An Algorithm Development and Computations of Tower Crane Foundation

Rizal Purnawan

Civil Engineer, Independent Researcher, Jakarta, Indonesia

Corresponding author. E-mail: rizalpurnawan23@gmail.com; ORCID: 0000-0001-8858-4036

5 July 2023

Abstract

This document presents an algorithm development for computing the foundation of tower cranes in Python. The theoretical framework of the algorithm will be presented in detail by employing mathematical formalism. A specific computational demonstration will also be presented.

1 Introduction

Computing tower crane (TC) foundation could be very complicated without a state of the art software which are mostly commercial. The complicated process is determining the forces at the footing of the TC which are transmitted into the foundation. In this paper, we attempt to develop an algorithm in Python (van Rossum, 2018) for computing the TC forces at the footing.

In our approach, we simplify the model of the TC as point vectors in Euclidean 3-space (Rudin, 1964; Roman, 2005). And at each point, we assign a force vector representing either the weight of the TC at the point or a working load at the point. This mathematical model is suitable for computer due to the superiority of computer in processing finite dimensional vector (Roman, 2005).

2 Mathematical Model of Tower Crane Structure

2.1 Predefined Mathematical Symbols

In this report, there will be several predefined mathematical symbols to be incorporated. We present the description of the symbols on the following list:

$\mathbb{N}$	The set of all natural numbers (Stoll, 1963; Judson, 2012)
$\mathbb{R}$	The set of all real numbers (Stoll, 1963; Judson, 2012)
$\mathbb{R}^3$	Three dimensional Euclidean space (Roman, 2005)
∞	Infinity (Stoll, 1963)
$\subset$	Set inclusion (Stoll, 1963)
$\in$	Set membership (Stoll, 1963)
$\forall$	Universal quantification in formal first order logic (Bergmann et al., 2014)
$\exists$	Existential quantification in formal first order logic (Bergmann et al., 2014)
$\mathbf{A}$	Any boldface upper case letter is a force vector of 3D Euclidean space (Strang, 2021)
$\mathbf{a}$	Any boldface lower case letter is a position vector of 2D or 3D Euclidean space (Strang, 2021)
$\|\cdot\|$	Standard Euclidean norm (Roman, 2005)
$\langle\cdot,\cdot\rangle$	Standard Euclidean inner product (Roman, 2005)
(a, b)	Open interval of real numbers with $a, b \in \mathbb{R}$ and $a < b$ (Rudin, 1964; Andre, 2020)
$[a, b]$	Closed interval of real numbers with $a, b \in \mathbb{R}$ and $a < b$ (Rudin, 1964; Andre, 2020)
$[n]$	Integer interval, equal to $\{0, 1, \ldots, n\}$, for some $n \in \mathbb{N}$.

If there exist other symbols not mentioned above, then the symbols are to be explained during the discussion.

2.2 Linear Algebraic Model of the Structure

In most TC structures in the market, the structural body of a TC are made of an assembly of parts or sections. The computational method in this analysis is by modelling these parts into point vectors in a vector space together with the respective force vectors, and the computation is conducted using computer programming. Thus, we necessarily create a vector space $V \subseteq \mathbb{R}^3$ (Roman, 2005) such that all the TC parts can be modelled as point vectors in V. And we call V a Tower Crane Space (TCS).

The orientation for the TCS shall be properly defined such that the TC model in the TCS can be physically understood. Suppose

$$\mathcal{B} := (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)\,,$$

with

$$\mathbf{e}_1 := \begin{bmatrix}1\\0\\0\end{bmatrix}\,, \quad \mathbf{e}_2 := \begin{bmatrix}0\\1\\0\end{bmatrix}\,, \quad \mathbf{e}_3 := \begin{bmatrix}0\\0\\1\end{bmatrix}\,,$$

is an ordered basis (Roman, 2005) of V. Note that $\mathcal{B}$ is an orthonormal basis (Strang, 2021) for V, since

$$\forall \alpha, \beta \in \{1,2,3\}\,,\ \alpha \neq \beta \iff \langle \mathbf{e}_\alpha, \mathbf{e} \rangle_V = 0$$

and

$$\forall \alpha \in \{1,2,3\}\,,\ \|\mathbf{e}_\alpha\|_V$$

hold, where $\langle \cdot, \cdot \rangle_V : V \times V \to V$ is the standard Euclidean innerproduct (Strang, 2021; Roman, 2005) on V and $\| \cdot \|_V : V \to \mathbb{R}$ is the standard Euclidean norm (Strang, 2021; Roman, 2005) on V. We designate $\mathbf{e}_1$ as well as $\mathbf{e}_2$ to represent the horizontal directions and $\mathbf{e}_3$ to represent the vertical direction in terms of physical orientation.

Now we model the TC parts as point vectors in V. Suppose there are N parts of the TC body, for some $N \in \mathbb{N}$. Then, let the map

$$\begin{aligned} \mathbf{c} : [N] &\to V \\ k &\mapsto \mathbf{c}_k \end{aligned}$$

represent a TC part in V. Note that $[N]$ is an integer interval, defined by

$$[N] := \{0, 1, \ldots, N\}\,.$$

As a convention, let $\mathbf{c}_0$ represent the TC support (the connection of the TC and the foundation). And also, $\mathbf{c}_0$ is not necessarily the origin of V. The values for $\mathbf{c}_1, \ldots, \mathbf{c}_N$ can be assigned from the TC brochure in use. For convenience, we denote each component of point vectors within the expression

$$\forall k \in [N] \forall \alpha \in \{1,2,3\}\,,\ c_{k,\alpha} := \langle \mathbf{e}_\alpha, \mathbf{c}_k \rangle_V\,.$$

The weight of each part of the TC as well as its attributes shall also be defined, and we define the weight as a force vector. We call the weight the dead load. Let

$$\begin{aligned} \mathbf{D} : [N] &\to V \\ k &\mapsto \mathbf{D}_k \end{aligned}$$

represent the weight of each part of the TC in V. Note that weights are of gravitaional direction, thus,

$$\forall k \in [N] \forall \alpha \in \{1,2,3\}\,,\ \alpha \in \{1,2\} \implies \langle \mathbf{e}_\alpha, \mathbf{D}_k \rangle_V = 0$$

and

$$\forall k \in [N]\,,\ \|\mathbf{D}_k\|_V = |\langle \mathbf{e}_3, \mathbf{D}_k \rangle_V|$$

necessarily hold.

Suppose $p \in \mathbb{N}$ such that $\mathbf{c}_p$ is the point vector representing the jib part in which the lifting load is located. And suppose $\mathbf{L}_L \in V$ is the force vector representing the lifting load. Note that $\mathbf{L}_L$ is also of gravitational direction. Thus,

$$\forall \alpha \in \{1,2,3\}\,,\ \alpha \in \{1,2\} \implies \langle \mathbf{e}_\alpha, \mathbf{L}_L \rangle_V = 0$$

and

$$\|\mathbf{L}_L\|_V = |\langle \mathbf{e}_3, \mathbf{L}_L \rangle_V| =: L_L$$

necessarily hold.

Suppose $b \in \mathbb{N}$ such that $\mathbf{c}_b$ is the point vector representing the counter jib part in which the counter balance is located. And suppose $\mathbf{L}_C \in V$ is the force vector representing the counter balance. Note that $\mathbf{L}_C$ is also of gravitational direction. Thus,

$$\forall \alpha \in \{1, 2, 3\}, \ \alpha \in \{1, 2\} \implies \langle \mathbf{e}_\alpha, \mathbf{L}_C \rangle_V = 0$$

and

$$\|\mathbf{L}_C\|_V = |\langle \mathbf{e}_3, \mathbf{L}_C \rangle_P| =: L_C$$

necessarily hold.

