Assignment 2

Theory Questions:

Problem 1: Color Theory (*10 points*)
In a three dimensional color space such as XYZ, any color C with coordinates (X, Y, Z) can be expressed as a linear combination of the primaries P_1, P_2, P_3 with coordinates (X_1, Y_1, Z_1), (X_2, Y_2, Z_2) and (X_3, Y_3, Z_3) respectively. This may be expressed as

$$C\,(X, Y, Z) = \alpha_1 * P_1\,(X_1, Y_1, Z_1) + \alpha_2 * P_2\,(X_2, Y_2, Z_2) + \alpha_3 * P_3(X_3, Y_3, Z_3)$$

In this question you are asked to show that similarly, the normalized *chromaticity* coordinates of C can *also* be expressed as a linear combination of the normalized *chromaticity* coordinates of P_1, P_2, P_3. Proceed by answering the following:

- Find the normalized chromaticity coordinates of P_1, P_2, and P_3 in terms of given known quantities (*2 points*)

Ans: x = X / (X+Y+Z)
y = Y / (X+Y+Z)
z = Z / (X+Y+Z)

- Express the normalized chromaticity coordinates of the color C in terms of the chromaticity coordinates of P_1, P_2, and P_3 (*4 points*)

Ans: – Let P1 = (X1,Y1,Z1) and P2 = (X2,Y2,Z2) be two different colors, and let P3 = (X3, Y3, Z3) fall on a line connecting P1 and P2.
– In this case, we know that
P3 =αP1+βP2
(X3,Y3,Z3) = α(X1,Y1,Z1)+β(X2,Y2,Z2) where α+β=1

- Hence prove that the chromaticity coordinates of any color C (which is a linear combination of primaries P_1, P_2, and P_3 in XYZ color space) can be represented also as a linear combination of the chromaticity coordinates of the respective primaries. (*4 points*)

Ans: Straight lines in (X, Y, Z) space project to straight lines in (x, y) chromaticity space.

Proof:
– Let P1 = (X1,Y1,Z1) and P2 = (X2,Y2,Z2) be two different colors, and let P3 = (X3, Y3, Z3) fall on a line connecting P1 and P2.
– In this case, we know that
P3 =αP1+βP2
(X3,Y3,Z3) = α(X1,Y1,Z1)+β(X2,Y2,Z2) where
α+β=1
– In order to show that (x3, y3) falls on a straight line
connecting (x1, y1) and (x2, y2), we must show that (x3, y3) = α′(x1, y1) + β′(x2, y2)
where

$\alpha' + \beta' = 1$

(x3, y3)

$= [(\alpha X1+\beta X2)/(X3 + Y3 + Z3), (\alpha Y1+\beta Y2)/(X3 + Y3 + Z3)]$

$= [\alpha X1/(X3 + Y3 + Z3) , \alpha Y1/(X3 + Y3 + Z3)] + [\beta X2/(X3 + Y3 + Z3) , \beta Y2/(X3 + Y3 + Z3)]$

$= (X1 + Y1 + Z1)/(X3 + Y3 + Z3) [\alpha X1/(X1 + Y1 + Z1), \alpha Y1/(X1 + Y1 + Z1)] +$
$(X2 + Y2 + Z2)/(X3 + Y3 + Z3) [\beta X2/(X2 + Y2 + Z2), \beta Y2/(X2 + Y2 + Z2)$

$= [\alpha(X1 + Y1 + Z1)/(X3 + Y3 + Z3)](x1,y1) + [\beta(X2 + Y2 + Z2)/(X3 + Y3 + Z3)](x2,y2)$

$= \alpha'(x1,y1)+\beta'(x2,y2$

Then α' and β' are given by

$\alpha' = \alpha(X1 + Y1 + Z1)/(\alpha(X1 + Y1 + Z1)+\beta(X2 + Y2 + Z2))$

$\beta' = \beta(X2 + Y2 + Z2)/(\alpha(X1 + Y1 + Z1)+\beta(X2 + Y2 + Z2))$

Hence, we have,

$\alpha' + \beta' = 1$

Problem 2: **Entropy Coding** (*10 points*)

Consider a communication system that gives out only two symbols X and Y. Assume that the parameterisation followed by the probabilities are $P(X) = x^2$ and $P(Y) = (1-x^2)$.

