**PH5720**

In this unit you will learn the determination of the numerical solution of differential equations using different approaches. To reduce the computation cost and achieve accurate numerical solution with small error, it is always advisable to compute the differential equation with larger stability region. You will study to solve the first order differential equations. There are several examples in the slide. You can write small program and see the numerical solution of the differential equation. The unit will focus on the following subsection.

- Ordinary Differential Equations: Initial Value Problems
- Ordinary Differential Equations: Boundary value Problems
- Partial Differential Equations



## Numerical methods for ODEs, Boundary value problems

We have solved the ordinary differential equation using initial value. However, additional conditions must be imposed on the solution to make it unique.

For the **initial value problem** (IVP), the conditions are specified at a single point, say $t_0$.

For **boundary value problem** (BVP), the conditions are specified at more than one point.

In $k^{th}$ order ODE, or equivalent first-order system, requires k-conditions.

For an ODEs, the conditions are typically specified at **two points**, namely the **endpoints** of some interval $[a, b]$, so we have two-point boundary value problem with boundary conditions at boundary values, i.e., $a$ and $b$.

## Numerical methods for ODEs, Boundary value problems

A general **first-order two-point boundary value problem** (BVP) for an ODE has the form.

$$x' = f(t, x), \qquad a < t < b$$

with boundary condition $g\big(x(a), x(b)\big) = 0$

where $f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^n$

**Boundary conditions** are **separated** if any given component of $g$ involves solution values only at $a$ or at $b$, but not both.

Boundary conditions are **linear** if they have the form

$$B_a\, x(a) + B_b\, x(b) = c$$

where $B_a, B_b \in \mathbb{R}^{n\times n}$ and $c \in \mathbb{R}^n$

Boundary value problem is **linear** if ODE and BC are both linear.



## Numerical methods for ODEs, BVP, Example

Two-point BVP for second order scalar ODE.

$$u'' = f(t, u, u'), \qquad a < t < b$$

with boundary condition (BC) $u(a) = \alpha$ and $u(b) = \beta$

is equivalent to ***first-order system*** of ODEs

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_2 \\ f(t, x_1, x_2) \end{bmatrix}, \qquad a < t < b$$

with separated linear BC

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(a) \\ x_2(a) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(b) \\ x_2(b) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

## Numerical methods for ODEs, BVP, Existence and uniqueness

Unlike IVP, with BVP we **can not begin** at initial point and continue solution step by step to nearby points.

Instead, **solution is determined everywhere simultaneously**, so existence and/or uniqueness **may not hold**.

For example,

$$u'' = -u \qquad\qquad 0 < t < b$$

with boundary condition (BC) $\qquad u(0) = 0$ and $u(b) = \beta$

with $b$ integer multiple of $\pi$, has infinitely many solutions if $\beta = 0$, but no solution if $\beta \neq 0$.



## Numerical methods for ODEs, BVP, Existence and uniqueness

In general, solvability of BVP,

$$x' = f(t, x), \qquad\qquad a < t < b$$

with boundary condition $\qquad g\big(x(a), x(b)\big) = 0$

depends on solvability of algebraic equation

$$g\big(y, x(b; y)\big) = 0$$

where $x(t; y)$ denotes solution to ODE with initial condition $x(a) = y$ for $x \in \mathbb{R}^n$

Solvability of later system is difficult to establish if $g$ is nonlinear.

## Numerical methods for ODEs, BVP, Existence and uniqueness

Assume Q is non-singular, define

$$\Phi(t) = Y(t)\, Q^{-1}$$

**Green's function**

$$G(t,s) = \begin{cases} \Phi(t)B_a(a)\Phi^{-1}(s) & a \le s \le t \\ \Phi(t)B_b(a)\Phi^{-1}(s) & a \le s \le t \end{cases}$$

The solution to BVP given by

$$y(t) = \Phi(t)c + \int_a^b G(t,s)\; b(s)\, ds$$

This result also gives absolute condition number for BVP

$$\kappa = max\{\|\Phi\|_\infty, \|G\|_\infty\}$$



## Numerical methods for ODEs, BVP, Conditioning and stability

**Conditioning or stability** of BVPs depends on the interplay between the growth of solution modes and the boundary conditions.

