# Chapter 2

# Fourier Transforms

## 2.1. INTEGRAL TRANSFORMS

The integral transform of a function $f(x)$ is defined by the equation

$$I\{f(x)\} = \bar{f}(s) = \int_a^b f(x)\, K(s, x)\, dx,$$

where $K(s, x)$ is a known function of $s$ and $x$, called the **kernel** of the transform; $s$ is called the **parameter** of the transform and $f(x)$ is called the **inverse transform** of $\bar{f}(s)$.

Some of the well-known transforms are given below:

(*i*) **Laplace Transform.** When $K(s, x) = e^{-sx}$, we have the Laplace transform of $f(x)$.

Thus $$L\{f(x)\} = \bar{f}(s) = \int_0^\infty f(x)\, e^{-sx}\, dx$$

(*ii*) **Fourier Transform.** When $K(s, x) = e^{isx}$, we have the Fourier transform of $f(x)$. Thus

$$F(s) = \int_{-\infty}^\infty f(x)\, e^{isx}\, dx$$

(*iii*) **Hankel Transform.** When $K(s, x) = x\, J_n(sx)$, we have the Hankel transform of $f(x)$.

Thus $$H_n(s) = \int_0^\infty f(x)\, x\, J_n(sx)\, dx$$

where $J_n(sx)$ is the Bessel function of the first kind and order $n$.

(*iv*) **Mellin Transform.** When $K(s, x) = x^{s-1}$, we have the Mellin transform of $f(x)$. Thus

$$M(s) = \int_0^\infty f(x)\, x^{s-1}\, dx$$

(*v*) **Fourier Sine Transform.** When $K(s, x) = \sin sx$, we have the Fourier sine transform of $f(x)$. Thus

$$F_s(s) = \int_0^\infty f(x) \sin sx\, dx$$

(*vi*) **Fourier Cosine Transform.** When $K(s, x) = \cos sx$, we have the Fourier cosine transform of $f(x)$. Thus

$$F_c(s) = \int_0^\infty f(x) \cos sx\, dx$$

We have already discussed Laplace transform and its applications to the solution of ordinary differential equations. In the present chapter, we shall discuss the Fourier integrals





and Fourier transforms which are useful in solving boundary value problems arising in engineering *e.g.* conduction of heat, theory of communication, wave propagation etc. Fourier series are helpful in problems involving periodic functions. However, in many practical problems, the function is non-periodic. A suitable representation for non-periodic functions can be obtained by considering the limiting form of Fourier series when the fundamental period is made infinite. In such case, the Fourier series becomes a Fourier integral which can be expressed in terms of Fourier transform which transforms a non-periodic function.

The effect of applying an integral transform to a partial differential equation is to reduce the number of independent variables by one. The choice of a particular transform is decided by the nature of the boundary conditions and the facility with which the transform can be inverted to give $f(x)$.

## 2.2. FOURIER INTEGRAL THEOREM

**Statement.** *If*

(*i*) *$f(x)$ satisfies Dirichlet's conditions in every interval $(-c, c)$ however large.*

(*ii*) *$\int_{-\infty}^\infty |f(x)|\, dx$ converges;*

then $$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \cos \lambda(t - x)\, dt\, d\lambda.$$

*The integral on the right hand side is called **Fourier Integral** of $f(x)$.*

**Proof.** Consider a function $f(x)$ satisfying Dirichlet's conditions in every interval $(-c, c)$, however large. Then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^\infty \left( a_n \cos \frac{n\pi x}{c} + b_n \sin \frac{n\pi x}{c} \right) \qquad ...(1)$$

where $a_0 = \frac{1}{c}\int_{-c}^{c} f(t)\, dt$, $a_n = \frac{1}{c}\int_{-c}^{c} f(t) \cos \frac{n\pi t}{c}\, dt$ and $b_n = \frac{1}{c}\int_{-c}^{c} f(t) \sin \frac{n\pi t}{c}\, dt$

Substituting the values of $a_0$, $a_n$ and $b_n$ in (1), we get

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\, dt + \frac{1}{c}\sum_{n=1}^\infty \int_{-c}^{c} \left[ \cos\frac{n\pi x}{c} \cos\frac{n\pi t}{c} + \sin\frac{n\pi x}{c} \sin\frac{n\pi t}{c} \right] f(t)\, dt$$

$$= \frac{1}{2c}\int_{-c}^{c} f(t)\, dt + \frac{1}{c}\sum_{n=1}^\infty \int_{-c}^{c} \cos\frac{n\pi(t - x)}{c} \cdot f(t)\, dt \qquad ...(2)$$

If we assume that the integral $\int_{-\infty}^\infty |f(x)|\, dx$ converges,

then $$\lim_{c \to \infty} \left[ \frac{1}{2c}\int_{-c}^{c} f(t)\, dt \right] = 0, \text{ since } \left| \frac{1}{2c}\int_{-c}^{c} f(t)\, dt \right| \le \frac{1}{2c}\int_{-\infty}^\infty |f(t)|\, dt$$

Putting $\frac{\pi}{c} = \Delta\lambda$, the second term in (2) becomes

$$\frac{1}{\pi}\sum_{n=1}^\infty \Delta\lambda \int_{-c}^{c} \cos\{n\Delta\lambda(t - x)\}\, f(t)\, dt$$

This is of the form $\sum_{n=1}^\infty F(n\Delta\lambda)\, \Delta\lambda$ whose limit as $\Delta\lambda \to 0$, is $\int_0^\infty F(\lambda) d\lambda$.

Hence as $c \to \infty$, (2) reduces to $$\boxed{f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t) \cos \lambda(t - x)\, dt\, d\lambda} \quad ...(3)$$

which is known as **Fourier Integral** of $f(x)$.

Equation (3) is true at a point of continuity. At a point of discontinuity the value of the integral on the right is

$$\tfrac{1}{2}[f(x+0) + f(x-0)].$$

## 2.3. FOURIER SINE AND COSINE INTEGRALS

We know that $\cos \lambda(t - x) = \cos \lambda t \cos \lambda x + \sin \lambda t \sin \lambda x$

$\therefore$ Fourier integral of $f(x)$ can be written as

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\,[\cos \lambda t \cos \lambda x + \sin \lambda t \sin \lambda x]\, dt\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \cos \lambda x \int_{-\infty}^\infty f(t) \cos \lambda t\, dt\, d\lambda + \frac{1}{\pi}\int_0^\infty \sin \lambda x \int_{-\infty}^\infty f(t) \sin \lambda t\, dt\, d\lambda \quad ...(4)$$

**When f(x) is an odd function,** $f(t) \cos \lambda t$ is odd while $f(t) \sin \lambda t$ is even. Thus the first integral in (4) vanishes and, we get

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty \sin \lambda x \int_0^\infty f(t) \sin \lambda t\, dt\, d\lambda} \quad ...(5)$$

This is called **Fourier sine integral.**

**When f(x) is an even function,** $f(t) \cos \lambda t$ is even while $f(t) \sin \lambda t$ is odd. Thus the second integral in (4) vanishes and, we get

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty \cos \lambda x \int_0^\infty f(t) \cos \lambda t\, dt\, d\lambda} \quad ...(6)$$

This is called **Fourier cosine integral.**

## 2.4. COMPLEX FORM OF FOURIER INTEGRAL

Since $\cos \lambda(t - x)$ is an even function of $\lambda$, we have from (3)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t) \cos \lambda(t - x)\, dt\, d\lambda \quad ...(7)$$

$$\left[\because\ 2\int_0^a f(x)\,dx = \int_{-a}^a f(x)\,dx, \text{if } f(x) \text{ is even}\right]$$

Also $\sin \lambda(t - x)$ is an odd function of $\lambda$, so that

$$0 = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t) \sin \lambda(t - x)\, dt\, d\lambda \quad ...(8)$$

$$\left[\because\ \int_{-a}^a f(x)\,dx = 0, \text{if } f(x) \text{ is odd}\right]$$



Multiplying (8) by $i$ and adding to (7), we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\,[\cos \lambda(t - x) + i \sin \lambda(t - x)]\, dt\, d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\, e^{i\lambda(t-x)}\, dt\, d\lambda = \boxed{\frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\lambda x}\int_{-\infty}^\infty f(t)\, e^{i\lambda t}\, dt\, d\lambda}$$

which is known as the **complex form of Fourier integral.**

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Express the function* $f(x) = \begin{cases} 1 & \text{for } |x| \le 1, \\ 0 & \text{for } |x| > 1, \end{cases}$

*as a Fourier integral. Hence evaluate* $\displaystyle\int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda}\, d\lambda$. (M.D.U. May 2008)

**Sol.** The Fourier integral for $f(x)$ is

$$\frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t) \cos \lambda(t - x)\, dt\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-1}^1 \cos \lambda(t - x)\, dt\, d\lambda \qquad \left[\because\ f(t) = \begin{Bmatrix} 1, & -1 < t < 1 \\ 0, & \text{otherwise} \end{Bmatrix}\right]$$

$$= \frac{1}{\pi}\int_0^\infty \left[\frac{\sin \lambda(t - x)}{\lambda}\right]_{-1}^1 d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin \lambda(1 - x) - \sin \lambda(-1 - x)}{\lambda}\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin \lambda(1 + x) + \sin \lambda(1 - x)}{\lambda}\, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda}\, d\lambda$$

$$\therefore \quad \int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda}\, d\lambda = \frac{\pi}{2} f(x) = \begin{cases} \frac{\pi}{2} & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}.$$

At $|x| = 1$, *i.e.*, $x = \pm 1$, $f(x)$ is discontinuous and the integral has the value

$$\frac{1}{2}\left(\frac{\pi}{2} + 0\right) = \frac{\pi}{4}.$$

**Note.** Putting $x = 0$, we get $\displaystyle\int_0^\infty \frac{\sin \lambda}{\lambda}\, d\lambda = \frac{\pi}{2}$ or $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$.

**Example 2.** *Express* $f(x) = \begin{cases} 1 & \text{for } 0 \le x \le \pi \\ 0 & \text{for } x > \pi \end{cases}$

*as a Fourier sine integral and hence evaluate*

$$\int_0^\infty \frac{1 - \cos(\pi\lambda)}{\lambda} \sin(x\lambda)\, d\lambda$$ (Kottayam 2005)

**Sol.** The Fourier sine integral for $f(x)$ is

$$\frac{2}{\pi}\int_0^{\infty} \sin(\lambda x)\int_0^{\infty} f(t)\sin(\lambda t)\,dt\,d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} \sin(\lambda x)\left[\int_0^{\pi} f(t)\sin(\lambda t)\,dt + \int_{\pi}^{\infty} f(t)\sin(\lambda t)\,dt\right]d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} \sin(\lambda x)\int_0^{\pi} \sin(\lambda t)\,dt\,d\lambda \qquad \text{[on substituting for } f(t)]$$

$$= \frac{2}{\pi}\int_0^{\infty} \sin(\lambda x)\left[-\frac{\cos(\lambda t)}{\lambda}\right]_0^{\pi} d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} \frac{1-\cos(\lambda\pi)}{\lambda}\sin(\lambda x)\,d\lambda$$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{1-\cos(\pi\lambda)}{\lambda}\sin(x\lambda)\,d\lambda$$

$$\Rightarrow \quad \int_0^{\infty} \frac{1-\cos(\pi\lambda)}{\lambda}\sin(x\lambda)\,d\lambda = \frac{\pi}{2}f(x) = \begin{cases} \dfrac{\pi}{2} & \text{for } 0 \le x < \pi \\ 0 & \text{for } x > \pi \end{cases}$$

At $x = \pi$, which is a point of discontinuity of $f(x)$, we have

$$f(x) = \frac{1}{2}[f(\pi - 0) + f(\pi + 0)] = \frac{1}{2}(1 + 0) = \frac{1}{2}$$

$$\therefore \quad \int_0^{\infty} \frac{1-\cos(\pi\lambda)}{\lambda}\sin(x\lambda)\,d\lambda = \frac{\pi}{2}\left(\frac{1}{2}\right) = \frac{\pi}{4}.$$

**Example 3.** *Using Fourier Integral representation, show that:*

$$\int_0^{\infty} \frac{\cos x\alpha + \alpha\sin x\alpha}{1+\alpha^2}\,d\alpha = \begin{cases} 0 & , \text{ if } x < 0 \\ \dfrac{\pi}{2} & , \text{ if } x = 0 \\ \pi e^{-x}, & \text{if } x > 0 \end{cases} \qquad \text{(M.D.U. Dec. 2009)}$$

**Sol.** Fourier Integral for

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ e^{-x}, & \text{if } x > 0 \end{cases}$$

is $\displaystyle \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda$

$$= \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{0} f(t)\cos\lambda(t-x)\,dt + \int_0^{\infty} f(t)\cos\lambda(t-x)\,dt\right]d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{0} 0\cos\lambda(t-x)\,dt + \int_0^{\infty} e^{-t}\cos\lambda(t-x)\,dt\right]d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\int_0^{\infty} e^{-t}\cos\lambda(t-x)\,dt\,d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\left[\frac{e^{-t}}{1+\lambda^2}\{-\cos\lambda(t-x) + \lambda\sin\lambda(t-x)\}\right]_0^{\infty} d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty} \frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\,d\lambda$$



$$= \frac{1}{\pi}\int_0^{\infty} \frac{\cos\alpha x + \alpha\sin\alpha x}{1+\alpha^2}\,d\alpha \qquad \text{(Replacing } \lambda \text{ by } \alpha)$$

$$\Rightarrow \quad \int_0^{\infty} \frac{\cos x\alpha + \alpha\sin x\alpha}{1+\alpha^2}\,d\alpha = \pi f(x) = \begin{cases} 0 & , \text{ if } x < 0 \\ \pi e^{-x}, & \text{if } x > 0 \end{cases}$$

When $x = 0$, $\displaystyle \int_0^{\infty} \frac{\cos x\alpha + \alpha\sin x\alpha}{1+\alpha^2}\,d\alpha = \int_0^{\infty} \frac{1}{1+\alpha^2}\,d\alpha = \left[\tan^{-1}\alpha\right]_0^{\infty} = \frac{\pi}{2}$

$$\therefore \quad \int_0^{\infty} \frac{\cos x\alpha + \alpha\sin x\alpha}{1+\alpha^2}\,d\alpha = \begin{cases} 0 & , \text{ if } x < 0 \\ \dfrac{\pi}{2} & , \text{ if } x = 0 \\ \pi e^{-x}, & \text{if } x > 0 \end{cases}$$

## EXERCISE 2.1

*Using Fourier integral representation, show that (1 – 9):*

1. $\displaystyle \int_0^{\infty} \frac{\lambda\sin x\lambda}{k^2+\lambda^2}\,d\lambda = \frac{\pi}{2}e^{-kx}, \; x > 0, k > 0$

2. $\displaystyle \int_0^{\infty} \frac{\cos x\omega}{1+\omega^2}\,d\omega = \frac{\pi}{2}e^{-x}, \; x \ge 0$

3. $\displaystyle \int_0^{\infty} \frac{\sin \pi\lambda \sin x\lambda}{1-\lambda^2}\,d\lambda = \begin{cases} \dfrac{\pi}{2}\sin x, & \text{when } 0 \le x \le \pi \\ 0, & \text{when } x > \pi \end{cases}$

4. $\displaystyle \int_0^{\infty} \frac{\cos\lambda x}{\lambda^2+a^2}\,d\lambda = \frac{\pi}{2a}e^{-ax}, \; a > 0, \; x \ge 0.$

5. $\displaystyle \int_0^{\infty} \left(\frac{\lambda^2+2}{\lambda^4+4}\right)\cos\lambda x\,d\lambda = \frac{\pi}{2}e^{-x}\cos x, \;\text{ if } x > 0.$

6. $\displaystyle \int_0^{\infty} \left(\frac{\lambda^3}{\lambda^4+4}\right)\sin\lambda x\,d\lambda = \frac{\pi}{2}e^{-x}\cos x, \;\text{ if } x > 0.$

7. $\displaystyle \int_0^{\infty} \frac{\cos\dfrac{\pi\lambda}{2}\cos\lambda x}{1-\lambda^2}\,d\lambda = \begin{cases} \dfrac{\pi}{2}\cos x, & \text{if } |x| < \dfrac{\pi}{2} \\ 0, & \text{if } |x| > \dfrac{\pi}{2} \end{cases}$

8. $\displaystyle \int_0^{\infty} \frac{\lambda\sin\lambda x}{(\lambda^2+\alpha^2)(\lambda^2+\beta^2)}\,d\lambda = \frac{\pi}{2}\left(\frac{e^{-\alpha x} - e^{-\beta x}}{\beta^2-\alpha^2}\right).$

9. Find Fourier sine integral representation of

$$f(x) = \begin{cases} 0, & 0 < x < 1 \\ k, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$$

where $k$ is a constant.

10. Find the Fourier integral representation for the following functions:

(*i*) $f(x) = \begin{cases} \dfrac{\pi}{2}\cos x, & |x| \le \pi \\ 0, & |x| > \pi \end{cases}$ (*ii*) $f(x) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$

(*iii*) $f(x) = e^{-|x|}, \; -\infty < x < \infty.$

**Answers**

9. $f(x) = \frac{2k}{\pi}\int_0^\infty \left(\frac{\cos\lambda - \cos 2\lambda}{\lambda}\right)\sin\lambda x\, d\lambda$

10. (i) $f(x) = \int_0^\infty \frac{\lambda \sin\lambda\pi}{1-\lambda^2}\cos\lambda x\, d\lambda$ (ii) $f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda a \cos\lambda x}{\lambda}\, d\lambda$

(iii) $f(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{1+\lambda^2}\cos\lambda x\, d\lambda.$

## 2.5. FOURIER TRANSFORMS AND INVERSION FORMULAE

(1) *Fourier sine transform and its inversion formula*

Fourier sine integral is $f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty f(t)\sin\lambda t\, dt\, d\lambda$

Replacing $\lambda$ by $s$, we get $f(x) = \frac{2}{\pi}\int_0^\infty \sin sx \int_0^\infty f(t)\sin st\, dt\, ds$

Denoting the value of the inner integral by $F_s(s)$, we have

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(s)\sin sx\, ds \qquad \ldots(1)$$

where $$F_s(s) = \int_0^\infty f(x)\sin sx\, dx \qquad \ldots(2)$$

The function $F_s(s)$ as defined by equation (2) is known as the **Fourier sine transform** of $f(x)$ in $0 < x < \infty$.

The function $f(x)$ as defined by equation (1) is called the **inverse Fourier sine transform** of $F_s(s)$. Thus equation (1) gives the inversion formula for the Fourier sine transform.

