Collinearity Equations
&
Direct Linear Transformation

Overview

- Objectives
- Rotation in Space
- Collinearity Equations
- DLT (Concept)
- From Collinearity to DLT
- IOPs and EOPs from the DLT Parameters
- Experimental Results
- Final Analysis

Objectives

- These slides summarize the back and forth transformation between the following models:
  - Collinearity Equations (CE)
  - Direct Linear Transformation (DLT)
- Those models give the mathematical relationship between the following coordinates:
  - Comparator/Machine Coordinates,
  - Image Coordinates, and
  - Ground Coordinates.

CE

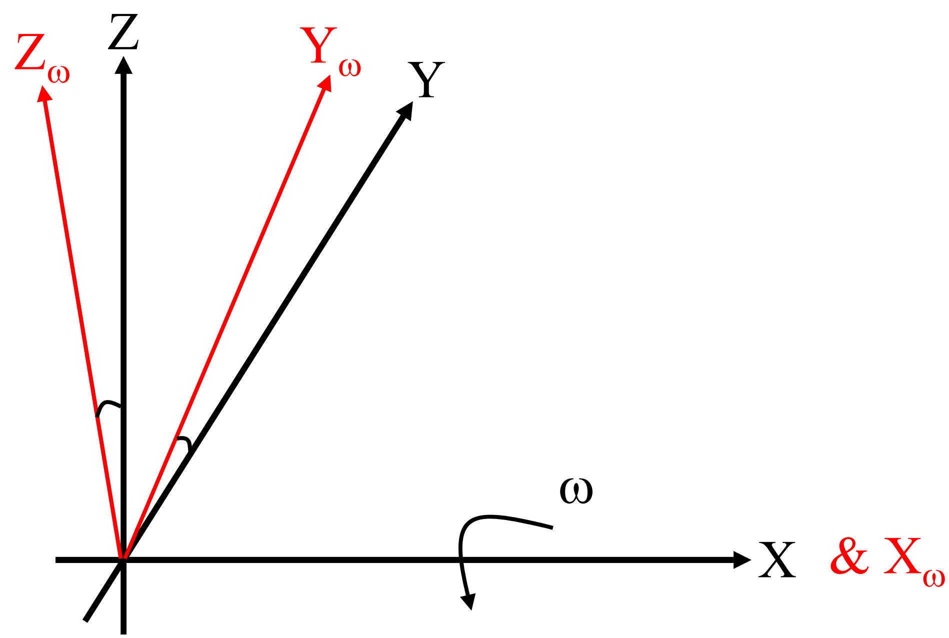
DLT

Rotation in Space

Rotation in Space

- Gives the rotational relationship between the ground and the image coordinate systems.
- The ground coordinate system (XYZ) is rotated until it becomes parallel to the image coordinate system (xyz).
- Standard rotation procedure:
  - Primary rotation ω around the X-axis – $XYZ \rightarrow X_\omega Y_\omega Z_\omega$
  - Secondary rotation ϕ around the Y_ω-axis – $X_\omega Y_\omega Z_\omega \rightarrow X_{\omega\phi} Y_{\omega\phi} Z_{\omega\phi}$
  - Tertiary rotation κ around the $Z_{\omega\phi}$-axis – $X_{\omega\phi} Y_{\omega\phi} Z_{\omega\phi} \rightarrow X_{\omega\phi\kappa} Y_{\omega\phi\kappa} Z_{\omega\phi\kappa}$ (xyz)

Primary Rotation (ω)

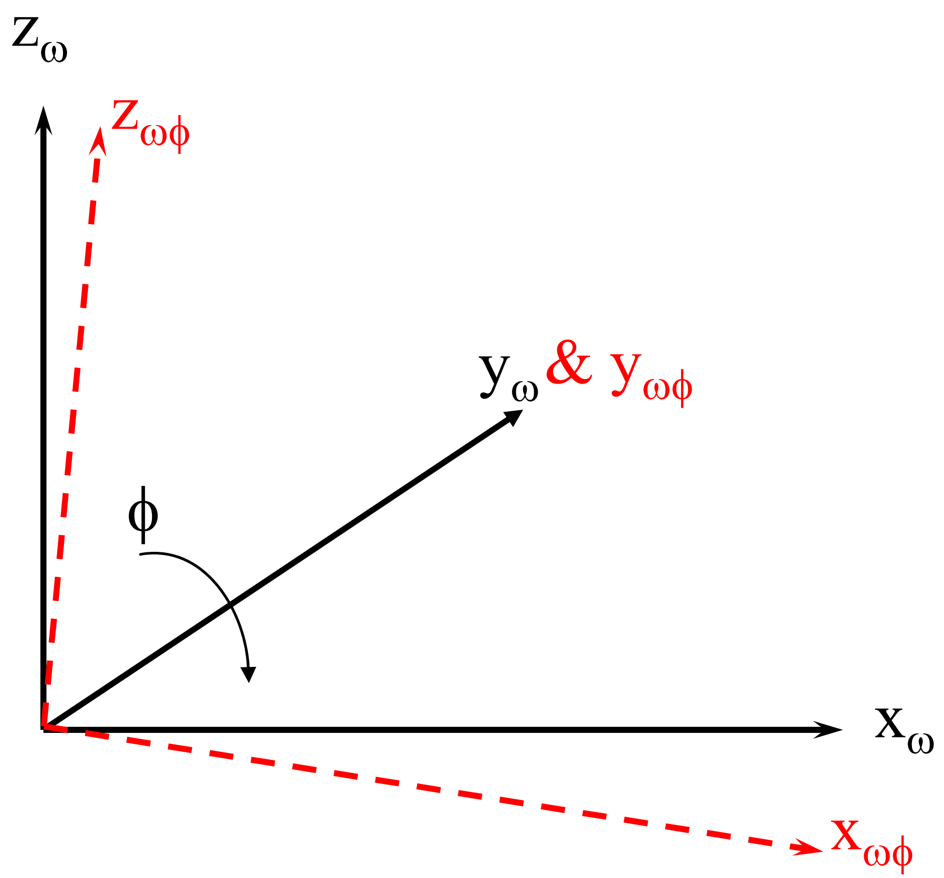


Primary Rotation (ω)