For an IVP, **instability** is associated with modes that **grow exponentially** as time increases.

For a BVP, the solution is determined everywhere simultaneously, so there is **no notion of "direction**" of integration in the interval [a; b],

Growth of modes **increasing with time increases** is limited by boundary conditions **at b**, and "growth" of a **decaying mode** is limited by the boundary condition **at a**.

For BVP to be well-conditioned, the **growing and decaying** modes must be controlled appropriately by the **boundary conditions imposed**.

## For ODEs, BVP, Numerical methods

For IVP, initial data supply all information necessary to begin numerical solution method at **initial point and step** forward from there.

For BVP, we have insufficient information to begin step-by-step numerical method, so numerical methods for solving BVPs are **more complicated** than those for solving IVPs.

We all consider four types of numerical methods for the two-point BVPs

Shooting

Finite difference

Collocation

Galerkin

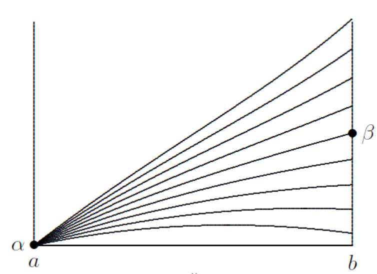


## Numerical methods for ODEs, BVP, Shooting method

In statement of two-point BVP, we are given the **initial** value $u(a)$

If we knew the value of $u'(a)$, then we would have an IVP that we could solve by methods discussed previously.

Lacking that information, we try sequence of increasingly accurate **guesses** until we find value for $u'(a)$ such that when we solve resulting IVP, approximate solution value at $t = b$ **matches desired boundary value**, $u(b) = \beta$.



## Numerical methods for ODEs, BVP, Shooting method

For given $\gamma$, value at $b$ of solution $u(b)$ to IVP

$$u'' = f(t, u, u'')$$

With initial conditions $u(a) = \alpha$ and $u'(a) = \gamma$

can be considered as function of $\gamma$, say $g(\gamma)$

Then BVP becomes problem of solving equation $g(\gamma) = \beta$

One-dimensional zero finder can be used to solve this scalar equation.



## Numerical methods for ODEs, BVP, Shooting method, Example

Consider two-point BVP, for second-order ODE

$$u'' = 6t \qquad 0 < t < 1$$

With BC $u(0) = 0$ and $u(1) = 1$

For each guess for $u'(0)$, we will integrate resulting IVP using classical **fourth-order Runge-Kutta** method to determine how close we come to hitting desired solution value at $t = 1$.

For simplicity of illustration, we will use step size $h = 0.5$ to integrate IVP from $t = 0$ to $t = 1$ in only two steps.

First, we **transform** second-order ODE into system of two first-order ODEs

$$x'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} x_2 \\ 6t \end{bmatrix}$$

## Numerical methods for ODEs, BVP, Shooting method, Example

We first try guess for initial slope of $x_2(0) = 1$

$$x^{(1)} = x^{(0)} + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$x^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{0.5}{6}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1.0 \\ 1.5 \end{bmatrix} + 2\begin{bmatrix} 1.375 \\ 1.500 \end{bmatrix} + \begin{bmatrix} 1.75 \\ 3.00 \end{bmatrix}\right) = \begin{bmatrix} 0.625 \\ 1.750 \end{bmatrix}$$

$$x^{(2)} = \begin{bmatrix} 0.625 \\ 1.75 \end{bmatrix} + \frac{0.5}{6}\left(\begin{bmatrix} 1.75 \\ 3.00 \end{bmatrix} + 2\begin{bmatrix} 2.5 \\ 4.5 \end{bmatrix} + 2\begin{bmatrix} 2.875 \\ 4.500 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