Another load that needs to be taken into account for a TC is the wind load. Suppose the map

$$\begin{aligned} \mathbf{W} : [N] &\to V \\ k &\mapsto \mathbf{W}_k \end{aligned}$$

represent the wind load at each part of the TC in V. Unlike the weights, the lifting load and the counter balance, the wind load can be of any direction in V. Thus, the value assignment of $\mathbf{W}_0, \mathbf{W}_1, \ldots, \mathbf{W}_N$ shall be made by the engineer in accordance with the applicable code. For convenience, we denote each component of the wind vectors by

$$\forall k \in [N] \forall \alpha \in \{1, 2, 3\}, \ W_{k,\alpha} := \langle \mathbf{e}_\alpha, \mathbf{W}_k \rangle_V .$$

The visual model of the TC is presented in figure 1. In the point B of figure 1, the black circles represent $\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_N$, the yellow arrows represent $\mathbf{D}_0, \mathbf{D}_1, \ldots, \mathbf{D}_N$, the red arrows represent $\mathbf{L}_L$ as well as $\mathbf{L}_C$ and the blue arrows represent $\mathbf{W}_0, \mathbf{W}_1, \ldots, \mathbf{W}_N$.

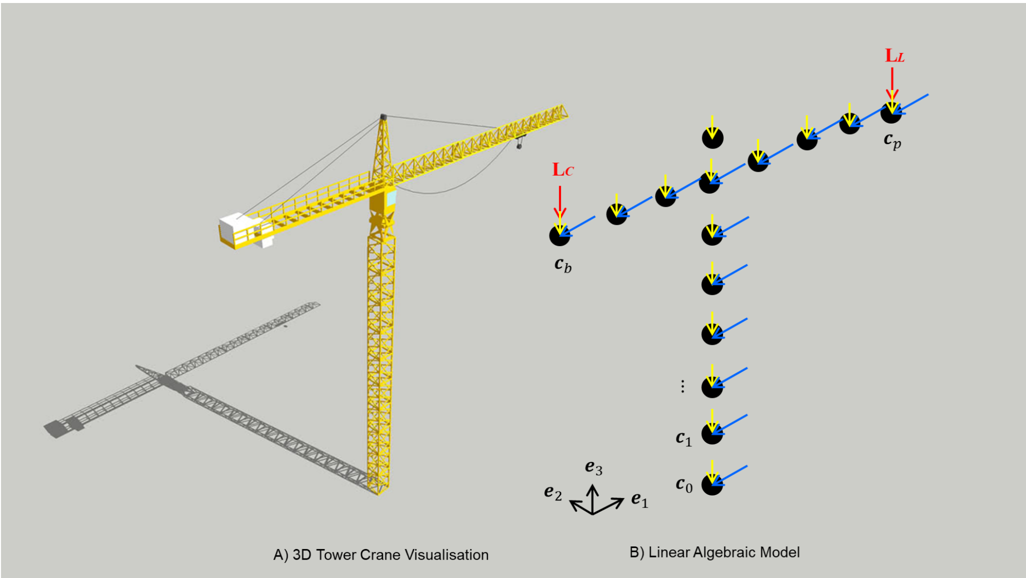

Figure 1: Visual representation of the linear algebraic model of TC

The force and moment reactions vectors at the support of the TC are denoted by $\mathbf{F}_r, \mathbf{M}_r \in V$ respectively. Follows from the force and moment equilibrium principle, we have expressions

$$\mathbf{F}_r + \mathbf{L}_L + \mathbf{L}_C + \sum_{k=0}^{N} (\mathbf{D}_k + \mathbf{W}_k) = \mathbf{0}$$

and

$$\mathbf{M}_r + (\mathbf{c}_0 - \mathbf{c}_0) \times \mathbf{F}_r + (\mathbf{c}_p - \mathbf{c}_0) \times \mathbf{L}_L + (\mathbf{c}_b - \mathbf{c}_0) \times \mathbf{L}_C + \sum_{k=0}^{N} (\mathbf{c}_k - \mathbf{c}_0) \times (\mathbf{D}_k + \mathbf{W}_k) = \mathbf{0} ,$$

where $\times$ is the cross product vector (Strang, 2021) in V. Then the force and the moment reactions vectors are given by

$$\mathbf{F}_r = -\left[\mathbf{L}_L + \mathbf{L}_C + \sum_{k=0}^{N} \mathbf{D}_k + \mathbf{W}_k \right] \tag{1}$$

and

$$\mathbf{M}_r = -\left[(\mathbf{c}_p - \mathbf{c}_0) \times \mathbf{L}_L + (\mathbf{c}_b - \mathbf{c}_0) \times \mathbf{L}_C + \sum_{k=0}^{N} (\mathbf{c}_k - \mathbf{c}_0) \times (\mathbf{D}_k + \mathbf{W}_k)\right] \tag{2}$$

respectively. And the components of the force and moment vectors of the support reaction are given by

$$\forall \alpha \in \{1, 2, 3\}\,,\ F_{r,\alpha} := \langle \mathbf{e}_\alpha, \mathbf{F}_r \rangle_V$$

and

$$\forall \alpha \in \{1, 2, 3\}\,,\ M_{r,\alpha} := \langle \mathbf{e}_\alpha, \mathbf{M}_r \rangle_V$$

respectively.

3 Algorithm of the Computation

We create an algorithm in `Python` (van Rossum, 2018) to conduct the computational modelling by implementing the linear algebraic model as presented on section 2.2. We also use `Numpy` (Harris et al., 2020), a Pythonic library for dealing with linear algebra-like data structures. Note that both `Python` and `Numpy` are of permissive software license, which means everyone can use these programmes for free.

The algorithms for computing the support reaction of the TC and the capacity of piles can be accessed at a GitHub repository via the following links:

1. `tc_solver`
   `https://github.com/rizalpurnawan23/Algorithms-in-Python/blob/main/tc_solver.py`

2. `pile_analysis`
   `https://github.com/rizalpurnawan23/Algorithms-in-Python/blob/main/pile_analysis.py`

4 Computing Support Reaction of Tower Crane

In this section, we will demonstrate the use of the algorithm for computing the support reaction of a TC in a project. The specification is given as follows:

1. D is the dead load. The dead load consists of the self weight of the TC as well as its attributes. The self weight of the TC is taken in accordance with the TC product specification in use. The counter weight will also be considered as a dead load, since it remains stationary while the lifting load may be removed or loaded.

2. L is the live load. We designate the crane's lifting load as the live load. The amount of the live load is the crane lifting capacity which is equal to 3000 kg ($\approx$ 30 kN) as described on attachment 2.

3. W_x, W_y are the wind loads in terms of $\mathbf{e}_1$ and $\mathbf{e}_2$ directions respectively. The wind loading is determined in accordance with the environmental condition at the site as well as SNI 1727:2020.

The load combinations for the design is taken in accordance with the project design basis as well as SNI 1727:2020, and are presented in table 1.

We divide the crane structure into 18 nodes where node 1 to node 5 are the tower sections, node 6 is the top section of the tower, node 7 to node 16 are the jib sections, node 17 is the hook block and node 18 is the counter jib section. And there are two additional nodes, node 19 and node 20 which are the nodes representing the point vectors of the lifting load and the counterweight respectively. The detailed values of the loading is presented on table 2. In this design, we define $\mathbf{c}_0$ as the origin of the TCS, that is, $\mathbf{c}_0 = 0\mathbf{e}_1 + 0\mathbf{e}_2 + 0\mathbf{e}_3$. And the dimensions of the pile cap in use are 6.2 $\times$ 6.2 m^2 of horizontal area and 1.5 m of thickness. For the wind load, the underlying wind speed is taken as 20 m/s, and the resulting wind load at each node is presented in table 2.

The computation is conducted using `JupyterLab` (Pérez and Granger, 2007), which can be accessed via the link below:

`https://github.com/rizalpurnawan23/My-Engineering-Projects/blob/main/Computational-Files/TC_Computation.ipynb`

The result of the computation is given by tables 3 and 4 for ASD and LRFD respectively.