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} X_\omega \\ Y_\omega \\ Z_\omega \end{bmatrix}$$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = R_\omega \begin{bmatrix} X_\omega \\ Y_\omega \\ Z_\omega \end{bmatrix}$$

Secondary Rotation (ϕ)

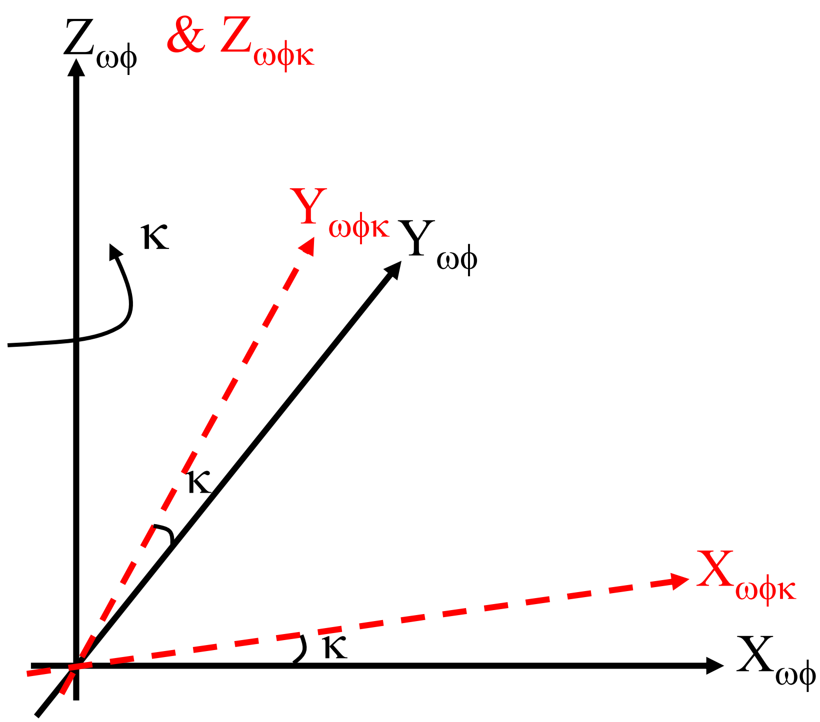


Secondary Rotation (ϕ)

$$\begin{bmatrix} X_\omega \\ Y_\omega \\ Z_\omega \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} X_{\omega\phi} \\ Y_{\omega\phi} \\ Z_{\omega\phi} \end{bmatrix}$$

$$\begin{bmatrix} X_\omega \\ Y_\omega \\ Z_\omega \end{bmatrix} = R_\phi \begin{bmatrix} X_{\omega\phi} \\ Y_{\omega\phi} \\ Z_{\omega\phi} \end{bmatrix}$$

Tertiary Rotation (κ)



Tertiary Rotation (κ)

$$\begin{bmatrix} X_{\omega\phi} \\ Y_{\omega\phi} \\ Z_{\omega\phi} \end{bmatrix} = \begin{bmatrix} \cos\kappa & -\sin\kappa & 0 \\ \sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{\omega\phi\kappa} \\ Y_{\omega\phi\kappa} \\ Z_{\omega\phi\kappa} \end{bmatrix}$$

$$\begin{bmatrix} X_{\omega\phi} \\ Y_{\omega\phi} \\ Z_{\omega\phi} \end{bmatrix} = R_{\kappa} \begin{bmatrix} X_{\omega\phi\kappa} \\ Y_{\omega\phi\kappa} \\ Z_{\omega\phi\kappa} \end{bmatrix}$$

Rotation in Space

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = R_{\omega} R_{\phi} R_{\kappa} \begin{bmatrix} X_{\omega\phi\kappa} \\ Y_{\omega\phi\kappa} \\ Z_{\omega\phi\kappa} \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

