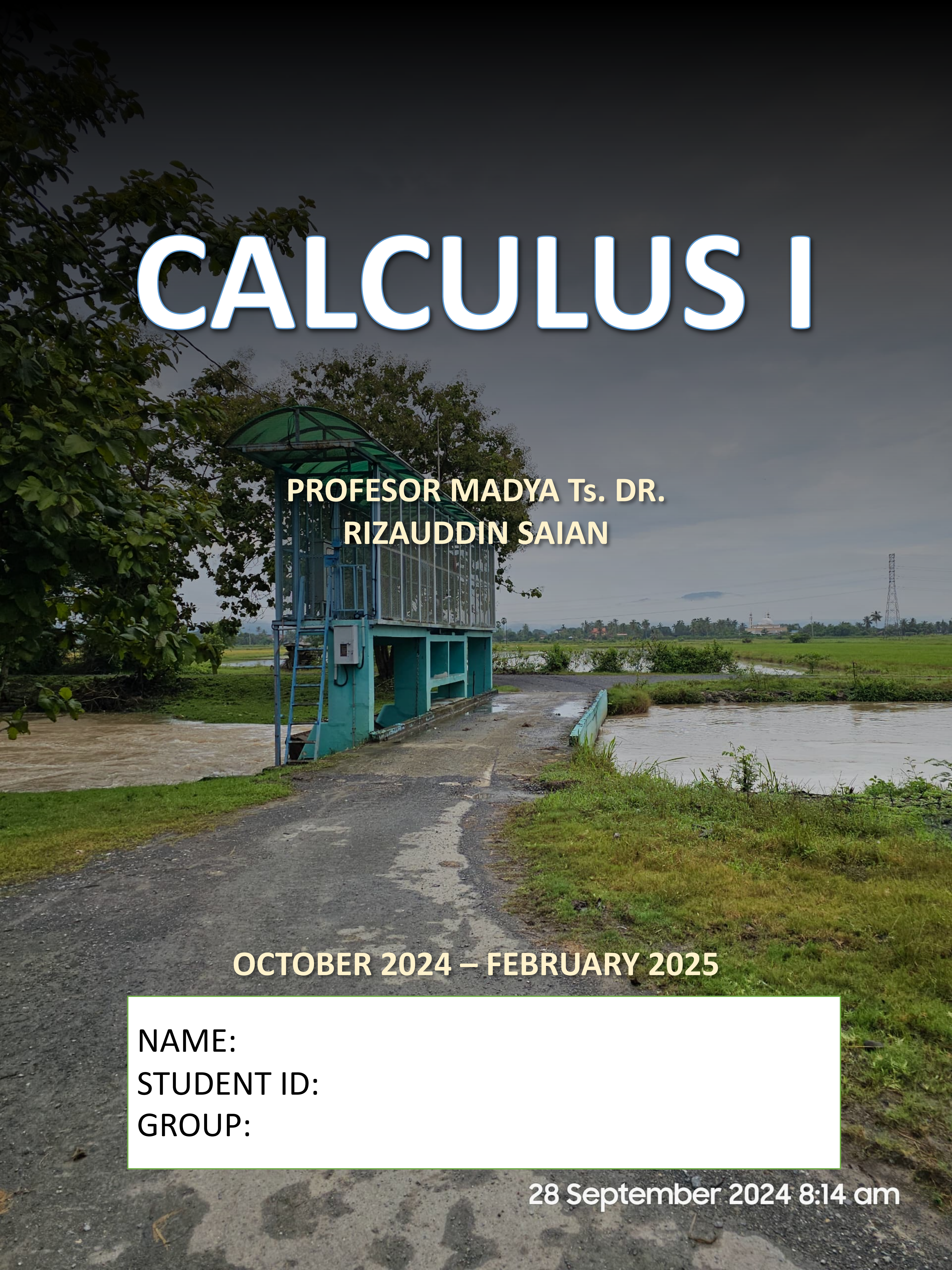

CALCULUS I

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OCTOBER 2024 – FEBRUARY 2025

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Calculus I Workbook

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Preface

Welcome to the world of Calculus I! This workbook is designed to be your trusted companion on your journey through the fundamental concepts of calculus. Whether you are a student gearing up for your first encounter with calculus, an educator looking for comprehensive teaching materials, or someone seeking to refresh their calculus skills, this workbook is tailored to meet your needs.

