} \begin{bmatrix} GY \\ MY \end{bmatrix} \begin{bmatrix} BY \\ YY \end{bmatrix} \begin{bmatrix} CY \\ RY \end{bmatrix} \begin{bmatrix} MY \\ GY \end{bmatrix} \begin{bmatrix} YY \\ BY \end{bmatrix} \end{array} \right\}$$

$$C_\circ^{\circ\bullet} = \left\{ \begin{array}{l} \begin{bmatrix} RR \\ R\bullet \end{bmatrix} \begin{bmatrix} RG \\ R\bullet \end{bmatrix} \begin{bmatrix} RB \\ R\bullet \end{bmatrix} \begin{bmatrix} RC \\ R\bullet \end{bmatrix} \begin{bmatrix} RM \\ R\bullet \end{bmatrix} \begin{bmatrix} RY \\ R\bullet \end{bmatrix} \begin{bmatrix} RR \\ \bullet R \end{bmatrix} \begin{bmatrix} GR \\ \bullet R \end{bmatrix} \begin{bmatrix} BR \\ \bullet R \end{bmatrix} \begin{bmatrix} CR \\ \bullet R \end{bmatrix} \begin{bmatrix} MR \\ \bullet R \end{bmatrix} \begin{bmatrix} YR \\ \bullet R \end{bmatrix} \begin{bmatrix} GR \\ G\bullet \end{bmatrix} \begin{bmatrix} GG \\ G\bullet \end{bmatrix} \begin{bmatrix} GB \\ G\bullet \end{bmatrix} \begin{bmatrix} GC \\ G\bullet \end{bmatrix} \begin{bmatrix} GM \\ G\bullet \end{bmatrix} \begin{bmatrix} GY \\ G\bullet \end{bmatrix} \\ \begin{bmatrix} RG \\ \bullet G \end{bmatrix} \begin{bmatrix} GG \\ \bullet G \end{bmatrix} \begin{bmatrix} BG \\ \bullet G \end{bmatrix} \begin{bmatrix} CG \\ \bullet G \end{bmatrix} \begin{bmatrix} MG \\ \bullet G \end{bmatrix} \begin{bmatrix} YG \\ \bullet G \end{bmatrix} \begin{bmatrix} BR \\ B\bullet \end{bmatrix} \begin{bmatrix} BG \\ B\bullet \end{bmatrix} \begin{bmatrix} BB \\ B\bullet \end{bmatrix} \begin{bmatrix} BC \\ B\bullet \end{bmatrix} \begin{bmatrix} BM \\ B\bullet \end{bmatrix} \begin{bmatrix} BY \\ B\bullet \end{bmatrix} \begin{bmatrix} RB \\ \bullet B \end{bmatrix} \begin{bmatrix} GB \\ \bullet B \end{bmatrix} \begin{bmatrix} BB \\ \bullet B \end{bmatrix} \begin{bmatrix} CB \\ \bullet B \end{bmatrix} \begin{bmatrix} MB \\ \bullet B \end{bmatrix} \begin{bmatrix} YB \\ \bullet B \end{bmatrix} \\ \begin{bmatrix} CR \\ C\bullet \end{bmatrix} \begin{bmatrix} CG \\ C\bullet \end{bmatrix} \begin{bmatrix} CB \\ C\bullet \end{bmatrix} \begin{bmatrix} CC \\ C\bullet \end{bmatrix} \begin{bmatrix} CM \\ C\bullet \end{bmatrix} \begin{bmatrix} CY \\ C\bullet \end{bmatrix} \begin{bmatrix} RC \\ \bullet C \end{bmatrix} \begin{bmatrix} GC \\ \bullet C \end{bmatrix} \begin{bmatrix} BC \\ \bullet C \end{bmatrix} \begin{bmatrix} CC \\ \bullet C \end{bmatrix} \begin{bmatrix} MC \\ \bullet C \end{bmatrix} \begin{bmatrix} YC \\ \bullet C \end{bmatrix} \begin{bmatrix} MR \\ M\bullet \end{bmatrix} \begin{bmatrix} MG \\ M\bullet \end{bmatrix} \begin{bmatrix} MB \\ M\bullet \end{bmatrix} \begin{bmatrix} MC \\ M\bullet \end{bmatrix} \begin{bmatrix} MM \\ M\bullet \end{bmatrix} \begin{bmatrix} MY \\ M\bullet \end{bmatrix} \\ \begin{bmatrix} RM \\ \bullet M \end{bmatrix} \begin{bmatrix} GM \\ \bullet M \end{bmatrix} \begin{bmatrix} BM \\ \bullet M \end{bmatrix} \begin{bmatrix} CM \\ \bullet M \end{bmatrix} \begin{bmatrix} MM \\ \bullet M \end{bmatrix} \begin{bmatrix} YM \\ \bullet M \end{bmatrix} \begin{bmatrix} YR \\ Y\bullet \end{bmatrix} \begin{bmatrix} YG \\ Y\bullet \end{bmatrix} \begin{bmatrix} YB \\ Y\bullet \end{bmatrix} \begin{bmatrix} YC \\ Y\bullet \end{bmatrix} \begin{bmatrix} YM \\ Y\bullet \end{bmatrix} \begin{bmatrix} YY \\ Y\bullet \end{bmatrix} \begin{bmatrix} RY \\ \bullet Y \end{bmatrix} \begin{bmatrix} GY \\ \bullet Y \end{bmatrix} \begin{bmatrix} BY \\ \bullet Y \end{bmatrix} \begin{bmatrix} CY \\ \bullet Y \end{bmatrix} \begin{bmatrix} MY \\ \bullet Y \end{bmatrix} \begin{bmatrix} YY \\ \bullet Y \end{bmatrix} \end{array} \right\}$$