Table 1: Load combinations

No. LC	Load Combination	LRFD/ASD
153	$1.4D$	LRFD
154	$1.2D + 1.6L$	LRFD
155	$1.2D + 1.0L + 1.0W_x$	LRFD
157	$1.2D + 1.0L + 1.0W_y$	LRFD
159	$0.9D + 1.0W_x$	LRFD
161	$0.9D + 1.0W_y$	LRFD
357	$1.0D$	ASD
358	$1.0D + 1.0L$	ASD
359	$1.0D + 0.75L(0.6W_x)$	ASD
361	$1.0D + 0.75L(0.6W_y)$	ASD
363	$0.9D + 0.9W_x$	ASD
365	$0.9D + 0.9W_y$	ASD

Table 2: Preliminary TC data from the brochure as well SNI 1727:2020 for wind loads

No.	Sections	Length	Comp. of Point Vectors			Self Weight	Components of $\mathbf{W}_x$			Components of $\mathbf{W}_y$		
		$\lVert\mathbf{c}_k\rVert_V$ (m)	$c_{k,1}$	$c_{k,2}$	$c_{k,3}$	$\lVert\mathbf{D}_k\rVert_V$ (kN)	$W_{x_{k,1}}$ (kN)	$W_{x_{k,2}}$ (kN)	$W_{x_{k,3}}$ (kN)	$W_{y_{k,1}}$ (kN)	$W_{y_{k,2}}$ (kN)	$W_{y_{k,3}}$ (kN)
1	Tower 1	5.920	0.000	0.000	2.960	45.000	0.078	0.024	0.000	0.024	0.078	0.000
2	Tower 2	5.920	0.000	0.000	8.880	45.000	0.533	0.160	0.000	0.160	0.533	0.000
3	Tower 3	5.920	0.000	0.000	14.800	45.000	0.652	0.196	0.000	0.196	0.652	0.000
4	Tower 4	5.920	0.000	0.000	20.720	45.000	0.706	0.212	0.000	0.212	0.706	0.000
5	Tower 5, Cab	5.920	0.000	0.000	26.640	54.080	0.744	0.223	0.000	0.223	0.744	0.000
6	Top Section	8.900	0.000	0.000	34.050	35.400	1.178	0.354	0.000	0.354	1.178	0.000
7	Jib 1	11.800	5.900	0.000	29.600	36.300	0.118	0.036	0.000	0.406	1.353	0.000
8	Jib 2	11.800	17.700	0.000	29.600	22.700	0.118	0.036	0.000	0.406	1.353	0.000
9	Jib 3	11.800	29.500	0.000	29.600	19.500	0.118	0.036	0.000	0.406	1.353	0.000
10	Jib 4	11.800	41.300	0.000	29.600	18.600	0.118	0.036	0.000	0.406	1.353	0.000
11	Jib 5	6.100	50.250	0.000	29.600	9.070	0.061	0.018	0.000	0.210	0.700	0.000
12	Jib 6	6.000	56.300	0.000	29.600	6.350	0.060	0.018	0.000	0.206	0.688	0.000
13	Jib 7	6.000	62.300	0.000	29.600	5.900	0.060	0.018	0.000	0.206	0.688	0.000
15	Jib 9	3.090	66.845	0.000	29.600	4.540	0.031	0.009	0.000	0.106	0.354	0.000
16	Jib A-Frame	6.570	71.675	0.000	29.600	3.600	0.066	0.020	0.000	0.226	0.754	0.000
17	Hook Block	0.740	71.675	0.000	29.600	5.000	0.007	0.002	0.000	0.025	0.085	0.000
18	Counterjib	21.100	-10.550	0.000	29.600	107.450	0.212	0.064	0.000	0.726	2.420	0.000
19	Lifting Load	0.000	71.675	0.000	29.600	0.000	0.000	0.000	0.000	0.000	0.000	0.000
20	Counterweight	0.000	-21.100	0.000	29.600	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 3: ASD support reaction

Load Combination	$F_{r,1}$ (kN)	$F_{r,2}$ (kN)	$F_{r,3}$ (kN)	$M_{r,1}$ (kN)	$M_{r,2}$ (kN)	$M_{r,3}$ (kN)
1.0D	0.000	0.000	1815.730	0.000	-3409.245	0.000
1.0D + 1.0L	0.000	0.000	1845.730	0.000	-5559.644	0.000
1.0D + 0.75D + 0.75(0.6Wx)	-2.092	-0.629	1838.230	15.100	-5072.250	-3.877
1.0D + 0.75D + 0.75(0.6Wy)	-1.697	-5.657	1838.230	155.753	-5068.766	-147.735
0.6D + 0.6Wx	-2.789	-0.839	1089.438	20.133	-2112.488	-5.170
0.6D + 0.6Wy	-2.263	-7.543	1089.438	207.671	-2107.842	-196.980

Table 4: LRFD support reaction

Load Combination	$F_{r,1}$ (kN)	$F_{r,2}$ (kN)	$F_{r,3}$ (kN)	$M_{r,1}$ (kN)	$M_{r,2}$ (kN)	$M_{r,3}$ (kN)
1.4D	0.000	0.000	2542.022	0.000	-4772.942	0.000
1.2D + 1.6L	0.000	0.000	2226.876	0.000	-7531.733	0.000
1.2D + 1.0L + 1.0Wx	-4.648	-1.398	2208.876	33.555	-6353.062	-8.616
1.2D + 1.0L + 1.0Wy	-3.772	-12.572	2208.876	346.119	-6345.318	-328.299
0.9D + 1.0Wx	-4.648	-1.398	1634.157	33.555	-3179.888	-8.616
0.9D + 1.0Wy	-3.772	-12.572	1634.157	346.119	-3172.145	-328.299

5 Foundation Design

From the support reaction of the TC, we can design the foundation. The soil capacity for the design is based on a soil investigation report.

5.1 Computing Number of Piles

With the data of support reaction, the number of piles shall be determined under the consideration of the pile compressive capacity, tensile capacity and lateral capacity.

It is necessary to construct a subspace of V. Then, let $\tilde{V} \subseteq \mathbb{R}^2$ be the subset of V such that $\tilde{\mathcal{B}} := (\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2)$ is a basis for $\tilde{V}$ with

$$\tilde{\mathbf{e}}_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \tilde{\mathbf{e}}_2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Suppose $K \in \mathbb{N}$ is the number of piles for the foundation. Let $\{\mathbf{p}_i\}_{i \in [K]} \subset \tilde{V}$ be the family of point vectors representing the piles on the horizontal plane of the foundation. For simplicity, suppose the centre of the family $\{\mathbf{p}_i\}_{i \in [K]}$ is at the origin of $\tilde{V}$. Then, given a probability space (Brémaud, 2020) $([K], \mathcal{F}, \mathrm{P})$ and a random vector (Brémaud, 2020) $\mathbf{P} : [K] \to \tilde{V}$ with $\tilde{V}$ equipped with the Borel σ-algebra (Salamon, 2016) in $\tilde{V}$ defined by

$$\forall i \in [K], \ \mathbf{P}(i) := \mathbf{p}_i .$$

And the expression

$$\mathrm{E}[\mathbf{P}] = \mathbf{0}_{\tilde{V}}$$

necessarily holds, where $\mathrm{E}[\mathbf{P}]$ is the expected value (Brémaud, 2020) of $\mathbf{P}$ and $\mathbf{0}_{\tilde{V}}$ is the origin of $\tilde{V}$. It is also desired that the adjacent piles have a uniform distance for the ease in constructions. Thus,

$$\exists s \in (0, \infty) \forall k \in [K], \ \min_{i \in [K] \setminus \{k\}} \|\mathbf{p}_k - \mathbf{p}_i\|_{\tilde{V}} = s$$

necessarily holds.