So we have hit $x_1(t) = 2$

instead of desired value $x_1(t) = 1$



## Numerical methods for ODEs, BVP, Shooting method, Example

We try again, this time with initial slope $x_2(0) = -1$

$$x^{(1)} = x^{(0)} + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$x^{(1)} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \frac{0.5}{6}\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -1.0 \\ 1.5 \end{bmatrix} + 2\begin{bmatrix} -0.625 \\ 1.500 \end{bmatrix} + \begin{bmatrix} -0.25 \\ 3.00 \end{bmatrix}\right)$$
$$= \begin{bmatrix} -0.375 \\ -0.250 \end{bmatrix}$$

$$x^{(2)} = \begin{bmatrix} -0.375 \\ -0.250 \end{bmatrix} + \frac{0.5}{6}\left(\begin{bmatrix} -0.25 \\ 3.00 \end{bmatrix} + 2\begin{bmatrix} 0.5 \\ 4.5 \end{bmatrix} + 2\begin{bmatrix} 0.875 \\ 4.500 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

So we have hit $x_1(t) = 0$

instead of desired value $x_1(t) = 1$

But we now have initial slope bracketed between $-1$ and $1$

## Numerical methods for ODEs, BVP, Shooting method, Example

We omit further iterations necessary to identify correct initial slope, which turns out to be $x_2(0) = 0$

$$x^{(1)} = x^{(0)} + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

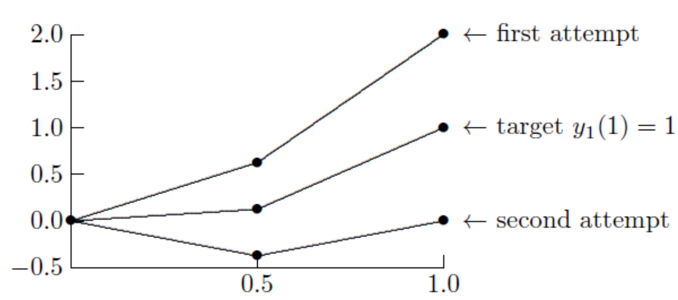
$$x^{(1)} = \begin{bmatrix}0\\0\end{bmatrix} + \frac{0.5}{6}\left(\begin{bmatrix}0\\0\end{bmatrix} + 2\begin{bmatrix}0.0\\1.5\end{bmatrix} + 2\begin{bmatrix}0.375\\1.500\end{bmatrix} + \begin{bmatrix}0.75\\3.00\end{bmatrix}\right) = \begin{bmatrix}0.125\\0.750\end{bmatrix}$$

$$x^{(2)} = \begin{bmatrix}0.125\\0.750\end{bmatrix} + \frac{0.5}{6}\left(\begin{bmatrix}0.75\\3.00\end{bmatrix} + 2\begin{bmatrix}1.5\\4.5\end{bmatrix} + 2\begin{bmatrix}1.875\\4.500\end{bmatrix} + \begin{bmatrix}3\\6\end{bmatrix}\right) = \begin{bmatrix}1\\3\end{bmatrix}$$

So we have indeed hit target solution value $x_1(1) = 1$.



## Numerical methods for ODEs, BVP, Shooting method, Example



## Numerical methods for ODEs, BVP, Multiple shooting

Simple **shooting** method inherits stability (or instability) of associated IVP, which may be **unstable even when BVP is stable**.

Such ill-conditioning may make it **difficult to achieve convergence** of iterative method for solving nonlinear equation.

Potential remedy is **multiple shooting**, in which the interval of integration $[a;\ b]$ is divided into subintervals and shooting is carried out on each subinterval separately.

Requiring continuity at internal mesh points provides BC for individual sub-problems.

Multiple shooting results in larger system of nonlinear equations to solve.



## Numerical methods for ODEs, BVP, Finite difference method

Finite difference method converts BVP into system of algebraic equations by replacing all derivatives with finite difference approximation.

For examples, to solve two-point BVP

$$u'' = f(t, u, u'') \qquad a < t < b$$

with BC $\quad u(a) = \alpha$ and $u'(a) = \gamma$

We introduce mesh points $t_i = a + ih, \qquad i = 0, 1, 2 \cdots n+1$

where $h = \frac{(b-a)}{n+1}$

We already have $x_0 = u(a) = \alpha$ and $x_{n+1} = u(b) = \beta$ from BC, and we seek approximate solution value $x_i \approx u(t_i)$ at each interior mesh point $t_i,\ i = 1, 2, \cdots n$.