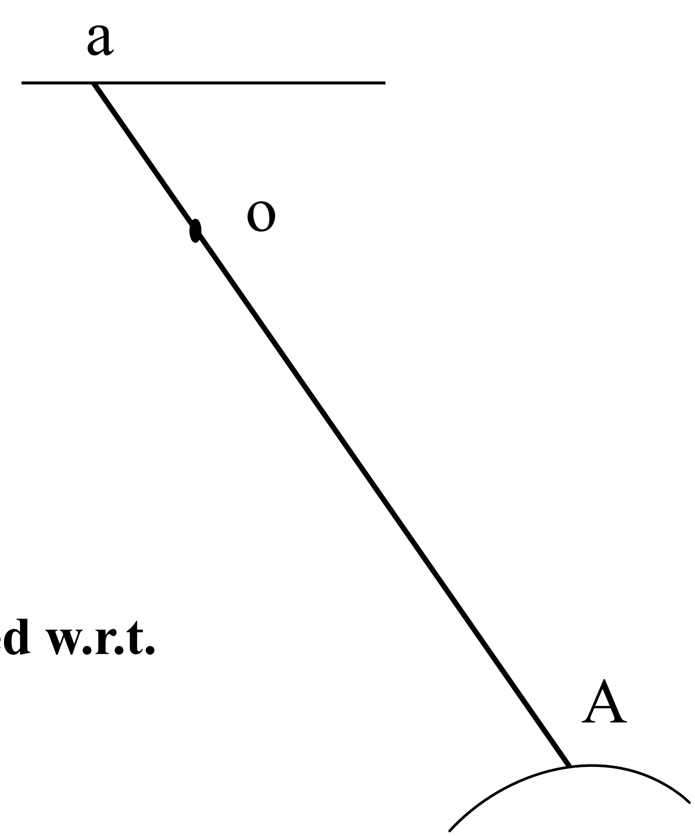
// to the ground coordinate system

// to the image coordinate system

Collinearity Equations

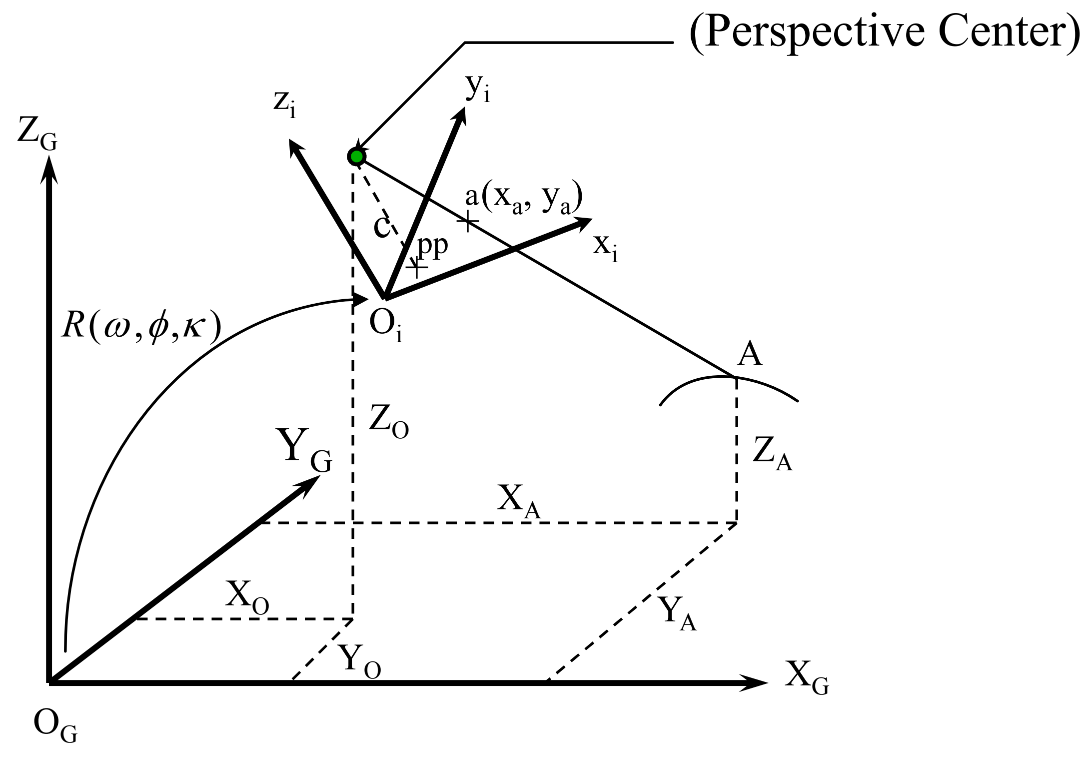
Collinearity Equations

$$\overrightarrow{\mathbf{oa}} = \lambda \ \ \overrightarrow{\mathbf{oA}}$$



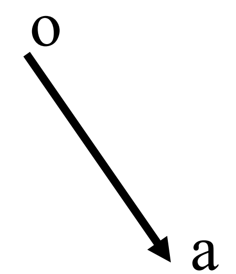
These vectors should be defined w.r.t. the same coordinate system.

Collinearity Equations



The vector connecting the perspective center to the image point

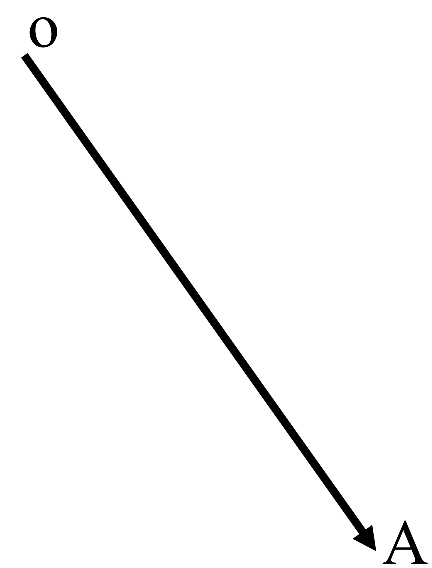
$$\vec{v}_i = \begin{bmatrix} x_a \\ y_a \\ 0 \end{bmatrix} - \begin{bmatrix} x_p \\ y_p \\ c \end{bmatrix} = \begin{bmatrix} x_a - x_p \\ y_a - y_p \\ -c \end{bmatrix}$$



w.r.t. the image coordinate system

The vector connecting the perspective center to the object point

$$\vec{V}_o = \begin{bmatrix} X_A \\ Y_A \\ Z_A \end{bmatrix} - \begin{bmatrix} X_O \\ Y_O \\ Z_O \end{bmatrix} = \begin{bmatrix} X_A - X_O \\ Y_A - Y_O \\ Z_A - Z_O \end{bmatrix}$$



w.r.t. the ground coordinate system

Collinearity Equations

$$\vec{v}_i = \lambda \; R^T(\omega, \phi, \kappa) \; \vec{V}_O$$

$$\begin{bmatrix} x_a - x_p \\ y_a - y_p \\ -c \end{bmatrix} = \lambda \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} X_A - X_O \\ Y_A - Y_O \\ Z_A - Z_O \end{bmatrix}$$

Where: λ is a scale factor (+ve).