**Note.** Some authors write the above formulae as:

$$F_s(s) \text{ or } F_s\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\, dx$$

and $$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx\, ds$$

(2) *Fourier cosine transform and its inversion formula* (Anna 2007)

Fourier cosine integral is $f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty f(t)\cos\lambda t\, dt\, d\lambda$

Replacing $\lambda$ by $s$, we get $f(x) = \frac{2}{\pi}\int_0^\infty \cos sx \int_0^\infty f(t)\cos st\, dt\, ds$

Denoting the value of the inner integral by $F_c(s)$, we have

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(s)\cos sx\, ds \qquad \ldots(3)$$

where $$F_c(s) = \int_0^\infty f(x)\cos sx\, dx \qquad \ldots(4)$$



The function $F_c(s)$ as defined by equation (4) is known as the **Fourier cosine transform** of $f(x)$ in $0 < x < \infty$.

The function $f(x)$ as defined by equation (3) is called the **inverse Fourier cosine transform** of $F_c(s)$. Thus equation (3) gives the inversion formula for the Fourier cosine transform.

**Note.** Some authors write the above formulae as:

$$F_c(s) \text{ or } F_c\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\, dx$$

and $$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\cos sx\, ds$$

(3) *Complex Fourier transform and its inversion formula* (Anna 2007)

Complex form of Fourier integral is $f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\lambda x}\int_{-\infty}^\infty f(t)e^{i\lambda t}\, dt\, d\lambda$

Replacing $\lambda$ by $s$, we get $f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-isx}\int_{-\infty}^\infty f(t)e^{ist}\, dt\, ds$

Denoting the value of the inner integral by F(s), we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty F(s)\, e^{-isx}\, ds \qquad \ldots(5)$$

where $$F(s) = \int_{-\infty}^\infty f(x)\, e^{isx}\, dx \qquad \ldots(6)$$

The function F(s) as defined by equation (6) is known as the **Fourier transform** of $f(x)$.

The function $f(x)$ as defined by equation (5) is called the **inverse Fourier transform** of F(s). Thus equation (5) gives the inversion formula for the Fourier transform.

**Note.** Some authors write the above formulae as:

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)\, e^{isx}\, dx$$

and $$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty F(s)\, e^{-isx}\, ds$$

(4) *Finite Fourier sine transform and its inversion formula*

The **finite Fourier sine transform** of $f(x)$ in $0 < x < c$ is defined as

$$F_s(n) = \int_0^c f(x)\sin\frac{n\pi x}{c}\, dx$$

where $n$ is an integer.

The function $f(x)$ is then called the **inverse finite Fourier sine transform** of $F_s(n)$ and is given by

$$f(x) = \frac{2}{c}\sum_{n=1}^\infty F_s(n)\sin\frac{n\pi x}{c}.$$

(5) *Finite Fourier cosine transform and its inversion formula*

The **finite Fourier cosine transform** of $f(x)$ in $0 < x < c$ is defined as

$$F_c(n) = \int_0^c f(x) \cos \frac{n\pi x}{c}\, dx$$

where $n$ is an integer.

The function $f(x)$ is then called the **inverse finite Fourier cosine transform** of $F_c(n)$ and is given by

$$f(x) = \frac{1}{c} F_c(0) + \frac{2}{c} \sum_{n=1}^{\infty} F_c(n) \cos \frac{n\pi x}{c}.$$

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Find the Fourier sine transform of $e^{-|x|}$. Hence evaluate $\int_0^\infty \frac{x \sin mx}{1+x^2} dx$.*

(V.T.U., 2007)

**Sol.** In the interval $(0, \infty)$, $x$ is positive so that $e^{-|x|} = e^{-x}$. Fourier sine transform of $f(x) = e^{-x}$ is given by

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin sx\, dx = \int_0^\infty e^{-x} \sin sx\, dx$$

$$= \left[\frac{e^{-x}}{1+s^2}(-\sin sx - s\cos sx)\right]_0^\infty = \frac{s}{1+s^2}$$

Using inversion formula for Fourier sine transform, we get

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s\{f(x)\} \sin sx\, ds \quad \text{or} \quad e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2} \sin sx\, ds$$

Replacing $x$ by $m$, we have $e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2} \sin ms\, ds = \frac{2}{\pi}\int_0^\infty \frac{x \sin mx}{1+x^2} dx$

Hence, $\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}.$

**Example 2.** *Find the Fourier sine transform of*

$$f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2-x, & \text{for } 1 < x < 2. \\ 0, & \text{for } x > 2 \end{cases}$$

**Sol.** Fourier transform of $f(x)$ is

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin sx\, dx$$



$$= \int_0^1 f(x) \sin sx\, dx + \int_1^2 f(x) \sin sx\, dx + \int_2^\infty f(x) \sin sx\, dx$$

$$= \int_0^1 x \sin sx\, dx + \int_1^2 (2-x) \sin sx\, dx + \int_2^\infty 0\, dx$$

$$= \left\{x.\frac{-\cos sx}{s} - 1.\frac{-\sin sx}{s^2}\right\}_0^1 + \left\{(2-x).\frac{-\cos sx}{s} - (-1).\frac{-\sin sx}{s^2}\right\}_1^2$$

$$= \left(-\frac{\cos s}{s} + \frac{\sin s}{s^2}\right) + \left(-\frac{\sin 2s}{s^2} + \frac{\cos s}{s} + \frac{\sin s}{s^2}\right)$$

$$= \frac{2\sin s - \sin 2s}{s^2} = \frac{2\sin s - 2\sin s\cos s}{s^2} = \frac{2\sin s(1-\cos s)}{s^2}.$$

**Example 3.** *Find the Fourier sine transform of $\frac{e^{-ax}}{x}$.* (P.T.U. 2006; M.D.U. 2005)

**Sol.** Fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$ is

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin sx\, dx$$

$$= \int_0^\infty \frac{e^{-ax}}{x} \sin sx\, dx = I \quad \text{(say)} \qquad \ldots(1)$$

Differentiating w.r.t. $s$, we have

$$\frac{dI}{ds} = \int_0^\infty \frac{\partial}{\partial s}\left(\frac{e^{-ax}}{x} \sin sx\right) dx$$

$$= \int_0^\infty \frac{e^{-ax}}{x}.x\cos sx\, dx = \int_0^\infty e^{-ax}\cos sx\, dx$$

$$= \left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx)\right]_0^\infty$$

$$= 0 - \frac{1}{a^2+s^2}(-a) = \frac{a}{s^2+a^2}$$

Integrating w.r.t. $s$, we get

$$I = \tan^{-1}\frac{s}{a} + c \qquad \ldots(2)$$

Now, when $s = 0$, from (1), $I = 0$

$\therefore$ From (2), $0 = 0 + c \Rightarrow c = 0$

$\therefore$ $I = \tan^{-1}\frac{s}{a}$ or $F_s\left\{\frac{e^{-ax}}{x}\right\} = \tan^{-1}\frac{s}{a}.$

**Example 4.** *Find Fourier sine transform of* $\frac{1}{x(x^2+a^2)}$. (M.D.U. Dec. 2008)

**Sol.** Fourier sine transform of $f(x) = \frac{1}{x(x^2+a^2)}$ is

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin sx \, dx$$

$$= \int_0^\infty \frac{\sin sx}{x(x^2+a^2)} dx = I \quad \text{(say)} \quad \ldots(1)$$

Differentiating w.r.t. $s$, we have

$$\frac{dI}{ds} = \int_0^\infty \frac{x \cos sx}{x(x^2+a^2)} dx = \int_0^\infty \frac{\cos sx}{x^2+a^2} dx \quad \ldots(2)$$

Differentiating again w.r.t. $s$, we have

$$\frac{d^2I}{ds^2} = \int_0^\infty \frac{-x \sin sx}{x^2+a^2} dx = \int_0^\infty \frac{-x^2 \sin sx}{x(x^2+a^2)} dx$$

$$= \int_0^\infty \frac{[a^2-(x^2+a^2)] \sin sx}{x(x^2+a^2)} dx = a^2 \int_0^\infty \frac{\sin sx}{x(x^2+a^2)} dx - \int_0^\infty \frac{\sin sx}{x} dx$$

$$= a^2 I - \frac{\pi}{2}$$

$$\Rightarrow \quad (D^2 - a^2) I = -\frac{\pi}{2}, \quad \text{where} \quad D = \frac{d}{ds}$$

Its A.E. is $\quad D^2 - a^2 = 0 \quad$ whence $\quad D = \pm a$

$$\text{C.F.} = c_1 e^{as} + c_2 e^{-as}$$

$$\text{P.I.} = \frac{1}{D^2 - a^2}\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} \cdot \frac{1}{D^2-a^2} e^{0s} = -\frac{\pi}{2} \cdot \frac{1}{-a^2} = \frac{\pi}{2a^2}$$

$$\therefore \quad I = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow \quad I = c_1 e^{as} + c_2 e^{-as} + \frac{\pi}{2a^2} \quad \ldots(3)$$

$$\therefore \quad \frac{dI}{ds} = ac_1 e^{as} - ac_2 e^{-as} \quad \ldots(4)$$

When $\quad s = 0$, from (1), $I = 0$

From (3), $\quad I = c_1 + c_2 + \frac{\pi}{2a^2}$

$$\therefore \quad c_1 + c_2 + \frac{\pi}{2a^2} = 0 \quad \ldots(5)$$



When $\quad s = 0$, from (2), $\frac{dI}{ds} = 1\left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^\infty = \frac{\pi}{2a}$

From (4), $\quad \frac{dI}{ds} = ac_1 - ac_2$

$$\therefore \quad ac_1 - ac_2 = \frac{\pi}{2a}$$

$$\text{or} \quad c_1 - c_2 - \frac{\pi}{2a^2} = 0 \quad \ldots(6)$$

Solving (5) and (6), $\quad c_1 = 0, \quad c_2 = -\frac{\pi}{2a^2}$

$$\therefore \quad I = -\frac{\pi}{2a^2} e^{-as} + \frac{\pi}{2a^2}$$

$$\text{or} \quad F_s\{f(x)\} = \frac{\pi}{2a^2}(1 - e^{-as}).$$

**Example 5.** *Find the Fourier sine transform of* $\frac{1}{x}$.

**Sol.** Fourier sine transform of $\frac{1}{x}$ is given by

$$F_s\left(\frac{1}{x}\right) = \int_0^\infty \frac{1}{x} \sin sx \, dx$$

Putting $sx = \theta$ *i.e.*, $x = \frac{\theta}{s}$, we have

$$F_s\left(\frac{1}{x}\right) = \int_0^\infty \frac{s}{\theta} \sin\theta \frac{d\theta}{s} = \int_0^\infty \frac{\sin\theta}{\theta} d\theta = \frac{\pi}{2}.$$

**Example 6.** *Find the Fourier sine and cosine transforms of* $x^{n-1}$, $n > 0$.
(M.D.U. Dec. 2011; Madras, 2006)

**Sol.** We know that

$$F_s(x^{n-1}) = \int_0^\infty x^{n-1} \sin sx \, dx \quad \ldots(1)$$

$$\text{and} \quad F_c(x^{n-1}) = \int_0^\infty x^{n-1} \cos sx \, dx \quad \ldots(2)$$

Now $\quad \Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt, \; n > 0$

Putting $t = ax$, $a > 0$, we have

$$\Gamma(n) = \int_0^\infty e^{-ax} (ax)^{n-1} a \, dx = a^n \int_0^\infty e^{-ax} x^{n-1} dx$$

$$\Rightarrow \quad \int_0^\infty e^{-ax}\, x^{n-1}\, dx = \frac{\Gamma(n)}{a^n}$$

Putting $a = is$, we have

$$\int_0^\infty e^{-isx}\, x^{n-1}\, dx = \frac{\Gamma(n)}{(is)^n}$$

$$\Rightarrow \quad \int_0^\infty (\cos sx - i\sin sx)\, x^{n-1}\, dx = \left(\frac{i}{i^2}\right)^n \frac{\Gamma(n)}{s^n} = (-i)^n \frac{\Gamma(n)}{s^n}$$

$$= \left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)^n \frac{\Gamma(n)}{s^n}$$

$$= \left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right)\frac{\Gamma(n)}{s^n}$$

Equating real and imaginary parts on both sides, we get

$$\int_0^\infty x^{n-1}\cos sx\, dx = \frac{\Gamma(n)}{s^n}\cos\frac{n\pi}{2}$$

and
$$\int_0^\infty x^{n-1}\sin sx\, dx = \frac{\Gamma(n)}{s^n}\sin\frac{n\pi}{2}$$

$\therefore$ From (1), $\quad F_s(x^{n-1}) = \dfrac{\Gamma(n)}{s^n}\sin\dfrac{n\pi}{2}$

and from (2), $\quad F_c(x^{n-1}) = \dfrac{\Gamma(n)}{s^n}\cos\dfrac{n\pi}{2}$

**Note.** Taking $n = \dfrac{1}{2}$, we get

$$F_s\left(\frac{1}{\sqrt{x}}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}}\sin\frac{\pi}{4} = \frac{\sqrt{\pi}}{\sqrt{s}}\cdot\frac{1}{\sqrt{2}} \qquad \left[\because\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \sqrt{\frac{\pi}{2s}}$$

$$F_c\left(\frac{1}{\sqrt{x}}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}}\cos\frac{\pi}{4} = \sqrt{\frac{\pi}{2s}}.$$

**Example 7.** *Find the Fourier cosine transform of* $e^{-x^2}$. (Rajasthan 2006)

**Sol.** Fourier cosine transform of $e^{-x^2}$ is given by

$$F_c\{e^{-x^2}\} = \int_0^\infty e^{-x^2}\cos sx\, dx = I \text{ (say)} \qquad \ldots(1)$$



Differentiating w.r.t. $s$, we have

$$\frac{dI}{ds} = -\int_0^\infty x e^{-x^2}\sin sx\, dx = \frac{1}{2}\int_0^\infty (\sin sx)(-2x\, e^{-x^2})\, dx$$

$$= \frac{1}{2}\left[\left\{\sin sx\, e^{-x^2}\right\}_0^\infty - s\int_0^\infty \cos sx\, e^{-x^2}\, dx\right]$$

(Integrating by parts)

$$= -\frac{s}{2}\int_0^\infty e^{-x^2}\cos sx\, dx = -\frac{s}{2}I$$

or
$$\frac{dI}{I} = -\frac{s}{2}\, ds$$

Integrating, we have $\log I = -\dfrac{s^2}{4} + \log A$ or $I = A\, e^{-\frac{s^2}{4}}$ $\qquad\ldots(2)$

Now when $s = 0$, from (1), $I = \displaystyle\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$

$\therefore$ From (2), $\quad \dfrac{\sqrt{\pi}}{2} = A$

Hence $\quad I = F_c\{e^{-x^2}\} = \dfrac{\sqrt{\pi}}{2}e^{-\frac{s^2}{4}}.$

**Note.** $\quad I = \displaystyle\int_0^\infty e^{-x^2}dx \quad$ Put $x^2 = t$ i.e., $x = \sqrt{t}$

$$= \int_0^\infty \frac{e^{-t}}{2\sqrt{t}}\, dt = \frac{1}{2}\int_0^\infty t^{-1/2}\, e^{-t} dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

**Example 8.** *Find the Fourier cosine transform of*

$$f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x \ge a \end{cases}$$

**Sol.** Fourier cosine transform of $f(x)$ is

$$F_c\{f(x)\} = \int_0^\infty f(x)\cos sx\, dx = \int_0^a f(x)\cos sx\, dx + \int_a^\infty f(x)\cos sx\, dx$$

$$= \int_0^a \cos x\cos sx\, dx + \int_a^\infty 0\, dx = \frac{1}{2}\int_0^a 2\cos x\cos sx\, dx + 0$$

$$= \frac{1}{2}\int_0^a [\cos(1+s)x + \cos(1-s)x]\, dx$$

$$= \frac{1}{2}\left[\frac{\sin(1+s)x}{1+s} + \frac{\sin(1-s)x}{1-s}\right]_0^a = \frac{1}{2}\left[\frac{\sin(1+s)a}{1+s} + \frac{\sin(1-s)a}{1-s}\right].$$

**Example 9.** *Find the Fourier cosine transform of* $f(x) = \dfrac{1}{1+x^2}$. (M.D.U. May 2011)

*Hence derive Fourier sine transform of* $\phi(x) = \dfrac{1}{x(1+x^2)}$.

**Sol.** Fourier cosine transform of $f(x) = \dfrac{1}{1+x^2}$ is

$$F_c\{f(x)\} = \int_0^\infty \frac{\cos sx}{1+x^2}\,dx = I \quad \text{(say)} \qquad \ldots(1)$$

Differentiating w.r.t. $s$, we have

$$\frac{dI}{ds} = \int_0^\infty \frac{-x\sin sx}{1+x^2}\,dx = \int_0^\infty \frac{-x^2 \sin sx}{x(1+x^2)}\,dx$$

$$= \int_0^\infty \frac{[1-(1+x^2)]\sin sx}{x(1+x^2)}\,dx = \int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx - \int_0^\infty \frac{\sin sx}{x}\,dx$$

$$= \int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx - \frac{\pi}{2} \qquad \ldots(2)$$

Differentiating again w.r.t. $s$, we have

$$\frac{d^2I}{ds^2} = \int_0^\infty \frac{x\cos sx}{x(1+x^2)}\,dx = \int_0^\infty \frac{\cos sx}{1+x^2}\,dx = I$$

$\Rightarrow$ $(D^2-1)\,I = 0$, where $D = \dfrac{d}{ds}$

Its A.E. is $D^2 - 1 = 0$ whence $D = \pm 1$

$\therefore$ $I = c_1 e^{s} + c_2 e^{-s}$ ...(3)

$$\frac{dI}{ds} = c_1 e^{s} - c_2 e^{-s} \qquad \ldots(4)$$

When $s = 0$, from (1), $I = \displaystyle\int_0^\infty \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^\infty = \frac{\pi}{2}$

From (3), $I = c_1 + c_2$

$\therefore$ $c_1 + c_2 = \dfrac{\pi}{2}$ ...(5)

When $s = 0$, from (2), $\dfrac{dI}{ds} = -\dfrac{\pi}{2}$

From (4), $\dfrac{dI}{ds} = c_1 - c_2$

$\therefore$ $c_1 - c_2 = -\dfrac{\pi}{2}$ ...(6)



Solving (5) and (6), $c_1 = 0$, $c_2 = \dfrac{\pi}{2}$

$\therefore$ $I = \dfrac{\pi}{2}e^{-s} \Rightarrow F_c\{f(x)\} = \dfrac{\pi}{2}e^{-s}$

Putting the values of $c_1$ and $c_2$ in (4)

$$\frac{dI}{ds} = -\frac{\pi}{2}e^{-s}$$

$\therefore$ From (2), $-\dfrac{\pi}{2}e^{-s} = \displaystyle\int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx - \frac{\pi}{2}$

$$\Rightarrow \quad \int_0^\infty \frac{\sin sx}{x(1+x^2)}\,dx = \frac{\pi}{2}(1-e^{-s})$$

$$\Rightarrow \quad F_s\{\phi(x)\} = \frac{\pi}{2}(1-e^{-s}).$$

**Note.** In example 9 above, we have proved that

$$\int_0^\infty \frac{\cos sx}{1+x^2}\,dx = \frac{\pi}{2}e^{-s}$$

Differentiating w.r.t. $s$, we get

$$\int_0^\infty \frac{\partial}{\partial s}\left(\frac{\cos sx}{1+x^2}\right)dx = -\frac{\pi}{2}e^{-s}$$

$$\Rightarrow \quad \int_0^\infty \frac{-x\sin sx}{1+x^2}\,dx = -\frac{\pi}{2}e^{-s}$$

$$\Rightarrow \quad \int_0^\infty \frac{x}{1+x^2}\sin sx\,dx = \frac{\pi}{2}e^{-s}$$

$$\Rightarrow \quad F_s\{\psi(x)\} = \frac{\pi}{2}e^{-s} \qquad \text{where } \psi(x) = \frac{x}{1+x^2}$$

$\Rightarrow$ Fourier sine transform of $\dfrac{x}{1+x^2}$ is $\dfrac{\pi}{2}e^{-s}$.