- Write down the entropy function and plot it as a function of *x*. From your plot, for what value of *x* does the Entropy become a minimum? (*2 points*)

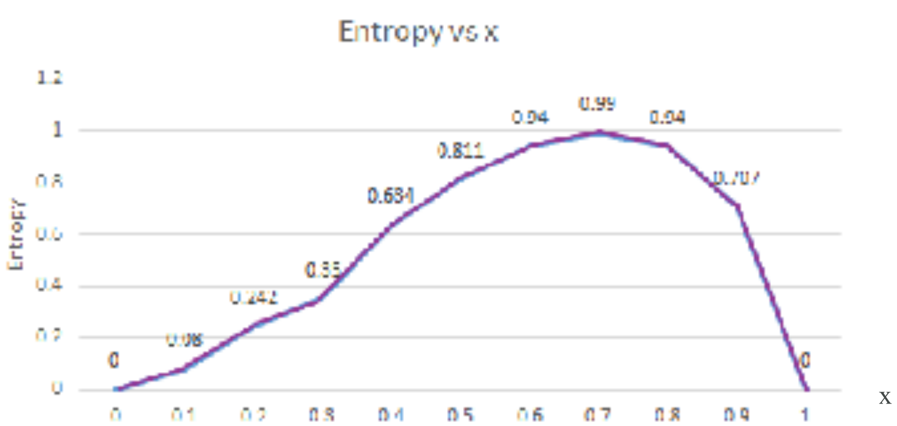
Ans:

Entropy function $H = -\sum^{N}_{i=1} P(i)*\log_2 P(i)$

Given $P(X) = x^2$ and $P(Y) = (1-x^2)$

Hence, $H = -(x^2*\log_2 x^2 + (1-x^2) *\log_2 (1-x^2))$

Entropy values as a function of x:



As we can see in the graph, the minimum values are obtained at x = 0 and x = 1.

- Although the plot visually gives you the value of *x* for which the entropy in minimum, can you now mathematically find out the value(s) for which the entropy is a minimum? (*4 points*)

Ans:

In this case, the entropy will be minimum when the symbols have the highest probability gap(such that some symbols are more likely to appear than others). This is possible when P(X) = 1 (making P(Y) = 0) or when P(Y) = 1(making P(X)=0).

When P(X) = 1, x = 1

When P(Y) = 1, x = 0

Hence the values of x for minimum entropy are 0,1.

- Can you do the same for the maximum, that is can you mathematically find out value(s) of *x* for which the value is a maximum? (*4 points*)

Ans:

The entropy is maximum when all symbols are equally probable.

Therefore, for max entropy P(X) = P(Y)

$x^2 = 1 - x^2$

x = 1/sqrt(2)

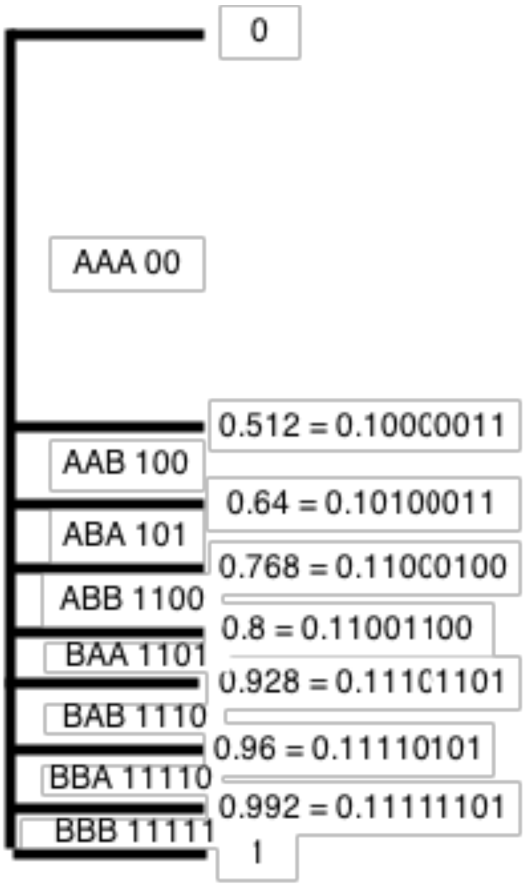
x = 0.707

Entropy is maximum when x = 0.707

Problem 3: Arithmetic Compression *(10 points)*
Consider two symbols, A and B, with the probability of occurrence of 0.8 and 0.2, respectively. The coding efficiency can be improved by combining N symbols at a time (called “symbol blocking”). Assume N = 3, so you are grouping symbols of 3 and giving them a unique code. (Assume that each symbol occurrence is independent of previous symbol occurrences).
Start by writing out different outcomes with their probabilities?

- Show the arrangement of symbols on the unit interval [0, 1] and determine the arithmetic code for the three-symbol sequence. *(4 points)*
  Ans:
  Totally 8 types of different outcomes shown in diagram below with following probabilities:
  AAA: 0.512;

AAB: 0.128;
ABA: 0.128;
ABB: 0.032;
BAA: 0.128;
BAB: 0.032;
BBA: 0.032;
BBB: 0.008.



- What is the average code word length? Is it optimum? *(1 points)*
  Ans: The average code word length is: (2+3+3+4+4+4+5+5) / 8 = 3.375. Yes, It is optimum.

- How many bits are required to code the message "ABABBAABBAAABBB" *(1 points)*
  Ans: The bits for the message are as follows:
"ABA(3)BBA(5)ABB(4)AAA(2)BBB(5)" is 3+5+4+2+5 = 19 bits.

- How could you do better than the above code length? *(4 points)*
  Ans: We can surely improve the above code length by preventing the shrinking when the interval bounds get too close.