## Numerical methods for ODEs, BVP, Finite difference method

We replace derivatives by finite difference approximations such as

$$u'(t) \approx \frac{x_{i+1} - x_{i-1}}{2h}$$

$$u''(t) \approx \frac{x_{i+1} - 2x_i + x_{i-1}}{h^2}$$

This yields system of equations

$$\frac{x_{i+1} - 2x_i + x_{i-1}}{h^2} = f\left(t_i, x_i, \frac{x_{i+1} - x_{i-1}}{2h}\right)$$

To be solved for unknowns $x_i$, $\quad i = 1, 2 \cdots n$

System of equations may be linear or non-linear, depending on whether $f$ is linear or nonlinear.



## Numerical methods for ODEs, BVP, Finite difference method

For these particular finite difference formulas, system to be solved is **tridiagonal**, which saves on both work and storage compared to general system of equations.

This is generally true of finite difference methods : they yield sparse systems because each equation involves few variations.

## Numerical methods for ODEs, BVP, Finite difference method, Example

Consider two-point BVP, for second-order ODE

$$u'' = 6t \qquad\qquad 0 < t < 1$$

With BC $\quad u(0) = 0$ and $u(1) = 1$

To keep computation to minimum, we compute approximate solution at one interior mesh point, $t = 0.5$, in interval $[0,1]$.

Including boundary points, we have three mesh points $t_0 = 0, t_1 = 0.5$ and $t_2 = 1$.

From BC, we know that $x_0 = u(t_0) = 0$ and $x_2 = u(t_2) = 1$ and we seek approximate solution $x_1 \approx u(t_1)$



## Numerical methods for ODEs, BVP, Finite difference method, Example

Replacing derivatives by standard finite difference approximations at $t_1$ gives equation

$$\frac{x_2 - 2x_1 + x_0}{h^2} = f\left(t_1, x_1, \frac{x_2 - x_0}{2h}\right)$$

Substituting boundary data, mesh size, and right hand side for this example we obtain

$$\frac{1 - 2x_1 + 0}{(0.5)^2} = 6t_1, \qquad t_1 = 0.5$$

$$4 - 8x_1 = 6(0.5) = 3$$

So that $x(0.5) \approx x_1 = 1/8 = 0.125$

## Numerical methods for ODEs, BVP, Finite difference method, Example

In a practical problem, much smaller step size and many more mesh points would be required to achieve acceptable accuracy.

we would therefore obtain system of equations to solve for approximate solution values at mesh points, rather than single equation as in this example.



## Numerical methods for ODEs, BVP, Collocation method

**Collocation** method approximates solution to BVP by finite linear combination of basis functions

For two-point BVP

$$u'' = f(t, u, u'') \qquad a < t < b$$

with BC $u(a) = \alpha$ and $u(b) = \beta$

we seek approximate solution of form

$$u(t) \approx v(t, x) = \sum_{i=1}^{n} x_i \phi_i(t)$$

where $\phi_i$ are basis functions defined on $[a, b]$ and $x$ is n-vector of parameters to be determined.

## Numerical methods for ODEs, BVP, Collocation method

Popular **choices of basis functions** include polynomials, B-splines, and trigonometric functions.

Basis functions with **global support**, such as polynomials or trigonometric functions, yield spectral method.

Basis functions with highly **localized support**, such as B-splines, yield finite element method.



## Numerical methods for ODEs, BVP, Collocation method

To determine the vector of parameters x, we define set of $n$ collection points $a = t_1 < \cdots\cdot < t_n = b$, at which **approximate solution** $v(t, x)$ to **forced to satisfy** the ODE and the boundary conditions.

**Common choices** of collection points include equally spaced points or **Chebyshev points**.

Suitably smooth basis functions can be differentiated analytically, so that approximate solution and its derivatives can be substituted into the ODE and BC to obtain system of algebraic equations for unknown parameters $x$.

## Numerical methods for ODEs, BVP, Collocation method, Example

Consider two-point BVP, for second-order ODE

$$u'' = 6t \qquad\qquad 0 < t < 1$$

With BC $\quad u(0) = 0$ and $u(1) = 1$

To keep computation to minimum, we use one interior collection point, $t = 0.5$, in interval $[0,1]$.