**Example 10.** *Find the Fourier transform of*

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$ (M.D.U. May 2011 ; Anna 2007)

*Hence evaluate*

(*i*) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\,ds$ (*ii*) $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

**Sol.** Fourier transform of $f(x)$ is given by

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \int_{-\infty}^{-a} f(x)\, e^{isx}\, dx + \int_{-a}^{a} f(x)\, e^{isx\, dx} + \int_{a}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \int_{-\infty}^{-a} 0 \, . \, e^{isx}\, dx + \int_{-a}^{a} 1 \, . \, e^{isx}\, dx + \int_{a}^{\infty} 0 \, . \, e^{isx}\, dx$$

$$= \left[\frac{e^{isx}}{is}\right]_{-a}^{a} = \frac{e^{ias} - e^{-ias}}{is}$$

$$= \frac{2}{s}\left(\frac{e^{ias} - e^{-ias}}{2i}\right) = \frac{2 \sin as}{s}, \qquad | \; s \neq 0$$

For $s = 0$, we find $F\{f(x)\} = 2a$

By inversion formula for Fourier transform, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F\{f(x)\} . \, e^{-isx}\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2 \sin as}{s} . \, e^{-isx}\, ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin as\,(\cos sx - i \sin sx)}{s}\, ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds - \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\sin as \sin sx}{s}\, ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds$$

[Second integral vanishes since the integrand is an odd function of $s$]

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds = \pi f(x) = \begin{cases} \pi, & |x| < a \\ 0, & |x| > a \end{cases} \qquad \text{...(1)}$$

Since the integrand in (1) is an even function of $s$, we have

$$2\int_{0}^{\infty} \frac{\sin as \cos sx}{s}\, ds = \begin{cases} \pi, & |x| < a \\ 0, & |x| > a \end{cases}$$

$$\Rightarrow \qquad \int_{0}^{\infty} \frac{\sin as \cos sx}{s}\, ds = \begin{cases} \pi/2, & |x| < a \\ 0, & |x| > a \end{cases}$$



Putting $x = 0$, $\quad \displaystyle\int_{0}^{\infty} \frac{\sin as}{s}\, ds = \begin{cases} \frac{\pi}{2}, & a > 0 \\ 0, & a < 0 \end{cases}$

Putting $a = 1$, $\quad \displaystyle\int_{0}^{\infty} \frac{\sin s}{s}\, ds = \frac{\pi}{2}$

$$\Rightarrow \qquad \int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

**Example 11.** *Find the Fourier transform of* $f(x) = \begin{cases} 1 - x^2, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1 \end{cases}$

*and use it to evaluate* $\displaystyle\int_{0}^{\infty} \left(\frac{x \cos x - \sin x}{x^3}\right) \cos \frac{x}{2}\, dx.$

(M.D.U. May 2006, Dec. 2006, Dec. 2007, Dec. 2010; Anna, 2006, 2007; V.T.U., 2006)

**Sol.** Fourier transform of $f(x)$ is given by

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

$$= \int_{-\infty}^{-1} f(x)\, e^{isx} dx + \int_{-1}^{1} f(x)\, e^{isx}\, dx + \int_{1}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \int_{-\infty}^{-1} 0\, dx + \int_{-1}^{1} (1 - x^2)\, e^{isx}\, dx + \int_{1}^{\infty} 0\, dx$$

$$= \left[(1 - x^2) . \frac{e^{isx}}{is} - (-2x)\frac{e^{isx}}{(is)^2} + (-2)\frac{e^{isx}}{(is)^3}\right]_{-1}^{1}$$

$$= \left[-\frac{2}{s^2}(e^{is} + e^{-is}) + \frac{2}{is^3}(e^{is} - e^{-is})\right]$$

$$= \left[-\frac{4}{s^2}\left(\frac{e^{is} + e^{-is}}{2}\right) + \frac{4}{s^3}\left(\frac{e^{is} - e^{-is}}{2i}\right)\right]$$

$$= 4\left[-\frac{\cos s}{s^2} + \frac{\sin s}{s^3}\right] = -\frac{4}{s^3}(s \cos s - \sin s)$$

By inversion formula for Fourier transform, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F\{f(x)\} . \, e^{-isx}\, ds$$

$$= -\frac{2}{\pi}\int_{-\infty}^{\infty} \left(\frac{s \cos s - \sin s}{s^3}\right)(\cos sx - i \sin sx)\, ds$$

$$= -\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{s\cos s - \sin s}{s^3}\right)\cos sx\,ds + \frac{2i}{\pi}\int_{-\infty}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\sin sx\,ds$$

$$= -\frac{4}{\pi}\int_0^\infty \left(\frac{s\cos s - \sin s}{s^3}\right)\cos sx\,ds$$

(Since the integrand in the first integral is even and that in the second integral is odd)

$$\Rightarrow \int_0^\infty \left(\frac{s\cos s - \sin s}{s^3}\right)\cos sx\,ds = -\frac{\pi}{4}f(x) = \begin{cases} -\frac{\pi}{4}(1-x^2), & \text{if } |x|<1 \\ 0, & \text{if } |x|>1\end{cases}$$

Putting $x = \frac{1}{2}$, we have $\int_0^\infty \left(\frac{s\cos s-\sin s}{s^3}\right)\cos\frac{s}{2}\,ds = -\frac{\pi}{4}\left(1-\frac{1}{4}\right) = -\frac{3\pi}{16}$

Hence $\int_0^\infty\left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx = -\frac{3\pi}{16}$.

**Example 12.** *Find the inverse Fourier transform of* $F(s) = e^{-|s|y}$.

**Sol.** The inverse Fourier transform of $F(s) = e^{-|s|y}$ is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,e^{-isx}\,ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|s|y}\,.\,e^{-isx}\,ds$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{sy}\,.\,e^{-isx}\,ds + \int_0^\infty e^{-sy}\,.\,e^{-isx}\,ds\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{s(y-ix)}\,ds + \int_0^\infty e^{-s(y+ix)}\,ds\right]$$

$$= \frac{1}{2\pi}\left[\left\{\frac{e^{s(y-ix)}}{y-ix}\right\}_{-\infty}^{0} + \left\{\frac{e^{-s(y+ix)}}{-(y+ix)}\right\}_0^{\infty}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{y-ix} + \frac{1}{y+ix}\right] = \frac{1}{2\pi}\left(\frac{2y}{y^2+x^2}\right)$$

$$= \frac{y}{\pi(y^2+x^2)}.$$

**Example 13.** *Solve the integral equation* $\int_0^\infty f(x)\cos px\,dx = \begin{cases} 1-p, & 0\le p\le 1 \\ 0, & p>1\end{cases}$

*Hence deduce that* $\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}$. (K.U.K. 2006)

**Sol.** Here $\int_0^\infty f(x)\cos px\,dx = F_c(p)$

$\therefore$ $F_c(p) = \begin{cases} 1-p, & 0\le p\le 1 \\ 0, & p>1\end{cases}$



By inversion formula for Fourier cosine transform, we have

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(p)\cos px\,dp$$

$$= \frac{2}{\pi}\left[\int_0^1 F_c(p)\cos px\,dp + \int_1^\infty F_c(p)\cos px\,dp\right]$$

$$= \frac{2}{\pi}\left[\int_0^1 (1-p)\cos px\,dp + \int_1^\infty 0\,dp\right] \quad \text{(Integrating by parts)}$$

$$= \frac{2}{\pi}\left[(1-p)\,.\,\frac{\sin px}{x} - (-1)\,.\,\frac{-\cos px}{x^2}\right]_0^1 = \frac{2}{\pi}\left[-\frac{\cos x}{x^2} + \frac{1}{x^2}\right] = \frac{2(1-\cos x)}{\pi x^2}.$$

**Deduction.** Since $\int_0^\infty f(x)\cos px\,dx = \begin{cases} 1-p, & 0\le p\le 1 \\ 0, & p>1\end{cases}$, where $f(x) = \frac{2(1-\cos x)}{\pi x^2}$

$$\therefore \quad \frac{2}{\pi}\int_0^\infty\left(\frac{1-\cos x}{x^2}\right)\cos px\,dx = \begin{cases} 1-p, & 0\le p\le 1 \\ 0, & p>1\end{cases}$$

When $p = 0$, we have $\frac{2}{\pi}\int_0^\infty \frac{1-\cos x}{x^2}\,dx = 1$ or $\int_0^\infty \frac{2\sin^2\frac{x}{2}}{x^2}\,dx = \frac{\pi}{2}$

Putting $\frac{x}{2} = t$ so that $dx = 2dt$, we have $\int_0^\infty \frac{\sin^2 t}{t^2} = \frac{\pi}{2}$.

**Example 14.** *Find the finite Fourier sine transform of*

$$f(x) = \begin{cases} \frac{2k}{l}x, & 0\le x\le \frac{l}{2} \\ \frac{2k}{l}(l-x), & \frac{l}{2}\le x\le l\end{cases}.$$

**Sol.** Finite Fourier sine transform of $f(x)$ in $0 \le x \le l$ is

$$F_s(n) = \int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \int_0^{l/2} f(x)\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \int_0^{l/2}\frac{2k}{l}x\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}\frac{2k}{l}(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2k}{l}\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx + \frac{2k}{l}\int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx \quad \text{(Integrating by parts)}$$

$$= \frac{2k}{l}\left[x\,.\,\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} - 1\,.\,\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_0^{l/2} + \frac{2k}{l}\left[(l-x)\,.\,\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} - (-1)\,.\,\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_{l/2}^{l}$$

$$= \frac{2k}{l}\left[-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]+\frac{2k}{l}\left[\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2k}{l}\cdot\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}=\frac{4kl}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

**Example 15.** *Find the finite Fourier cosine transform of*

$$f(x)=\left(1-\frac{x}{\pi}\right)^2,\ 0\le x<\pi.$$

**Sol.** Finite Fourier cosine transform of $f(x)$ in $0\le x\le\pi$ is

$$F_c(n)=\int_0^\pi f(x)\cos\frac{n\pi x}{\pi}\,dx=\int_0^\pi\left(1-\frac{x}{\pi}\right)^2\cos nx\,dx \qquad \text{(Integrating by par}$$

$$=\left[\left(1-\frac{x}{\pi}\right)^2\cdot\frac{\sin nx}{n}-2\left(1-\frac{x}{\pi}\right)\left(-\frac{1}{\pi}\right)\cdot\frac{-\cos nx}{n^2}+2\left(-\frac{1}{\pi}\right)\left(-\frac{1}{\pi}\right)\cdot\frac{-\sin nx}{n^3}\right]_0^\pi$$

$$=\frac{2}{\pi n^2}\ \text{when } n\ne 0.$$

If $n=0$, then $F_c(0)=\int_0^\pi\left(1-\frac{x}{\pi}\right)^2dx$ $\qquad [\because\ \cos 0x = 1$

$$=\left[\frac{\left(1-\frac{x}{\pi}\right)^3}{3x\left(-\frac{1}{\pi}\right)}\right]_0^\pi=-\frac{\pi}{3}(0-1)=\frac{\pi}{3}$$

$$\therefore\qquad F_c(n)=\begin{cases}\dfrac{2}{\pi n^2}, & \text{if } n=1,2,3,\ldots\\[2mm] \dfrac{\pi}{3}, & \text{if } n=0.\end{cases}$$

**Example 16.** *Find $f(x)$, if $F_s(n)=\dfrac{1-\cos n\pi}{n^2\pi^2}$, where $0\le x\le\pi$.*

**Sol.** Here $F_s(n)=\dfrac{1-\cos n\pi}{n^2\pi^2}$ is the finite Fourier sine transform of $f(x)$ in $0\le x\le\pi$.

$\therefore$ $f(x)$ = inverse finite Fourier sine transform of $F_s(n)$

$$=\frac{2}{c}\sum_{n=1}^{\infty}F_s(n)\sin\frac{n\pi x}{c}=\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi}{n^2\pi^2}\right)\sin nx \qquad (\because\ \text{here } c=$$

$$=\frac{2}{\pi^3}\sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi}{n^2}\right)\sin nx.$$



## EXERCISE 2.2

1. Find the Fourier sine and cosine transform of the function $f(x)=e^{-x}$ and hence show that

$$\int_0^\infty\frac{x\sin mx}{1+x^2}dx=\frac{\pi}{2}e^{-m}\quad\text{and}\quad\int_0^\infty\frac{\cos mx}{1+x^2}dx=\frac{\pi}{2}e^{-m}.$$ (Anna, 2006)

2. Find the Fourier sine transform of

(i) $f(x)=\begin{cases}\sin x, & 0<x<a\\ 0, & x>a\end{cases}$ (Anna, 2007)

(ii) $f(x)=\begin{cases}0, & 0\le x<a\\ x, & a\le x\le b\\ 0, & x>b\end{cases}$

3. Find the Fourier cosine transform of

(i) $f(x)=e^{-x}+e^{-2x}, x>0$ (M.D.U. May, 2006)

(ii) $f(x)=e^{-\frac{x^2}{2}}$

(iii) $f(x)=\begin{cases}x, & \text{for } 0<x<1\\ 2-x, & \text{for } 1<x<2\\ 0, & \text{for } x>2\end{cases}$ (J.N.T.U. 2006)

(iv) $f(x)=2e^{-5x},+5e^{-2x}$

(v) $f(x)=\begin{cases}x, & 0<x<\frac{1}{2}\\ 1-x, & \frac{1}{2}<x<1\\ 0, & x>1\end{cases}$

(vi) $\left(1-\frac{x}{\pi}\right)^2$ (P.T.U. 2006)

4. Find the Fourier sine and cosine transforms of

(i) $f(x)=\begin{cases}1, & \text{for } 0\le x<a\\ 0, & \text{for } x\ge a\end{cases}$

(ii) $f(x)=e^{-ax}, a>0$

[**Note.** Remember the results of Q. 4.]

5. Find the Fourier transforms of the following functions:

(i) $f(x)=\begin{cases}x, & |x|<a\\ 0, & |x|>a\end{cases}$

(ii) $f(x)=\begin{cases}x^2, & |x|<a\\ 0, & |x|>a\end{cases}$

(iii) $f(x)=e^{-|x|}$

(iv) $f(x)=e^{-\frac{x^2}{2}}, -\infty<x<\infty.$ (K.U.K. Dec. 2010; M.D.U. May 2007)

(v) $f(x)=\begin{cases}e^{ikx}, & a<x<b\\ 0, & x<a \text{ and } x>b\end{cases}$

(vi) $f(x)=\begin{cases}0, & x<\alpha\\ 1, & \alpha<x<\beta\\ 0, & x>\beta\end{cases}$

6. (i) Find the Fourier transform of $f(x)=\begin{cases}1 & \text{for } |x|<1\\ 0 & \text{for } |x|>1\end{cases}$

Hence evaluate $\displaystyle\int_0^\infty\frac{\sin x}{x}dx$. (K.U.K. 2009; Madras 2006; V.T.U. 2006)

(ii) Find the Fourier transform of the function

$$f(x)=\begin{cases}1+\dfrac{x}{a}, & \text{for } -a<x<0\\[2mm] 1-\dfrac{x}{a}, & \text{for } 0<x<a\\[2mm] 0, & \text{otherwise}\end{cases}$$

(*iii*) **Find the Fourier transform of the function**

$$f(t) = \begin{cases} 1, & \text{for } -2 < t < -1 \\ 2, & \text{for } -1 < t < 1 \\ 1, & \text{for } \quad 1 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$

**7.** **Using inverse Fourier sine transform, find $f(x)$ if**

(*i*) $F_s(\lambda) = \frac{1}{\lambda} e^{-a\lambda}$ (*M.D.U. 2009*) (*ii*) $F_s(\lambda) = \frac{\lambda}{1+\lambda^2}$.

**8.** **Find the finite Fourier sine and cosine transforms of the following functions:**

(*i*) $f(x) = 2x,\ 0 \le x \le 4$ (*ii*) $f(x) = x(l-x),\ 0 \le x \le l$

(*iii*) $f(x) = x^2,\ 0 \le x \le 2$ (*iv*) $f(x) = a\left(1 - \frac{x}{l}\right),\ 0 \le x \le l$

(*v*) $f(x) = \begin{cases} kx, & 0 \le x \le \frac{\pi}{2} \\ k(\pi - x), & \frac{\pi}{2} \le x \le \pi \end{cases}$.

**9.** **Find the finite Fourier sine transform of the following functions:**

(*i*) $f(x) = \cos x,\ 0 \le x \le \pi$ (*ii*) $f(x) = \begin{cases} \frac{2x}{3}, & 0 \le x \le \frac{\pi}{3} \\ \frac{\pi - x}{3}, & \frac{\pi}{3} \le x \le \pi \end{cases}$.

**10.** **If $f(x) = \sin kx$, where $0 \le x \le \pi$ and $k$ is a positive integer, show that**

$$F_s(n) = \begin{cases} 0, & \text{if } n \ne k \\ \frac{\pi}{2}, & \text{if } n = k \end{cases}$$

**11.** **Find the finite Fourier cosine transform of the following functions:**

(*i*) $f(x) = \sin x,\ 0 \le x \le \pi$ (*ii*) $f(x) = \begin{cases} kx, & 0 \le x \le \frac{l}{2} \\ k(l-x), & \frac{l}{2} \le x \le l \end{cases}$.