Collinearity Equations

$$x_a = x_p - c\,\frac{r_{11}(X_A - X_O) + r_{21}(Y_A - Y_O) + r_{31}(Z_A - Z_O)}{r_{13}(X_A - X_O) + r_{23}(Y_A - Y_O) + r_{33}(Z_A - Z_O)}$$

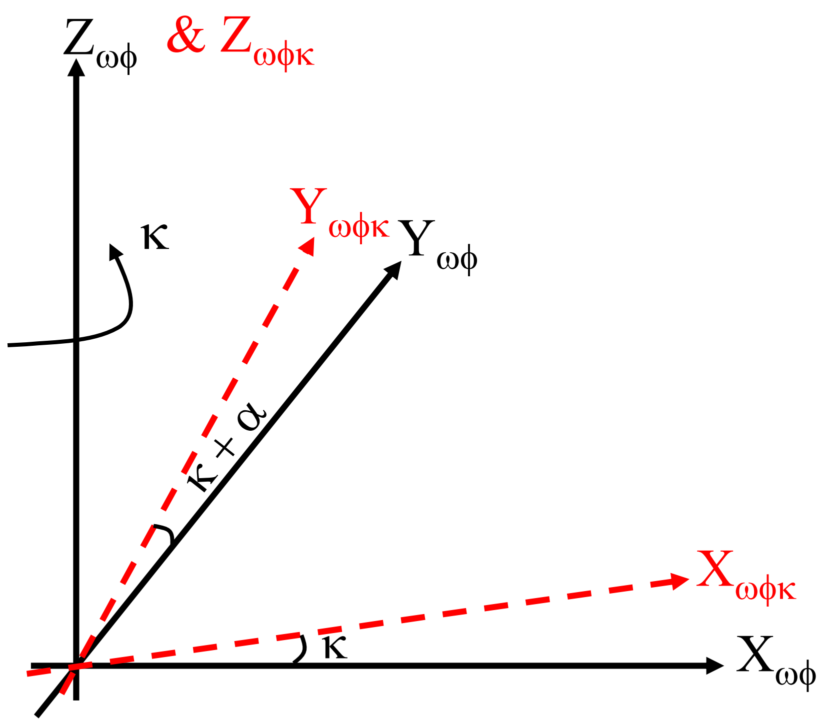
$$y_a = y_p - c\,\frac{r_{12}(X_A - X_O) + r_{22}(Y_A - Y_O) + r_{32}(Z_A - Z_O)}{r_{13}(X_A - X_O) + r_{23}(Y_A - Y_O) + r_{33}(Z_A - Z_O)}$$

From Collinearity to DLT

Collinearity Equations → DLT

- DLT combines:
  - Collinearity Equations
  - Affine Transformation
    - Rotation Angle:
      - Rotation angle between the image and comparator/machine coordinate systems
      - Compensated for through the κ rotation
    - Non Orthogonality (α):
      - Non-Orthogonality between the comparator/machine axes
    - Two Scale Factors (S_x, S_y):
      - Non uniform scale along the axes of the comparator coordinate system
    - Two Shifts:
      - Shift between the origins of the comparator/machine and image coordinate systems
      - Compensated for through x_p and y_p

Tertiary Rotation (κ) & Non-Orthogonality (α)



Tertiary Rotation (κ) & Non-Orthogonality (α)

$$\begin{bmatrix} X_{\omega\phi} \\ Y_{\omega\phi} \\ Z_{\omega\phi} \end{bmatrix} = \begin{bmatrix} \cos\kappa & -\sin(\kappa+\alpha) & 0 \\ \sin\kappa & \cos(\kappa+\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{\omega\phi\kappa} \\ Y_{\omega\phi\kappa} \\ Z_{\omega\phi\kappa} \end{bmatrix}$$

$$\sin(\kappa+\alpha) = \sin\kappa\ \cos\alpha + \cos\kappa\ \sin\alpha = \sin\kappa + \alpha\cos\kappa$$

$$\cos(\kappa+\alpha) = \cos\kappa\ \cos\alpha - \sin\kappa\ \sin\alpha = \cos\kappa - \alpha\ \sin\kappa$$

$$\begin{bmatrix} X_{\omega\phi} \\ Y_{\omega\phi} \\ Z_{\omega\phi} \end{bmatrix} = \begin{bmatrix} \cos\kappa & -\sin\kappa - \alpha\cos\kappa & 0 \\ \sin\kappa & \cos\kappa - \alpha\sin\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{\omega\phi\kappa} \\ Y_{\omega\phi\kappa} \\ Z_{\omega\phi\kappa} \end{bmatrix}$$

Tertiary Rotation (κ) & Non-Orthogonality (α)

$$\begin{bmatrix} \cos\kappa & -\sin\kappa - \alpha\cos\kappa & 0 \\ \sin\kappa & \cos\kappa - \alpha\sin\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\kappa & -\sin\kappa & 0 \\ \sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = R_\omega R_\phi R_\kappa \begin{bmatrix} 1 & -\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{\omega\phi\kappa} \\ Y_{\omega\phi\kappa} \\ Z_{\omega\phi\kappa} \end{bmatrix} = R \begin{bmatrix} 1 & -\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{\omega\phi\kappa} \\ Y_{\omega\phi\kappa} \\ Z_{\omega\phi\kappa} \end{bmatrix}$$

Tertiary Rotation (κ) & Non-Orthogonality (α)

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = R \begin{bmatrix} 1 & -\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{\omega\phi\kappa} \\ Y_{\omega\phi\kappa} \\ Z_{\omega\phi\kappa} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} X_{\omega\phi\kappa} \\ Y_{\omega\phi\kappa} \\ Z_{\omega\phi\kappa} \end{bmatrix} = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

// to the image coordinate system // to the ground coordinate system

Collinearity Equations → DLT

$$\begin{bmatrix} x - x_p \\ y - y_p \\ -c \end{bmatrix} = \lambda \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X - X_O \\ Y - Y_O \\ Z - Z_O \end{bmatrix}$$

$$\begin{bmatrix} x - x_p \\ y - y_p \\ -c \end{bmatrix} = \lambda \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} X - X_O \\ Y - Y_O \\ Z - Z_O \end{bmatrix}$$

- Affine Transformation Parameters compensated for:
  - Two shifts, rotation and non-orthogonality angles

Collinearity Equations → DLT
Scale Factor

$$\begin{bmatrix} (x-x_p)/s_x \\ (y-y_p)/s_y \\ -c \end{bmatrix} = \lambda \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X-X_O \\ Y-Y_O \\ Z-Z_O \end{bmatrix}$$

- Divide both sides by (-c)

$$\begin{bmatrix} -(x-x_p)/(cs_x) \\ -(y-y_p)/(cs_y) \\ 1 \end{bmatrix} = -\lambda / c \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X-X_O \\ Y-Y_O \\ Z-Z_O \end{bmatrix}$$