Table A2 (*continued*)

$$C_{\circ}^{\bullet\circ}=\left\{\begin{array}{l}
\begin{bmatrix} R\bullet \\ RR \end{bmatrix}\begin{bmatrix} R\bullet \\ RG \end{bmatrix}\begin{bmatrix} R\bullet \\ RB \end{bmatrix}\begin{bmatrix} R\bullet \\ RC \end{bmatrix}\begin{bmatrix} R\bullet \\ RM \end{bmatrix}\begin{bmatrix} R\bullet \\ RY \end{bmatrix}\begin{bmatrix} \bullet R \\ RR \end{bmatrix}\begin{bmatrix} \bullet R \\ GR \end{bmatrix}\begin{bmatrix} \bullet R \\ BR \end{bmatrix}\begin{bmatrix} \bullet R \\ CR \end{bmatrix}\begin{bmatrix} \bullet R \\ MR \end{bmatrix}\begin{bmatrix} \bullet R \\ YR \end{bmatrix}\begin{bmatrix} G\bullet \\ GR \end{bmatrix}\begin{bmatrix} G\bullet \\ GG \end{bmatrix}\begin{bmatrix} G\bullet \\ GB \end{bmatrix}\begin{bmatrix} G\bullet \\ GC \end{bmatrix}\begin{bmatrix} G\bullet \\ GM \end{bmatrix}\begin{bmatrix} G\bullet \\ GY \end{bmatrix} \\
\begin{bmatrix} \bullet G \\ RG \end{bmatrix}\begin{bmatrix} \bullet G \\ GG \end{bmatrix}\begin{bmatrix} \bullet G \\ BG \end{bmatrix}\begin{bmatrix} \bullet G \\ CG \end{bmatrix}\begin{bmatrix} \bullet G \\ MG \end{bmatrix}\begin{bmatrix} \bullet G \\ YG \end{bmatrix}\begin{bmatrix} B\bullet \\ BR \end{bmatrix}\begin{bmatrix} B\bullet \\ BG \end{bmatrix}\begin{bmatrix} B\bullet \\ BB \end{bmatrix}\begin{bmatrix} B\bullet \\ BC \end{bmatrix}\begin{bmatrix} B\bullet \\ BM \end{bmatrix}\begin{bmatrix} B\bullet \\ BY \end{bmatrix}\begin{bmatrix} \bullet B \\ RB \end{bmatrix}\begin{bmatrix} \bullet B \\ GB \end{bmatrix}\begin{bmatrix} \bullet B \\ BB \end{bmatrix}\begin{bmatrix} \bullet B \\ CB \end{bmatrix}\begin{bmatrix} \bullet B \\ MB \end{bmatrix}\begin{bmatrix} \bullet B \\ YB \end{bmatrix} \\
\begin{bmatrix} C\bullet \\ CR \end{bmatrix}\begin{bmatrix} C\bullet \\ CG \end{bmatrix}\begin{bmatrix} C\bullet \\ CB \end{bmatrix}\begin{bmatrix} C\bullet \\ CC \end{bmatrix}\begin{bmatrix} C\bullet \\ CM \end{bmatrix}\begin{bmatrix} C\bullet \\ CY \end{bmatrix}\begin{bmatrix} \bullet C \\ RC \end{bmatrix}\begin{bmatrix} \bullet C \\ GC \end{bmatrix}\begin{bmatrix} \bullet C \\ BC \end{bmatrix}\begin{bmatrix} \bullet C \\ CC \end{bmatrix}\begin{bmatrix} \bullet C \\ MC \end{bmatrix}\begin{bmatrix} \bullet C \\ YC \end{bmatrix}\begin{bmatrix} M\bullet \\ MR \end{bmatrix}\begin{bmatrix} M\bullet \\ MG \end{bmatrix}\begin{bmatrix} M\bullet \\ MB \end{bmatrix}\begin{bmatrix} M\bullet \\ MC \end{bmatrix}\begin{bmatrix} M\bullet \\ MM \end{bmatrix}\begin{bmatrix} M\bullet \\ MY \end{bmatrix} \\
\begin{bmatrix} \bullet M \\ RM \end{bmatrix}\begin{bmatrix} \bullet M \\ GM \end{bmatrix}\begin{bmatrix} \bullet M \\ BM \end{bmatrix}\begin{bmatrix} \bullet M \\ CM \end{bmatrix}\begin{bmatrix} \bullet M \\ MM \end{bmatrix}\begin{bmatrix} \bullet M \\ YM \end{bmatrix}\begin{bmatrix} Y\bullet \\ YR \end{bmatrix}\begin{bmatrix} Y\bullet \\ YG \end{bmatrix}\begin{bmatrix} Y\bullet \\ YB \end{bmatrix}\begin{bmatrix} Y\bullet \\ YC \end{bmatrix}\begin{bmatrix} Y\bullet \\ YM \end{bmatrix}\begin{bmatrix} Y\bullet \\ YY \end{bmatrix}\begin{bmatrix} \bullet Y \\ RY \end{bmatrix}\begin{bmatrix} \bullet Y \\ GY \end{bmatrix}\begin{bmatrix} \bullet Y \\ BY \end{bmatrix}\begin{bmatrix} \bullet Y \\ CY \end{bmatrix}\begin{bmatrix} \bullet Y \\ MY \end{bmatrix}\begin{bmatrix} \bullet Y \\ YY \end{bmatrix}
\end{array}\right\}$$

$$C_{\circ}^{\bullet\bullet}=\left\{\begin{bmatrix} R\bullet \\ R\bullet \end{bmatrix}\begin{bmatrix} G\bullet \\ G\bullet \end{bmatrix}\begin{bmatrix} B\bullet \\ B\bullet \end{bmatrix}\begin{bmatrix} C\bullet \\ C\bullet \end{bmatrix}\begin{bmatrix} M\bullet \\ M\bullet \end{bmatrix}\begin{bmatrix} Y\bullet \\ Y\bullet \end{bmatrix}\begin{bmatrix} \bullet R \\ \bullet R \end{bmatrix}\begin{bmatrix} \bullet G \\ \bullet G \end{bmatrix}\begin{bmatrix} \bullet B \\ \bullet B \end{bmatrix}\begin{bmatrix} \bullet C \\ \bullet C \end{bmatrix}\begin{bmatrix} \bullet M \\ \bullet M \end{bmatrix}\begin{bmatrix} \bullet Y \\ \bullet Y \end{bmatrix}\right\}$$