**12.** **Find $f(x)$, if**

(*i*) $F_s(n) = \frac{2l}{n\pi} \sin^2 \frac{n\pi}{4},\ 0 \le x \le l$ (*ii*) $F_s(n) = \frac{1}{n^2} \sin \frac{n\pi}{3},\ 0 \le x \le \pi$

(*iii*) $F_s(n) = \frac{2l^3}{n^3\pi^3}(1 - \cos n\pi),\ 0 \le x \le l$

(*iv*) $F_c(n) = \frac{cl}{n\pi} \sin \frac{n\pi a}{l}$ **and** $F_c(0) = ac$, **where** $0 \le x \le l$

(*v*) $F_c(n) = -\frac{l^3}{n^2\pi^2}(1 + \cos n\pi)$ **and** $F_c(0) = \frac{l^3}{6}$, **where** $0 \le x \le l$

(*vi*) $F_c(n) = \frac{\cos\left(\frac{2n\pi}{3}\right)}{(2n+1)^2}$, **where** $0 \le x \le 1$.



**13.** Solve the following integral equations:

(*i*) $\int_0^\infty f(x) \sin tx\, dx = \begin{cases} 1, & 0 \le t < 1 \\ 2, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases}$ (*M.D.U. May 2006*)

(*ii*) $\int_0^\infty f(x) \cos \lambda x\, dx = e^{-\lambda},\ \lambda > 0$ (*M.D.U. Dec. 2011; Anna 2007*)

(*iii*) $\int_0^\infty f(x) \sin \lambda x\, dx = \begin{cases} 1-\lambda, & 0 \le \lambda \le 1 \\ 0, & \lambda \ge 1 \end{cases}$ (*iv*) $\int_0^\infty f(x) \sin \lambda x\, dx = \frac{\lambda}{\lambda^2 + k^2}$.

## Answers

**2.** (*i*) $\frac{1}{2}\left[\frac{\sin(1-s)a}{1-s} - \frac{\sin(1+s)a}{1+s}\right]$ (*ii*) $\frac{a \cos sa - b \cos sb}{s} + \frac{\sin sb - \sin sa}{s^2}$

**3.** (*i*) $\frac{6+3s^2}{4+5s^2+s^4}$ (*ii*) $\sqrt{\frac{\pi}{2}}\, e^{-\frac{s^2}{2}}$

(*iii*) $\frac{2\cos s(1-\cos s)}{s^2}$ (*iv*) $10\left(\frac{1}{s^2+25} + \frac{1}{s^2+4}\right)$

(*v*) $\frac{1}{s^2}\left(2\cos\frac{s}{2} - 1 - \cos s\right)$ (*vi*) $\frac{2}{\pi s^2}$

**4.** (*i*) $\frac{1-\cos as}{s}, \frac{\sin as}{s}$ (*ii*) $\frac{s}{s^2+a^2}, \frac{a}{s^2+a^2}$

**5.** (*i*) $2i\left(\frac{\sin as - as\cos as}{s^2}\right)$ (*ii*) $\frac{2}{s^3}\,[(a^2s^2 - 2)\sin as + 2as\cos as]$

(*iii*) $\frac{2}{1+s^2}$ (*iv*) $\sqrt{2\pi}\, e^{-\frac{s^2}{2}}$

(*v*) $\frac{i}{(k+s)}\,[e^{i(k+s)a} - e^{i(k+s)b}]$ (*vi*) $\frac{i}{s}(e^{is\alpha} - e^{is\beta})$

**6.** (*i*) $\frac{2\sin s}{s};\ \frac{\pi}{2}$ (*ii*) $\frac{2}{as^2}(1 - \cos as)$

(*iii*) $\frac{2}{s}\sin s(1 + 2\cos s)$

**7.** (*i*) $f(x) = \frac{2}{\pi}\tan^{-1}\frac{x}{a}$ (*ii*) $f(x) = e^{-x}$

**8.** (*i*) $-\frac{32}{n\pi}\cos n\pi = -\frac{32(-1)^n}{n\pi};\ \frac{32[(-1)^n - 1]}{n^2\pi^2},\ F_c(0) = 16$

(*ii*) $\frac{2l^3}{n^3\pi^3}[1 - (-1)^n];\ -\frac{l^3}{n^2\pi^2}[(-1)^n + 1],\ F_c(0) = \frac{l^3}{6}$

(*iii*) $\frac{16}{n^3\pi^3}[(-1)^n - 1] - \frac{8}{n\pi}(-1)^n;\ \frac{16}{n^2\pi^2}(-1)^n,\ F_c(0) = \frac{8}{3}$

(*iv*) $\frac{al}{n\pi};\ \frac{al}{n^2\pi^2}[1 - (-1)^n],\ F_c(0) = \frac{al}{2}$

(*v*) $\frac{2k}{n^2}\sin\frac{n\pi}{2};\ \frac{k}{n^2}\left[2\cos\frac{n\pi}{2} - (-1)^n - 1\right],\ F_c(0) = \frac{k\pi^2}{4}$

9. (*i*) $\frac{n}{n^2-1}[1+(-1)^n]$ (*ii*) $\frac{1}{n^2}\sin\frac{n\pi}{3}$

11. (*i*) $-\frac{1}{n^2-1}[1+(-1)^n], F_c(0)=2$ (*ii*) $\frac{kl^2}{n^2\pi^2}\left[2\cos\frac{n\pi}{2}-1-(-1)^n\right], F_c(0)=\frac{kl^2}{4}$

12. (*i*) $f(x)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin^2\frac{n\pi}{4}\sin\frac{n\pi x}{l}$ (*ii*) $f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin nx$

(*iii*) $f(x)=\frac{4l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}(1-\cos n\pi)\sin\frac{n\pi x}{l}$ (*iv*) $f(x)=\frac{ac}{l}+\frac{2c}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi a}{l}\cos\frac{n\pi x}{l}$

(*v*) $f(x)=\frac{l^2}{6}-\frac{2l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(1+\cos n\pi)\cos\frac{n\pi x}{l}$

(*vi*) $f(x)=1+2\sum_{n=1}^{\infty}\frac{\cos\frac{2n\pi}{3}}{(2n+1)^2}\cos n\pi x$

13. (*i*) $f(x)=\frac{2}{\pi x}(1+\cos x-2\cos 2x)$ (*ii*) $f(x)=\frac{2}{\pi(1+x^2)}$

(*iii*) $f(x)=\frac{2}{\pi}\left(\frac{x-\sin x}{x^2}\right)$ (*iv*) $f(x)=e^{-kx}, x>0.$

## 2.6. PROPERTIES OF FOURIER TRANSFORMS

1. **Linearity Property.** *If F(s) and G(s) are Fourier transforms of f(x) and g(x) respectively, then*

$$\mathbf{F[af(x)+bg(x)]=aF(s)+bG(s)}$$

*where a and b are constants.*

**Proof.** By definition of Fourier transform, we have

$$F(s)=F[f(x)]=\int_{-\infty}^{\infty}f(x)\,.\,e^{isx}\,dx$$

and

$$G(s)=F[g(x)]=\int_{-\infty}^{\infty}g(x)\,.\,e^{isx}\,dx$$

$$\therefore\quad F[af(x)+bg(x)]=\int_{-\infty}^{\infty}[af(x)+bg(x)]\,e^{isx}dx$$

$$=a\int_{-\infty}^{\infty}f(x)\,.\,e^{isx}dx+b\int_{-\infty}^{\infty}g(x)\,.\,e^{isx}\,dx$$

$$=aF(s)+bG(s).$$

**Cor.** (*i*) If $F_s(s)$ and $G_s(s)$ are the Fourier sine transforms of $f(x)$ and $g(x)$ respectively, then

$$F_s[af(x)+bg(x)]=aF_s(s)+bG_s(s)$$

where $a$ and $b$ are constants.

(*ii*) If $F_c(s)$ and $G_c(s)$ are the Fourier cosine transforms of $f(x)$ and $g(x)$ respectively, then

$$F_c[af(x)+bg(x)]=aF_c(s)+bG_c(s)$$

where $a$ and $b$ are constants.



**2. Change of scale property** (Similarity Theorem). *If F(s) is the complex Fourier transform of f(x), then*

$$\mathbf{F[f(ax)]=\frac{1}{a}F\left(\frac{s}{a}\right),\ a\neq 0.}\qquad\text{(Anna 2006, 2007)}$$

**Proof.** By definition of complex Fourier transform, we have

$$F(s)=F[f(x)]=\int_{-\infty}^{\infty}f(x)\,.\,e^{isx}\,dx\qquad\ldots(1)$$

$$\therefore\quad F[f(ax)]=\int_{-\infty}^{\infty}f(ax)\,.\,e^{isx}\,dx\qquad\ldots(2)$$

Putting $ax=t$ *i.e.,* $x=\frac{t}{a}$, we have $dx=\frac{dt}{a}$.

When $x\to-\infty, t\to-\infty$ and when $x\to\infty, t\to\infty$

$\therefore$ From (2) we have

$$F[f(ax)]=\int_{-\infty}^{\infty}f(t)\,e^{ist/a}\,.\,\frac{dt}{a}$$

$$=\frac{1}{a}\int_{-\infty}^{\infty}f(t)\,e^{i(s/a)t}\,dt=\frac{1}{a}F\left(\frac{s}{a}\right)\qquad\text{[by (2)]}$$

**Cor.** (*i*) If $F_s(s)$ is the Fourier sine transform of $f(x)$, then

$$F_s[f(ax)]=\frac{1}{a}F_s\left(\frac{s}{a}\right).$$

(*ii*) If $F_c(s)$ is the Fourier cosine transform of $f(x)$, then

$$F_c[f(ax)]=\frac{1}{a}F_c\left(\frac{s}{a}\right).$$

**3. Shifting Property.** *If F(s) is the complex Fourier transform of f(x), then*

$$\mathbf{F[f(x-a)]=e^{isa}F(s).}\qquad\text{(Anna 2007)}$$

**Proof.** By definition of complex Fourier transform, we have

$$F(s)=F[f(x)]=\int_{-\infty}^{\infty}f(x)\,.\,e^{isx}\,dx$$

$$\therefore\quad F[f(x-a)]=\int_{-\infty}^{\infty}f(x-a)\,e^{isx}\,dx\qquad\ldots(1)$$

Putting $x-a=t$ *i.e.,* $x=a+t$, we have $dx=dt$

When $x\to-\infty, t\to-\infty$ and when $x\to\infty, t\to\infty$

$\therefore$ From (1) we have

$$F[f(x-a)]=\int_{-\infty}^{\infty}f(t)\,e^{is(a+t)}\,dt=\int_{-\infty}^{\infty}f(t)\,.\,e^{isa}\,.\,e^{ist}\,dt$$

$$=e^{isa}\int_{-\infty}^{\infty}f(t)\,.\,e^{ist}\,dt=e^{isa}\,F(s).$$

**3. (a) Shifting on time axis.** *If F(s) is the complex Fourier transform of f(t) and $t_0$ is any real number, then* (M.D.U. May 2009)

$$\mathbf{F[f(t-t_0)] = e^{ist_0}F(s).}$$

**Proof.** By definition of complex Fourier transform, we have

$$F(s) = F[f(t)] = \int_{-\infty}^{\infty} f(t)\,.\,e^{ist}\,dt \qquad ...(1)$$

$$\therefore \qquad F[f(t-t_0)] = \int_{-\infty}^{\infty} f(t-t_0)\,e^{ist}\,dt$$

Putting $t - t_0 = T$ *i.e.,* $t = t_0 + T$, we have $dt = dT$

When $t \to -\infty$, $T \to -\infty$ and when $t \to \infty$, $T \to \infty$

$\therefore$ From (1), we have

$$F[f(t-t_0)] = \int_{-\infty}^{\infty} f(T)\,e^{is(t_0+T)}\,dT$$

$$= \int_{-\infty}^{\infty} f(T)\,e^{ist_0}\,.\,e^{isT}\,dT = e^{ist_0}\int_{-\infty}^{\infty} f(T)\,e^{isT}\,dT$$

$$= e^{ist_0}\,F(s).$$

**Remark.** Inverse Fourier transform of $e^{ist_0}\,F(s)$ is $f(t-t_0)$.

**3. (b) Shifting on frequency axis.** *If F(s) is the complex Fourier transform of f(t), and $s_0$ is any real number, then* (M.D.U. May 2009)

$$\mathbf{F[e^{is_0t}\,f(t)] = F(s+s_0).}$$

**Proof.** By definition of complex Fourier transform, we have

$$F(s) = F[f(t)] = \int_{-\infty}^{\infty} f(t)\,.\,e^{ist}\,dt$$

$$\therefore \qquad F[e^{is_0t}\,f(t)] = \int_{-\infty}^{\infty} e^{is_0t}\,f(t)\,.\,e^{ist}\,dt$$

$$= \int_{-\infty}^{\infty} f(t)\,.\,e^{i(s+s_0)t}\,dt = F(s+s_0).$$

**Remark.** Inverse Fourier transform of $F(s+s_0)$ is $e^{is_0t}f(t)$.

**4. Modulation Theorem.** *If F(s) is the complex Fourier transform of f(x), then*

$$\mathbf{F[f(x)\cos ax] = \frac{1}{2}[F(s+a) + F(s-a)].}$$

**Proof.** By definition of complex Fourier transform, we have

$$F(s) = F[f(x)] = \int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$$



$$\therefore \qquad F[f(x)\cos ax] = \int_{-\infty}^{\infty} f(x)\cos ax\,.\,e^{isx}\,dx$$

$$= \int_{-\infty}^{\infty} f(x)\left(\frac{e^{iax}+e^{-iax}}{2}\right)e^{isx}\,dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(x)\,e^{i(s+a)x}\,dx + \int_{-\infty}^{\infty} f(x)\,.\,e^{i(s-a)x}\,dx\right]$$

$$= \frac{1}{2}\,[F(s+a) + F(s-a)].$$

**Cor.** If $F_s(s)$ and $F_c(s)$ are Fourier sine and cosine transforms of $f(x)$ respectively, then

(*i*) $F_s[f(x)\cos ax] = \dfrac{1}{2}[F_s(s+a) + F_s(s-a)]$

(*ii*) $F_s[f(x)\sin ax] = \dfrac{1}{2}[F_c(s-a) - F_c(s+a)]$

(*iii*) $F_c[f(x)\cos ax] = \dfrac{1}{2}[F_c(s+a) + F_c(s-a)]$

(*iv*) $F_c[f(x)\sin ax] = \dfrac{1}{2}[F_s(s+a) - F_s(s-a)]$.

**5. If $F_s(s)$ and $F_c(s)$ are Fourier sine and cosine transforms of f(x) respectively, then**

(*i*) $$\mathbf{F_s\{x\,f(x)\} = -\frac{d}{ds}\{F_c(s)\}}$$

(*ii*) $$\mathbf{F_c\{x\,f(x)\} = \frac{d}{ds}\{F_s(s)\}}$$

**Proof.** (*i*) $$\frac{d}{ds}\{F_c(s)\} = \frac{d}{ds}\left\{\int_0^{\infty} f(x)\cos sx\,dx\right\}$$

$$= \int_0^{\infty} f(x)\,(-x\sin sx)\,dx$$

$$= -\int_0^{\infty} \{x\,f(x)\}\sin sx\,dx$$

$$= -F_s\{x\,f(x)\}$$

$$\Rightarrow \qquad F_s\{x\,f(x)\} = -\frac{d}{ds}\{F_c(s)\}$$

(*ii*) $$\frac{d}{ds}\{F_s(s)\} = \frac{d}{ds}\left\{\int_0^{\infty} f(x)\sin sx\,dx\right\}$$

$$= \int_0^{\infty} f(x)\,(x\cos sx)\,dx$$

$$= \int_0^\infty [x f(x)] \cos sx \, dx$$
$$= F_c [x f(x)]$$
$$\Rightarrow \quad F_c [x f(x)] = \frac{d}{ds} [F_s (s)].$$

**Example 1.** *Find the Fourier transform of $e^{-x^2}$. Hence find the Fourier transform of*

(i) $e^{-ax^2}$, $(a > 0)$ (M.D.U. May 2009) (ii) $e^{-\frac{x^2}{2}}$

(iii) $e^{-4(x-3)^2}$ (iv) $e^{-x^2} \cos 2x$

**Sol.** Fourier transform of $f(x) = e^{-x^2}$ is given by

$$F[f(x)] = \int_{-\infty}^{\infty} f(x) \cdot e^{isx} \, dx = \int_{-\infty}^{\infty} e^{-x^2} \cdot e^{isx} dx$$

$$= \int_{-\infty}^{\infty} e^{-(x^2 - isx)} \, dx = \int_{-\infty}^{\infty} e^{-\left\{\left(x - \frac{is}{2}\right)^2 + \frac{s^2}{4}\right\}} dx$$

$$= e^{-\frac{s^2}{4}} \int_{-\infty}^{\infty} e^{-\left(x - \frac{is}{2}\right)^2} dx$$

$$= e^{-\frac{s^2}{4}} \int_{-\infty}^{\infty} e^{-z^2} dz \quad \text{where } z = x - \frac{is}{2}$$

$$= 2e^{-\frac{s^2}{4}} \int_0^{\infty} e^{-z^2} dz = 2e^{-\frac{s^2}{4}} \cdot \frac{\sqrt{\pi}}{2}$$

$$= \sqrt{\pi} \, e^{-\frac{s^2}{4}} = F(s)$$

(i) $$e^{-ax^2} = e^{-(\sqrt{a}x)^2} = f(\sqrt{a}x)$$

By change of scale property, we have

$$F[f(\sqrt{a}x)] = \frac{1}{\sqrt{a}} F\left(\frac{s}{\sqrt{a}}\right)$$

$$\Rightarrow \quad F(e^{-ax^2}) = \frac{1}{\sqrt{a}} \sqrt{\pi} \, e^{-\frac{1}{4}\left(\frac{s}{\sqrt{a}}\right)^2} = \sqrt{\frac{\pi}{a}} \, e^{-\frac{s^2}{4a}}$$

(ii) Putting $a = \frac{1}{2}$ in deduction (i), we have

$$F\left(e^{-\frac{x^2}{2}}\right) = \sqrt{2\pi} \, e^{-s^2/2}$$



(iii) $e^{-4x^2} = e^{-(2x)^2} = f(2x)$

By change of scale property, we have

$$F[f(2x)] = \frac{1}{2} F\left(\frac{s}{2}\right)$$

$$\Rightarrow \quad F[e^{-4x^2}] = \frac{\sqrt{\pi}}{2} e^{-\frac{1}{4}\left(\frac{s}{2}\right)^2} = \frac{\sqrt{\pi}}{2} e^{-\frac{s^2}{16}}$$

By shifting property $F[f(x - a)] = e^{isa} F(s)$

$$\therefore \quad F[e^{-4(x-3)^2}] = e^{3is} \cdot \frac{\sqrt{\pi}}{2} e^{-\frac{s^2}{16}} = \frac{\sqrt{\pi}}{2} e^{\left(3is - \frac{s^2}{16}\right)}$$