This workbook is carefully crafted to guide you through the essential topics of Calculus I, starting from the basic principles of limits and continuity and progressing to the heart of calculus: differentiation and integration. Each chapter is structured to provide a clear explanation of key concepts, accompanied by exercises that reinforce your understanding. The exercises are designed to challenge you and encourage active learning, allowing you to practice and master the skills necessary to solve calculus problems with confidence.

Key Features of This Workbook:

1. **Comprehensive Coverage:** This workbook covers all the fundamental concepts of Calculus I, ensuring that you have a strong foundation for advanced calculus and related subjects.

2. **Clarity and Accessibility:** Complex topics are explained in a clear and concise manner, making the material accessible to learners of all levels.

3. **Practice Exercises:** Ample practice exercises are provided throughout the workbook, ranging from basic to advanced levels of difficulty.

Remember, learning calculus is a gradual process that requires patience, practice, and perseverance. By working through this workbook diligently, you will not only grasp the principles of calculus but also develop the confidence to apply your knowledge to diverse challenges.

Best of luck in your studies, and may your exploration of calculus be both rewarding and enlightening!

Happy Learning Calculus!

Acknowledgements

As I embark on the task of acknowledging those whose support and inspiration have been instrumental in the creation of this workbook, I find myself deeply grateful to the remarkable individuals who have shaped my journey in mathematics education.

First and foremost, I express my heartfelt gratitude to my wife, whose unwavering support, encouragement, and patience have been my constant pillars. Her belief in my work and her boundless love have provided me with the strength to undertake this endeavor. To my daughters, who have brought immense joy and laughter into our lives, thank you for your understanding during the long hours spent crafting these pages. Your presence has been my source of inspiration, reminding me of the importance of education for future generations.

I extend my sincere appreciation to my teachers, whose passion for teaching ignited the spark of curiosity within me. Their dedication to nurturing young minds has been a guiding light, and I am forever indebted for their wisdom and guidance. It is through their teachings that I found my love for mathematics, a passion I aim to instill in others through this workbook.

To my previous students, your enthusiasm, questions, and thirst for knowledge have been the driving force behind this project. Your engagement in the classroom has challenged me to find innovative ways to explain complex concepts, and your success stories continue to inspire me. Each interaction with you has reinforced my belief in the transformative power of education.

I am also grateful for the support and encouragement I have received from colleagues, friends, and mentors. Your insights and discussions have enriched my understanding of calculus and teaching methodologies, shaping the content of this workbook.

Lastly, I extend my thanks to the countless authors, researchers, and educators whose contributions to the field of mathematics have paved the way for innovative teaching approaches. Your work has been a wellspring of knowledge and ideas, shaping the content of this workbook and enriching the learning experience for readers.

To all of you, I offer my deepest gratitude. This workbook stands as a testament to the collective effort and shared passion for education. May it serve as a valuable resource for learners, igniting the same love for calculus that has been kindled in me by the remarkable individuals in my life.

With heartfelt appreciation,
Rizauddin Saian
8 Oct 2023

Contents

1 Functions, Limits and Continuity

1.1 Find the Domain of a Function

Find the domain of the following functions.