$$C_{\bullet}^{\circ\circ}=\left\{\begin{array}{l}
\begin{bmatrix} RR \\ CC \end{bmatrix}\begin{bmatrix} RG \\ CM \end{bmatrix}\begin{bmatrix} RB \\ CY \end{bmatrix}\begin{bmatrix} RC \\ CR \end{bmatrix}\begin{bmatrix} RM \\ CG \end{bmatrix}\begin{bmatrix} RY \\ CB \end{bmatrix}\begin{bmatrix} RR \\ CC \end{bmatrix}\begin{bmatrix} GR \\ MC \end{bmatrix}\begin{bmatrix} BR \\ YC \end{bmatrix}\begin{bmatrix} CR \\ RC \end{bmatrix}\begin{bmatrix} MR \\ GC \end{bmatrix}\begin{bmatrix} YR \\ BC \end{bmatrix}\begin{bmatrix} GR \\ MC \end{bmatrix}\begin{bmatrix} GG \\ MM \end{bmatrix}\begin{bmatrix} GB \\ MY \end{bmatrix}\begin{bmatrix} GC \\ MR \end{bmatrix}\begin{bmatrix} GM \\ MG \end{bmatrix}\begin{bmatrix} GY \\ MB \end{bmatrix} \\
\begin{bmatrix} RG \\ CM \end{bmatrix}\begin{bmatrix} GG \\ MM \end{bmatrix}\begin{bmatrix} BG \\ YM \end{bmatrix}\begin{bmatrix} CG \\ RM \end{bmatrix}\begin{bmatrix} MG \\ GM \end{bmatrix}\begin{bmatrix} YG \\ BM \end{bmatrix}\begin{bmatrix} BR \\ YC \end{bmatrix}\begin{bmatrix} BG \\ YM \end{bmatrix}\begin{bmatrix} BB \\ YY \end{bmatrix}\begin{bmatrix} BC \\ YR \end{bmatrix}\begin{bmatrix} BM \\ YG \end{bmatrix}\begin{bmatrix} BY \\ YB \end{bmatrix}\begin{bmatrix} RB \\ CY \end{bmatrix}\begin{bmatrix} GB \\ MY \end{bmatrix}\begin{bmatrix} BB \\ YY \end{bmatrix}\begin{bmatrix} CB \\ RY \end{bmatrix}\begin{bmatrix} MB \\ GY \end{bmatrix}\begin{bmatrix} YB \\ BY \end{bmatrix} \\
\begin{bmatrix} CR \\ RC \end{bmatrix}\begin{bmatrix} CG \\ RM \end{bmatrix}\begin{bmatrix} CB \\ RY \end{bmatrix}\begin{bmatrix} CC \\ RR \end{bmatrix}\begin{bmatrix} CM \\ RG \end{bmatrix}\begin{bmatrix} CY \\ RB \end{bmatrix}\begin{bmatrix} RC \\ CR \end{bmatrix}\begin{bmatrix} GC \\ MR \end{bmatrix}\begin{bmatrix} BC \\ YR \end{bmatrix}\begin{bmatrix} CC \\ RR \end{bmatrix}\begin{bmatrix} MC \\ GR \end{bmatrix}\begin{bmatrix} YC \\ BR \end{bmatrix}\begin{bmatrix} MR \\ GC \end{bmatrix}\begin{bmatrix} MG \\ GM \end{bmatrix}\begin{bmatrix} MB \\ GY \end{bmatrix}\begin{bmatrix} MC \\ GR \end{bmatrix}\begin{bmatrix} MM \\ GG \end{bmatrix}\begin{bmatrix} MY \\ GB \end{bmatrix} \\
\begin{bmatrix} RM \\ CG \end{bmatrix}\begin{bmatrix} GM \\ MG \end{bmatrix}\begin{bmatrix} BM \\ YG \end{bmatrix}\begin{bmatrix} CM \\ RG \end{bmatrix}\begin{bmatrix} MM \\ GG \end{bmatrix}\begin{bmatrix} YM \\ BG \end{bmatrix}\begin{bmatrix} YR \\ BC \end{bmatrix}\begin{bmatrix} YG \\ BM \end{bmatrix}\begin{bmatrix} YB \\ BY \end{bmatrix}\begin{bmatrix} YC \\ BR \end{bmatrix}\begin{bmatrix} YM \\ BG \end{bmatrix}\begin{bmatrix} YY \\ BB \end{bmatrix}\begin{bmatrix} RY \\ CB \end{bmatrix}\begin{bmatrix} GY \\ MB \end{bmatrix}\begin{bmatrix} BY \\ YB \end{bmatrix}\begin{bmatrix} CY \\ RB \end{bmatrix}\begin{bmatrix} MY \\ GB \end{bmatrix}\begin{bmatrix} YY \\ BB \end{bmatrix}
\end{array}\right\}$$