(iv) By modulation theorem,

$$F[f(x) \cos ax] = \frac{1}{2} [F(s + a) + F(s - a)]$$

$$\therefore \quad F[e^{-x^2} \cos 2x] = \frac{1}{2}\left[\sqrt{\pi} \, e^{-\frac{1}{4}(s+2)^2} + \sqrt{\pi} \, e^{-\frac{1}{4}(s-2)^2}\right]$$

$$= \frac{\sqrt{\pi}}{2}\left[e^{-\frac{1}{4}(s+2)^2} + e^{-\frac{1}{4}(s-2)^2}\right].$$

**Example 2.** *Find the Fourier sine and cosine transform of $x \, e^{-ax}$.* (Madras 2006)

**Sol.** Let us first find the Fourier sine and cosine transforms of $e^{-ax}$.

$$F_s (e^{-ax}) = \int_0^\infty e^{-ax} \sin sx \, dx = \left[\frac{e^{-ax}}{a^2 + s^2}(-a \sin sx - s \cos sx)\right]_0^\infty = \frac{s}{a^2 + s^2}$$

and $$F_c (e^{-ax}) = \int_0^\infty e^{-ax} \cos sx \, dx = \left[\frac{e^{-ax}}{a^2 + s^2}(-a \cos sx + s \sin sx)\right]_0^\infty = \frac{a}{a^2 + s^2}$$

$$\therefore \quad F_s (xe^{-ax}) = -\frac{d}{ds} [F_c (e^{-ax})] = -\frac{d}{ds}\left(\frac{a}{a^2 + s^2}\right) = \frac{2as}{(a^2 + s^2)^2}$$

and $$F_c(xe^{-ax}) = \frac{d}{ds}\left\{F_s (e^{-ax})\right\} = \frac{d}{ds}\left(\frac{s}{a^2 + s^2}\right)$$

$$= \frac{(a^2 + s^2) \cdot 1 - s(2s)}{(a^2 + s^2)^2} = \frac{a^2 - s^2}{(a^2 + s^2)^2}.$$

## 2.7. CONVOLUTION

The convolution of two functions $f(x)$ and $g(x)$ over the interval $(-\infty, \infty)$ is defined as

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u) \, g(x - u) \, du.$$

## 2.8. CONVOLUTION THEOREM FOR FOURIER TRANSFORMS (or Falting Theorem)

(Anna 2007; M.D.U. 2007, 2008, Dec. 2010, Dec. 2011; K.U.K. 2009, Dec. 2010)

*The Fourier transform of the convolution of f(x) and g(x) is equal to the product of the Fourier transforms of f(x) and g(x), i.e.,*

$$\mathbf{F[f(x) * g(x)] = F[f(x)].\ F[g(x)] = F(s)G(s).}$$

**Proof.** By definition of Fourier transform

$$F(s) = F[f(x)] = \int_{-\infty}^{\infty} f(x) . e^{isx}\, dx,$$

$$G(s) = F[g(x)] = \int_{-\infty}^{\infty} g(x) . e^{isx}\, dx$$

and by definition of convolution,

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

$$\therefore \quad F[f(x) * g(x)] = \int_{-\infty}^{\infty} [f(x) * g(x)]\, e^{isx} dx$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du\right] e^{isx} dx$$

$$= \int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} g(x-u)\, e^{isx}\, dx\right] du$$

(By changing the order of integration)

Putting $x - u = t$ *i.e.*, $x = u + t$, we have $dx = dt$.

When $x \to -\infty$, $t \to -\infty$ and when $x \to \infty$, $t \to \infty$

$$\therefore \quad F[f(x) * g(x)] = \int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} g(t)\, e^{is(u+t)}\, dt\right] du$$

$$= \int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} g(t) . e^{isu} . e^{ist}\, dt\right] du$$

$$= \left[\int_{-\infty}^{\infty} f(u)\, e^{isu}\, du\right]\left[\int_{-\infty}^{\infty} g(t)\, e^{ist}\, dt\right]$$

$$= \left[\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx\right]\left[\int_{-\infty}^{\infty} g(x) . e^{isx}\, dx\right]$$

$$= F[f(x)]\ F[g(x)] = F(s)\ G(s).$$

**Remark.** The following properties of convolution can be easily proved.

(*i*) $f(x) * g(x) = g(x) * f(x)$

(*ii*) $f(x) * [g(x) * h(x)] = [f(x) * g(x)] * h(x)$

(*iii*) $f(x) * [g(x) + h(x)] = f(x) * g(x) + f(x) * h(x)$



## 2.9. RELATION BETWEEN FOURIER AND LAPLACE TRANSFORMS

(P.T.U. 2005)

*If* $\quad f(t) = \begin{cases} e^{-xt} g(t), & t > 0 \\ 0, & t < 0 \end{cases}$, *then* $F[f(t)] = L[g(t)]$.

**Proof.** By definition of Fourier transform, we have

$$F[f(t)] = \int_{-\infty}^{\infty} f(t) e^{ist}\, dt$$

$$= \int_{-\infty}^{0} 0.e^{ist}\, dt + \int_{0}^{\infty} e^{-xt} g(t) . e^{ist}\, dt \qquad \text{[By def. of } f(t)\text{]}$$

$$= \int_{0}^{\infty} e^{-(x-is)t} g(t)\, dt$$

$$= \int_{0}^{\infty} e^{-pt} g(t)\, dt, \quad \text{where } p = x - is$$

$$= L[g(t)].$$

## 2.10. PARSEVAL'S IDENTITY FOR FOURIER TRANSFORMS

*If the Fourier transforms of f(x) and g(x) are F(s) and G(s) respectively, then*

(*i*) $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} F(s)\, \overline{G}(s)\, ds = \int_{-\infty}^{\infty} f(x)\, \overline{g}(x)\, dx$

(*ii*) $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$.

*where bar stands for the complex conjugate.*

**Proof.** (*i*) By definition of Fourier transform

$$F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x) . e^{isx}\, dx$$

and $$G(s) = F\{g(x)\} = \int_{-\infty}^{\infty} g(x) . e^{isx}\, dx$$

By inversion formula for Fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s) . e^{-isx}\, ds$$

and $$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(s) . e^{-isx}\, ds$$

$$\therefore \quad \int_{-\infty}^{\infty} f(x)\, \overline{g}(x)\, dx = \int_{-\infty}^{\infty} f(x)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G}(s) . e^{isx}\, ds\right] dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G}(s)\left[\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx\right] ds$$

(Changing the order of integration)

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G}(s)\, [F(s)]\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{F}(s)\,\overline{\mathrm{G}}(s)\,ds.$$

(*ii*) By definition of Fourier transform

$$\mathrm{F}(s) = \mathrm{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$$

By inversion formula for Fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{F}(s)\,e^{-isx}\,ds$$

$$\therefore \quad \int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} f(x)\,\bar{f}(x)\,dx$$

$$= \int_{-\infty}^{\infty} f(x)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\mathrm{F}}(s)\,e^{isx}\,ds\right]dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\mathrm{F}}(s)\left[\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx\right]ds$$

(Changing the order of integration)

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\mathrm{F}}(s)\,[\mathrm{F}(s)]\,ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathrm{F}(s)|^2\,ds.$$

**Remark.** The following Parseval's identities for Fourier sine and cosine transforms can be easily proved:

(*i*) $\dfrac{2}{\pi}\displaystyle\int_0^{\infty} \mathrm{F}_s(s)\,\mathrm{G}_s(s)\,ds = \int_0^{\infty} f(x)\,g(x)\,dx$

(*ii*) $\dfrac{2}{\pi}\displaystyle\int_0^{\infty} [\mathrm{F}_s(s)]^2\,ds = \int_0^{\infty} [f(x)]^2\,dx$

(*iii*) $\dfrac{2}{\pi}\displaystyle\int_0^{\infty} \mathrm{F}_c(s)\,\mathrm{G}_c(s)\,ds = \int_0^{\infty} f(x)\,g(x)\,dx$

(*iv*) $\dfrac{2}{\pi}\displaystyle\int_0^{\infty} [\mathrm{F}_c(s)]^2\,ds = \int_0^{\infty} [f(x)]^2\,dx.$

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Using Parseval's identities, prove that*

(*i*) $\displaystyle\int_0^{\infty} \frac{dt}{(4+t^2)(9+t^2)} = \frac{\pi}{60}$ (*ii*) $\displaystyle\int_0^{\infty} \frac{t^2}{(4+t^2)(9+t^2)}\,dt = \frac{\pi}{10}$

(*iii*) $\displaystyle\int_0^{\infty} \frac{dt}{(4+t^2)^2} = \frac{\pi}{32}$ (*iv*) $\displaystyle\int_0^{\infty} \frac{t^2}{(9+t^2)^2}\,dt = \frac{\pi}{12}.$

**Sol.** Let $f(x) = e^{-2x}$ and $g(x) = e^{-3x}$



Then $\quad \mathrm{F}_c(s) = \dfrac{2}{4+s^2}, \quad \mathrm{G}_c(s) = \dfrac{3}{9+s^2}$

$$\mathrm{F}_s(s) = \frac{s}{4+s^2}, \quad \mathrm{G}_s(s) = \frac{s}{9+s^2}$$

$$\left[\because\ f(x) = e^{-ax} \Rightarrow \mathrm{F}_c(s) = \frac{a}{a^2+s^2},\ \mathrm{F}_s(s) = \frac{s}{a^2+s^2}\right]$$

(*i*) Using Parseval's identity for Fourier cosine transforms, *i.e.*,

$$\frac{2}{\pi}\int_0^{\infty} \mathrm{F}_c(s)\,\mathrm{G}_c(s)\,ds = \int_0^{\infty} f(x)\,g(x)\,dx$$

We have $\quad \dfrac{2}{\pi}\displaystyle\int_0^{\infty}\left(\frac{2}{4+s^2}\right)\left(\frac{3}{9+s^2}\right)ds = \int_0^{\infty} e^{-2x}\,.\,e^{-3x}\,dx$

$$\Rightarrow \quad \frac{12}{\pi}\int_0^{\infty} \frac{ds}{(4+s^2)(9+s^2)} = \int_0^{\infty} e^{-5x}\,dx$$

$$= \left[\frac{e^{-5x}}{-5}\right]_0^{\infty} = -\frac{1}{5}(0-1) = \frac{1}{5}$$

$$\Rightarrow \quad \int_0^{\infty} \frac{ds}{(4+s^2)(9+s^2)} = \frac{\pi}{60}$$

$$\therefore \quad \int_0^{\infty} \frac{dt}{(4+t^2)(9+t^2)} = \frac{\pi}{60}.$$

(*ii*) Using Parseval's identity for fourier sine transforms, *i.e.*,

$$\frac{2}{\pi}\int_0^{\infty} \mathrm{F}_s(s)\,\mathrm{G}_s(s)\,ds = \int_0^{\infty} f(x)\,g(x)\,dx$$

We have $\quad \dfrac{2}{\pi}\displaystyle\int_0^{\infty}\left(\frac{s}{4+s^2}\right)\left(\frac{s}{9+s^2}\right)ds = \int_0^{\infty} e^{-2x}\,.\,e^{-3x}\,dx$

$$\Rightarrow \quad \frac{2}{\pi}\int_0^{\infty} \frac{s^2}{(4+s^2)(9+s^2)}\,ds = \int_0^{\infty} e^{-5x}\,dx$$

$$= \left[\frac{e^{-5x}}{-5}\right]_0^{\infty} = -\frac{1}{5}(0-1) = \frac{1}{5}$$

$$\Rightarrow \quad \int_0^{\infty} \frac{s^2}{(4+s^2)(9+s^2)}\,ds = \frac{\pi}{10}$$

$$\therefore \quad \int_0^{\infty} \frac{t^2}{(4+t^2)(9+t^2)}\,dt = \frac{\pi}{10}.$$

(*iii*) Using Parseval's identity for Fourier cosine transform, *i.e.*,

$$\frac{2}{\pi}\int_0^{\infty} [F_c(s)]^2\,ds = \int_0^{\infty} [f(x)]^2\,dx$$

We have $\quad \dfrac{2}{\pi}\displaystyle\int_0^{\infty}\left(\dfrac{2}{4+s^2}\right)^2 ds = \int_0^{\infty}\left(e^{-2x}\right)^2 dx$

$$\Rightarrow \quad \frac{8}{\pi}\int_0^{\infty}\frac{ds}{(4+s^2)^2} = \int_0^{\infty} e^{-4x}\, dx$$

$$= \left[\frac{e^{-4x}}{-4}\right]_0^{\infty} = -\frac{1}{4}(0-1) = \frac{1}{4}$$

$$\Rightarrow \quad \int_0^{\infty}\frac{ds}{(4+s^2)^2} = \frac{\pi}{32}$$

$$\therefore \quad \int_0^{\infty}\frac{dt}{(4+t^2)^2} = \frac{\pi}{32}.$$

(*iv*) Using Parseval's identity for Fourier sine transform, *i.e.*,

$$\frac{2}{\pi}\int_0^{\infty}[G_s(s)]^2\, ds = \int_0^{\infty}[g(x)]^2\, dx$$

We have $\quad \dfrac{2}{\pi}\displaystyle\int_0^{\infty}\left(\dfrac{s}{9+s^2}\right)^2 ds = \int_0^{\infty}(e^{-3x})^2\, dx$

$$\Rightarrow \quad \frac{2}{\pi}\int_0^{\infty}\frac{s^2}{(9+s^2)^2}\, ds = \int_0^{\infty} e^{-6x}\, dx$$

$$= \left[\frac{e^{-6x}}{-6}\right]_0^{\infty} = -\frac{1}{6}(0-1) = \frac{1}{6}$$

$$\Rightarrow \quad \int_0^{\infty}\frac{s^2}{(9+s^2)^2}\, ds = \frac{\pi}{12}$$

$$\therefore \quad \int_0^{\infty}\frac{t^2}{(9+t^2)^2}\, dt = \frac{\pi}{12}.$$

**Example 2.** *Using Parseval's identity, prove that*

$$\int_0^{\infty}\frac{\sin at}{t(a^2+t^2)}\, dt = \frac{\pi}{2a^2}(1-e^{-a^2}).$$

**Sol.** Let $\quad f(x) = e^{-ax},\ a>0 \quad \text{and} \quad g(x) = \begin{cases}1, & 0<x<a\\ 0, & x>a\end{cases}$

then $\quad F_c(s) = \dfrac{a}{a^2+s^2}$ and $G_c(s) = \dfrac{\sin as}{s}$.

Using Parseval's identity for Fourier cosine transforms, *i.e.*,

$$\frac{2}{\pi}\int_0^{\infty} F_c(s)\, G_c(s)\, ds = \int_0^{\infty} f(x)\, g(x)\, dx$$

We have $\quad \dfrac{2}{\pi}\displaystyle\int_0^{\infty}\left(\dfrac{a}{a^2+s^2}\right)\left(\dfrac{\sin as}{s}\right) ds = \int_0^{\infty} e^{-ax}\,.\, g(x)\, dx$



$$\Rightarrow \quad \frac{2a}{\pi}\int_0^{\infty}\frac{\sin as}{s(a^2+s^2)}\, ds = \int_0^{a} e^{-ax}\,.\, g(x)\, dx + \int_a^{\infty} e^{-ax}\,.\, g(x)\, dx$$

$$= \int_0^{a} e^{-ax}\,.\,1\, dx + \int_a^{\infty} e^{-ax}\,.\,0\, dx$$

$$= \left[\frac{e^{-ax}}{-a}\right]_0^{a} + 0 = -\frac{1}{a}(e^{-a^2}-1) = \frac{1}{a}(1-e^{-a^2})$$

$$\Rightarrow \quad \int_0^{\infty}\frac{\sin as}{s(a^2+s^2)}\, ds = \frac{\pi}{2a^2}(1-e^{-a^2})$$

$$\therefore \quad \int_0^{\infty}\frac{\sin at}{t(a^2+t^2)}\, dt = \frac{\pi}{2a^2}(1-e^{-a^2}).$$

**Example 3.** *Find the Fourier transform of*

$$f(x) = \begin{cases}1-|x|, & |x|<1\\ 0, & |x|>1\end{cases}$$

*and hence find the value of* $\displaystyle\int_0^{\infty}\frac{\sin^4 t}{t^4}\, dt.$ (M.D.U., Dec. 2010)

**Sol.** Fourier transform of $f(x)$ is given by

$$F(f(x)) = \int_{-\infty}^{\infty} f(x)e^{isx}\, dx$$

$$= \int_{-\infty}^{-1} f(x)e^{isx}\, dx + \int_{-1}^{1} f(x)\, e^{isx}\, dx + \int_1^{\infty} f(x)\, e^{isx}\, dx$$

$$= \int_{-\infty}^{-1} 0\, dx + \int_{-1}^{1}(1-|x|)\, e^{isx}\, dx + \int_1^{\infty} 0\, dx$$

$$= \int_{-1}^{1}(1-|x|)(\cos sx + i\sin sx)\, dx$$

$$= \int_{-1}^{1}(1-|x|)\cos sx\, dx + \int_{-1}^{1}(1-|x|)(i\sin sx)\, dx$$

$$= 2\int_0^{1}(1-|x|)\cos sx\, dx + 0$$

[$\because$ $(1-|x|)\cos sx$ is an even function of $x$ where as $(1-|x|)\sin sx$ is an odd function of $x$]

$$= 2\int_0^{1}(1-x)\cos sx\, dx \qquad [\because\ |x| = x \text{ where } x>0]$$

$$= 2\left[(1-x)\,.\,\frac{\sin sx}{s} - (-1)\left(-\frac{\cos sx}{s^2}\right)\right]_0^1$$

$$= 2\left[-\frac{\cos s}{s^2} + \frac{1}{s^2}\right] = 2\left(\frac{1-\cos s}{s^2}\right) = F(s)$$

Using Parseval's identity for Fourier transform.

*i.e.*,
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

We have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4(1-\cos s)^2}{s^4}\, ds = \int_{-1}^{1} (1-|x|)^2\, dx$$

$$\Rightarrow \quad \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{(1-\cos s)^2}{s^4}\, ds = \int_{-1}^{1} (1-|x|)^2\, dx$$

$$\Rightarrow \quad \frac{4}{\pi}\int_{0}^{\infty} \frac{(1-\cos s)^2}{s^4}\, ds = 2\int_{0}^{1} (1-|x|)^2\, dx$$

($\because$ Both integrands are even functions)

$$\Rightarrow \quad \frac{4}{\pi}\int_{0}^{\infty} \frac{(2\sin^2 s/2)^2}{s^4}\, ds = 2\int_{0}^{1} (1-x)^2\, dx$$

$$\Rightarrow \quad \frac{16}{\pi}\int_{0}^{\infty} \frac{\sin^4 s/2}{s^4}\, ds = 2\left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{2}{3}$$

Putting $\frac{s}{2} = t$ *i.e.*, $s = 2t$, we have

$$\frac{16}{\pi}\int_{0}^{\infty} \frac{\sin^4 t}{16t^4}(2dt) = \frac{2}{3}$$

$$\Rightarrow \quad \int_{0}^{\infty} \frac{\sin^4 t}{t^4}\, dt = \frac{\pi}{3}.$$

## EXERCISE 2.3

1. Verify convolution theorem for $f(x) = g(x) = e^{-x^2}$. *(M.D.U. May 2011)*
2. Using Parseval's identities, prove that

   (*i*) $\int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$ (*ii*) $\int_0^{\infty} \frac{x^2}{(x^2+1)^2}\, dx = \frac{\pi}{4}$.