1 $f(x) = x + 4$

Ans: $(-\infty, +\infty)$

2 $f(x) = x^2 + 4x + 5$

Ans: $(-\infty, +\infty)$

3 $f(x) = 4$

Ans: $(-\infty, +\infty)$

4 $f(x) = \frac{1}{4}$

Ans: $(-\infty, +\infty)$

5 $f(x) = -\sqrt{3}$

Ans: $(-\infty, +\infty)$

6 $f(x) = 0$

Ans: $(-\infty, +\infty)$

7 $f(x) = x^3 - 5$

Ans: $(-\infty, +\infty)$

8 $f(x) = \frac{2}{x+3}$

Ans: $(-\infty, -3) \cup (-3, +\infty)$

9 $f(x) = \frac{2}{x^2+3}$

Ans: $(-\infty, +\infty)$

10 $f(x) = \sqrt{x+3}$

Ans: $[-3, +\infty)$

11 $f(x) = \sqrt{x^2+3}$

Ans: $(-\infty,+\infty)$

12 $f(x) = \sqrt{x^2+2x-8}$

Ans: $(-\infty,-4] \cup [2,+\infty)$

13 $f(x) = \log(x+3)$

Ans: $(-3,+\infty)$

14 $f(x) = \log\left(x^2+3\right)$

Ans: $(-\infty,+\infty)$

15 $f(x) = \log(x^2+2x-8)$

Ans: $(-\infty,-4) \cup (2,+\infty)$

16 $f(x) = 2^{x+3}$

Ans: $(-\infty,+\infty)$

17 $f(x) = \dfrac{\sqrt{x+3}}{x-2}$

Ans: $[-3,2) \cup (2,+\infty)$

18 $f(x) = \dfrac{x+3}{\log(x-2)}$

Ans: $(2,3) \cup (3,+\infty)$

19 $f(x) = \dfrac{\sqrt{x+3}}{x^2-16}$

Ans: $[-3,4) \cup (4,+\infty)$

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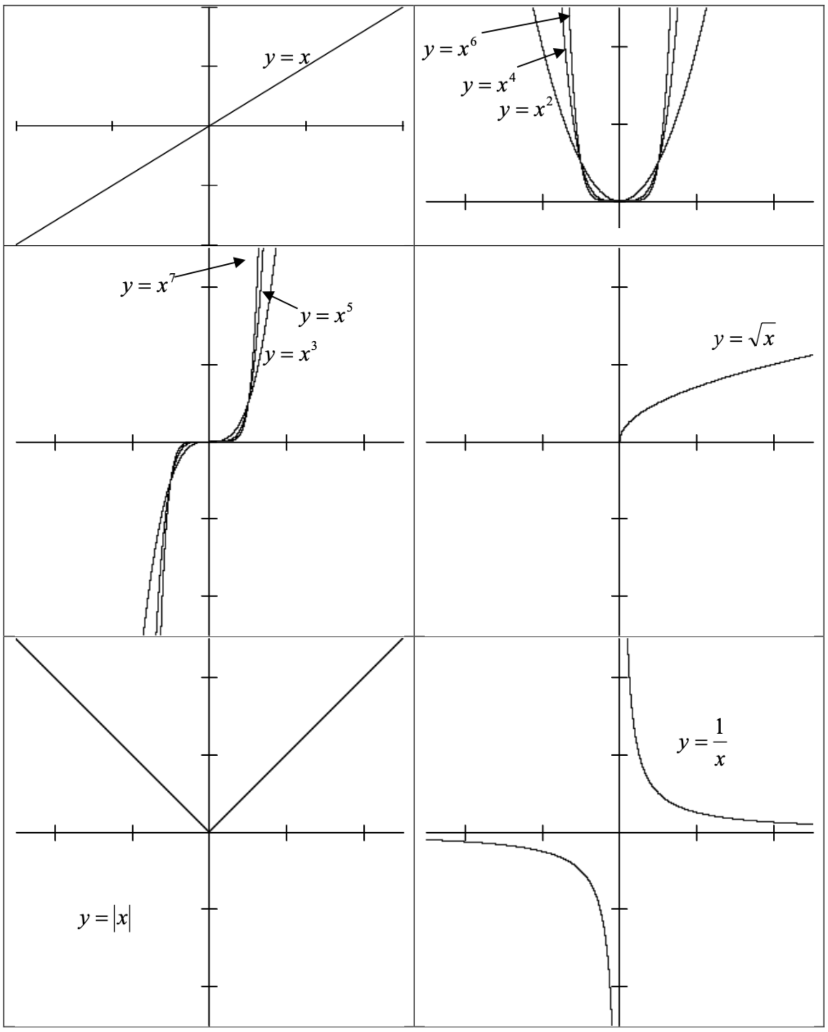
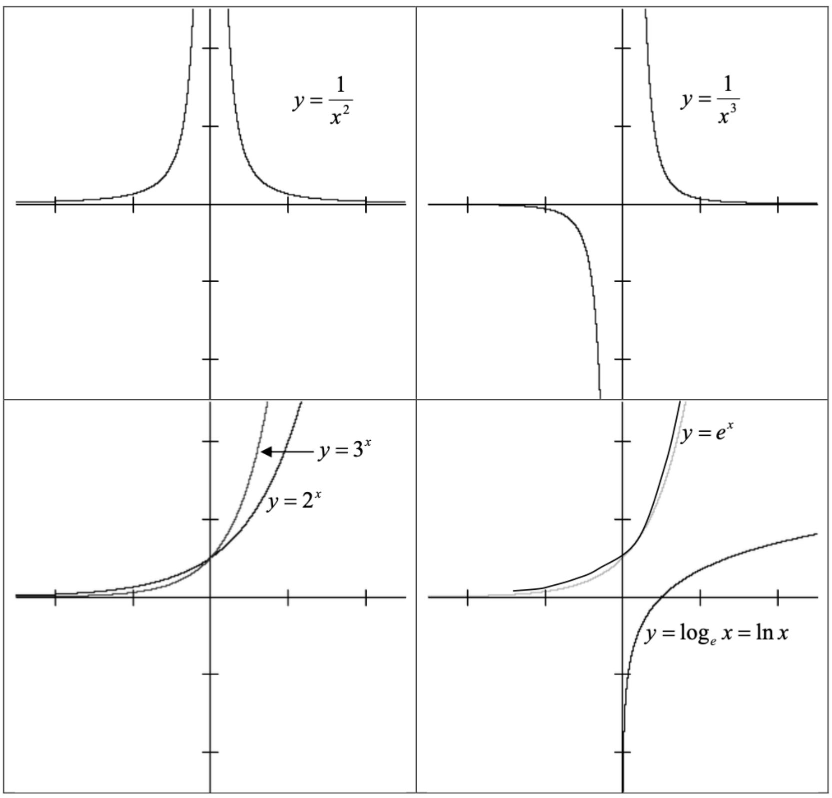
1.2 Graphs of Functions