$$C_{\bullet}^{\circ\bullet}=\left\{\begin{array}{l}
\begin{bmatrix} RR \\ C\bullet \end{bmatrix}\begin{bmatrix} RG \\ C\bullet \end{bmatrix}\begin{bmatrix} RB \\ C\bullet \end{bmatrix}\begin{bmatrix} RC \\ C\bullet \end{bmatrix}\begin{bmatrix} RM \\ C\bullet \end{bmatrix}\begin{bmatrix} RY \\ C\bullet \end{bmatrix}\begin{bmatrix} RR \\ \bullet C \end{bmatrix}\begin{bmatrix} GR \\ \bullet C \end{bmatrix}\begin{bmatrix} BR \\ \bullet C \end{bmatrix}\begin{bmatrix} CR \\ \bullet C \end{bmatrix}\begin{bmatrix} MR \\ \bullet C \end{bmatrix}\begin{bmatrix} YR \\ \bullet C \end{bmatrix}\begin{bmatrix} GR \\ M\bullet \end{bmatrix}\begin{bmatrix} GG \\ M\bullet \end{bmatrix}\begin{bmatrix} GB \\ M\bullet \end{bmatrix}\begin{bmatrix} GC \\ M\bullet \end{bmatrix}\begin{bmatrix} GM \\ M\bullet \end{bmatrix}\begin{bmatrix} GY \\ M\bullet \end{bmatrix} \\
\begin{bmatrix} RG \\ \bullet M \end{bmatrix}\begin{bmatrix} GG \\ \bullet M \end{bmatrix}\begin{bmatrix} BG \\ \bullet M \end{bmatrix}\begin{bmatrix} CG \\ \bullet M \end{bmatrix}\begin{bmatrix} MG \\ \bullet M \end{bmatrix}\begin{bmatrix} YG \\ \bullet M \end{bmatrix}\begin{bmatrix} BR \\ Y\bullet \end{bmatrix}\begin{bmatrix} BG \\ Y\bullet \end{bmatrix}\begin{bmatrix} BB \\ Y\bullet \end{bmatrix}\begin{bmatrix} BC \\ Y\bullet \end{bmatrix}\begin{bmatrix} BM \\ Y\bullet \end{bmatrix}\begin{bmatrix} BY \\ Y\bullet \end{bmatrix}\begin{bmatrix} RB \\ \bullet Y \end{bmatrix}\begin{bmatrix} GB \\ \bullet Y \end{bmatrix}\begin{bmatrix} BB \\ \bullet Y \end{bmatrix}\begin{bmatrix} CB \\ \bullet Y \end{bmatrix}\begin{bmatrix} MB \\ \bullet Y \end{bmatrix}\begin{bmatrix} YB \\ \bullet Y \end{bmatrix} \\
\begin{bmatrix} CR \\ R\bullet \end{bmatrix}\begin{bmatrix} CG \\ R\bullet \end{bmatrix}\begin{bmatrix} CB \\ R\bullet \end{bmatrix}\begin{bmatrix} CC \\ R\bullet \end{bmatrix}\begin{bmatrix} CM \\ R\bullet \end{bmatrix}\begin{bmatrix} CY \\ R\bullet \end{bmatrix}\begin{bmatrix} RC \\ \bullet R \end{bmatrix}\begin{bmatrix} GC \\ \bullet R \end{bmatrix}\begin{bmatrix} BC \\ \bullet R \end{bmatrix}\begin{bmatrix} CC \\ \bullet R \end{bmatrix}\begin{bmatrix} MC \\ \bullet R \end{bmatrix}\begin{bmatrix} YC \\ \bullet R \end{bmatrix}\begin{bmatrix} MR \\ G\bullet \end{bmatrix}\begin{bmatrix} MG \\ G\bullet \end{bmatrix}\begin{bmatrix} MB \\ G\bullet \end{bmatrix}\begin{bmatrix} MC \\ G\bullet \end{bmatrix}\begin{bmatrix} MM \\ G\bullet \end{bmatrix}\begin{bmatrix} MY \\ G\bullet \end{bmatrix} \\
\begin{bmatrix} RM \\ \bullet G \end{bmatrix}\begin{bmatrix} GM \\ \bullet G \end{bmatrix}\begin{bmatrix} BM \\ \bullet G \end{bmatrix}\begin{bmatrix} CM \\ \bullet G \end{bmatrix}\begin{bmatrix} MM \\ \bullet G \end{bmatrix}\begin{bmatrix} YM \\ \bullet G \end{bmatrix}\begin{bmatrix} YR \\ B\bullet \end{bmatrix}\begin{bmatrix} YG \\ B\bullet \end{bmatrix}\begin{bmatrix} YB \\ B\bullet \end{bmatrix}\begin{bmatrix} YC \\ B\bullet \end{bmatrix}\begin{bmatrix} YM \\ B\bullet \end{bmatrix}\begin{bmatrix} YY \\ B\bullet \end{bmatrix}\begin{bmatrix} RY \\ \bullet B \end{bmatrix}\begin{bmatrix} GY \\ \bullet B \end{bmatrix}\begin{bmatrix} BY \\ \bullet B \end{bmatrix}\begin{bmatrix} CY \\ \bullet B \end{bmatrix}\begin{bmatrix} MY \\ \bullet B \end{bmatrix}\begin{bmatrix} YY \\ \bullet B \end{bmatrix}
\end{array}\right\}$$