- Affine Transformation Parameters compensated for:
  - Two Shifts, rotation and non-orthogonality angles
  - Two scale factors

Collinearity Equations → DLT

- $cs_x \rightarrow c_x$, $cs_y \rightarrow c_y$ & $-\lambda/c \rightarrow \lambda$`.

$$\begin{bmatrix} -(x-x_p)/(c_x) \\ -(y-y_p)/(c_y) \\ 1 \end{bmatrix} = \lambda' \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X-X_O \\ Y-Y_O \\ Z-Z_O \end{bmatrix}$$

$$\begin{bmatrix} -1/c_x & 0 & 0 \\ 0 & -1/c_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (x-x_p) \\ (y-y_p) \\ 1 \end{bmatrix} = \lambda' \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X-X_O \\ Y-Y_O \\ Z-Z_O \end{bmatrix}$$

$$\begin{bmatrix} (x-x_p) \\ (y-y_p) \\ 1 \end{bmatrix} = \lambda' \begin{bmatrix} -c_x & 0 & 0 \\ 0 & -c_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X-X_O \\ Y-Y_O \\ Z-Z_O \end{bmatrix}$$

Collinearity Equations → DLT

$$\begin{bmatrix} (x-x_p) \\ (y-y_p) \\ 1 \end{bmatrix} = \lambda' \begin{bmatrix} -c_x & -\alpha\, c_x & 0 \\ 0 & -c_y & 0 \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X-X_O \\ Y-Y_O \\ Z-Z_O \end{bmatrix}$$

$$\begin{bmatrix} (x-x_p) \\ (y-y_p) \\ 1 \end{bmatrix} = \lambda' \begin{bmatrix} -c_x & -\alpha\, c_x & 0 \\ 0 & -c_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} X-X_O \\ Y-Y_O \\ Z-Z_O \end{bmatrix}$$

$$\begin{bmatrix} (x-x_p) \\ (y-y_p) \\ 1 \end{bmatrix} = \lambda' \begin{bmatrix} -c_x\, r_{11} - \alpha\, c_x r_{12} & -c_x\, r_{21} - \alpha\, c_x r_{22} & -c_x\, r_{31} - \alpha\, c_x r_{32} \\ -c_y\, r_{12} & -c_y\, r_{22} & -c_y\, r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} X-X_O \\ Y-Y_O \\ Z-Z_O \end{bmatrix}$$

Collinearity Equations → DLT

$$\begin{bmatrix} (x-x_p) \\ (y-y_p) \\ 1 \end{bmatrix} = \lambda' \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} [X - X_O]$$

$$M_1 = \begin{bmatrix} -c_x r_{11} - \alpha c_x r_{12} & -c_x r_{21} - \alpha c_x r_{22} & -c_x r_{31} - \alpha c_x r_{32} \end{bmatrix} = \begin{bmatrix} -c_x & -\alpha c_x & 0 \end{bmatrix} R^T$$

$$M_2 = \begin{bmatrix} -c_y r_{12} & -c_y r_{22} & -c_y r_{32} \end{bmatrix} = \begin{bmatrix} 0 & -c_y & 0 \end{bmatrix} R^T$$

$$M_3 = \begin{bmatrix} r_{13} & r_{23} & r_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} R^T$$

$$X = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$X_O = \begin{bmatrix} X_O \\ Y_O \\ Z_O \end{bmatrix}$$

Collinearity Equations → DLT

$$x = x_p + \frac{M_1 (X - X_O)}{M_3 (X - X_O)}$$

$$y = y_p + \frac{M_2 (X - X_O)}{M_3 (X - X_O)}$$

$$x = \frac{[x_p M_3 + M_1](X - X_O)}{M_3 (X - X_O)}$$

$$y = \frac{[y_p M_3 + M_2](X - X_O)}{M_3 (X - X_O)}$$

$$x = \frac{\left\{\begin{bmatrix} 0 & 0 & x_p \end{bmatrix} + \begin{bmatrix} -c_x & -\alpha\, c_x & 0 \end{bmatrix}\right\} R^T (X - X_O)}{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} R^T (X - X_O)}$$

$$y = \frac{\left\{\begin{bmatrix} 0 & 0 & y_p \end{bmatrix} + \begin{bmatrix} 0 & -c_y & 0 \end{bmatrix}\right\} R^T (X - X_O)}{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} R^T (X - X_O)}$$

Collinearity Equations → DLT

$$x = \frac{\begin{bmatrix} -c_x & -\alpha\, c_x & x_p \end{bmatrix} R^T (X - X_O)}{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} R^T (X - X_O)}$$

$$y = \frac{\begin{bmatrix} 0 & -c_y & y_p \end{bmatrix} R^T (X - X_O)}{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} R^T (X - X_O)}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} -c_x & -\alpha\, c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X - X_O \\ Y - Y_O \\ Z - Z_O \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda \{K\, R^T\, X - K\, R^T\, X_O\}$$

Collinearity Equations → DLT

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda\, K\, R^T \begin{bmatrix} I_3 & -X_O \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$K = \begin{bmatrix} -c_x & -\alpha\, c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \equiv \{Calibration\ Matrix\}$$

Collinearity Equations → DLT

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} L_1 & L_2 & L_3 & L_4 \\ L_5 & L_6 & L_7 & L_8 \\ L_9 & L_{10} & L_{11} & L_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$D = \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x & -\alpha\, c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} R^T$$

$$\begin{bmatrix} L_4 \\ L_8 \\ L_{12} \end{bmatrix} = -\lambda \begin{bmatrix} -c_x & -\alpha\, c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X_O \\ Y_O \\ Z_O \end{bmatrix}$$

DLT $\rightarrow$ IOP & EOP

Approach # 1

DLT → IOP & EOP

$$\begin{bmatrix} L_4 \\ L_8 \\ L_{12} \end{bmatrix} = -\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} \begin{bmatrix} X_O \\ Y_O \\ Z_O \end{bmatrix}$$

$$\begin{bmatrix} X_O \\ Y_O \\ Z_O \end{bmatrix} = -\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix}^{-1} \begin{bmatrix} L_4 \\ L_8 \\ L_{12} \end{bmatrix}$$