   [**Hint.** Use Parseval's identity for Fourier sine and cosine transforms of $f(x) = e^{-x}$]
3. Using Parseval's identity, show that $\int_0^{\infty} \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$. *(M.D.U. 2005)*

   [**Hint.** Use Parseval's identity for Fourier cosine transforms of $f(x) = e^{-ax}$, $g(x) = e^{-bx}$]



4. If $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ and $F(s) = \frac{2\sin as}{s}$, $(s \neq 0)$, then prove that $\int_0^{\infty} \frac{\sin^2 ax}{x^2}\, dx = \frac{\pi a}{2}$.
5. Using Parseval's identity, prove that

   (*i*) $\int_0^{\infty} \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}$ (*ii*) $\int_0^{\infty} \frac{1-\cos 2x}{x^2}\, dx = \pi$.
6. Using Parseval's identity, prove that

   (*i*) $\int_0^{\infty} \left(\frac{1-\cos x}{x}\right)^2 dx = \frac{\pi}{2}$ (*ii*) $\int_0^{\infty} \frac{\sin^4 t}{t^2}\, dt = \frac{\pi}{4}$.

## 2.11. FOURIER TRANSFORMS OF THE DERIVATIVES OF A FUNCTION

The Fourier transform of the function $u(x, t)$ is given by

$$F[u(x,t)] = \int_{-\infty}^{\infty} u e^{isx}\, dx$$

(*i*) **Fourier transform of** $\frac{\partial \mathbf{u}}{\partial \mathbf{x}}$

Suppose $u \to 0$ as $x \to \pm\infty$, then the Fourier transform of $\frac{\partial u}{\partial x}$ is given by

$$F\left[\frac{\partial u}{\partial x}\right] = \int_{-\infty}^{\infty} \frac{\partial u}{\partial x}.e^{isx}\, dx = \int_{-\infty}^{\infty} e^{isx}.\frac{\partial u}{\partial x}\, dx \qquad \text{(Integrating by parts)}$$

$$= \left[e^{isx}.u\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} ise^{isx}.u\, dx$$

$$= 0 - is\int_{-\infty}^{\infty} u.e^{isx}\, dx \qquad [\because\ u \to 0 \text{ as } x \to \pm\infty]$$

$$= -is\, F(u)$$

Hence $\quad F\left[\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right] = -\mathbf{is\, F(u)}.$

(*ii*) **Fourier transform of** $\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}$

Suppose $u$ and $\frac{\partial u}{\partial x} \to 0$ as $x \to \pm\infty$, then the Fourier transform of $\frac{\partial^2 u}{\partial x^2}$ is given by

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2}.e^{isx}\, dx = \int_{-\infty}^{\infty} e^{isx}.\frac{\partial^2 u}{\partial x^2}\, dx$$

(Applying general rule of integration by parts)

$$= \left[e^{isx}.\frac{\partial u}{\partial x} - ise^{isx}.u\right]_{-\infty}^{\infty} + (is)^2\int_{-\infty}^{\infty} e^{isx}.u\, dx$$

$$= 0 - s^2 \int_{-\infty}^{\infty} u \,.\, e^{isx}\, dx \qquad \left[\because \; u \text{ and } \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \pm\infty\right]$$

$$= - s^2 F(u).$$

Hence $\quad \mathbf{F}\left[\dfrac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = - \mathbf{s}^2 \mathbf{F(u)} = (- \mathbf{is})^2\, \mathbf{F(u)}.$

**(iii) Fourier transform of $\dfrac{\partial^n \mathbf{u}}{\partial \mathbf{x}^n}$**

Suppose $u, \dfrac{\partial u}{\partial x}, \dfrac{\partial^2 u}{\partial x^2}, \ldots\ldots, \dfrac{\partial^{n-1} u}{\partial x^{n-1}} \to 0$ as $x \to \pm\infty$, then the Fourier transform of $\dfrac{\partial^n u}{\partial x^n}$ is given by

$$F\left[\frac{\partial^n u}{\partial x^n}\right] = \int_{-\infty}^{\infty} \frac{\partial^n u}{\partial x^n} \,.\, e^{isx}\, dx = \int_{-\infty}^{\infty} e^{isx} \,.\, \frac{\partial^n u}{\partial x^n}\, dx$$

(Applying general rule of integration by parts)

$$= \left[e^{isx} \,.\, \frac{\partial^{n-1} u}{\partial x^{n-1}} - ise^{isx} \,.\, \frac{\partial^{n-2} u}{\partial x^{n-2}} + (is)^2\, e^{isx} \,.\, \frac{\partial^{n-3} u}{\partial x^{n-3}} - \ldots\ldots + (- is)^{n-1}\, u\right]_{-\infty}^{\infty}$$

$$+ (- is)^n \int_{-\infty}^{\infty} e^{isx} \,.\, u\, dx$$

$$= 0 + (- is)^n \int_{-\infty}^{\infty} u \,.\, e^{isx}\, dx = (- is)^n\, F(u)$$

Hence $\quad \mathbf{F}\left[\dfrac{\partial^n \mathbf{u}}{\partial \mathbf{x}^n}\right] = (- \mathbf{is})^n\, \mathbf{F(u)}.$

## 2.12. FOURIER SINE AND COSINE TRANSFORMS OF $\dfrac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}$

The Fourier sine and cosine transforms of the function $u(x, t)$ are given by

$$F_s[u(x, t)] = \int_0^{\infty} u \sin sx\, dx$$

and $\quad F_c[u(x, t)] = \displaystyle\int_0^{\infty} u \cos sx\, dx.$

**(i) Fourier sine transform of $\dfrac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}$**

Suppose $u$ and $\dfrac{\partial u}{\partial x} \to 0$ as $x \to \infty$, then the Fourier transform of $\dfrac{\partial^2 u}{\partial x^2}$ is given by

$$F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} \,.\, \sin sx\, dx = \int_0^{\infty} \sin sx \,.\, \frac{\partial^2 u}{\partial x^2}\, dx \qquad \text{(Integrating by parts)}$$



$$= \left[\sin sx \,.\, \frac{\partial u}{\partial x}\right]_0^{\infty} - \int_0^{\infty} (s \cos sx) \,.\, \frac{\partial u}{\partial x}\, dx$$

$$= 0 - s \int_0^{\infty} (\cos sx)\, \frac{\partial u}{\partial x}\, dx \qquad \left[\because \; \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty\right]$$

(Integrating again by parts)

$$= - s \left[\{\cos sx \,.\, u\}_0^{\infty} - \int_0^{\infty} (- s \sin sx) \,.\, u\, dx\right]$$

$$= - s \left[0 - (u)_{x=0} + s \int_0^{\infty} u \sin sx\, dx\right] \qquad [\because \; u \to 0 \text{ as } x \to \infty]$$

$$= s(u)_{x=0} - s^2 F_s(u)$$

Hence $\mathbf{F_s}\left[\dfrac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = \mathbf{s(u)_{x=0}} - \mathbf{s}^2 \mathbf{F_s(u)}.$

**(ii) Fourier cosine transform of $\dfrac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}$**

Suppose $u$ and $\dfrac{\partial u}{\partial x} \to 0$ as $x \to \infty$, then the Fourier transform of $\dfrac{\partial^2 u}{\partial x^2}$ is given by

$$F_c\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} \,.\, \cos sx\, dx = \int_0^{\infty} \cos sx \,.\, \frac{\partial^2 u}{\partial x^2}\, dx \qquad \text{(Integrating by parts)}$$

$$= \left[\cos sx \,.\, \frac{\partial u}{\partial x}\right]_0^{\infty} - \int_0^{\infty} - s \sin sx \,.\, \frac{\partial u}{\partial x}\, dx$$

$$= \left[0 - \left(\frac{\partial u}{\partial x}\right)_{x=0}\right] + s \int_0^{\infty} \sin sx\, \frac{\partial u}{\partial x}\, dx \qquad \left[\because \; \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty\right]$$

$$= - \left(\frac{\partial u}{\partial x}\right)_{x=0} + s \int_0^{\infty} \sin sx \,.\, \frac{\partial u}{\partial x}\, dx \qquad \text{(Integrating again by parts)}$$

$$= - \left(\frac{\partial u}{\partial x}\right)_{x=0} + s \left[\{\sin sx \,.\, u\}_0^{\infty} - \int_0^{\infty} s \cos sx \,.\, u\, dx\right]$$

$$= - \left(\frac{\partial u}{\partial x}\right)_{x=0} - s^2 \int_0^{\infty} u \cos sx\, dx \qquad [\because \; u \to 0 \text{ as } x \to \infty]$$

$$= - \left(\frac{\partial u}{\partial x}\right)_{x=0} - s^2\, F_c(u)$$

Hence $\mathbf{F_c}\left[\dfrac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = - \left(\dfrac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)_{\mathbf{x=0}} - \mathbf{s}^2 \mathbf{F_c(u)}.$

## 2.13. APPLICATION OF FOURIER TRANSFORMS TO BOUNDARY VALUE PROBLEMS

Fourier transforms are very useful in solving boundary value problems. We take Fourier transform of the given partial differential equation using given boundary and initial conditions. The required solution is then obtained by taking corresponding inverse transform. The choice of particular transform to be employed depends on the boundary conditions of the problem.

(*i*) *If the interval is* $-\infty < x < \infty$ and if boundary conditions are

$$u \text{ and } \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \pm\infty$$

use infinite Fourier transform.

(*ii*) *If the interval is* $0 < x < \infty$ and

(*a*) boundary conditions are $u$ and $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$ and $u(x, t) = 0$ or $f(t)$ at $x = 0$ and for all $t$, use Fourier sine transform.

(*b*) boundary conditions are $u$ and $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$ and $\frac{\partial u}{\partial x} = 0$ or $f(t)$ at $x = 0$ and for all $t$, use Fourier cosine transform.

For the interval $0 < x < \infty$, we always assume $u$ and $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$, even if it is not given in the problem.

(*iii*) *If the interval is* $0 < x < L$ and

(*a*) boundary conditions are $u(0, t) = u(L, t) = 0$ for all $t$, use finite Fourier sine transform.

(*b*) boundary conditions are $\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0$ for all $t$, use finite Fourier cosine transform.

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Solve the equation* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $x > 0, t > 0$

*subject to the conditions*

(*i*) $u = 0$, *when* $x = 0, t > 0$ (*ii*) $u = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$, *when* $t = 0$

*and* (*iii*) $u(x, t)$ *is bounded.*

**Sol.** Given $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, x > 0, t > 0 \quad ...(1)$$

Boundary condition is $u(0, t) = 0$



Initial conditions are $u(x, 0) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$ ...(2)

and $u(x, t)$ is bounded.

Since $u(0, t)$ is given, we take Fourier sine transform of both sides of (1). Thus

$$F_s\left(\frac{\partial u}{\partial t}\right) = F_s\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$\Rightarrow \int_0^\infty \frac{\partial u}{\partial t} \sin sx \, dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx \, dx. \text{ Integrating by parts}$$

$$\Rightarrow \frac{d}{dt}\int_0^\infty u \sin sx \, dx = \left[\sin sx \cdot \frac{\partial u}{\partial x}\right]_0^\infty - \int_0^\infty s \cos sx \cdot \frac{\partial u}{\partial x} \cdot dx$$

$$= 0 - s\int_0^\infty \cos sx \cdot \frac{\partial u}{\partial x} \cdot dx$$

$$\left[\because \ u(x, t) \text{ is bounded } \therefore \ \frac{\partial u}{\partial x} \to 0 \text{ when } x \to \infty\right]$$

$$= -s\left[\left\{\cos sx \cdot u\right\}_0^\infty - \int_0^\infty - s \sin sx \cdot u \, dx\right]$$

$$= -s\left[0 + s\int_0^\infty u \sin sx \, dx\right]$$

$$[\because \text{ when } x \to \infty, u \to 0 \text{ and when } x = 0, u = u(0, t) = 0]$$

$$\Rightarrow \frac{d}{dt}\int_0^\infty u \sin sx \, dx = -s^2 \int_0^\infty u \sin sx \, dx$$

$$\Rightarrow \frac{d\bar{u}_s}{dt} = -s^2 \bar{u}_s \quad \text{where } \bar{u}_s = \bar{u}_s(s, t) = F_s[u(x, t)]$$

Separating the variables,

$$\frac{d\bar{u}_s}{\bar{u}_s} = -s^2 dt$$

Integrating $\log \bar{u}_s = -s^2 t + \log c$

$$\Rightarrow \log_e\left(\frac{\bar{u}_s}{c}\right) = -s^2 t \text{ or } \bar{u}_s = ce^{-s^2 t} \quad ...(3)$$

Putting $t = 0$ in (3),

$$c = \bar{u}_s(s, 0) = F_s[u(x, 0)] = \int_0^\infty u(x, 0) \sin sx \, dx$$

$$= \int_0^1 u(x, 0) \sin sx \, dx + \int_1^\infty u(x, 0) \sin sx \, dx$$

$$= \int_0^1 1 . \sin sx \, dx + \int_1^\infty 0 . \sin sx \, dx \qquad \text{[from (2)]}$$

$$= \left[ -\frac{\cos sx}{s} \right]_0^1 = \frac{1 - \cos s}{s}$$

$\therefore$ From (3), $\quad \bar{u}_s (s, t) = \left( \frac{1 - \cos s}{s} \right) e^{-s^2 t}$

Taking its inverse Fourier sine transform, we get

$$u(x, t) = \frac{2}{\pi} \int_0^\infty \left( \frac{1 - \cos s}{s} \right) e^{-s^2 t} \sin sx \, ds$$

which is the required solution.

**Example 2.** *Solve* $\frac{\partial V}{\partial t} = K \frac{\partial^2 V}{\partial x^2}$ *for* $x > 0$, $t > 0$ *under the boundary conditions* $V = V_0$ *when* $x = 0$, $t > 0$ *and the initial condition* $V = 0$ *when* $t = 0$, $x > 0$.

**Sol.** Given $\quad \frac{\partial V}{\partial t} = K \frac{\partial^2 V}{\partial x^2}, \quad x > 0, t > 0 \qquad \ldots(1)$

Boundary condition is $\quad V(0, t) = V_0, t > 0$

Initial condition is $\quad V(x, 0) = 0, \; x > 0 \qquad \ldots(2)$

Since $V(0, t)$ is given, we take Fourier sine transform of both sides of (1). Thus

$$F_s \left( \frac{\partial V}{\partial t} \right) = F_s \left( K \frac{\partial^2 V}{\partial x^2} \right)$$

$$\Rightarrow \quad \int_0^\infty \frac{\partial V}{\partial t} \sin sx \, dx = K \int_0^\infty \frac{\partial^2 V}{\partial x^2} \sin sx \, dx . \qquad \text{Integrating by parts}$$

$$\Rightarrow \quad \frac{d}{dt} \int_0^\infty V \sin sx \, dx = K \left[ \left\{ \sin sx . \frac{\partial V}{\partial x} \right\}_0^\infty - \int_0^\infty s \cos sx . \frac{\partial V}{\partial x} dx \right]$$

$$= K \left[ 0 - s \int_0^\infty \cos sx . \frac{\partial V}{\partial x} dx \right] \qquad \left[ \because \frac{\partial V}{\partial x} \to 0 \text{ when } x \to \infty \right]$$

$$= - Ks \left[ \left\{ \cos sx . V \right\}_0^\infty - \int_0^\infty - s \sin sx . V dx \right]$$

$$= - Ks \left[ - V_0 + s \int_0^\infty V \sin sx \, dx \right]$$

$[\because$ when $x \to \infty$, $V \to 0$ and when $x = 0$, $V = V(0, t) = V_0]$



$$\Rightarrow \quad \frac{d}{dt} \int_0^\infty V \sin sx \, dx = KsV_0 - Ks^2 \int_0^\infty V \sin sx \, dx$$

$$\Rightarrow \quad \frac{d\bar{V}_s}{dt} = KsV_0 - Ks^2 \bar{V}_s \quad \text{where } \bar{V}_s = \bar{V}_s(s, t) = F_s[V(x, t)]$$

$$\Rightarrow \quad \frac{d\bar{V}_s}{dt} + Ks^2 \bar{V}_s = KsV_0$$

which is a linear differential equation.

$$\text{I.F.} = e^{\int Ks^2 dt} = e^{Ks^2 t}$$

$\therefore$ Its solution is

$$\bar{V}_s \, e^{Ks^2 t} = \int K s V_0 . e^{Ks^2 t} \, dt + c$$

$$= KsV_0 . \frac{e^{Ks^2 t}}{Ks^2} + c$$

or $\quad \bar{V}_s = \frac{V_0}{s} + ce^{-Ks^2 t} \qquad \ldots(3)$

Putting $t = 0$ in (3), we have $\bar{V}_s (s, 0) = \frac{V_0}{s} + c$

$$\Rightarrow \quad c = - \frac{V_0}{s} + \bar{V}_s (s, 0) = - \frac{V_0}{s} + F_s [V(x, 0)]$$

$$= - \frac{V_0}{s} + \int_0^\infty 0 \sin sx \, dx \qquad \text{[from (2)]}$$

$$= - \frac{V_0}{s}$$

$\therefore$ From (3), $\bar{V}_s (s, t) = \frac{V_0}{S} \left( 1 - e^{-Ks^2 t} \right)$

Taking its inverse Fourier sine transform, we get

$$V(x, t) = \frac{2}{\pi} \int_0^\infty \frac{V_0}{s} \left( 1 - e^{-Ks^2 t} \right) \sin sx \, ds$$

$$= \frac{2V_0}{\pi} \left[ \int_0^\infty \frac{\sin sx}{s} ds - \int_0^\infty \frac{e^{-Ks^2 t}}{s} \sin sx \, ds \right]$$

$$= \frac{2V_0}{\pi} \left[ \frac{\pi}{2} - \int_0^\infty \frac{e^{-Ks^2 t}}{s} \sin sx \, ds \right] \qquad \left[ \because \int_0^\infty \frac{\sin sx}{s} ds = \frac{\pi}{2} \right]$$

or $\quad V(x, t) = V_0 \left[ 1 - \frac{2}{\pi} \int_0^\infty \frac{e^{-Ks^2 t}}{s} \sin sx \, ds \right]$

which is the required solution.