$$C_{\bullet}^{\bullet\circ}=\left\{\begin{array}{l}
\begin{bmatrix} R\bullet \\ CR \end{bmatrix}\begin{bmatrix} R\bullet \\ CG \end{bmatrix}\begin{bmatrix} R\bullet \\ CB \end{bmatrix}\begin{bmatrix} R\bullet \\ CC \end{bmatrix}\begin{bmatrix} R\bullet \\ CM \end{bmatrix}\begin{bmatrix} R\bullet \\ CY \end{bmatrix}\begin{bmatrix} \bullet R \\ RC \end{bmatrix}\begin{bmatrix} \bullet R \\ GC \end{bmatrix}\begin{bmatrix} \bullet R \\ BC \end{bmatrix}\begin{bmatrix} \bullet R \\ CC \end{bmatrix}\begin{bmatrix} \bullet R \\ MC \end{bmatrix}\begin{bmatrix} \bullet R \\ YC \end{bmatrix}\begin{bmatrix} G\bullet \\ MR \end{bmatrix}\begin{bmatrix} G\bullet \\ MG \end{bmatrix}\begin{bmatrix} G\bullet \\ MB \end{bmatrix}\begin{bmatrix} G\bullet \\ MC \end{bmatrix}\begin{bmatrix} G\bullet \\ MM \end{bmatrix}\begin{bmatrix} G\bullet \\ MY \end{bmatrix} \\
\begin{bmatrix} \bullet G \\ RM \end{bmatrix}\begin{bmatrix} \bullet G \\ GM \end{bmatrix}\begin{bmatrix} \bullet G \\ BM \end{bmatrix}\begin{bmatrix} \bullet G \\ CM \end{bmatrix}\begin{bmatrix} \bullet G \\ MM \end{bmatrix}\begin{bmatrix} \bullet G \\ YM \end{bmatrix}\begin{bmatrix} B\bullet \\ YR \end{bmatrix}\begin{bmatrix} B\bullet \\ YG \end{bmatrix}\begin{bmatrix} B\bullet \\ YB \end{bmatrix}\begin{bmatrix} B\bullet \\ YC \end{bmatrix}\begin{bmatrix} B\bullet \\ YM \end{bmatrix}\begin{bmatrix} B\bullet \\ YY \end{bmatrix}\begin{bmatrix} \bullet B \\ RY \end{bmatrix}\begin{bmatrix} \bullet B \\ GY \end{bmatrix}\begin{bmatrix} \bullet B \\ BY \end{bmatrix}\begin{bmatrix} \bullet B \\ CY \end{bmatrix}\begin{bmatrix} \bullet B \\ MY \end{bmatrix}\begin{bmatrix} \bullet B \\ YY \end{bmatrix} \\
\begin{bmatrix} C\bullet \\ RR \end{bmatrix}\begin{bmatrix} C\bullet \\ RG \end{bmatrix}\begin{bmatrix} C\bullet \\ RB \end{bmatrix}\begin{bmatrix} C\bullet \\ RC \end{bmatrix}\begin{bmatrix} C\bullet \\ RM \end{bmatrix}\begin{bmatrix} C\bullet \\ RY \end{bmatrix}\begin{bmatrix} \bullet C \\ RR \end{bmatrix}\begin{bmatrix} \bullet C \\ GR \end{bmatrix}\begin{bmatrix} \bullet C \\ BR \end{bmatrix}\begin{bmatrix} \bullet C \\ CR \end{bmatrix}\begin{bmatrix} \bullet C \\ MR \end{bmatrix}\begin{bmatrix} \bullet C \\ YR \end{bmatrix}\begin{bmatrix} M\bullet \\ GR \end{bmatrix}\begin{bmatrix} M\bullet \\ GG \end{bmatrix}\begin{bmatrix} M\bullet \\ GB \end{bmatrix}\begin{bmatrix} M\bullet \\ GC \end{bmatrix}\begin{bmatrix} M\bullet \\ GM \end{bmatrix}\begin{bmatrix} M\bullet \\ GY \end{bmatrix} \\
\begin{bmatrix} \bullet M \\ RG \end{bmatrix}\begin{bmatrix} \bullet M \\ GG \end{bmatrix}\begin{bmatrix} \bullet M \\ BG \end{bmatrix}\begin{bmatrix} \bullet M \\ CG \end{bmatrix}\begin{bmatrix} \bullet M \\ MG \end{bmatrix}\begin{bmatrix} \bullet M \\ YG \end{bmatrix}\begin{bmatrix} Y\bullet \\ BR \end{bmatrix}\begin{bmatrix} Y\bullet \\ BG \end{bmatrix}\begin{bmatrix} Y\bullet \\ BB \end{bmatrix}\begin{bmatrix} Y\bullet \\ BC \end{bmatrix}\begin{bmatrix} Y\bullet \\ BM \end{bmatrix}\begin{bmatrix} Y\bullet \\ BY \end{bmatrix}\begin{bmatrix} \bullet Y \\ RB \end{bmatrix}\begin{bmatrix} \bullet Y \\ GB \end{bmatrix}\begin{bmatrix} \bullet Y \\ BB \end{bmatrix}\begin{bmatrix} \bullet Y \\ CB \end{bmatrix}\begin{bmatrix} \bullet Y \\ MB \end{bmatrix}\begin{bmatrix} \bullet Y \\ YB \end{bmatrix}
\end{array}\right\}$$

$$C_{\bullet}^{\bullet\bullet}=\left\{\begin{bmatrix} R\bullet \\ C\bullet \end{bmatrix}\begin{bmatrix} G\bullet \\ M\bullet \end{bmatrix}\begin{bmatrix} B\bullet \\ Y\bullet \end{bmatrix}\begin{bmatrix} C\bullet \\ R\bullet \end{bmatrix}\begin{bmatrix} M\bullet \\ G\bullet \end{bmatrix}\begin{bmatrix} Y\bullet \\ B\bullet \end{bmatrix}\begin{bmatrix} \bullet R \\ \bullet C \end{bmatrix}\begin{bmatrix} \bullet G \\ \bullet M \end{bmatrix}\begin{bmatrix} \bullet B \\ \bullet Y \end{bmatrix}\begin{bmatrix} \bullet C \\ \bullet R \end{bmatrix}\begin{bmatrix} \bullet M \\ \bullet G \end{bmatrix}\begin{bmatrix} \bullet Y \\ \bullet B \end{bmatrix}\right\}$$

Table A3
The collections of $C_{\circ}^{c_1c_2}$ and $C_{\bullet}^{c_1c_2}$ in Construction 3.

$$C_{\circ}^{\circ\circ}=\left\{\begin{bmatrix} RR \\ RG \end{bmatrix}\begin{bmatrix} RG \\ RR \end{bmatrix}\begin{bmatrix} RR \\ GR \end{bmatrix}\begin{bmatrix} GR \\ RR \end{bmatrix}\begin{bmatrix} GR \\ GG \end{bmatrix}\begin{bmatrix} GG \\ GR \end{bmatrix}\begin{bmatrix} RG \\ GG \end{bmatrix}\begin{bmatrix} GG \\ RG \end{bmatrix}\right\} \quad C_{\circ}^{\circ\bullet}=\left\{\begin{bmatrix} RR \\ R\bullet \end{bmatrix}\begin{bmatrix} RG \\ R\bullet \end{bmatrix}\begin{bmatrix} RR \\ \bullet R \end{bmatrix}\begin{bmatrix} GR \\ \bullet R \end{bmatrix}\begin{bmatrix} GR \\ G\bullet \end{bmatrix}\begin{bmatrix} GG \\ G\bullet \end{bmatrix}\begin{bmatrix} RG \\ \bullet G \end{bmatrix}\begin{bmatrix} GG \\ \bullet G \end{bmatrix}\right\}$$

$$C_{\circ}^{\bullet\circ}=\left\{\begin{bmatrix} R\bullet \\ RR \end{bmatrix}\begin{bmatrix} R\bullet \\ RG \end{bmatrix}\begin{bmatrix} \bullet R \\ RR \end{bmatrix}\begin{bmatrix} \bullet R \\ GR \end{bmatrix}\begin{bmatrix} G\bullet \\ GR \end{bmatrix}\begin{bmatrix} G\bullet \\ GG \end{bmatrix}\begin{bmatrix} \bullet G \\ RG \end{bmatrix}\begin{bmatrix} \bullet G \\ GG \end{bmatrix}\right\} \quad C_{\circ}^{\bullet\bullet}=\left\{\begin{bmatrix} R\bullet \\ R\bullet \end{bmatrix}\begin{bmatrix} G\bullet \\ G\bullet \end{bmatrix}\begin{bmatrix} \bullet R \\ \bullet R \end{bmatrix}\begin{bmatrix} \bullet G \\ \bullet G \end{bmatrix}\right\}$$