No Sign Ambiguity

DLT → IOP & EOP

$$D\,D^T = (\lambda\,K\,R^T)\,(\lambda\,K\,R^T)^T = \lambda^2\,KK^T = \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} \begin{bmatrix} L_1 & L_5 & L_9 \\ L_2 & L_6 & L_{10} \\ L_3 & L_7 & L_{11} \end{bmatrix}$$

$$D\,D^T = \lambda^2 \begin{bmatrix} -c_x & -\alpha\,c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -c_x & 0 & 0 \\ -\alpha\,c_x & -c_y & 0 \\ x_p & y_p & 1 \end{bmatrix}$$

$$(D\,D^T)_{3\times 3} = L_9^2 + L_{10}^2 + L_{11}^2 = \lambda^2$$

$$\lambda = \pm\sqrt{L_9^2 + L_{10}^2 + L_{11}^2}\ \{Sign\ \ Ambiguity\}$$

DLT → IOP & EOP

$$D\,D^T = \lambda^2 \begin{bmatrix} -c_x & -\alpha\, c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -c_x & 0 & 0 \\ -\alpha\, c_x & -c_y & 0 \\ x_p & y_p & 1 \end{bmatrix} = \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} \begin{bmatrix} L_1 & L_5 & L_9 \\ L_2 & L_6 & L_{10} \\ L_3 & L_7 & L_{11} \end{bmatrix}$$

$$(D\,D^T)_{3\times 1} = L_9\,L_1 + L_{10}\,L_2 + L_{11}\,L_3 = \lambda^2\ x_p$$

$$x_p = {(L_9\,L_1 + L_{10}\,L_2 + L_{11}\,L_3)} \Big/ {(L_9^2 + L_{10}^2 + L_{11}^2)}$$

No Sign Ambiguity

DLT → IOP & EOP

$$D\,D^T = \lambda^2 \begin{bmatrix} -c_x & -\alpha\,c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -c_x & 0 & 0 \\ -\alpha\,c_x & -c_y & 0 \\ x_p & y_p & 1 \end{bmatrix} = \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} \begin{bmatrix} L_1 & L_5 & L_9 \\ L_2 & L_6 & L_{10} \\ L_3 & L_7 & L_{11} \end{bmatrix}$$

$$(D\,D^T)_{3\times 2} = L_9\,L_5 + L_{10}\,L_6 + L_{11}\,L_7 = \lambda^2\ y_p$$

$$y_p = {(L_9\,L_5 + L_{10}\,L_6 + L_{11}\,L_7)} \Big/ {(L_9^2 + L_{10}^2 + L_{11}^2)}$$

No Sign Ambiguity

DLT → IOP & EOP

$$D\,D^T = \lambda^2 \begin{bmatrix} -c_x & -\alpha\, c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -c_x & 0 & 0 \\ -\alpha\, c_x & -c_y & 0 \\ x_p & y_p & 1 \end{bmatrix} = \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} \begin{bmatrix} L_1 & L_5 & L_9 \\ L_2 & L_6 & L_{10} \\ L_3 & L_7 & L_{11} \end{bmatrix}$$

$$(D\,D^T)_{2\times 2} = L_5^2 + L_6^2 + L_7^2 = \lambda^2\ (y_p^2 + c_y^2)$$

$$c_y = \left[L_5^2 + L_6^2 + L_7^2 \big/ (L_9^2 + L_{10}^2 + L_{11}^2) - y_p^2 \right]^{0.5}$$

No Sign Ambiguity

DLT → IOP & EOP

$$D\,D^T = \lambda^2 \begin{bmatrix} -c_x & -\alpha\, c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -c_x & 0 & 0 \\ -\alpha\, c_x & -c_y & 0 \\ x_p & y_p & 1 \end{bmatrix} = \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} \begin{bmatrix} L_1 & L_5 & L_9 \\ L_2 & L_6 & L_{10} \\ L_3 & L_7 & L_{11} \end{bmatrix}$$

$$(D\,D^T)_{1\times 2} = L_1\,L_5 + L_2\,L_6 + L_3\,L_7 = \lambda^2\ (\alpha\, c_x\, c_y + x_p\, y_p)$$

$$\alpha\, c_x = 1/c_y \left[L_1\,L_5 + L_2\,L_6 + L_3\,L_7 \Big/ (L_9^2 + L_{10}^2 + L_{11}^2) - x_p\, y_p \right]$$

No Sign Ambiguity

DLT → IOP & EOP

$$D\,D^T = \lambda^2 \begin{bmatrix} -c_x & -\alpha\, c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -c_x & 0 & 0 \\ -\alpha\, c_x & -c_y & 0 \\ x_p & y_p & 1 \end{bmatrix} = \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} \begin{bmatrix} L_1 & L_5 & L_9 \\ L_2 & L_6 & L_{10} \\ L_3 & L_7 & L_{11} \end{bmatrix}$$

$$(D\,D^T)_{1\times1} = L_1^2 + L_2^2 + L_3^2 = \lambda^2\ (c_x^2 + \alpha^2\ c_x^2 + x_p^2)$$

$$c_x = \left[{L_1^2 + L_2^2 + L_3^2} \Big/ {(L_9^2 + L_{10}^2 + L_{11}^2)} - \alpha^2\ c_x^2 - x_p^2 \right]^{0.5}$$

No Sign Ambiguity

DLT → IOP & EOP

$$\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x & -\alpha\, c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

$$L_9 = \lambda\ r_{13} = \lambda\ \sin\phi$$

Sign Ambiguity

Collinearity Equations

- Objective: Resolve the sign ambiguity in λ

$$\begin{bmatrix} x - x_p \\ y - y_p \\ -c \end{bmatrix} = S \begin{bmatrix} r_{11}(X - X_O) + r_{21}(Y - Y_O) + r_{31}(Z - Z_O) \\ r_{12}(X - X_O) + r_{22}(Y - Y_O) + r_{32}(Z - Z_O) \\ r_{13}(X - X_O) + r_{23}(Y - Y_O) + r_{33}(Z - Z_O) \end{bmatrix}$$