1.3 Limits

1. One sided limits:

   (a) the limit of $f(x)$ as x approaches a from the right is L.

   $$\lim_{x\to a^+} f(x) = L$$

   (b) the limit of $f(x)$ as x approaches a from the left is L.

   $$\lim_{x\to a^-} f(x) = L$$

2. Two sided limits:

   (a) What is the limit of the following function as x approaches a?

   $$\lim_{x\to a} f(x) = L \Longleftrightarrow \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

   (b) If there is no sign in the limit notation, then you're being asked for a two-sided limit. You only have a two-sided limit if your left and right limits agree. The existence of a limit from the left-hand side does not imply that you have a right-sided limit. When we say that something has a limit, then we mean that it has an actual numeric value.

3. Infinite limits

   (a) Increases without bound

   $$\lim_{x\to a} f(x) = +\infty$$

   (b) Decreases without bound

   $$\lim_{x\to a} f(x) = -\infty$$

4. Limits at infinity

   (a) $\lim_{x\to +\infty} f(x) = L$: $f(x) \to L$ as $x \to +\infty$

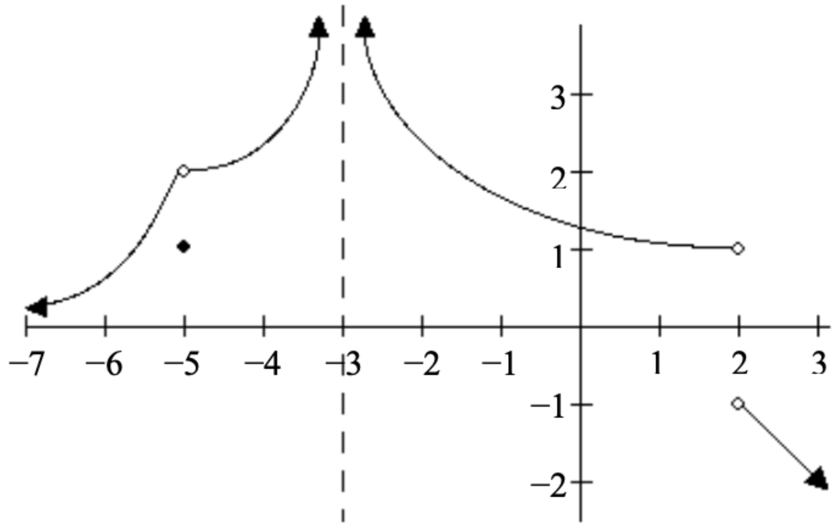
   (b) $\lim_{x\to -\infty} f(x) = L$: $f(x) \to L$ as $x \to -\infty$

5. Vertical asymptote

   If $\left(\lim_{x\to a^-} f(x) = +\infty \text{ or } \lim_{x\to a^-} f(x) = -\infty\right)$ and $\left(\lim_{x\to a^+} f(x) = +\infty \text{ or } \lim_{x\to a^+} f(x) = -\infty\right)$, then **the line** $x = a$ is called the **vertical asymptote** of the graph of a function f.