**Example 3.** *The temperature u in a semi-infinite rod $0 \le x < \infty$ is determined by the differential equation* $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ *subject to the conditions:*

*(i)* $u = 0$ *when* $t = 0, x \ge 0$.

*(ii)* $\frac{\partial u}{\partial x} = -\mu$ *(a constant) when* $x = 0$ *and* $t > 0$.

*Determine the temperature formula.*

**Sol.** Given $$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad ...(1)$$

Boundary condition is $\frac{\partial u}{\partial x} = -\mu$ when $x = 0, t > 0$

Initial condition is $u(x, 0) = 0$ ...(2)

Since $\frac{\partial u}{\partial x}$ at $x = 0$ is given, we take Fourier cosine transform of both sides of (1). Thus

$$F_c\left(\frac{\partial u}{\partial t}\right) = F_c\left(k\frac{\partial^2 u}{\partial x^2}\right)$$

$\Rightarrow$ $\int_0^\infty \frac{\partial u}{\partial t} \cos sx\, dx = k\int_0^\infty \frac{\partial^2 u}{\partial x^2}\cos sx\, dx$. Integrating by parts

$$\Rightarrow \quad \frac{d}{dt}\int_0^\infty u\cos sx\, dx = k\left[\left\{\cos sx \cdot \frac{\partial u}{\partial x}\right\}_0^\infty - \int_0^\infty - s\sin sx \cdot \frac{\partial u}{\partial x}\, dx\right]$$

$$= k\left[0 - (-\mu) + s\int_0^\infty \sin sx \frac{\partial u}{\partial x}\, dx\right]$$

$$\left[\because \frac{\partial u}{\partial x} \to 0 \text{ when } x \to \infty \text{ and } \frac{\partial u}{\partial x} = -\mu \text{ when } x = 0\right]$$

$$= k\left[\mu + s\left\{\left|\sin sx \cdot u\right|_0^\infty\right\} - \int_0^\infty s\cos sx \cdot u\, dx\right]$$

$$= k\left[\mu + s(0 - s\int_0^\infty s\cos sx \cdot u\, dx\right]$$

$$[\because \quad u \to 0 \text{ when } x \to \infty]$$

$$\Rightarrow \quad \frac{d}{dt}\int_0^\infty u\cos sx\, dx = k\mu - ks^2\int_0^\infty u\cos sx\, dx$$

$$\Rightarrow \quad \frac{d\overline{u}_c}{dt} = k\mu - ks^2\overline{u}_c \qquad \text{where } \overline{u}_c = \overline{u}_c(s, t) = F_c[u(x, t)]$$



$$\Rightarrow \quad \frac{d\overline{u}_c}{dt} + ks^2\overline{u}_c = k\mu$$

which is a linear differential equation.

$$\text{I.F.} = e^{\int ks^2 dt} = e^{ks^2 t}$$

$\therefore$ Its solution is

$$\overline{u}_c \cdot e^{ks^2 t} = \int k\mu \cdot e^{ks^2 t}\, dt + c = k\mu \cdot \frac{e^{ks^2 t}}{ks^2} + c$$

or $$\overline{u}_c = \frac{\mu}{s^2} + ce^{-ks^2 t} \quad ...(3)$$

Putting $t = 0$ in (3), we have

$$\overline{u}_c(s, 0) = \frac{\mu}{s^2} + c$$

$$\Rightarrow \quad c = -\frac{\mu}{s^2} + \overline{u}_c(s, 0) = -\frac{\mu}{s^2} + F_c[u(x, 0)]$$

$$= -\frac{\mu}{s^2} + \int_0^\infty 0\cos sx\, dx \quad \text{[from (2)]}$$

$$= -\frac{\mu}{s^2}$$

$\therefore$ From (3), $\overline{u}_c(s, t) = \frac{\mu}{s^2}\left(1 - e^{-ks^2 t}\right)$

Taking its inverse Fourier cosine transform, we get

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \frac{\mu}{s^2}\left(1 - e^{-ks^2 t}\right)\cos sx\, ds$$

or $$u(x, t) = \frac{2\mu}{\pi}\int_0^\infty \frac{\cos sx}{s^2}\left(1 - e^{-ks^2 t}\right) ds$$

which is the required solution.

**Example 4.** *If the initial temperature of an infinite bar is given by*

$$\theta(x) = \begin{cases} \theta_0 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases},$$

*determine the temperature at any instant x and at any instant t.* (M.D.U. 2006)

**Sol.** To determine the temperature $\theta(x, t)$, we have to solve the one-dimensional heat-flow equation

$$\frac{\partial \theta}{\partial t} = c^2\frac{\partial^2 \theta}{\partial x^2}, \quad t > 0 \quad ...(1)$$

subject to the initial condition $$\theta(x, 0) = \begin{cases} \theta_0 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases} \quad ...(2)$$

Taking Fourier transform of (1), we get

$$\int_{-\infty}^{\infty} \frac{\partial\theta}{\partial t} e^{isx}\, dx = c^2 \int_{-\infty}^{\infty} \frac{\partial^2\theta}{\partial x^2} e^{isx}\, dx$$

or $$\frac{d}{dt}\int_{-\infty}^{\infty} \theta e^{isx}\, dx = c^2(-s^2\bar{\theta})$$

or $$\frac{d\bar{\theta}}{dt} = -c^2 s^2 \bar{\theta} \text{ where } \bar{\theta} = \bar{\theta}(s,t) = F[\theta(x,t) \quad \text{...(3)}$$

Now taking the Fourier transform of (2), we get

$$\bar{\theta}(s,0) = \int_{-\infty}^{\infty} \theta(x,0) e^{isx}\, dx = \int_{-a}^{a} \theta_0 e^{isx}\, dx = \theta_0 \left[\frac{e^{isx}}{is}\right]_{-a}^{a}$$

$$= \theta_0 \left[\frac{e^{isa} - e^{-isa}}{is}\right] = \frac{2\theta_0}{s}\left[\frac{e^{isa} - e^{-isa}}{2i}\right]$$

$$= \frac{2\theta_0 \sin sa}{s} \quad \text{...(4)}$$

From (3), $\dfrac{d\bar{\theta}}{\bar{\theta}} = -c^2s^2dt$

Integrating $\log \bar{\theta} = -c^2s^2t + \log A$ or $\bar{\theta} = Ae^{-c^2s^2t}$

Since $\bar{\theta} = \dfrac{2\theta_0 \sin sa}{s}$ when $t = 0$, from (4), we get $A = \dfrac{2\theta_0 \sin sa}{s}$

$$\therefore \quad \bar{\theta} = \frac{2\theta_0 \sin sa}{s} e^{-c^2s^2t}$$

Taking its inverse Fourier transform, we get

$$\theta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\theta_0 \sin as}{s} . e^{-c^2s^2t} . e^{-isx}\, ds$$

$$= \frac{\theta_0}{\pi}\int_{-\infty}^{\infty} \frac{\sin as}{s} . e^{-c^2s^2t} (\cos xs - i \sin xs)\, ds$$

$$= \frac{\theta_0}{\pi}\left[\int_{-\infty}^{\infty} \frac{\sin as}{s} . e^{-c^2s^2t} \cos xs\, ds - i\int_{-\infty}^{\infty} \frac{\sin as}{s} . e^{-c^2s^2t} \sin xs\, ds\right]$$

$$= \frac{\theta_0}{\pi}\int_{-\infty}^{\infty} \frac{\sin as}{s} . e^{-c^2s^2t} \cos xs\, ds$$

(The second integral vanishes since its integrand is an odd function)

$$= \frac{2\theta_0}{\pi}\int_{0}^{\infty} \frac{\sin as}{s} . e^{-c^2s^2t} \cos xs\, ds = \frac{\theta_0}{\pi}\int_{0}^{\infty} \frac{e^{-c^2s^2t}}{s} . 2 \sin as \cos xs\, ds$$

$$= \frac{\theta_0}{\pi}\int_{0}^{\infty} e^{-c^2s^2t}\left(\frac{\sin (a+x)s + \sin (a-x)s}{s}\right) ds$$

which is the required solution.



**Example 5.** *Use the method of Fourier transform to determine the displacement y(x, t) of an infinite string, given that the string is initially at rest and that the initial displacement is f(x), $(-\infty < x < \infty)$.*

**Sol.** The equation for the vibration of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{...(1)}$$

subject to the initial conditions

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad \text{and} \quad y(x,0) = f(x) \quad \text{...(2)}$$

Taking Fourier transform of (1), we get

$$\frac{d^2\bar{y}}{dt^2} = c^2(-s^2\bar{y}) \qquad \text{where} \quad \bar{y} = F[y(x,t)]$$

or $$\frac{d^2\bar{y}}{dt^2} + c^2s^2\bar{y} = 0$$

Its solution is $\bar{y} = A \cos cst + B \sin cst$ ...(3)

where A, B are constants.

Now taking Fourier transform of (2), we get

$$\frac{\partial\bar{y}}{\partial t} = 0 \quad \text{and} \quad \bar{y} = F(s) \quad \text{when } t = 0$$

Putting $t = 0$, $\bar{y} = F(s)$ in (3), we get $A = F(s)$

Also $$\frac{\partial\bar{y}}{\partial t} = -csA \sin cst + csB \cos cst$$

Putting $t = 0$, $\dfrac{\partial\bar{y}}{\partial t} = 0$, we get $B = 0$

$$\therefore \quad \bar{y} = F(s) \cos cst$$

Taking inverse Fourier transform, we get

$$y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s) \cos cst . e^{-isx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\left[\frac{e^{icst} + e^{-icst}}{2}\right] . e^{-isx}\, dx$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} [F(s) e^{-is(x-ct)} + F(s) e^{-is(x+ct)}]\, dx$$

$$= \frac{1}{2}[f(x-ct) + f(x+ct)] \qquad \left[\because f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s) e^{-isx}\, dx\right]$$

**Example 6.** *Use the complex form of Fourier transform to show that*

$$u = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(u)\, e^{-\frac{(x-u)^2}{4t}}\, du$$

*is the solution of the boundary value problem*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty, t > 0 ; \qquad u = f(x) \text{ when } t = 0.$$

**Sol.** Given $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \ldots(1)$$

$$\ldots(2)$$

Initial condition is $u(x, 0) = f(x)$

Taking Fourier transform of both sides of (1)

$$\int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{isx}\, dx = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{isx}\, dx$$

$$\Rightarrow \quad \frac{d}{dt}\int_{-\infty}^{\infty} u e^{isx} dx = \left[e^{isx} . \frac{\partial u}{\partial x}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} ise^{isx} . \frac{\partial u}{\partial x} dx$$

$$= 0 - is \int_{-\infty}^{\infty} e^{isx} . \frac{\partial u}{\partial x} dx \qquad \left[\because \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \pm \infty\right]$$

$$= -is\left[\left\{e^{isx} . u\right\}_{-\infty}^{\infty} - \int_{-\infty}^{\infty} ise^{isx} . u\, dx\right]$$

$$= -is\left[0 - is\int_{-\infty}^{\infty} u\, e^{isx}\, dx\right] \qquad [\because\ u \to 0 \text{ as } x \to \pm \infty]$$

$$\Rightarrow \quad \frac{d}{dt}\int_{-\infty}^{\infty} ue^{isx}\, dx = i^2 s^2 \int_{-\infty}^{\infty} ue^{isx}\, dx$$

$$\Rightarrow \quad \frac{d\bar{u}}{dt} = -s^2 \bar{u}$$

$$\Rightarrow \quad \frac{d\bar{u}}{\bar{u}} = -s^2 dt$$

Integrating $\quad \log \bar{u} = -s^2 t + \log c$

$$\Rightarrow \quad \bar{u} = ce^{-s^2 t} \qquad \ldots(3)$$

Putting $\quad t = 0,\ \bar{u}(s, 0) = c$

$$\therefore \quad c = \bar{u}(s, 0) = \int_{-\infty}^{\infty} u(x, 0)\, e^{isx} dx$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{isx} dx \qquad [\text{from } (2)]$$

$\therefore$ From (3), $$\bar{u} = e^{-s^2 t} \int_{-\infty}^{\infty} f(x)\, e^{isx} dx$$

$$= e^{-s^2 t} \int_{-\infty}^{\infty} f(u) e^{isu} du \qquad [\text{by changing the variable } x \text{ to } u]$$



Taking its inverse Fourier transform, we get

$$u(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[e^{-s^2 t}\int_{-\infty}^{\infty} f(u)\, e^{isu} du\right] e^{-isx} ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-s^2 t} f(u) e^{isu} . e^{-isx}\, du\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-s^2 t} f(u) e^{isu - isx}\, ds\, du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-s^2 t + is(u-x)}\, ds\right] du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-t\left[s^2 + is\left(\frac{x-u}{t}\right)\right]}\, ds\right] du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-t\left[s^2 + 2s . \frac{i(x-u)}{2t} + \frac{i^2(x-u)^2}{4t^2} - \frac{i^2(x-u)^2}{4t^2}\right]}\, ds\right] du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-t\left[s + i\frac{x-u}{2t}\right]^2 - \frac{(x-u)^2}{4t}}\, ds\right] du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-t\left[s + i\frac{x-u}{2t}\right]^2} . e^{-\frac{(x-u)^2}{4t}}\, ds\right] du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\, e^{-\frac{(x-u)^2}{4t}}\left[\int_{-\infty}^{\infty} e^{-t\left[s + i\frac{x-u}{2t}\right]^2}\, ds\right] du$$

Put $\sqrt{t}\left(s + i\dfrac{x-u}{2t}\right) = y$, then $\sqrt{t}\, ds = dy$ or $ds = \dfrac{dy}{\sqrt{t}}$

$$\therefore \quad u(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u) . e^{-\frac{(x-u)^2}{4t}}\left[\int_{-\infty}^{\infty} e^{-y^2}\frac{dy}{\sqrt{t}}\right] du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f(u)\, e^{-\frac{(x-u)^2}{4t}}}{\sqrt{t}} \sqrt{\pi}\, du \qquad \left[\because \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}\right]$$

$$= \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(u)\, e^{-\frac{(x-u)^2}{4t}}\, du.$$

**Example 7.** *Using suitable transforms, solve the differential equation* $\frac{\partial^2 V}{\partial x^2} = \frac{\partial V}{\partial t}$, $0 \le x \le \pi,\ t \ge 0$ *where* $V(0, t) = 0 = V(\pi, t)$ *and* $V(x, 0) = V_0$ *constant.*

**Sol.** Given $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2},\ 0 \le x \le \pi,\ t \ge 0$ ...(1)

Boundary conditions are $V(0, t) = 0 = V(\pi, t)$

Initial condition is $V(x, 0) = V_0$ ...(2)

Since the interval $0 \le x \le \pi$ is finite and $V(0, t)$ and $V(\pi, t)$ are given, we use finite Fourier sine transform.

Let $\overline{V}_s(n, t)$ denote finite Fourier sine transform of $V(x, t)$, then

$$\overline{V}_s(n, t) = \int_0^\pi V(x, t) \sin\left(\frac{n\pi x}{\pi}\right) dx = \int_0^\pi V(x, t) \sin nx\, dx$$

Taking finite Fourier sine transform of both sides of (1), we have

$$\int_0^\pi \frac{\partial V}{\partial t} \sin nx\, dx = \int_0^\pi \frac{\partial^2 V}{\partial x^2} \sin nx\, dx$$

$$\Rightarrow \quad \frac{d}{dt}\int_0^\pi V \sin nx\, dx = \left[\sin nx \cdot \frac{\partial V}{\partial x}\right]_0^\pi - \int_0^\pi n \cos nx \cdot \frac{\partial V}{\partial x}\, dx$$

$$= 0 - n\int_0^\pi \cos nx \cdot \frac{\partial V}{\partial x}\, dx$$

$$= -n\left[\left\{\cos nx \cdot V\right\}_0^\pi - \int_0^\pi -n \sin nx \cdot V dx\right]$$

$$= -n\left[0 + n\int_0^\pi V \sin nx\, dx\right] \qquad [\because\ V = 0 \text{ at } x = 0 \text{ and}$$

$$\Rightarrow \quad \frac{d}{dt}\int_0^\pi V \sin nx\, dx = -n^2\int_0^\pi V \sin nx\, dx$$

$$\Rightarrow \quad \frac{d\overline{V}_s}{dt} = -n^2 \overline{V}_s$$

$$\Rightarrow \quad \frac{d\overline{V}_s}{\overline{V}_s} = -n^2 dt$$

Integrating, $\log \overline{V}_s = -n^2 t + \log c$

$\therefore \quad \overline{V}_s = c e^{-n^2 t}$



Putting $t = 0$ in (3),

$$c = \overline{V}_s(n, 0) = \int_0^\pi V(x, 0) \sin nx\, dx$$

$$= \int_0^\pi V_0 \sin nx\, dx \qquad \text{[from (2)]}$$

$$= V_0\left[-\frac{\cos nx}{n}\right]_0^\pi = \frac{V_0}{n}[1 - \cos n\pi]$$

$$= \frac{V_0}{n}[1 - (-1)^n]$$

$\therefore$ From (3), $\overline{V}_s(n, t) = \frac{V_0}{n}[1 - (-1)^n]\, e^{-n^2 t}$

Taking its inverse finite Fourier sine transform, we get

$$V(x, t) = \frac{2}{\pi}\sum_{n=1}^\infty \overline{V}_s(n, t) \sin nx$$

$$= \frac{2}{\pi}\sum_{n=1}^\infty \frac{V_0}{n}[1 - (-1)^n]\, e^{-n^2 t} \sin nx$$

$$= \frac{2V_0}{\pi}\sum_{n=1,3,5,\ldots}^\infty \frac{1}{n}(2)\, e^{-n^2 t} \sin nx$$

$$[\because\ 1 - (-1)^n = 0 \text{ for } n = 2, 4, 6, \ldots]$$

or $$V(x, t) = \frac{4V_0}{\pi}\sum_{n=1,3,5,\ldots}^\infty \frac{1}{n} e^{-n^2 t} \sin nx$$

which is the required solution.

**Example 8.** *Using finite Fourier transform, find the solution of the wave equation* $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ *subject to the conditions :*

$u(0, t) = u(\pi, t) = 0,\ t > 0$

$u(x, 0) = 3 \sin x + 4 \sin 4x$ *and* $u_t(x, 0) = 0$ *for* $0 < x < \pi$.