Including boundary points, we have three collection points $t_0 = 0, t_1 = 0.5$ and $t_2 = 1$. So we will be able to determine three parameters.

As basis functions we use first three monomials, so approximate solution has form $v(t, x) = x_1 + x_2 t + x_3 t^2$



## Numerical methods for ODEs, BVP, Collocation method, Example

The derivatives of this approximate solution function with respect to $t$ are given by

$$v'(t, x) = x_2 + 2x_3 t$$

$$v''(t, x) = 2x_3$$

Requiring the ODE to be satisfied at the interior collocation point $t_2 = 0.5$ gives the equation

$$v''(t_2, x) = f(t_2, v(t_2, x), v'(t_2, x))$$

$$2x_3 = 6t_2 = 6(0.5) = 3$$

Boundary condition to be satisfied at $t_1 = 0$ gives the equation

$$x_1 + x_2 t_1 + x_3 t_1^2 = x_1 = 0$$

## Numerical methods for ODEs, BVP, Collocation method, Example

Boundary condition to be satisfied at $t_3 = 1$ gives the equation

$$x_1 + x_2 t_3 + x_3 t_3^2 = x_1 + x_2 + x_3 = 1$$

Solving this system of three equations in three unknowns gives

$$x_1 = 0, \qquad x_2 = -0.5, \qquad x_3 = 1.5$$

So that the approximate solution function is given by the quadratic polynomial trivially solved to obtain

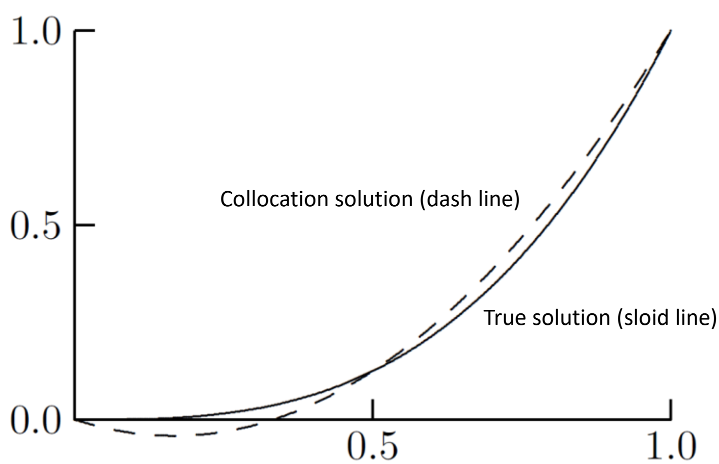
$$u(t) \approx v(t) = -0.5t + 1.5t^2$$

At the interior collocation point, $t_2 = 0.5$, where we forced $v$ to satisfy the ODE, we have the approximate solution value

$$u(0.5) \approx v(0.5, x) = 0.125$$



## Numerical methods for ODEs, BVP, Collocation method, Example



## Numerical methods for ODEs, BVP, Galerkin method,

Rather than forcing residual to be zero at finite number of points, as in collocation, we could instead minimize the residual over the entire interval of integration.

For example, for scalar **Poisson equation** in one dimension,

$$u'' = f(t) \qquad\qquad a < t < b$$

with homogeneous BC $\qquad u(a) = 0$ and $u(b) = 0$

substitute approximate solution of form

$$u(t) \approx v(t, x) = \sum_{i=1}^{n} x_i \phi_i(t)$$

into ODE, where $\phi_i$ are basis functions defined on $[a, b]$ and $x$ is n-vector of parameters to be determined.