$$C_{\bullet}^{\circ\circ}=\left\{\begin{bmatrix} RR \\ GG \end{bmatrix}\begin{bmatrix} RG \\ GR \end{bmatrix}\begin{bmatrix} RR \\ GG \end{bmatrix}\begin{bmatrix} GR \\ RG \end{bmatrix}\begin{bmatrix} GR \\ RG \end{bmatrix}\begin{bmatrix} GG \\ RR \end{bmatrix}\begin{bmatrix} RG \\ GR \end{bmatrix}\begin{bmatrix} GG \\ RR \end{bmatrix}\right\} \quad C_{\bullet}^{\circ\bullet}=\left\{\begin{bmatrix} RR \\ G\bullet \end{bmatrix}\begin{bmatrix} RG \\ G\bullet \end{bmatrix}\begin{bmatrix} RR \\ \bullet G \end{bmatrix}\begin{bmatrix} GR \\ \bullet G \end{bmatrix}\begin{bmatrix} GR \\ R\bullet \end{bmatrix}\begin{bmatrix} GG \\ R\bullet \end{bmatrix}\begin{bmatrix} RG \\ \bullet R \end{bmatrix}\begin{bmatrix} GG \\ \bullet R \end{bmatrix}\right\}$$

$$C_{\bullet}^{\bullet\circ}=\left\{\begin{bmatrix} R\bullet \\ GR \end{bmatrix}\begin{bmatrix} R\bullet \\ GG \end{bmatrix}\begin{bmatrix} \bullet R \\ RG \end{bmatrix}\begin{bmatrix} \bullet R \\ GG \end{bmatrix}\begin{bmatrix} G\bullet \\ RR \end{bmatrix}\begin{bmatrix} G\bullet \\ RG \end{bmatrix}\begin{bmatrix} \bullet G \\ RR \end{bmatrix}\begin{bmatrix} \bullet G \\ GR \end{bmatrix}\right\} \quad C_{\bullet}^{\bullet\bullet}=\left\{\begin{bmatrix} R\bullet \\ G\bullet \end{bmatrix}\begin{bmatrix} \bullet R \\ \bullet G \end{bmatrix}\begin{bmatrix} G\bullet \\ R\bullet \end{bmatrix}\begin{bmatrix} \bullet G \\ \bullet R \end{bmatrix}\right\}$$

Table A4
The matrices of cover images of $A^{c_1c_2c_3}$ of (2, 3)-CBW-EVCS in Construction 4.

$$A^{\circ\circ\circ}=\left\{\begin{bmatrix}R\\G\\B\end{bmatrix}\begin{bmatrix}B\\R\\G\end{bmatrix}\begin{bmatrix}G\\B\\R\end{bmatrix}\right\},\quad A^{\circ\circ\bullet}=\left\{\begin{bmatrix}R\\G\\\bullet\end{bmatrix}\begin{bmatrix}B\\R\\\bullet\end{bmatrix}\begin{bmatrix}G\\B\\\bullet\end{bmatrix}\right\},\quad A^{\circ\bullet\circ}=\left\{\begin{bmatrix}R\\\bullet\\B\end{bmatrix}\begin{bmatrix}B\\\bullet\\G\end{bmatrix}\begin{bmatrix}G\\\bullet\\R\end{bmatrix}\right\},\quad A^{\circ\bullet\bullet}=\left\{\begin{bmatrix}R\\\bullet\\\bullet\end{bmatrix}\begin{bmatrix}B\\\bullet\\\bullet\end{bmatrix}\begin{bmatrix}G\\\bullet\\\bullet\end{bmatrix}\right\}$$

$$A^{\bullet\circ\circ}=\left\{\begin{bmatrix}\bullet\\G\\B\end{bmatrix}\begin{bmatrix}\bullet\\R\\G\end{bmatrix}\begin{bmatrix}\bullet\\B\\R\end{bmatrix}\right\},\quad A^{\bullet\circ\bullet}=\left\{\begin{bmatrix}\bullet\\G\\\bullet\end{bmatrix}\begin{bmatrix}\bullet\\R\\\bullet\end{bmatrix}\begin{bmatrix}\bullet\\B\\\bullet\end{bmatrix}\right\},\quad A^{\bullet\bullet\circ}=\left\{\begin{bmatrix}\bullet\\\bullet\\B\end{bmatrix}\begin{bmatrix}\bullet\\\bullet\\G\end{bmatrix}\begin{bmatrix}\bullet\\\bullet\\R\end{bmatrix}\right\},\quad A^{\bullet\bullet\bullet}=\left\{\begin{bmatrix}\bullet\\\bullet\\\bullet\end{bmatrix}\begin{bmatrix}\bullet\\\bullet\\\bullet\end{bmatrix}\begin{bmatrix}\bullet\\\bullet\\\bullet\end{bmatrix}\right\}$$

Table A5
The matrices of cover images of $A^{c_1c_2c_3c_4}$ in Example 4.

$$A^{\circ\circ\circ\circ}=\left\{\begin{bmatrix}R\bullet\\G\bullet\\B\bullet\\\bullet R\end{bmatrix}\begin{bmatrix}\bullet R\\R\bullet\\G\bullet\\B\bullet\end{bmatrix}\begin{bmatrix}B\bullet\\\bullet R\\R\bullet\\G\bullet\end{bmatrix}\begin{bmatrix}G\bullet\\B\bullet\\\bullet R\\R\bullet\end{bmatrix}\right\},\quad A^{\circ\circ\circ\bullet}=\left\{\begin{bmatrix}R\bullet\\G\bullet\\B\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}\bullet R\\R\bullet\\G\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}B\bullet\\\bullet R\\R\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}G\bullet\\B\bullet\\\bullet R\\\bullet\bullet\end{bmatrix}\right\},\quad A^{\circ\circ\bullet\circ}=\left\{\begin{bmatrix}R\bullet\\G\bullet\\\bullet\bullet\\\bullet R\end{bmatrix}\begin{bmatrix}\bullet R\\R\bullet\\\bullet\bullet\\B\bullet\end{bmatrix}\begin{bmatrix}B\bullet\\\bullet R\\\bullet\bullet\\G\bullet\end{bmatrix}\begin{bmatrix}G\bullet\\B\bullet\\\bullet\bullet\\R\bullet\end{bmatrix}\right\}$$