**Sol.** Given $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2},\quad 0 \le x \le \pi,\ t > 0$ ...(1)

and $u(0, t) = u(\pi, t) = 0,\ t > 0$

$$\left.\begin{array}{l} u(x, 0) = 3 \sin x + 4 \sin 4x \\ u_t(x, 0) = 0 \end{array}\right\} \text{for } 0 < x < \pi \qquad \text{...(2)}$$

Taking finite Fourier sine transform of both sides of (1), we have

$$\int_0^\pi \frac{\partial^2 u}{\partial t^2} \sin nx\, dx = a^2 \int_0^\pi \frac{\partial^2 u}{\partial x^2} \sin nx\, dx$$

$$\Rightarrow \quad \frac{d^2}{dt^2}\int_0^\pi u \sin nx\, dx = a^2\left[\left\{\sin nx . \frac{\partial u}{\partial x}\right\}_0^\pi - \int_0^\pi n \cos nx . \frac{\partial u}{\partial x}\, dx\right]$$

$$= a^2\left[0 - n\int_0^\pi \cos nx \frac{\partial u}{\partial x}\, dx\right]$$

$$= -a^2 n\left[\left\{\cos nx . u\right\}_0^\pi - \int_0^\pi - n \sin nx . u\, dx\right]$$

$$= -a^2 n\left[0 + n\int_0^\pi u \sin nx\, dx\right] \qquad [\because \ u(0, t) = u(\pi, t) = 0]$$

$$= -a^2 n^2 \int_0^\pi u \sin nx\, dx$$

$$\Rightarrow \quad \frac{d^2 \bar{u}_s}{dt^2} + a^2 n^2 \bar{u}_s = 0$$

where $\quad \bar{u}_s = \bar{u}_s(n, t) = \int_0^\pi u(x, t) \sin\left(\frac{n\pi x}{\pi}\right) dx = \int_0^\pi u(x, t) \sin nx\, dx$

A.E. is $D^2 + a^2 n^2 = 0 \ \Rightarrow \ D = \pm\, ian$

$\therefore \quad \bar{u}_s = c_1 \cos ant + c_2 \sin ant \qquad$ ...(3)

Put $t = 0$ in (3), $\quad \bar{u}_s(n, 0) = c_1$

$$\Rightarrow \quad c_1 = \bar{u}_s(n, 0) = \int_0^\pi u(x, 0) \sin nx\, dx$$

$$= \int_0^\pi (3 \sin x + 4 \sin 4x) \sin nx\, dx \qquad \text{[from (2)]}$$

$$= 3\int_0^\pi \sin nx \sin x\, dx + 4\int_0^\pi \sin nx \sin 4x\, dx$$

$$= \frac{3}{2}\int_0^\pi 2 \sin nx \sin x\, dx + 2\int_0^\pi 2 \sin nx \sin 4x\, dx$$

$$= \frac{3}{2}\int_0^\pi [\cos(n-1)x - \cos(n+1)x]\, dx + 2\int_0^\pi [\cos(n-4)x - \cos(n+4)x]\, dx$$



$$= \frac{3}{2}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^\pi + 2\left[\frac{\sin(n-4)x}{n-4} - \frac{\sin(n+4)x}{n+4}\right]_0^\pi$$

$= 0$ except for $n = 1$ and $n = 4$

For $n = 1$, $\quad c_1 = \bar{u}_s(n, 0) = \int_0^\pi u(x, 0) \sin x\, dx$

$$= \int_0^\pi (3 \sin x + 4 \sin 4x) \sin x\, dx$$

$$= 3\int_0^\pi \sin^2 x\, dx + 2\int_0^\pi 2 \sin 4x \sin x dx$$

$$= 3\int_0^\pi \frac{1 - \cos 2x}{2}\, dx + 2\int_0^\pi (\cos 3x - \cos 5x)\, dx$$

$$= \frac{3}{2}\left[x - \frac{\sin 2x}{2}\right]_0^\pi + 2\left[\frac{\sin 3x}{3} - \frac{\sin 5x}{5}\right]_0^\pi = \frac{3}{2}\pi$$

For $n = 4$, $\quad c_1 = \bar{u}_s(n, 0) = \int_0^\pi u(x, 0) \sin 4x\, dx$

$$= \int_0^\pi (3 \sin x + 4 \sin 4x) \sin 4x\, dx$$

$$= \frac{3}{2}\int_0^\pi 2 \sin 4x \sin x\, dx + 4\int_0^\pi \sin^2 4x\, dx$$

$$= \frac{3}{2}\int_0^\pi (\cos 3x - \cos 5x) dx + 4\int_0^\pi \frac{1 - \cos 8x}{2}\, dx$$

$$= \frac{3}{2}\left[\frac{\sin 3x}{3} - \frac{\sin 5x}{5}\right]_0^\pi + 2\left[x - \frac{\sin 8x}{8}\right]_0^\pi = \pi$$

$$\therefore \quad c_1 = \begin{cases} \dfrac{3\pi}{2} & \text{for } n = 1 \\ 2\pi & \text{for } n = 4 \end{cases}$$

Again from (3), $\quad \frac{\partial \bar{u}_s}{\partial t} = -c_1 an \sin ant + c_2 an \cos ant$

Putting $t = 0$, $\quad \frac{\partial \bar{u}_s}{\partial t} = 0 \qquad \left[\because \text{ when } t = 0, \frac{\partial u}{\partial t} = 0 \ \therefore \ \frac{\partial \bar{u}_s}{\partial t} = 0 \text{ for } t = 0\right.$

$0 = c_2 an \ \Rightarrow \ c_2 = 0$

$\therefore$ (3) reduces to $\bar{u}_s = c_1 \cos ant$

where $\ c_1 = \frac{3\pi}{2}$ for $n = 1$ and $\ c_1 = 2\pi$ for $n = 4$

Taking inverse finite Fourier sine transform of (4), we have

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \bar{u}_s(n,t)\sin nx = \frac{2}{\pi}\sum_{n=1}^{\infty} c_1 \cos ant \sin nx$$

$$= \frac{2}{\pi}\left[\underset{(\text{for } n=1)}{\frac{3\pi}{2}\cos at \sin x} + \underset{(\text{for } n=4)}{2\pi \cos 4at \sin 4x}\right]$$

or $$u(x,t) = 3\cos at \sin x + 4\cos 4at \sin 4x$$

which is the required solution.

**Example 9.** *Using finite Fourier transform, solve* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ *subject to the conditions:*

*(a)* $u_x(0,t) = u_x(6,t) = 0, \quad 0 < x < 6,\ t > 0$

*(b)* $u(x,0) = x(6-x), \quad 0 < x < 6.$

**Sol.** Since the boundary conditions are $u_x(0,t) = u_x(6,0) = 0$, we take finite Fourier cosine transform.

Let $\bar{u}_c(n,t)$ denote finite Fourier cosine transform of $u(x,t)$, then

$$\bar{u}_c(n,t) = \int_0^6 u(x,t)\cos\left(\frac{n\pi x}{6}\right)dx \qquad [\because \ L = 6]$$

Taking finite Fourier cosine transform of both sides of the given equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

we get $$\int_0^6 \frac{\partial u}{\partial t}\cos\frac{n\pi x}{6}dx = \int_0^6 \frac{\partial^2 u}{\partial x^2}\cos\frac{n\pi x}{6}dx$$

$$\Rightarrow \frac{d}{dt}\int_0^6 u\cos\frac{n\pi x}{6}dx = \left[\cos\frac{n\pi x}{6}\cdot\frac{\partial u}{\partial x}\right]_0^6 - \int_0^6 -\frac{n\pi}{6}\sin\frac{n\pi x}{6}\cdot\frac{\partial u}{\partial x}dx$$

$$= 0 + \frac{n\pi}{6}\int_0^6 \sin\frac{n\pi x}{6}\cdot\frac{\partial u}{\partial x}dx \left[\because \ \frac{\partial u}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = 6\right]$$

$$= \frac{n\pi}{6}\left[\left\{\sin\frac{n\pi x}{6}\cdot u\right\}_0^6 - \int_0^6 \frac{n\pi}{6}\cos\frac{n\pi x}{6}\cdot u\,dx\right]$$

$$= \frac{n\pi}{6}\left[0 - \frac{n\pi}{6}\int_0^6 u\cos\frac{n\pi x}{6}dx\right]$$

$$\Rightarrow \frac{d}{dt}\int_0^6 u\cos\frac{n\pi x}{6}dx = -\frac{n^2\pi^2}{36}\int_0^6 u\cos\frac{n\pi x}{6}dx$$

$$\Rightarrow \frac{d\bar{u}_c}{dt} = -\frac{n^2\pi^2}{36}\bar{u}_c$$

$$\Rightarrow \frac{d\bar{u}_c}{\bar{u}_c} = -\frac{n^2\pi^2}{36}dt$$



Integrating, $$\log \bar{u}_c = -\frac{n^2\pi^2}{36}t + \log A \qquad \text{where } A = A(n)$$

$$\Rightarrow \bar{u}_c = Ae^{-\frac{n^2\pi^2}{36}t} \Rightarrow \bar{u}_c(n,t) = A(n)e^{-\frac{n^2\pi^2}{36}t} \qquad \ldots(1)$$

Putting $t = 0$, $\bar{u}_c(n,0) = A(n)$

$\therefore$ $A(n) = \bar{u}_c(n,0)$ = finite Fourier cosine transform of $u(x,0)$

$$= \int_0^6 u(x,0)\cos\frac{n\pi x}{6}dx = \int_0^6 x(6-x)\cos\frac{n\pi x}{6}dx \qquad \ldots(2)$$

$$= \left[(6x - x^2)\cdot\frac{\sin\frac{n\pi x}{6}}{\frac{n\pi}{6}} - (6-2x)\cdot\frac{-\cos\frac{n\pi x}{6}}{\left(\frac{n\pi}{6}\right)^2} + (-2)\cdot\frac{-\sin\frac{n\pi x}{6}}{\left(\frac{n\pi}{6}\right)^3}\right]_0^6$$

$$= -6\cdot\left(\frac{6}{n\pi}\right)^2\cos n\pi - 6\cdot\left(\frac{6}{n\pi}\right)^2$$

$$= -\frac{216}{n^2\pi^2}(1 + \cos n\pi)$$

$\therefore$ From (1), $$\bar{u}_c = -\frac{216}{n^2\pi^2}(1+\cos n\pi)\,e^{-\frac{n^2\pi^2}{36}t} \qquad \ldots(3)$$

To find inverse finite Fourier cosine transform, we need $\bar{u}_c(0,t)$.

From (1), $\bar{u}_c(0,t) = A(0)$

From (2), $A(0) = \bar{u}_c(0,0)$

$$= \int_0^6 x(6-x)dx = \left[3x^2 - \frac{x^3}{3}\right]_0^6$$

$$= 108 - 72 = 36$$

$\therefore$ $\bar{u}_c(0,t) = 36$

Taking inverse finite Fourier cosine transform of (3), we have

$$u(x,t) = \frac{1}{L}\bar{u}_c(0,t) + \frac{2}{L}\sum_{n=1}^{\infty}\bar{u}_c(n,t)\cos\frac{n\pi x}{L}$$

$$= \frac{1}{6}(36) + \frac{2}{6}\sum_{n=1}^{\infty} -\frac{216}{n^2\pi^2}(1+\cos n\pi)\,e^{-\frac{n^2\pi^2}{36}t}\cos\frac{n\pi x}{6} \qquad [\because \ L = 6]$$

or $$u(x,t) = 6 - \frac{72}{\pi^2}\sum_{n=1}^{\infty}\frac{1+\cos n\pi}{n^2}e^{-\frac{n^2\pi^2 t}{36}}\cos\frac{n\pi x}{6}$$

which is the required solution.

## EXERCISE 2.4

1. Solve the partial differential equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, x > 0 , t > 0$$

subject to the following conditions:

(a) $u(0, t) = 0, t > 0$ (b) $u(x, 0) = e^{-x}, x > 0$

(c) $u$ and $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$. (M.D.U. Dec. 2005, Dec. 2006)

2. Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, x > 0 , t > 0$ where $u(x, t)$ satisfies the conditions:

(a) $\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0, t > 0$ (b) $u(x, 0) = \begin{cases} x, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$

(c) $| u(x, t) | < M$.

3. The initial temperature along the length of an infinite bar is given by

$$u(x, 0) = \begin{cases} 2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

If the temperature $u(x, t)$ satisfies the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty, t > 0$, find the temperature at any point of the bar at any time $t$. (M.D.U. Dec. 2005)

4. Determine the distribution of temperature in the semi-infinite medium $x \ge 0$, when the end $x = 0$ is maintained at zero temperature and the initial distribution of temperature is $f(x)$.

5. (a) If the flow of heat is linear so that the variation of $\theta$ (temperature) with $z$ and $y$ may be neglected and if it is assumed that no heat is generated in the medium, then solve the differential equation

$$\frac{\partial \theta}{\partial t} = k\frac{\partial^2 \theta}{\partial x^2}$$

by using Fourier transform, where $-\infty < x < \infty$ and $\theta = f(x)$ when $t = 0$, $f(x)$ being a function of $x$.

(b) If the initial temperature of an infinite bar is given by

$$\mu(x, 0) = \begin{cases} 1, & \text{for } -c < x < c \\ 0, & \text{otherwise} \end{cases}$$

determine the temperature of the infinite bar at any point $x$ and at any time $t > 0$. (U.P.T.U. 2005)

6. Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, 0 < x < \pi, t > 0$ subject to the conditions

(a) $u(x, 0) = 1, 0 < x < \pi$ (b) $u(0, t) = u(\pi, t) = 0, t > 0$

using appropriate Fourier transform.

7. Use finite Fourier transform to solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ given that

$$u(0, t) = 0, u(\pi, t) = 0, u(x, 0) = 2x, 0 < x < \pi, t > 0.$$



8. Use finite Fourier transform to solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, 0 < x < 4, t > 0$$ (Rajasthan 2006)

subject to the conditions

(a) $u(x, 0) = 2x, \quad 0 < x < 4$ (b) $u(0, t) = u(4, t) = 0, t > 0.$

9. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, 0 < x < 6, t > 0$ subject to the conditions

(a) $u(0, t) = u(6, t) = 0, t > 0$ (b) $u(x, 0) = \begin{cases} 1, & 0 < x < 3 \\ 0, & 3 < x < 6 \end{cases}$.

10. Solve $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$, given

(a) $u(0, t) = u(\pi, t) = 0$ for $t > 0$ (b) $u(x, 0) = \frac{1}{10}\sin x + \frac{1}{100}\sin 4x$

(c) $u_t(x, 0) = 0$ for $0 < x < \pi$.

11. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, 0 < x < 6, t > 0$ subject to conditions $u_x(0, t) = u_x(6, t) = 0, u(x, 0) = 2x$.

12. Solve by using finite Fourier transform

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2}, 0 \le x \le 2$$

subject to the conditions:

$$u(0, t) = u(2, t) = 0, u(x, 0) = x(2 - x) \text{ and } u_t(x, 0) = 0.$$

### Answers

1. $u(x, t) = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2} e^{-2s^2 t}\sin sx\, ds$

2. $u(x, t) = \frac{2}{\pi}\int_0^\infty \left(\frac{s\sin s + \cos s - 1}{s^2}\right) e^{-s^2 t}\cos sx ds$

3. $u(x, t) = \frac{2}{\pi}\int_0^\infty e^{-s^2 t}\left(\frac{\sin(1+x)s + \sin(1-x)s}{s}\right) ds$

4. $u(x, t) = \frac{2}{\pi}\int_0^\infty \bar{f}_s(s)\, e^{-c^2 s^2 t}\sin sx\, ds$ where $\bar{f}_s(s) = F_s[f(x)]$

5. (a) $\theta(x, t) = \frac{1}{2\pi}\int_{-\infty}^\infty \bar{f}(s)\, e^{-ks^2 t - isx} ds$, where $\bar{f}(s) = F[f(x)]$

(b) $\mu(x, t) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{2}{s}\sin cs e^{-k^2 s^2 t} . e^{-isx}\, ds = \frac{1}{\pi}\int_0^\infty \frac{e^{-k^2 s^2 t}}{s}[\sin(c+x)s + \sin(c-x)s]\, ds$

6. $u(x, t) = \frac{2}{\pi}\sum_{n=1}^\infty \frac{1-\cos n\pi}{n} e^{-n^2 t}\sin nx$ or $\frac{4}{\pi}\sum_{n=1,3,5,\ldots}^\infty \frac{1}{n} e^{-n^2 t}\sin nx$

7. $u(x, t) = 4 \sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n+1} e^{-n^2 t} \sin nx$

8. $u(x, t) = \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-\frac{kn^2\pi^2 t}{16}} \sin \frac{n\pi x}{4}$

9. $u(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}\left(1 - \cos \frac{n\pi}{2}\right) e^{-\frac{n^2\pi^2 t}{36}} \sin \frac{n\pi x}{6}$

10. $u(x, t) = \frac{1}{10} \cos 2t \sin x + \frac{1}{100} \cos 8t \sin 4x$

11. $u(x, t) = 6 + \frac{24}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} e^{-\frac{n^2\pi^2 t}{36}} \cos \frac{n\pi x}{6}$

12. $u(x, t) = \frac{16}{\pi^3} \sum_{n=1}^{\infty} \frac{1 - \cos n\pi}{n^3} \cos \frac{3n\pi t}{2} \sin \frac{n\pi x}{2}$.

## 2.14. FOURIER TRANSFORM OF AN INTEGRAL

**Theorem.** Let $f(t)$ be piecewise continuous on every interval $[-l, l]$ and $\int_{-\infty}^{\infty} |f(t)|dt$ converge. Let $F[f(t)] = F(s)$ and $F(s)$ satisfies $F(0) = 0$. Then

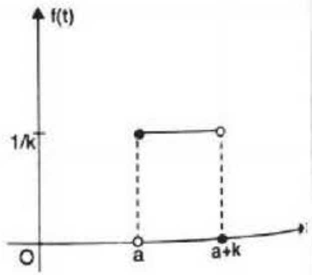
$$F\left[\int_{-\infty}^{t} f(T)\,dT\right] = \frac{1}{is} F(s).$$

## 2.15. FOURIER TRANSFORM OF DIRAC-DELTA FUNCTION

Dirac-delta function (or unit impulse function) $\delta(t - a)$ is defined as

$$\delta(t - a) = \lim_{k \to 0} \delta_k(t - a) \text{ where}$$

$$\delta_k(t - a) = \begin{cases} 0, & \text{for } t < a \\ \frac{1}{k}, & \text{for } a \le t < a + k \\ 0, & \text{for } t \ge a + k \end{cases}$$




$$F\{\delta(t - a)\} = \int_{-\infty}^{\infty} \delta(t - a) e^{ist}\, dt$$

$$= \lim_{k \to 0} \int_{a}^{a+k} \frac{1}{k} e^{ist}\, dt$$

$$= \lim_{k \to 0} \frac{1}{k} \left[\frac{e^{ist}}{is}\right]_{a}^{a+k}$$

$$= \lim_{k \to 0} \frac{e^{is(a+k)} - e^{isa}}{isk} = \lim_{k \to 0} e^{isa} \left(\frac{e^{isk} - 1}{isk}\right)$$

$$= e^{isa} \times 1 \qquad \left[\because \lim_{x \to 0} \frac{e^x - 1}{x} = 1\right]$$

$$= e^{isa}.$$