DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS
PH2140 Mathematics on the Computer
Take-Home/Projects (30 Marks)

03 Nov 2017 Due: 07 Nov 2017

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1. (**Codes in Python [6 Marks]**) Consider the logistic map:

$$x_{n+1} = r\, x_n(1 - x_n).$$

When $r = 4$, start from any x_0 in the interval $(0, 1)$ and iterate say $N = 10^6$ times to get the orbit $x_0, \cdots x_N$.

(a) Plot the normalized distribution of the $\{x_i\}$. Now vary the x_0 for 6 different instances in all and plot the resulting distributions.

(b) Write a code to choose uniformly, randomly, x_0, say $K = 100$ times and iterate say $N = 10000$ times each and create *one* ensemble and plot its normalized distribution. Compare, on the plot, the answer you get with the expected analytical one

$$p(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}.$$

The values of K and N are indicative. Your code should be such that these can change (see below).

(c) Investigate such distributions for some 6 different values of r that lie in the interval $3.5 < r < 4$.

(d) Again, let x_0 be some random number in $(0, 1)$. Iterate the logistic map $N = 1000$ times for some r in $[0, 4]$ and extract the last 500 iterates (discard the first 500 as "transient"). Plot r *vs* all the 500 iterates as you vary r in $0 \le r \le 4$ in *one plot*. That is you will stack all the last 500 points on the y-axis when the x-axis is the r value.

You will write **two codes** that can be interrogated with a "help" command. **First code** for parts (a, b, c) which will be a function that will take as input r, N and K and number of bins *nbins* for histogramming, will output a data file with histogram values of x_i as well as save a figure of the normalized distribution in a PDF format. The figure for $r = 4$ alone should also include the analytical prediction.

Annotate the figure with axes labels and titles and legends that contain details like the value of r. **Second code** for part (d) that will (use function again) take as input the number of r values in the interval $(0, 4)$, the number of interates in total and the number of transient points discarded, and save the resulting figure in a PDF file. Again annotate the figure appropriately.

2. (**Codes in Python [6 Marks]**) Examine two kinds of random walks in 2 dimensions: (a) when the increments are by one unit along $\pm x$ or $\pm y$ with equal probability. In other words the walker goes "up", "down", "left" or "right" with equal probability and (b) when the increments along x and y are random standard normal numbers. In the latter case the walk is not restricted to lie on a uniform grid. Write Python codes with function inputs as number of walker steps, T, in both cases. Return the walker paths and plot them.

   Now consider that there are N independent walkers all starting from the origin. Measure the mean and standard deviation of the walker as function of number of steps, and examine the formula

   $$\langle r^2 \rangle \ = \ Dt^{\alpha}$$

   where r is the distance from the origin, t is the number of steps, and D and α are constants. Find also the distribution of r for some times such as $t = 10$, $t = 100$.

   Examine and implement "turtle graphics" to visualize the two walks in real time. In three dimensional walks, how does the exponent α behave? What if the normal distribution is replaced by any other zero-mean symmetric distribution? Explore. Also explore how the walk behaves when the normal distribution is replaced by the standard Cauchy distribution.

3. (**Python or Mathematica [6 Marks]**) The Hamiltonian for some two spin-1/2 system is given by

   $$H = J\sigma^z \otimes \sigma^z + h_z(\sigma^z \otimes I_2 + I_2 \otimes \sigma^z) + h_x(\sigma^x \otimes I_2 + I_2 \otimes \sigma^x) + h_y(\sigma^y \otimes I_2 + I_2 \otimes \sigma^y).$$

   Here each of the matrices $\sigma^{x,y,z}$ are Pauli matrices, I_2 is the 2×2 identity matrix, and the symbol $\otimes$ refers to the "Kronecker product". Check out the definition of the Kronecker product and the Numpy/Mathematica command for it.

   (a) Write a code that will take as input J, h_x, h_y, h_z, and return the eigenvalues and the eigenvectors of H.

(b) Take the case $J = 1$, $h_y = 0$, $h_x = h \sin\theta$ and $h_z = h \cos\theta$. Write a code that will tabulate and store in a file the eigenvalues as a function of θ for some fixed values of h (suggested 3 values: $h = 0.5, 1.0, 1.5$) and for many values of θ in $[0, \pi/2]$.