$$A^{\circ\circ\bullet\bullet}=\left\{\begin{bmatrix}R\bullet\\G\bullet\\\bullet\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}\bullet R\\R\bullet\\\bullet\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}B\bullet\\\bullet R\\\bullet\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}G\bullet\\B\bullet\\\bullet\bullet\\\bullet\bullet\end{bmatrix}\right\},\quad A^{\circ\bullet\circ\circ}=\left\{\begin{bmatrix}R\bullet\\\bullet\bullet\\B\bullet\\\bullet R\end{bmatrix}\begin{bmatrix}\bullet R\\\bullet\bullet\\G\bullet\\B\bullet\end{bmatrix}\begin{bmatrix}B\bullet\\\bullet\bullet\\R\bullet\\G\bullet\end{bmatrix}\begin{bmatrix}G\bullet\\\bullet\bullet\\\bullet R\\R\bullet\end{bmatrix}\right\},\quad A^{\circ\bullet\circ\bullet}=\left\{\begin{bmatrix}R\bullet\\\bullet\bullet\\B\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}\bullet R\\\bullet\bullet\\G\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}B\bullet\\\bullet\bullet\\R\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}G\bullet\\\bullet\bullet\\\bullet R\\\bullet\bullet\end{bmatrix}\right\}$$

$$A^{\circ\bullet\bullet\circ}=\left\{\begin{bmatrix}R\bullet\\\bullet\bullet\\\bullet\bullet\\\bullet R\end{bmatrix}\begin{bmatrix}\bullet R\\\bullet\bullet\\\bullet\bullet\\B\bullet\end{bmatrix}\begin{bmatrix}B\bullet\\\bullet\bullet\\\bullet\bullet\\G\bullet\end{bmatrix}\begin{bmatrix}G\bullet\\\bullet\bullet\\\bullet\bullet\\R\bullet\end{bmatrix}\right\},\quad A^{\circ\bullet\bullet\bullet}=\left\{\begin{bmatrix}R\bullet\\\bullet\bullet\\\bullet\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}\bullet R\\\bullet\bullet\\\bullet\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}B\bullet\\\bullet\bullet\\\bullet\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}G\bullet\\\bullet\bullet\\\bullet\bullet\\\bullet\bullet\end{bmatrix}\right\},\quad A^{\bullet\circ\circ\circ}=\left\{\begin{bmatrix}\bullet\bullet\\G\bullet\\B\bullet\\\bullet R\end{bmatrix}\begin{bmatrix}\bullet\bullet\\R\bullet\\G\bullet\\B\bullet\end{bmatrix}\begin{bmatrix}\bullet\bullet\\\bullet R\\R\bullet\\G\bullet\end{bmatrix}\begin{bmatrix}\bullet\bullet\\B\bullet\\\bullet R\\R\bullet\end{bmatrix}\right\}$$

$$A^{\bullet\circ\circ\bullet}=\left\{\begin{bmatrix}\bullet\bullet\\G\bullet\\B\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}\bullet\bullet\\R\bullet\\G\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}\bullet\bullet\\\bullet R\\R\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}\bullet\bullet\\B\bullet\\\bullet R\\\bullet\bullet\end{bmatrix}\right\},\quad A^{\bullet\circ\bullet\circ}=\left\{\begin{bmatrix}\bullet\bullet\\G\bullet\\\bullet\bullet\\\bullet R\end{bmatrix}\begin{bmatrix}\bullet\bullet\\R\bullet\\\bullet\bullet\\B\bullet\end{bmatrix}\begin{bmatrix}\bullet\bullet\\\bullet R\\\bullet\bullet\\G\bullet\end{bmatrix}\begin{bmatrix}\bullet\bullet\\B\bullet\\\bullet\bullet\\R\bullet\end{bmatrix}\right\},\quad A^{\bullet\circ\bullet\bullet}=\left\{\begin{bmatrix}\bullet\bullet\\G\bullet\\\bullet\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}\bullet\bullet\\R\bullet\\\bullet\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}\bullet\bullet\\\bullet R\\\bullet\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}\bullet\bullet\\B\bullet\\\bullet\bullet\\\bullet\bullet\end{bmatrix}\right\}$$

$$A^{\bullet\bullet\circ\circ}=\left\{\begin{bmatrix}\bullet\bullet\\\bullet\bullet\\B\bullet\\\bullet R\end{bmatrix}\begin{bmatrix}\bullet\bullet\\\bullet\bullet\\G\bullet\\B\bullet\end{bmatrix}\begin{bmatrix}\bullet\bullet\\\bullet\bullet\\R\bullet\\G\bullet\end{bmatrix}\begin{bmatrix}\bullet\bullet\\\bullet\bullet\\\bullet R\\R\bullet\end{bmatrix}\right\},\quad A^{\bullet\bullet\circ\bullet}=\left\{\begin{bmatrix}\bullet\bullet\\\bullet\bullet\\B\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}\bullet\bullet\\\bullet\bullet\\G\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}\bullet\bullet\\\bullet\bullet\\R\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}\bullet\bullet\\\bullet\bullet\\R\bullet\\\bullet\bullet\end{bmatrix}\right\},\quad A^{\bullet\bullet\bullet\circ}=\left\{\begin{bmatrix}\bullet\bullet\\\bullet\bullet\\\bullet\bullet\\\bullet R\end{bmatrix}\begin{bmatrix}\bullet\bullet\\\bullet\bullet\\\bullet\bullet\\B\bullet\end{bmatrix}\begin{bmatrix}\bullet\bullet\\\bullet\bullet\\\bullet\bullet\\G\bullet\end{bmatrix}\begin{bmatrix}\bullet\bullet\\\bullet\bullet\\\bullet\bullet\\R\bullet\end{bmatrix}\right\}$$

$$A^{\bullet\bullet\bullet\bullet}=\left\{\begin{bmatrix}\bullet\bullet\\\bullet\bullet\\\bullet\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}\bullet\bullet\\\bullet\bullet\\\bullet\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}\bullet\bullet\\\bullet\bullet\\\bullet\bullet\\\bullet\bullet\end{bmatrix}\begin{bmatrix}\bullet\bullet\\\bullet\bullet\\\bullet\bullet\\\bullet\bullet\end{bmatrix}\right\}$$