## Numerical methods for ODEs, BVP, Galerkin method,

Using the least squares method, we minimize the function

$$F(x) = \frac{1}{2}\int_a^b r(t, x)^2\, dt$$

by setting each component of its gradient to zero,

This yields symmetric system of linear algebraic equations $Ax = b$, where

$$a_{ij} = \int_a^b \phi_j''(t)\phi_i''(t)dt\,, \qquad b_i = \int_a^b f(t)\phi_i''(t)dt$$

whose solution gives vector of parameters $x$

## Numerical methods for ODEs, BVP, Galerkin method,

More generally, weighted residual method forces the residual to be orthogonal to each of a given set of weight functions (or test functions), $w_i$,

$$\int_a^b r(t,x)\, w_i(t)\ dt = 0, \qquad i = 1, 2, \cdots, n$$

This yields symmetric system of linear algebraic equations $Ax = b$, where

$$a_{ij} = \int_a^b \phi_j''(t)\, w_i(t) dt\,, \qquad b_i = \int_a^b f(t) w_i(t) dt$$

whose solution gives vector of parameters $x$



## Numerical methods for ODEs, BVP, Galerkin method,

The matrix $A$ resulting from a weighted residual method is generally not symmetric. Entries of the matrix involve second derivatives of the basis functions.

Both drawbacks are overcome by **Galerkin method**, in which the weight functions are chosen to be the same as the basis functions, i.e., $w_i = \phi_i$, $i = 1, 2, \cdots, n$.

Orthogonality condition then becomes

$$\int_a^b r(t,x)\, \phi_i(t)\ dt = 0, \qquad i = 1, 2, \cdots, n$$

$$\int_a^b v''(t,x)\, \phi_i(t) dt = \int_a^b f(t) \phi_i(t) dt\,, \qquad i = 1, 2, \cdots, n$$

## Numerical methods for ODEs, BVP, Galerkin method,

Degree of differentiability required can be reduced using integration by parts, which gives,

$$\int_a^b v''(t,x)\,\phi_i(t)dt = v'(t)\phi_i(t)\Big|_a^b - \int_a^b v'(t)\phi_i'(t)dt$$

$$= v'(b)\phi_i(b) - v'(a)\phi_i(a) - \int_a^b v'(t)\phi_i'(t)dt$$

Assuming the basis functions $\phi_i$ satisfy the homogeneous boundary conditions, so that $\phi_i(0) = \phi_i(1) = 0$ the orthogonality condition then becomes

$$-\int_a^b v'(t)\phi_i'(t)dt = \int_a^b f(t)\phi_i(t)dt\,, \qquad i = 1, 2,\cdots, n$$



## Numerical methods for ODEs, BVP, Galerkin method,

This yields system of linear algebraic equations $Ax = b$, where

$$a_{ij} = -\int_a^b \phi_j'(t)\phi_i'(t)dt\,, \qquad b_i = \int_a^b f(t)\phi_i'(t)dt$$

whose solution gives vector of parameters $x$

$A$ is symmetric and involves only first derivatives of the basis functions.

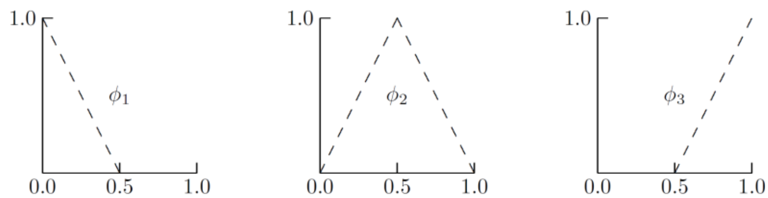
## Numerical methods for ODEs, BVP, Galerkin method, Example

Consider again two-point BVP, for second-order ODE

$$u'' = 6t \qquad\qquad 0 < t < 1$$

With BC $\quad u(0) = 0$ and $u(1) = 1$

We will approximate the solution by a piecewise linear polynomial, for which the B-splines of degree 1 **("hat" functions)** form a suitable set of basis functions.




To keep computation to a minimum, we again use the same three mesh points, but now they become the knots in the piecewise linear polynomial approximation.