5.1.1 Axial Capacity of Piles

In terms of axial force, every single pile in the pile group shall be sufficient in withstanding the axial force from the upper structure. We can compute the extreme axial force applied to a single pile in a group by

$$P := \frac{|\langle \mathbf{e}_3, \mathbf{F}_r \rangle_V|}{K} \pm \frac{p_{1_{max}} |\langle \mathbf{e}_2, \mathbf{M}_r \rangle_V|}{\sum_{i \in [K]} \langle \tilde{\mathbf{e}}_1, \mathbf{p}_i \rangle_{\tilde{V}}^2} \pm \frac{p_{2_{max}} |\langle \mathbf{e}_1, \mathbf{M}_r \rangle_V|}{\sum_{i \in [K]} \langle \tilde{\mathbf{e}}_2, \mathbf{p}_i \rangle_{\tilde{V}}^2}, \tag{3}$$

where

$$p_{1_{max}} := \max_{i \in [K]} |\langle \tilde{\mathbf{e}}_1, \mathbf{p}_i \rangle_{\tilde{V}}| \tag{4}$$

and

$$p_{2_{max}} := \max_{i \in [K]} |\langle \tilde{\mathbf{e}}_2, \mathbf{p}_i \rangle_{\tilde{V}}| . \tag{5}$$

Since $\mathbf{F}_r$ is the support reaction, then the axial force is a compression if $P > 0$, and is a tension if $P < 0$. Note that the axial capacity includes compression and tension. Suppose $C_{all}, T_{all} \in (0, \infty)$ are the allowable compression and tension respectively which shall be determined in accordane with the soil report in use. The allowable compression is taken as the sum of the allowable end bearing and the allowable skin friction. And the allowable end bearing is the ultimate end bearing divided by the factor of safety of the end bearing $SF_p > 1$, while the allowable skin friction is the the ultimate skin friction divided by the factor of safety of the skin friction $\mathrm{SF}_f > 1$. Thus, we have

$$C_{all} = \frac{Q_f}{\mathrm{SF}_f} + \frac{Q_p}{\mathrm{SF}_p} .$$

On the other hand, the allowable tension is taken as 70% of the allowable skin friction, that is,

$$T_{all} = 70\% \frac{Q_f}{\mathrm{SF}_f} = \frac{7 Q_f}{10 \mathrm{SF}_f} .$$

From the expression above, we obtain

$$Q_f = \frac{10 \mathrm{SF}_f T_{all}}{7} . \tag{6}$$

And by substituting the expression above to the expression for C_{all}, we obtain

$$Q_p = \left(C_{all} - \frac{Q_f}{\mathrm{SF}_f} \right) \mathrm{SF}_p = \left(C_{all} - \frac{10 T_{all}}{7} \right) \mathrm{SF}_p \,. \tag{7}$$

The safety factors in use are given by $\mathrm{SF}_f = \mathrm{SF}_p = 2.5$.

The capacity of a single pile in a group shall be computed as the multiplication of the corresponding efficiency factor and the allowable capacity. The efficiency factor is given in accordance with (Sayed and Bakeer, 1992) by

$$E_a = 1 - (1 - \eta' \kappa) \frac{\sum_{i \in [K]} Q_{f_k}}{\sum_{i \in [K]} Q_{f_k} + Q_{p_k}}$$

where $\eta' \in [0.6, 2.5]$ is the geometric efficiency parameter, and $\kappa \in [0.4, 0.9]$ is the group interaction factor. Note that we have a uniform pile dimension in this project. Thus, by substituting equations 6 and 7 to the expression above under this consideration, we obtain

$$\begin{aligned} E_a &= 1 - (1 - \eta' \kappa) \frac{K Q_f}{K (Q_f + Q_p)} \\ &= 1 - (1 - \eta' \kappa) \frac{Q_f}{Q_f + Q_p} \\ &= 1 - (1 - \eta' \kappa) \frac{1}{1 + \left(\frac{7}{10} \frac{C_{all}}{T_{all}} \right) \frac{\mathrm{SF}_p}{\mathrm{SF}_f}} \end{aligned}$$

Since in our project $\mathrm{SF}_f = \mathrm{SF}_p$, then

$$E_a = 1 - (1 - \eta' \kappa) \frac{10}{7} \frac{T_{all}}{C_{all}} \,. \tag{8}$$

We declare that the pile is sufficient against the compressive force if and only if $P > 0$ and

$$r_c := \frac{P}{E_a C_{all}} = \frac{P}{C_{all} - (1 - \eta' \kappa) \frac{10}{7} T_{all}} \leq 1 \,. \tag{9}$$

And the pile is sufficient against the tensile force if and only if $P < 0$ and

$$r_t := \frac{-P}{E_a C_{all}} = \frac{-P}{T_{all} - (1 - \eta' \kappa) \frac{10}{7} \frac{T_{all}^2}{C_{all}}} \leq 1 \,. \tag{10}$$

5.1.2 Lateral Capacity

The lateral force applied to every single pile can be given by the norm of the linear projection (Strang, 2021) of the force at the support reaction onto the vector $\mathbf{e}_1 + \mathbf{e}_2$ divided by the number of piles. Thus, we have

$$\begin{aligned} H &= \frac{\| \mathrm{proj}_{\mathbf{e}_1 + \mathbf{e}_2} \mathbf{F}_r \|_V}{K} \\ &= \frac{1}{K} \left\| \frac{\langle \mathbf{e}_1 + \mathbf{e}_2, \mathbf{F}_r \rangle_V}{\langle \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 \rangle_V} (\mathbf{e}_1 + \mathbf{e}_2) \right\|_V \\ &= \frac{1}{K} \frac{|\langle \mathbf{e}_1 + \mathbf{e}_2, \mathbf{F}_r \rangle_V|}{|\langle \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 \rangle_V|} \| \mathbf{e}_1 + \mathbf{e}_2 \|_V \\ &= \frac{1}{K} \frac{|\langle \mathbf{e}_1 + \mathbf{e}_2, \mathbf{F}_r \rangle_V|}{\| \mathbf{e}_1 + \mathbf{e}_2 \|_V^2} \| \mathbf{e}_1 + \mathbf{e}_2 \|_V \\ &= \frac{1}{K} \frac{|\langle \mathbf{e}_1 + \mathbf{e}_2, \mathbf{F}_r \rangle_V|}{\| \mathbf{e}_1 + \mathbf{e}_2 \|_V} \\ &= \frac{|\langle \mathbf{e}_1 + \mathbf{e}_2, \mathbf{F}_r \rangle_V|}{K \sqrt{2}} \,. \end{aligned} \tag{11}$$

Suppose $H_{all} \in (0, \infty)$ is the allowable lateral capacity provided in the soil report. The pile group efficiency factor for the lateral force is given in accordance with (Zhao and Stolarski, 1999) by

$$E_l = \begin{cases} 0.5 + 0.1 \frac{S}{D} & : 1 \leq S/D < 5 \\ 1 & : S/D \geq 5 \end{cases} \,. \tag{12}$$

And we declare that the pile is sufficient against the lateral force if and only if

$$r_h := \frac{H}{E_l H_{all}} = \frac{|\langle \mathbf{e}_1 + \mathbf{e}_2, \mathbf{F}_r \rangle_V|}{E_l H_{all} K \sqrt{2}} \leq 1 \tag{13}$$

is satisfied.

5.1.3 Analysis Result for Number of Piles

The computation for the pile capacity is conducted using the algorithm written in `Python`, as mentioned in section 3. The factor of safety used for the piles is given by $\mathrm{SF} = \mathrm{SF}_f = \mathrm{SF}_p = 1.75$, recalling that this construction is temporary. From the computation, we have 9 piles configured in 3 × 3 with the spacing of adjacent piles being 2.4 meters and the pile diameter being 400 mm. For the axial efficiency factor, η' and κ are given by $\eta' = 0.6$ and $\kappa = 0.4$. The pile capacity including the efficiency factors will be computed inside the algorithm `tc_solver`. The result of the computation is presented in table 5.