(c) For the case in part (b) above, plot all the four eigenvalues as function of θ on the *same* graph. Also plot the eigenvector intensities (modulus squared of vector elements) at $\theta = 0$, $\theta = \pi/4$ and $\theta = \pi/2$. Export all figures as PDF files also.

4. (**Codes in Mathematica [4 Marks]**) Consider two bodies of masses m_1, m_2 moving under the influence of a central potential $V(r)$, r being the magnitude of distance between the two bodies. This problem can effectively be reduced to that of a single mass $m = m_1 m_2/(m_1 + m_2)$ moving about a fixed center of force, which will be taken as the origin of the coordinate system. The corresponding equations of motion are as follows:

$$mr^2 \frac{d\theta}{dt} = l; \quad m\frac{d^2 r}{dt^2} - \frac{l^2}{mr^3} = f(r),$$

where, θ is the azimuthal angle, l is the angular momentum (which is a constant of motion) and $f(r) = -\frac{\partial V}{\partial r}$ is the central force. Use `Mathematica` to analyze the following aspects of such central-force potentials.

(a) Solve the above differential equations numerically for the Newtonian gravitational potential $V_N(r) = -\frac{k}{r}$, and plot: (i) $r(\theta)$ (ii) $t(\theta)$.

(b) For $V_N(r)$, find, by explicit integration, the area ΔA swept by the radius vector between times t and $t + \Delta$. Show, through relevant plots, that this area is independent of t and depends linearly on Δ.

5. (**Codes in Mathematica [8 Marks]**) Consider a system of two simple pendulums attached end to end. Say l_1 and l_2 are their respective lengths, m_1, m_2 are the corresponding point-masses attached at the end of each string, and, θ_1, θ_2 are their respective displacements measured from the vertical axis. The equations of motion of such a *double pendulum* are given by

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) = \frac{\partial \mathcal{L}}{\partial \theta_1}, \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) = \frac{\partial \mathcal{L}}{\partial \theta_2},$$

where,

$$\begin{aligned}\mathcal{L} \;=\; & \frac{1}{2}(m_1 + m_2) l_1^2 \left(\frac{d\theta_1}{dt}\right)^2 + \frac{1}{2} m_2 l_2 \left(\frac{d\theta_2}{dt}\right)^2 + m_2 l_1 l_2 \frac{d\theta_1}{dt}\frac{d\theta_2}{dt}\cos(\theta_2 - \theta_1) \\ & + (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2\end{aligned}$$

is the **Lagrangian** of the system (g is the acceleration due to gravity).

(a) Solve the equations of motion numerically for some choice of initial conditions and hence plot $\theta_1(t)$ and $\theta_2(t)$ as a function of time t.

(b) In the small angle approximation ($\sin\theta_1 \approx \theta_1, \sin\theta_2 \approx \theta_2$), you can linearize the system of equations, to get the following form:

$$\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \frac{1}{\alpha} M \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \frac{1}{\alpha} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.$$

The eigenvectors of the matrix M describe the *normal modes* of vibration of the double pendulum and the eigenvalues give the normal mode frequencies. Obtain the normal modes of the system. Plot the high and low frequency modes, and an arbitrary sum of the two, as a function of time.

(c) A dynamical system is said to be **chaotic** if the dynamics are *sensitive to initial conditions.* By plotting different trajectories ($(\theta_1(t), \theta_2(t)$ as a function of time) of the system, show that the double pendulum exhibits chaotic behaviour for large initial displacements. In particular, show that the difference between the displacements $\theta_1^{(1)}(t), \theta_1^{(2)}(t)$ at time t corresponding to two different initial conditions $\theta_1^{(}1)(0), \theta_1^{(2)}(0)$ can be exponentially large:

$$\begin{aligned} |\theta_1^{(1)}(t) - \theta_1^{(2)}(t)| &\approx e^{\lambda_1 t}|\theta_1^{(1)}(0) - \theta_1^{(2)}(0)|, \\ |\theta_2^{(1)}(t) - \theta_2^{(2)}(t)| &\approx e^{\lambda_2 t}|\theta_2^{(1)}(0) - \theta_2^{(2)}(0)|. \end{aligned}$$

(d) Finally, use the `Animate` command in Mathematica to create an animation of the double pendulum for different initial conditions.