References

[1] G. Ateniese, C. Blundo, A. De Santis, D.R. Stinson, Extended capabilities for visual cryptography, Theoret. Comput. Sci. 250 (2001) 143–161.
[2] S. Cimato, R. De Prisco, A. De Santis, Optimal colored threshold visual cryptography schemes, Des. Codes Cryptogr. 35 (2005) 311–335.
[3] S. Cimato, R. De Prisco, A. De Santis, Probabilistic visual cryptography schemes, Comput. J. 49 (2006) 97–107.
[4] S. Cimato, A. De Santis, A.L. Ferrara, B. Masucci, Ideal contrast visual cryptography schemes with reversing, Inform. Process. Lett. 93 (2005) 199–206.
[5] S. Cimato, R. De Prisco, A. De Santis, Colored visual cryptography without color darkening, Theoret. Comput. Sci. 374 (2007) 261–276.
[6] S. Cimato, R. De Prisco, A. De Santis, Visual cryptography for color images, in: Visual Cryptography and Secret Image Sharing, CRC Press, ISBN 978-1-4398-3721-4, 2012, pp. 31–56, Chap. 2.
[7] R. De Prisco, A. De Santis, Color visual cryptography schemes for black and white secret images, Theoret. Comput. Sci. 510 (2013) 62–86.
[8] R. De Prisco, A. De Santis, On the relation of random grid and deterministic visual cryptography, IEEE Trans. Inform. Forensics Secur. 9 (2014) 653–665.
[9] Y.C. Hou, Visual cryptography for color image, Pattern Recognit. 36 (2003) 1619–1629.
[10] Y.C. Hou, Z.Y. Quan, C.F. Tsai, A.Y. Tseng, Block-based progressive visual secret sharing, Inform. Sci. 233 (2013) 290–304.
[11] R. Ito, H. Kuwakado, H. Tanaka, Image size invariant visual cryptography, IEICE Trans. Fundam. Electron. Commun. Comput. Sci. E 82-A (1999) 2172–2177.
[12] O. Kafri, E. Keren, Encryption of pictures and shapes by random grids, Opt. Lett. 12 (1987) 377–379.
[13] H. Kuwakado, H. Tanaka, Size-reduced visual secret sharing scheme, IEICE Trans. Fundam. Electron. Commun. Comput. Sci. E 87-A (2004) 1193–1197.
[14] F. Liu, C.K. Wu, X.J. Lin, Some extensions on threshold visual cryptography schemes, Comput. J. 53 (2010) 107–119.
[15] F. Liu, C. Wu, X.J. Lin, Cheating immune visual cryptography scheme, IET Inf. Secur. 5 (2011) 51–59.

[16] M. Naor, A. Shamir, Visual cryptography, in: Advances in Cryptology, EUROCRYPT'94, in: LNCS, vol. 950, 1994, pp. 1–12.
[17] S.J. Shyu, Efficient visual secret sharing scheme for color images, Pattern Recognit. 39 (2006) 866–880.
[18] S.J. Shyu, H.W. Jiang, General constructions for threshold multiple-secret visual cryptography schemes, IEEE Trans. Inform. Forensics Secur. 8 (2013) 733–743.
[19] E.R. Verheul, H.C.A. Van Tilborg, Constructions and properties of k out of n visual secret sharing scheme, Des. Codes Cryptogr. 11 (1997) 179–196.
[20] D. Wang, T. Song, L. Dong, C.N. Yang, Optimal contrast greyscale visual cryptography schemes with reversing, IEEE Trans. Inform. Forensics Secur. 8 (2013) 2059–2072.
[21] D. Wang, F. Yi, X. Li, On general construction for extended visual cryptography schemes, Pattern Recognit. 42 (2009) 3071–3082.
[22] D. Wang, F. Yi, X. Li, Probabilistic visual secret sharing schemes for grey-scale images and color images, Inform. Sci. 181 (2011) 2189–2208.
[23] C.N. Yang, C.S. Laih, New colored visual secret sharing schemes, Des. Codes Cryptogr. 20 (2000) 325–335.
[24] C.N. Yang, New visual secret sharing schemes using probabilistic method, Pattern Recogn. Lett. 25 (2004) 481–494.
[25] C.N. Yang, T.S. Chen, Reduce shadow size in aspect ratio invariant visual secret sharing schemes using a square block-wise operation, Pattern Recognit. 39 (2006) 1300–1314.
[26] C.N. Yang, T.S. Chen, Extended visual secret sharing schemes: improving the shadow image quality, Int. J. Pattern Recognit. Artif. Intell. 21 (2007) 879–898.
[27] C.N. Yang, T.S. Chen, Colored visual cryptography scheme based on additive color mixing, Pattern Recognit. 41 (2008) 3114–3129.
[28] C.N. Yang, P.W. Chen, H.W. Shih, C. Kim, Aspect ratio invariant visual cryptography by image filtering and resizing, Pers. Ubiquitous Comput. 17 (2013) 843–850.
[29] C.N. Yang, H.W. Shih, C.C. Wu, L. Harn, k Out of n region incrementing scheme in visual cryptography, IEEE Trans. Circuits Syst. Video Technol. 22 (2012) 779–810.
[30] C.N. Yang, D. Wang, Property analysis of XOR based visual cryptography, IEEE Trans. Circuits Syst. Video Technol. 24 (2014) 189–197.
[31] C.N. Yang, C.C. Wu, D. Wang, A discussion on the relationship between probabilistic visual cryptography and random grid, Inform. Sci. 278 (2014) 141–173.
[32] C.N. Yang, Y.Y. Yang, New extended visual cryptography schemes with clearer shadow images, Inform. Sci. 271 (2014) 246–263.