Table 5: Result of pile computation

Load Combination	$F_{r,1}$ (kN)	$F_{r,2}$ (kN)	$F_{r,3}$ (kN)	$M_{r,1}$ (kNm)	$M_{r,2}$ (kNm)	$M_{r,3}$ (kNm)	P (kN)	$-P$ (kN)	H (kN)
1.0D	0.000	0.000	1815.730	0.000	-3409.245	0.000	438.500	35.010	0.000
1.0D + 1.0L	0.000	0.000	1845.730	0.000	-5559.644	0.000	591.170	181.010	0.000
1.0D + 0.75D + 0.75(0.6Wx)	-2.092	-0.629	1838.230	15.100	-5072.250	-3.877	557.540	149.040	0.240
1.0D + 0.75D + 0.75(0.6Wy)	-1.697	-5.657	1838.230	155.753	-5068.766	-147.735	567.060	158.570	0.660
0.6D + 0.6Wx	-2.789	-0.839	1089.438	20.133	-2112.488	-5.170	269.150	27.050	0.320
0.6D + 0.6Wy	-2.263	-7.543	1089.438	207.671	-2107.842	-196.980	281.850	39.750	0.880

Load Combination	C_{all} (kN)	T_{all} (kN)	H_{all} (kN)	r_c	r_t	r_h	Satisfiability
1.0D	651.390	184.840	54.000	0.673	0.189	0.000	OK
1.0D + 1.0L	651.390	184.840	54.000	0.908	0.979	0.000	OK
1.0D + 0.75D + 0.75(0.6Wx)	651.390	184.840	54.000	0.856	0.806	0.004	OK
1.0D + 0.75D + 0.75(0.6Wy)	651.390	184.840	54.000	0.871	0.858	0.012	OK
0.6D + 0.6Wx	651.390	184.840	54.000	0.413	0.146	0.006	OK
0.6D + 0.6Wy	651.390	184.840	54.000	0.433	0.215	0.016	OK

5.2 Pile Cap Thickness

We have predetermined the thickness of the pile cap, arbitrarily 1.5 meters according to experiences. Now, this pile cap thickness needs to be checked against the breakout due to the TC and the punching shear of the piles.

5.2.1 Tensile Breakout Resistance

The thickness of the pile cap shall be sufficient in withstanding the breakout from the legs of the TC. The configuration of the legs of the TC is presented in figure 2.

Note that in our design, the TC has 4 legs. Suppose $\{\mathbf{t}_i\}_{i=1}^4 \subset \tilde{V}$ is the family of point vectors representing the legs of the TC in $\tilde{V}$. For simplicity, we assume that the centre of $\mathbf{t}_1, \ldots, \mathbf{t}_4$ is at the origin of $\tilde{V}$. Suppose a probability space (Brémaud, 2020) $(\{1,2,3,4\}, \mathcal{F}_{\{1,2,3,4\}}, \mathrm{P})$. Then given a random vector $\mathbf{T} : \{1,2,3,4\} \to \tilde{V}$ with $\tilde{V}$ equipped with a Borel σ-algebra (Salamon, 2016) in $\tilde{V}$ defined by

$$\forall i \in \{1,2,3,4\}\,,\ \mathbf{T}(i) := \mathbf{t}_i\,,$$

the expression

$$\mathrm{E}[\mathbf{T}] = \mathbf{0}_{\tilde{V}}$$

necessarily holds, where $\mathrm{E} : \mathbf{T} \mapsto \mathrm{E}[\mathbf{T}]$ is the expected value (Brémaud, 2020) of $\mathbf{T}$ and $\mathbf{0}_{\tilde{V}}$ is the origin of $\tilde{V}$. Based on the legs configuration in figure 2, the expressions

$$\forall i \in \{1,2,3,4\}\,,\ t_{1_{max}} = \max_{j \in \{1,2,3,4\}} |\langle \tilde{\mathbf{e}}_1, \mathbf{t}_j \rangle_{\tilde{V}}| = |\langle \tilde{\mathbf{e}}_1, \mathbf{t}_i \rangle_{\tilde{V}}| = \frac{\ell_1}{2}$$

and

$$\forall i \in \{1,2,3,4\}\,,\ t_{2_{max}} = \max_{j \in \{1,2,3,4\}} |\langle \tilde{\mathbf{e}}_2, \mathbf{t}_j \rangle_{\tilde{V}}| = |\langle \tilde{\mathbf{e}}_2, \mathbf{t}_i \rangle_{\tilde{V}}| = \frac{\ell_2}{2}$$

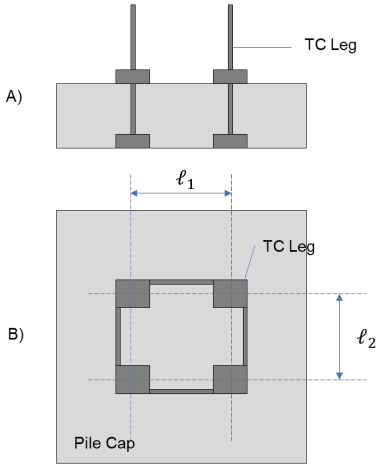

Figure 2: Configuration of the legs of the TC

necessarily hold. Then the maximum breakout force is given by

$$\begin{aligned} N_{ua} &= \max\left\{ -\left(\frac{\langle \mathbf{e}_3, \mathbf{F}_r\rangle_V}{4} - \frac{t_{1_{max}} |\langle \mathbf{e}_2, \mathbf{M}_r\rangle_V|}{\sum_{i=1}^4 \langle \mathbf{e}_1, \mathbf{t}_i\rangle_{\tilde{V}}^2} - \frac{t_{2_{max}} |\langle \mathbf{e}_1, \mathbf{M}_r\rangle_V|}{\sum_{i=1}^4 \langle \mathbf{e}_2, \mathbf{t}_i\rangle_{\tilde{V}}^2} \right),\ 0 \right\} \\ &= \max\left\{ -\left(\frac{F_{r,3}}{4} - \frac{\ell_1}{2} \frac{M_{r,2}}{\sum_{i=1}^4 \frac{\ell_1^2}{4}} - \frac{\ell_2}{2} \frac{M_{r,1}}{\sum_{i=1}^4 \frac{\ell_2^2}{4}} \right),\ 0 \right\} \\ &= \max\left\{ -\left(\frac{F_{r,3}}{4} - \frac{\ell_1}{2} \frac{M_{r,2}}{\ell_1^2} - \frac{\ell_2}{2} \frac{M_{r,1}}{\ell_2^2} \right),\ 0 \right\} \\ &= \max\left\{ -\left(\frac{F_{r,3}}{4} - \frac{M_{r,2}}{2\ell_1} - \frac{M_{r,1}}{2\ell_2} \right),\ 0 \right\} \end{aligned} \tag{14}$$

Note that $\mathbf{F}_r, \mathbf{M}_r$ are taken from the LRFD combinations. Thus, N_{ua} is the ultimate maximum breakout force.

The nominal concrete breakout strength is given in accordance with SNI 2847:2019 section 17.4.2.1 by

$$N_{cb} = \frac{A_{Nc}}{A_{Nco}} \psi_{ed,N} \psi_{c,N} \psi_{cp,N} N_b\,, \tag{15}$$

where $\psi_{ed,N}, \psi_{c,N}, \psi_{cp,N}$ are breakout edge, breakout cracking, breakout splitting factors given in accordance with SNI 2847:2019 sections 17.4.2.5 to 14.4.2.7, A_{Nc} is the projected concrete failure area that is approximated as the base of the rectilinear geometrical shape that results from projecting the failure surface outward $1.5h_{ef}$ from the centre line of the anchor, and A_{Nco} is given in accordance with SNI 2847:2019 section 17.4.2.1 by

$$A_{Nco} = (2 \cdot 1.5h_{ef})^2 = 9h_{ef}^2\,,$$

and N_b the basic concrete breakout strength given in accordance with SNI 2847:2019 section 17.4.2.2 by

$$N_b = k_c \lambda_a \sqrt{f_c' h_{ef}^3}\,. \tag{16}$$

In addition, h_{ef} is the effective embedded depth of anchors, k_c is a factor given by

$$k_c = \begin{cases} 10 & : \text{ for cast in situ} \\ 7 & : \text{ for post installation} \end{cases},$$

and λ_a is the modification factor which is taken as $\lambda_a = 1.0$ since we use a normal weight concrete material.