6. Horizontal asymptote If $\lim_{x\to +\infty} f(x) = L$ or $\lim_{x\to -\infty} f(x) = L$, then **the line** $y = L$ is called the **horizontal asymptote** of the graph of a function f

Consider the graph of a function



Find:

$\boxed{\mathbf{1}}$ $f(2)$

Ans: undefined

$\boxed{\mathbf{2}}$ $\lim\limits_{x\to 2^-} f(x)$

Ans: 1

$\boxed{\mathbf{3}}$ $\lim\limits_{x\to 2^+} f(x)$

Ans: −1

$\boxed{\mathbf{4}}$ $\lim\limits_{x\to 2} f(x)$

Ans: undefined

$\boxed{\mathbf{5}}$ $f(-5)$

Ans: 1

$\boxed{\mathbf{6}}$ $\lim\limits_{x\to -5^-} f(x)$

Ans: 2

$\boxed{\mathbf{7}}$ $\lim\limits_{x\to -5^+} f(x)$

Ans: 2

$\boxed{\mathbf{8}}$ $\lim\limits_{x\to -5} f(x)$

Ans: 2

$\boxed{\mathbf{9}}$ $f(-3)$

Ans: undefined

$\boxed{\mathbf{10}}$ $\lim\limits_{x\to -3^-} f(x)$

Ans: $+\infty$

$\boxed{\mathbf{11}}$ $\lim\limits_{x\to -3^+} f(x)$

Ans: $+\infty$

$\boxed{\mathbf{12}}$ $\lim\limits_{x\to -3} f(x)$

Ans: $+\infty$

$\boxed{\mathbf{13}}$ $\lim\limits_{x\to -\infty} f(x)$

Ans: 0

$\boxed{\mathbf{14}}$ $\lim\limits_{x\to +\infty} f(x)$

Ans: $-\infty$

$\boxed{\mathbf{15}}$ The vertical asymptote

Ans: $x = -3$

1.4 Computing Limits

Evaluate the following limits.

$\boxed{\mathbf{1}}$ $\lim_{x \to 3} (x+2)$

Ans: 5

$\boxed{\mathbf{2}}$ $\lim_{x \to 3} \left(x^2 + x + 2\right)$

Ans: 14

$\boxed{\mathbf{3}}$ $\lim_{x \to 3} \frac{x+2}{4-x}$

Ans: 5

$\boxed{\mathbf{4}}$ $\lim_{x \to 4} (4-x)$

Ans: 0

$\boxed{\mathbf{5}}$ $\lim_{x \to 4} \frac{4-x}{x-2}$

Ans: 0

$\boxed{\mathbf{6}}$ $\lim_{x \to 4} \frac{x-2}{4-x}$

Ans: does not exists

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1.5 Computing Limits - Case $\lim_{x\to a} f(x) = \frac{0}{0}$

Evaluate the following limits.

1 $\lim_{x\to -3} \dfrac{x^2+5x+6}{x+3}$

Ans: -1

2 $\lim_{x\to 0} \dfrac{x}{\sqrt{x+4}-2}$

Ans: 4

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1.6 Computing Limits at ∞

1.6.1 Odd vs Even Positive Integer Powers

Evaluate the following limits.