- Since the scale factor is always +ve

$$r_{13}(X - X_O) + r_{23}(Y - Y_O) + r_{33}(Z - Z_O) \Rightarrow -ve$$

- Assuming that the origin (0, 0, 0) is visible in the imagery

$$-r_{13} X_O - r_{23} Y_O - r_{33} Z_O \Rightarrow -ve$$

DLT → IOP & EOP

$$\begin{bmatrix} L_4 \\ L_8 \\ L_{12} \end{bmatrix} = -\lambda \begin{bmatrix} -c_x & -\alpha\, c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} R^T \begin{bmatrix} X_O \\ Y_O \\ Z_O \end{bmatrix}$$

- By choosing $L_{12} = 1$.

$$L_{12} = -\lambda\ (r_{13}\, X_O + r_{23}\, Y_O + r_{33}\, X_O)$$

$$1 = \lambda\ (-r_{13}\, X_O - r_{23}\, Y_O - r_{33}\, X_O)$$

$$\lambda = 1 \Big/ (-r_{13}\, X_O - r_{23}\, Y_O - r_{33}\, X_O)$$

λ is Negative

$$\lambda = -\sqrt{L_9^2 + L_{10}^2 + L_{11}^2}$$

DLT → IOP & EOP

$$L_9 = \lambda \, r_{13} = \lambda \, \sin\phi$$

$$\sin\phi = L_9 \Big/ -\sqrt{L_9^2 + L_{10}^2 + L_{11}^2}$$

- No sign Ambiguity

DLT → IOP & EOP

$$\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x & -\alpha\, c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

$$L_{10} = \lambda\ r_{23} = -\lambda\ \sin\omega\ \cos\phi$$

$$L_{11} = \lambda\ r_{33} = \lambda\ \cos\omega\ \cos\phi$$

$$\tan\omega = {-L_{10}}/{L_{11}}$$

No Sign Ambiguity

DLT → IOP & EOP

$$\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x & -\alpha\, c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \lambda \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix}^{-1} \begin{bmatrix} -c_x & -\alpha\, c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix}$$

- Retrieve κ

$$\cos\kappa = {r_{11}}\big/{\cos\phi}$$

DLT $\rightarrow$ IOP & EOP

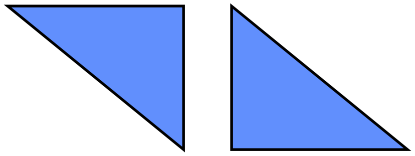
Approach # 2

Matrix Factorization

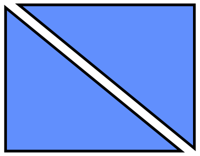
DLT → IOP (Factorization # 1)

$$D\,D^T = (\lambda\,K\,R^T)(\lambda\,K\,R^T)^T = \lambda^2\,KK^T = \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} \begin{bmatrix} L_1 & L_5 & L_9 \\ L_2 & L_6 & L_{10} \\ L_3 & L_7 & L_{11} \end{bmatrix}$$

$$D\,D^T = \lambda^2 \begin{bmatrix} -c_x & -\alpha\,c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -c_x & 0 & 0 \\ -\alpha\,c_x & -c_y & 0 \\ x_p & y_p & 1 \end{bmatrix}$$

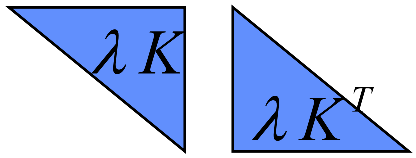


- Cholesky Decomposition of $DD^T \rightarrow \lambda K$ (Calibration Matrix)? Wrong



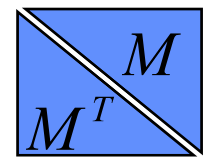
DLT → IOP (Factorization # 2)

$$N = D\,D^T$$



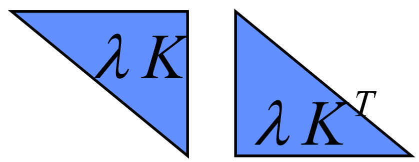
$$CHO(N^{-1}) = M$$

$$M^T\,M = N^{-1}$$



$$N^{-1} = M^T\,M$$

$$N = M^{-1}\,M^{T^{-1}} = \lambda^2\;K\;K^T$$



$$\lambda\,K = [CHO(\{DD^T\}^{-1})]^{-1}$$

DLT → Rotation Angles

$$\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x & -\alpha\, c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \lambda \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix}^{-1} \begin{bmatrix} -c_x & -\alpha\, c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix}$$

- Using the rotation matrix R, one can derive the individual rotation angles ω, ϕ and κ.

Experimental Results

Input Data

	Exp. 1	Exp. 2
x_p(mm)	0.0	20.0
y_p(mm)	0.0	20.0
c_x(mm)	150.0	150.0
c_y(mm)	140.0	140.0

	Exp. 1	Exp. 2
X_O(m)	1000.0	1000.0
Y_O(m)	1000.0	1000.0
Z_O(m)	2000.0	2000.0
ω(°)	3.0	3.0
ϕ(°)	3.0	3.0
κ(°)	3.0	3.0

Object Space Points
Experiments 1 & 2

Pt # 1	-200.0(m)	-200.0(m)	100.0(m)
Pt # 2	-200.0(m)	2200.0(m)	100.0(m)
Pt # 3	2200.0(m)	2200.0(m)	100.0(m)
Pt # 4	2200.0(m)	-200.0(m)	100.0(m)
Pt # 5	2200.0(m)	1000.0(m)	100.0(m)
Pt # 6	200.0(m)	1000.0(m)	100.0(m)
Pt # 7	900.0(m)	2000.0(m)	050.0(m)
Pt # 8	1100.0(m)	100.0(m)	150.0(m)