The breakout edge factor is given by

$$\psi_{ed,N} = \begin{cases} 1.0 & : c_{a,min} \geq 1.5h_{ef} \\ 0.7 + 0.3\frac{c_{a,min}}{1.5h_{ef}} & : c_{a,min} < 1.5h_{ef} \end{cases},$$

where $c_{a,min}$ is the least edge distance to the centre of the anchor. The breakout cracking factor is given by

$$\psi_{c,N} = \begin{cases} 1.25 & : \text{ for cast in situ} \\ 1.4 & : \text{ for post installation} \end{cases}.$$

The breakout splitting factor is given by

$$\psi_{cp,N} = \begin{cases} 1.0 & : c_{a,min} \geq c_{ac} \\ \max\left\{\frac{c_{a,min}}{c_{a,c}}, \frac{1.5h_{ef}}{c_{ac}}\right\} & : c_{a,min} < c_{ac} \end{cases}$$

and c_{ac} is given in accordance with SNI 2847:2019 section 17.7.6 and is taken as $c_{ac} = 2h_{ef}$.

The breakout strength is sufficient if the expression

$$r_{cb} := \frac{N_{ua}}{\phi N_{cb}} \leq 1\,, \tag{17}$$

where ϕ is taken in accordance with SNI 2847:2019 section 17.3.3 by $\phi = 0.75$.

In our design, we have $\ell_1 = \ell_2 = 2.15$ m, and we take h_{ef} as 70% of the total thickness of the pile cap, thus, $h_{ef} = 0.7\ 1.5 = 1.05$ m. And consequently, $c_{ac} = 2h_{ef} = 2.1$ m and $A_{Nco} = 9.1809$ m^2. And we have

$$c_{a,min} = c_{a1} = c_{a2} = \frac{5.6 - 2.15}{2} = 1.725 \text{ m}\,,$$

The projected concrete failure are is given by

$$A_{Nc} = (c_{a,min} + 1.5h_{ef})^2 = (1.725 + 1.5 \cdot 1.05)^2 = 10.4976 \text{ m}^2.$$

And in addition, the construction will be casted in situ. Thus, $k_c = 10$. And the breakout factors are given by

$$\psi_{ed,N} = 1.0\,, \quad \psi_{c,N} = 1.25\,, \quad \psi_{cp,N} = \max\left\{\frac{1.725}{2.1}, \frac{1.5 \cdot 1.05}{2.1}\right\} = 0.821\,.$$

The basic concrete breakout strength is given by

$$N_b = 10 \cdot 1.0 \cdot \sqrt{29 \cdot 1010^3} = 1832242.47 \text{ N}.$$

And the factored nominal strength of the breakout is given by

$$\phi N_{cb} = 0.75\frac{10.4976}{9.1809} \cdot 1.0 \cdot 1.25 \cdot 0.8211832242.47 = 0.75 \cdot 4120987.547 = 3090740.66 \text{ N} = 3090.74 \text{ kN}.$$

The computation result is given in table 6.

Table 6: Computation of pile cap breakout

Load Combination	$F_{r,3}$ (kN)	$M_{r,1}$ (kNm)	$M_{r,2}$ (kNm)	N_{ua} (kN)	ϕN_{cb} (kN)	r_{cb}
1.4D	2542.02	0.00	-4772.94	474.48	3090.74	0.15
1.2D + 1.6L	2226.88	0.00	-7531.73	1194.85	3090.74	0.39
1.2D + 1.0L + 1.0Wx	2208.88	33.56	-6353.06	933.04	3090.74	0.30
1.2D + 1.0L + 1.0Wy	2208.88	346.12	-6345.32	1003.93	3090.74	0.32
0.9D + 1.0Wx	1634.16	33.56	-3179.89	338.77	3090.74	0.11
0.9D + 1.0Wy	1634.16	346.12	-3172.15	409.66	3090.74	0.13

From table 6 we obtain that the maximum value for r_{cb} is given by $r_{cb} = 0.39$. Thus, we conclude that with the used pile thickness, the breakout capacity is sufficient.

5.2.2 Punching Shear

The pile cap thickness shall also be evaluated against punching shears. Punching shears consists of the one way shear and the two way shear.

(**One-Way Shear**). For the one-way shear, the concrete shear capacity is given in accordance with SNI 2847:2019 table 22.5.5.1, and is summarised by an expression

$$V_{c,1} = 0.17\lambda\sqrt{f_c'}b_{0,1}d\,, \tag{18}$$

where $b_{0,1}$ is the critical perimeter of the possible one-way shear. It is illustrated in figure 3, where d is the distance of the extreme compressive fibre of the pile cap and the tensile rebars, D_p is the pile diameter and s_e is the distance from the exterior pile to the nearest edge of the pile cap. Note that one-way shear only occurs at the extrerior pile.

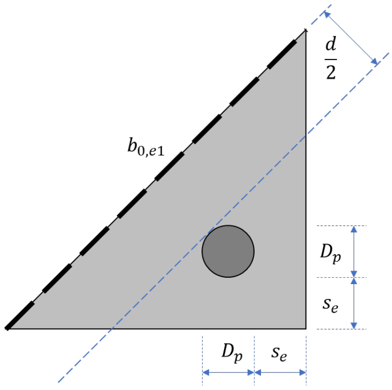

Figure 3: Possible critical perimeter of one-way shear

By observing figure 3, we obtain

$$b_{0,1} = 2\left(\frac{d}{2} + D_p + s_e\sqrt{2}\right) = d + 2D_p + s_e\sqrt{8}\,. \tag{19}$$

In our design, the pile cap width for both sides is 6.2 m, there are 3 $\times$ 3 piles with adjacent distance equal to 2.4 m. Thus,

$$s_e = \frac{6.2 - (3-1)2.4}{2} = 0.7 \text{ m}.$$

The ultimate punching shear force is given by $V_u = P$, where P is computed in accordance with equation 3 with $\mathbf{F}_r$ and $\mathbf{M}_r$ taken from the LRFD combinations. The pile cap is considered sufficient against the one-way shear if

$$r_{v,1} = \frac{V_u}{\phi_v V_c} \leq 1 \tag{20}$$

is satisfied, where $\phi_v = 0.75$ is the reduction factor for shears given in accordance with SNI 2847:2019 section 21.2.1.

(**Two-Way Shear**). For the two-way shear, the property to be controlled is the shear stress capacity of the concrete $v_{c,2}$ as stated in SNI 2847:2019 section 22.6.5.1. And it is given by

$$v_{c,2} = 0.33\lambda\sqrt{f_c'}\,. \tag{21}$$

In exchange for the ultimate shear force, one shall consider the ultimate shear stress v_u along the possible two-way critical shear perimeter $b_{0,2}$. Then, the ultimate shear stress is given by

$$v_u = \frac{V_u}{b_{0,2}d}\,. \tag{22}$$

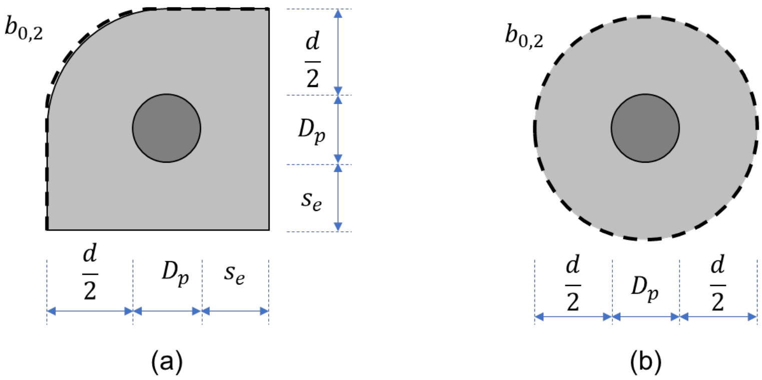

Figure 4: Two-way shear; (a) for exterior pile, (b) for interior pile

The perimeter of the two-way critical shear path is illustrated in figure 4. According to the illustration, we obtain

$$b_{0,2} = \begin{cases} 2s_e + \frac{\pi}{4}(d + D_p) = 2s_e + \frac{\pi(d+D_p)}{4} & : \text{ for exterior piles} \\ \pi(d + D_p) & : \text{ for interior piles} \end{cases} . \tag{23}$$