## Numerical methods for ODEs, BVP, Galerkin method, Example

Thus, we seek an approximate solution of the form

$$u(t) \approx v(t,x) = x_1\phi_1(t) + x_2\phi_2(t) + x_3\phi_3(t)$$

From BC, we must have $\qquad x_1 = 0 \qquad$ and $\qquad x_3 = 1$

To determine the remaining parameter $x_2$, we impose the **Galerkin orthogonality condition** on the interior basis function $\phi_2$ and obtain the equation,

$$-\sum_{j=1}^{3}\left(\int_0^1 \phi_j'(t)\phi_2'(t)dt\right)x_j = \int_0^1 6t\phi_2(t)dt$$

Or, upon evaluating these simple integrals analytically

$$2x_1 - 4x_2 + 2x_3 = 3/2$$

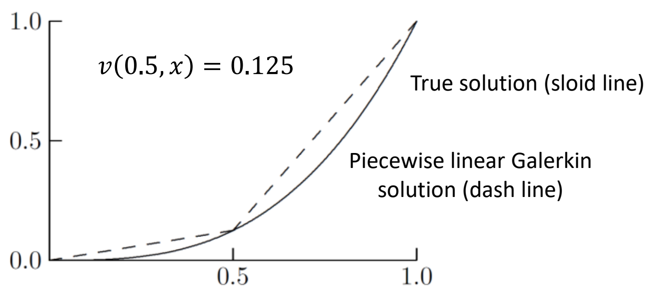
## Numerical methods for ODEs, BVP, Galerkin method, Example

From BC, we must have $x_1 = 0$ and $x_3 = 1$

$$2x_1 - 4x_2 + 2x_3 = 3/2$$

Substituting the known values for $x_1$ and $x_3$ then gives $x_2 = 1/8$ for the remaining unknown parameter, so piecewise linear approximate solution is

$$u(t) \approx v(t, x) = 0.125\phi_2(t) + \phi_3(t)$$






## Numerical methods for ODEs, BVP, Galerkin method, Example

More realistic problem would have many more **interior mesh points** and **basis functions**, and correspondingly many parameters to be determined.

Resulting system of equations would be much larger but still sparse, and therefore relatively easy to solve, provided **local basis functions**, such as "hat" functions are used.

Resulting approximate solution function is less smooth than true solution, but nevertheless becomes **more accurate** as **more mesh points** are used.

## Numerical methods for ODEs, BVP, Eigenvalue problem

A standard eigenvalue problem for a second-order scalar BVP has the form

$$u'' = \lambda f(t, u, u') \qquad a < t < b$$

with BC $\quad u(a) = \alpha \quad$ and $\quad u(b) = \beta$

where we seek not only solution $u$ but also parameter $\lambda$

The (possibly complex) scalar $\lambda$ is called an eigenvalue and the solution $u$ an **eigenfunction** for this **two-point boundary value problem**.

Discretization of an eigenvalue problem for an ODE results in an algebraic eigenvalue problem whose solution approximates that of the original problem.



## Numerical methods for ODEs, BVP, Eigenvalue problem, Example

Consider linear two-point BVP

$$u'' = \lambda g(t) u \qquad a < t < b$$

with BC $\quad u(a) = 0 \quad$ and $\quad u(b) = 0$

Introduce discrete mesh points $t_i$ in interval $[a, b]$, with mesh spacing $h$ and use standard finite difference approximation for second derivative to obtained algebraic system

$$\frac{x_{i+1} - 2x_i + x_{i-1}}{h^2} = \lambda g_i x_i$$

where $x_i = u(t_i)$, $g_i = g(t_i)$

From BC $\quad x_0 = u(a) = 0 \qquad x_{n+1} = u(b) = 0$

## Numerical methods for ODEs, BVP, Eigenvalue problem, Example

Assuming $g_i \neq 0$, divide equation $i$ by $g_i$ for $i = 1, 2, \cdots, n$, to obtain linear system $Ax = \lambda x$, where $n \times n$ matrix $A$ has **tridiagonal form**

$$A = \frac{1}{h^2}\begin{bmatrix} -2/g_1 & 1/g_1 & 0 & \cdots & 0 \\ 1/g_2 & -2/g_2 & 1/g_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1/g_{n-1} & -2/g_{n-1} & 1/g_{n-1} \\ 0 & \cdots & 0 & 1/g_n & -2/g_n \end{bmatrix}$$

This standard algebraic eigenvalue problem can be solved from $(A - \lambda I)x = 0$

This homogeneous system of linear equations has a nonzero solution $x$ if, and only if, its matrix is singular.

The $\det(A - \lambda I)$ is a polynomial of degree $n$ in $\lambda$, called the characteristic polynomial of $A$, and its **roots** are the eigenvalues of $A$.