DLT Parameters

0.07499730	0.00413615	-0.00371907	-71.69530893
-0.00366841	0.06998742	0.00386040	-74.03981347
-0.00002624	0.00002620	-0.00049998	**1.0**

Experiment # 1

0.07447252	0.00466020	-0.01371871	-51.69530893
-0.00419319	0.07051148	-0.00613924	-54.03981347
-0.00002624	0.00002620	-0.00049998	**1.0**

Experiment # 2

Calibration Matrix (Factorization # 1)

-150.00062621	-0.00001489	-0.00040343
0.00000000	-140.00058695	-0.00076907
0.00000000	0.00000000	1.00000000

Experiment # 1

-150.00067543	-0.00016177	19.99967926
0.00000000	-140.00069981	19.99931466
0.00000000	0.00000000	1.00000000

Experiment # 2

Calibration Matrix (Factorization # 2)

150.00062621	0.00001489	-0.00040343
0.00000000	140.00058695	-0.00076907
0.00000000	0.00000000	1.00000000

Experiment # 1

150.00067543	0.00016177	19.99967926
0.00000000	140.00069981	19.99931466
0.00000000	0.00000000	1.00000000

Experiment # 2

Rotation Matrix (Factorization # 1)

0.99726081	-0.05226448	0.05233836
0.05499963	0.99711789	-0.05225857
-0.04945624	0.05499402	0.99726112

Experiment # 1

0.99726083	-0.05226412	0.05233836
0.05499927	0.99711791	-0.05225857
-0.04945626	0.05499400	0.99726112

Experiment # 2

Rotation Matrix (Factorization # 2)

0.99726081	-0.05226448	-0.05233836
0.05499963	0.99711789	0.05225857
-0.04945624	0.05499402	-0.99726112

Experiment # 1

0.99726083	-0.05226412	-0.05233836
0.05499927	0.99711791	0.05225857
-0.04945626	0.05499400	-0.99726112

Experiment # 2

RR^T (Factorization # 1)

1.0	**0.0**	**0.0**
0.0	**1.0**	**0.0**
0.0	**0.0**	**1.0**

Experiment # 1

1.0	**0.0**	**0.0**
0.0	**1.0**	**0.0**
0.0	**0.0**	**1.0**

Experiment # 2

RR^T (Factorization # 2)

1.0	**0.0**	**0.0**
0.0	**1.0**	**0.0**
0.0	**0.0**	**1.0**

Experiment # 1

1.0	**0.0**	**0.0**
0.0	**1.0**	**0.0**
0.0	**0.0**	**1.0**

Experiment # 2

Location of the Perspective Center

	Experiment 1	Experiment 2
X_O (m)	999.99998323	1000.00011584
Y_O (m)	1000.00004851	1000.00005583
Z_O (m)	2000.00000342	1999.99999685

Analysis

Perspective Center

$$\begin{bmatrix} L_4 \\ L_8 \\ L_{12} \end{bmatrix} = - \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{12} \end{bmatrix} \begin{bmatrix} X_O \\ Y_O \\ Z_O \end{bmatrix}$$

$$L_1 X_O + L_2 Y_O + L_3 Z_O = -L_4$$
$$L_5 X_O + L_6 Y_O + L_7 Z_O = -L_8$$
$$L_9 X_O + L_{10} Y_O + L_{11} Z_O = -L_{12}$$

- (X_O, Y_O, Z_O) is the intersection point of three different planes whose surface normals are (L_1, L_2, L_3), (L_5, L_6, L_7) and (L_9, L_{10}, L_{11}), respectively.

Perspective Center

$$\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x & -\alpha c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

- Assuming:
  - $x_p \approx 0.0$ and $y_p \approx 0.0$
  - $-\alpha c_x \approx 0.0$

$$\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x r_{11} & -c_x r_{21} & -c_x r_{31} \\ -c_y r_{12} & -c_y r_{22} & -c_y r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

- The three surfaces are orthogonal to each other.
  - This would lead to better intersection.

Perspective Center

$$\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x & -\alpha\, c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

- Assuming:
  - $x_p \neq 0.0$ and $y_p \neq 0.0$
  - $-\alpha c_x \approx 0.0$

$$\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x\, r_{11} + x_p\, r_{13} & -c_x\, r_{21} + x_p\, r_{23} & -c_x\, r_{31} + x_p\, r_{33} \\ -c_y\, r_{12} + y_p\, r_{13} & -c_y\, r_{22} + y_p\, r_{23} & -c_y\, r_{32} + y_p\, r_{33} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

- As x_p and y_p increase, the surface normals become almost parallel.
  - This would lead to weak intersection.

Rotation Angles

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \lambda \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix}^{-1} \begin{bmatrix} -c_x & -\alpha\, c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix}$$

- Assuming:
  - $x_p \approx 0.0$ and $y_p \approx 0.0$
  - $-\alpha c_x \approx 0.0$

$$\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x\, r_{11} & -c_x\, r_{21} & -c_x\, r_{31} \\ -c_y\, r_{12} & -c_y\, r_{22} & -c_y\, r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

- The rows of D are not correlated:
  - They are orthogonal to each other.
- L^{-1} is well defined.

Rotation Angles

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \lambda \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix}^{-1} \begin{bmatrix} -c_x & -\alpha\, c_x & x_p \\ 0 & -c_y & y_p \\ 0 & 0 & 1 \end{bmatrix}$$

- Assuming:
  - $x_p \neq 0.0$ and $y_p \neq 0.0$
  - $-\alpha c_x \approx 0.0$

$$\begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} = \lambda \begin{bmatrix} -c_x\, r_{11} + x_p\, r_{13} & -c_x\, r_{21} + x_p\, r_{23} & -c_x\, r_{31} + x_p\, r_{33} \\ -c_y\, r_{12} + y_p\, r_{13} & -c_y\, r_{22} + y_p\, r_{23} & -c_y\, r_{32} + y_p\, r_{33} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

- The rows of D tend to be highly correlated.
- L^{-1} is not well defined.