Equation 22 shows that $v_u \propto b_{0,2}^{-1}$. Thus, the smaller $b_{0,2}$ gets the larger v_u gets. We will prove that $b_{0,2}$ for exterior piles is smaller than that of interior piles. In this analysis, we may take $d = 0.8t_p = 1.2$ meters. Since $D_p = 0.4$ meters, then $d = 4D_p$. On the other hand, we already have that $s_e = 0.7$ meters. Thus, $s_e < 2D_p$. And we obtain

$$\begin{aligned} \pi(d + D_p) &= \frac{3\pi(d + D_p)}{4} + \frac{\pi(d + D_p)}{4} \\ &> \frac{9(d + D_p)}{4} + \frac{\pi(d + D_p)}{4} \\ &> 2(d + D_p) + \frac{\pi(d + D_p)}{4} \\ &= 5(2D_p) + \frac{\pi(d + D_p)}{4} \\ &> 5s_e + \frac{\pi(d + D_p)}{4} \\ &> 2s_e + \frac{\pi(d + D_p)}{4} \end{aligned}$$

which proves that $b_{0,2}$ for exterior piles is smaller than that of interior piles. Thus, the governing $b_{0,2}$ is given as the smallest one, that is,

$$b_{0,2} = 2s_e + \frac{\pi(d + D_p)}{4} .$$

And the pile cap is considered sufficient against two-way shear if

$$r_{v,2} = \frac{v_u}{\phi_v v_{c,2}} \leq 1 \tag{24}$$

is satisfied.

The computational result of the pile cap punching shear is given in table 7. The result shows that the pile cap is sufficient against punching shears. And we can confirm that the thickness of the pile cap is sufficient.

Table 7: Computational result of punching shear

Load Combination	One-Way Shear					Two-Way Shear					Sat.
	$b_{0,1}$ (mm)	$V_{c,1}$ (kN)	$\phi_v V_{c,1}$ (kN)	V_u (kN)	$r_{v,1}$	$b_{0,2}$ (mm)	$v_{c,2}$ (MPa)	$\phi_v v_{c,2}$ (MPa)	v_u (MPa)	$r_{v,2}$	
1.4D	3979.899	4372.212	3279.159	613.900	0.187	2656.637	1.777	1.333	0.193	0.144	OK
1.2D + 1.6L	3979.899	4372.212	3279.159	770.470	0.235	2656.637	1.777	1.333	0.242	0.181	OK
1.2D + 1.0L + 1.0Wx	3979.899	4372.212	3279.159	688.950	0.210	2656.637	1.777	1.333	0.216	0.162	OK
1.2D + 1.0L + 1.0Wy	3979.899	4372.212	3279.159	710.110	0.217	2656.637	1.777	1.333	0.223	0.167	OK
0.9D + 1.0Wx	3979.899	4372.212	3279.159	404.730	0.123	2656.637	1.777	1.333	0.127	0.095	OK
0.9D + 1.0Wy	3979.899	4372.212	3279.159	425.900	0.130	2656.637	1.777	1.333	0.134	0.100	OK

5.3 Rebar Design of Pile Cap

Throughout this subsection, let us assume $\alpha, \beta \in \{1, 2\}$. Since our design has a rectangular pile configuration, then let $K_1, K_2 \in \mathbb{N}$ such that K_1 lines of piles in the direction of $\mathbf{e}_1$ and K_2 lines of piles in the direction of $\mathbf{e}_2$. Consequently,

$$K = K_1 K_2 .$$

And suppose $\mathbf{t} \in \tilde{V}$ is a point vector representing the corresponding TC leg. And the ultimate moment of the pile cap is given by

$$\alpha \neq \beta \implies M_{u,\alpha} = K_\alpha P_u(p_{\beta max} - |\langle \tilde{\mathbf{e}}_\beta, \mathbf{t} \rangle_{\tilde{V}}|) - \frac{\gamma_c \ell_\alpha}{2}(s_e + p_{\beta_{max}} - |\langle \tilde{\mathbf{e}}_\beta, \mathbf{t} \rangle_{\tilde{V}}|)^2 , \tag{25}$$

where γ_c is the unit weight of concrete and ℓ_β is the length of the pile cap in the direction of $\tilde{\mathbf{e}}_\beta$. The pile axial force P_u can be computed from equation 3 with $\mathbf{F}_r$ and $\mathbf{M}_r$ computed from the LRFD combinations.

Now, the flexural capacity of the reinforced concrete (RC) shall be evaluated. The nominal moment of the RC is given in accordance with SNI 2847:2019 by

$$M_{n,\alpha} = f_y A_{s,\alpha} \left(d_\alpha - \frac{a_\alpha}{2} \right) , \tag{26}$$

where f_y is the rebar yield strength, $A_{s,\alpha}$ is the cross sectional area of the tensile rebars in the direction of $\tilde{\mathbf{e}}_\alpha$, d_α is the distance from the extreme compressive fibre of the concrete to the centre of the tensile rebars orthogonal to the neutral axis of the stress diagram of the concrete fibre in the direction of $\tilde{\mathbf{e}}_\alpha$ and a_α is the the height of the equivalent rectangular concrete compressive fibre according to (Whitney, 1937) in the direction of $\tilde{\mathbf{e}}_\alpha$. And follows from the force couple theorem (Du-Bois, 1902), we have

$$f_y A_{s,\alpha} = 0.85 f_c' a_\alpha b_\alpha ,$$

where b_α is the width of the pile cap in the direction of $\tilde{\mathbf{e}}_\alpha$. And note that $b_\alpha = \ell_\beta$. By algebraic manipulation, we obtain

$$a_\alpha = \frac{f_y A_{s,\alpha}}{0.85 f_c' b_\alpha} . \tag{27}$$

By substituting equation 27 to equation 26, we obtain

$$M_{n,\alpha} = f_y A_{s,\alpha} \left(d_\alpha - \frac{f_y A_{s,\alpha}}{1.7 f_c' b_\alpha} \right) . \tag{28}$$

If we observe equation 28, it is inferred that equation 28 can be expressed as a polynomial of degree two with $A_{s,\alpha}$ being the variable. Then, we obtain

$$\frac{f_y^2}{1.7 f_c' b_\alpha} A_{s,\alpha}^2 - f_y d_\alpha A_{s,\alpha} + M_{n,\alpha} = 0 . \tag{29}$$

The quadratic form gives a solution

$$A_{s,\alpha} = \frac{0.85 f_c' b_\alpha d_\alpha}{f_y} \left[1 - \sqrt{1 - \frac{2 M_{n,\alpha}}{0.85 f_c' b_\alpha d_\alpha^2}} \right] . \tag{30}$$

For a more simple expression, suppose

$$m_\alpha := \frac{f_y}{0.85 f_c'} , \quad R_{n,\alpha} := \frac{M_{n,\alpha}}{b_\alpha d_\alpha^2} .$$

Then the quadratic solution can also be given in terms of the rebars ratio by

$$\rho_\alpha := \frac{A_{s,\alpha}}{b_\alpha d_\alpha} = \left[1 - \sqrt{1 - \frac{2 m_\alpha R_{n,\alpha}}{f_y}} \right] . \tag{31}$$

Follows from SNI 2847:2019 secton 8.6.1.1, the required rebars $A_{s,\alpha}$ shall satisfy

$$A_{s,\min} \le A_{s,\alpha} \le A_{s,\max} , \tag{32}$$

where

$$A_{s,\min} := \begin{cases} 0.002 A_g & : f_y < 420 \text{ MPa} \\ \max \left\{ \frac{0.0018 \cdot 420}{f_y} A_g , \ 0.0014 A_g \right\} & : f_y \ge 420 \text{ MPa} \end{cases} \tag{33}$$

and

$$A_{s,\max} := 0.75 \rho_b b_\alpha d_\alpha \tag{34}$$

with ρ_b being the rebars ratio at the balanced state, that is, the state in which the tensile rebars reach the yield strain. The expression for ρ_b is given by

$$\rho_b = \frac{0.85 f_c' \beta_1}{f_y} \frac{600}{600 + f_y}$$

where β_1 is the Whitney factor (Whitney, 1937), and is given in accordance with SNI 2847:2019 section 22.2.2.4.3 by

$$\beta_1 := \begin{cases} 0.85 & : 17 \text{ MPa} \le f_c' \le 28 \text{ MPa} \\ 0.85 - \frac{0.05(f_c' - 28)}{7} & : 28 \text{ MPa} < f_c' < 55 \text{ MPa} \\ 0.65 & : f_c' \ge 55 \text{ MPa} \end{cases} . \tag{35}$$

The computation for the rebars is presented on the following page. The actual calculation is conducted in a Google colab sheet, which can be accessed via the link below:

`https://colab.research.google.com/drive/174dPJ045FHscF68txfltp3aYkvEhIHxx?usp=sharing`

CALCULATION OF REBAR FOUNDATION

Material Properties

Concrete compressive strength	$f'_c =$	29.00	MPa	
Rebar yield strength	$f_y =$	420.00	MPa	
Whitney factor	$\beta_1 =$	0.84		(Eq. 35)
Reduction factor for moment	$\phi =$	0.90		(SNI 2847:2019)

Geometrical Properties of Foundation

Number of piles in direction 1	$K_1 =$	3.00		
Number of piles in direction 2	$K_2 =$	3.00		
Pile cap thickness	$t_p =$	1500.00	mm	
Concrete cover	$cov =$	100.00	mm	
Rebar diameter	$D =$	25.00	mm	
Dist. from top of PC to centre of tensile rebar:				
	$d = t_p - cov - 0.5D =$	1388.00	mm	
Pile cap width in direction 1	$\ell_1 =$	6200.00	mm	
Pile cap width in direction 2	$\ell_2 =$	6200.00	mm	
TC leg distance from origin in direction 1	$\lvert\langle \mathbf{e}_1, \mathbf{t}\rangle_{\tilde{V}}\rvert =$	1075.00	mm	
TC leg distance from origin in direction 2	$\lvert\langle \mathbf{e}_2, \mathbf{t}\rangle_{\tilde{V}}\rvert =$	1075.00	mm	

Ultimate Moment

Maximum ultimate axial force of pile	$P_u =$	770.47	kN	(1.2D + 1.6L)
Ultimate moment in direction 1	$M_{u,1} =$	2604.99	kN	(Eq. 25)
Ultimate moment in direction 2	$M_{u,2} =$	2604.99	kN	(Eq. 25)

Rebar Design in Direction 1

Nominal moment	$M_{n,1} = M_{u,1}/\phi =$	2894.43	kNm	
Required rebar area	$A_{s,1} =$	4991.52	mm^2	(Eq. 30)
Minimum rebar area	$A_{s,\min,1} =$	16740.00	mm^2	(Eq. 33)
Maximum rebar area	$A_{s,\max,1} =$	187740.78	mm^2	(Eq. 34)

$$A_{s,1} < A_{s,\min} \models A_{s,1} := A_{s,\min}$$

Required number of rebars $\quad n_1 = \left\lceil \frac{A_{s,1}}{0.25\pi D^2} \right\rceil =$ 35

Rebar configuration in use : **D25–150**

Actual number of rebars $\quad n_{act,1} = \left\lfloor \frac{b_1}{150} \right\rfloor =$ 41

$$n_1 = 35 < 41 = n_{act,1} \quad \models \quad \textbf{SATISFIABLE}$$

Rebar Design in Direction 2

Nominal moment	$M_{n,2} = M_{u,2}/\phi =$	2894.43	kNm	
Required rebar area	$A_{s,2} =$	4991.52	mm^2	(Eq. 30)
Minimum rebar area	$A_{s,\min,2} =$	16740.00	mm^2	(Eq. 33)
Maximum rebar area	$A_{s,\max,2} =$	187740.78	mm^2	(Eq. 34)

$$A_{s,2} < A_{s,\min} \models A_{s,2} := A_{s,\min}$$

Required number of rebars $\quad n_2 = \left\lceil \frac{A_{s,2}}{0.25\pi D^2} \right\rceil =$ 35

Rebar configuration in use : **D25–150**

Actual number of rebars $\quad n_{act,2} = \left\lfloor \frac{b_2}{150} \right\rfloor =$ 41

$$n_2 = 35 < 41 = n_{act,2} \quad \models \quad \textbf{SATISFIABLE}$$

6 Conclusion

We have developed an algorithm for computing the support reaction of a tower crane (TC). We have presented the mathematical framework of the algorithm and we have also presented a demonstration in computing the support reaction of the TC.

The TC has a jib of length 71.68 meters and a tower section of height 29.6 meters. The result of the computation shows that a pile foundation with 9 piles of diameter 400 mm with an adjacent piles distance equal to 2.4 meters attains a factor of safety $\mathrm{SF} = 1.75$, which is deemed sufficient for a temporary construction. The pile cap design with a thickness of 1.5 meters and a horizontal dimension of 6.2×6.2 square meters is sufficient against the breakout force from the crane operation as well as the punching shear. The pile cap is equipped with a flexural rebars on both tensile and compreessive fibres with a configuration of D25–150 for both layers of fibre. The summary of the result is given in table 8.

Table 8: Summary of the result

Item	Measure	Unit
Pile diameter	400	mm
Number of piles	9	pcs
Distance of adjacent piles	2.4	m
Pile cap thickness	1.5	m
Pile cap width	6.2	m
Pile cap length	6.2	m
Rebar configuration (for both directions and both layers)	D25–150	
Rebar diameter	25	mm
Distance between adjacent rebars	150	mm

References

Andre, R. (2020). *Point Set Topology with Topics*. Author, Waterloo.

Bergmann, M., Moor, J., and Nelson, J. (2014). *The Logic Book (Sixth Edition)*. McGraw-Hill, New York.

Brémaud, P. (2020). *Probability Theory and Stochastic Processes*. Springer Nature, Cham, Switzerland.

Du-Bois, A. J. (1902). *The Mechanics of Engineering*. John Wiley & Sons, New York.

Harris, C. R., Millman, K. J., van der Walt, S. J., Gommers, R., Virtanen, P., Cournapeau, D., Wieser, E., Taylor, J., Berg, S., Smith, N. J., Kern, R., Picus, M., Hoyer, S., van Kerkwijk, M. H., Brett, M., Haldane, A., del Río, J. F., Wiebe, M., Peterson, P., Gérard-Marchant, P., Sheppard, K., Reddy, T., Weckesser, W., Abbasi, H., Gohlke, C., and Oliphant, T. E. (2020). Array programming with NumPy. *Nature*, 585(7825):357–362, DOI: `10.1038/s41586-020-2649-2`, `https://doi.org/10.1038/s41586-020-2649-2`.

Judson, T. W. (2012). *Abstract Algebra Theory and Applications*. Free Software Foundation.

Pérez, F. and Granger, B. E. (2007). IPython: a system for interactive scientific computing. *Computing in Science and Engineering*, 9(3):21–29, ISSN: `"1521-9615"`, DOI: `10.1109/MCSE.2007.53`, `"https://ipython.org"`.

Roman, S. (2005). *Advanced Linear Algebra (Second Edition)*. Springer, New York.

Rudin, W. (1964). *Principles of Mathematical Analysis*. McGraw-Hill, New York.

Salamon, D. A. (2016). *Measure and Integration*. European Mathematical Society, Zurich.

Sayed, S. M. and Bakeer, R. M. (1992). Efficiency formula for pile groups. *Journal of Geotechnical Engineering, ASCE*, 118(2):278–299.

Stoll, R. R. (1963). *Set Theory and Logic*. Dover, New York.

Strang, G. (2021). *Introduction to Linear Algebra (Edition 5)*. Wellesley-Cambridge Press, Cambridge, UK.

van Rossum, G. (2018). *Python Tutorial: Release 3.7.0*. Python Software Foundation.

Whitney, C. S. (1937). *Design of Reinforced Concrete Members Under Flexure and Combined Flexure and Direct Compression*. *ACI Journal*, March-April; 33:483–498.

Zhao, Z. and Stolarski, H. K. (1999). Stability of pile groups. Technical Report MN/RC - 1999-31, University of Minnesota and Minnesota Department of Transportation, Minnesota.