1 $\lim\limits_{x\to 2} x$

Ans: 2

2 $\lim\limits_{x\to +\infty} x$

Ans: $+\infty$

3 $\lim\limits_{x\to -2} x$

Ans: -2

4 $\lim\limits_{x\to -\infty} x$

Ans: $-\infty$

5 $\lim\limits_{x\to 2} 3$

Ans: 3

6 $\lim\limits_{x\to +\infty} 3$

Ans: 3

7 $\lim\limits_{x\to -2} x^3$

Ans: -8

8 $\lim\limits_{x\to -\infty} x^3$

Ans: $-\infty$

9 $\lim\limits_{x\to -2} x^2$

Ans: $+4$

10 $\lim\limits_{x\to -\infty} x^2$

Ans: $+\infty$

11 $\lim\limits_{x\to -2} (-x^2)$

Ans: -4

12 $\lim\limits_{x\to -\infty} (-x^2)$

Ans: $-\infty$

13 $\lim\limits_{x\to -2} (-x^3)$

Ans: $+8$

14 $\lim\limits_{x\to -\infty} (-x^3)$

Ans: $+\infty$

15 $\lim\limits_{x\to +\infty} (-3x)$

Ans: $-\infty$

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1.6.2 Polynomials

Limits at ∞ for polynomials matches limits at ∞ of its highest degree term.

Evaluate the following limits.

1 $\lim_{x\to+\infty}(3x^4+2x+3)$

Ans: $+\infty$

2 $\lim_{x\to+\infty}(-3x^4+2x+3)$

Ans: $-\infty$

3 $\lim_{x\to-\infty}(3x^4+2x+3)$

Ans: $+\infty$

4 $\lim_{x\to-\infty}(-3x^4+2x+3)$

Ans: $-\infty$

5 $\lim_{x\to-\infty}(2x^3+2x+3)$

Ans: $-\infty$

6 $\lim_{x\to-\infty}(-2x^3+2x+3)$

Ans: $+\infty$

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1.6.3 Rational Functions

1. Divide the numerator and denominator by the highest power x that occurs in the denominator.
2. $\displaystyle\lim_{x\to+\infty}\frac{1}{x} = \lim_{x\to-\infty}\frac{1}{x} = 0$

Evaluate the following limits.

1 $\displaystyle\lim_{x\to+\infty}\frac{2x+4}{5x-3}$

Ans: $\frac{2}{5}$

2 $\displaystyle\lim_{x\to-\infty}\frac{2x^2+4}{5x-3}$

Ans: $-\infty$

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1.6.4 Radical Functions

1. Find the limit first, and subsequently, calculate the square root.
2. $\lim_{x\to+\infty} \frac{1}{x} = \lim_{x\to-\infty} \frac{1}{x} = 0$

Evaluate the following limits.

1 $\lim_{x\to+\infty} \sqrt{\frac{2x+4}{5x-3}}$

Ans: $\sqrt{\frac{2}{5}}$

2 $\lim_{x\to+\infty} \frac{\sqrt{x^2+1}}{2x-3}$

Ans: $\frac{1}{2}$

3 $\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{2x-3}$

Ans: $-\frac{1}{2}$

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1.6.5 Indeterminate Form of Type $\infty - \infty$

Evaluate the following limits.

$\boxed{\mathbf{1}}$ $\lim\limits_{x \to +\infty} \left(\sqrt{x^2 - 3x} - x \right)$

Ans: $-\frac{3}{2}$

$\boxed{\mathbf{2}}$ $\lim\limits_{x \to +\infty} \left(\sqrt{x^2 + 3} - x \right)$

Ans: 0

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1.7 Limits of Trigonometric Functions

Evaluate the following limits.

1 $\lim_{x\to 0}(\sin(x))$

Ans: 0

2 $\lim_{x\to 0}(\cos(x+1))$

Ans: $\cos(1)$

3 $\lim_{x\to +\infty}\left(\cos\left(\frac{1}{x}\right)\right)$

Ans: 1

1. $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x}{\sin x} = 1$

2. $\lim_{x\to 0} \frac{\cos x - 1}{x} = \lim_{x\to 0} \frac{x}{1-\cos x} = 0$

Evaluate the following limits.

$\boxed{\mathbf{4}}$ $\lim_{x\to 0} \left(\frac{\sin(3x)}{x}\right)$

Ans: 3

$\boxed{\mathbf{5}}$ $\lim_{x\to 0} \left(\frac{\sin(3x) - x\cos(3x)}{x}\right)$

Ans: 2

Evaluate the following limits.

6 $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\pi + x\cos(x)}{2x - \sin(x)}$

Ans: $\frac{\pi-1}{\pi}$

7 $\lim\limits_{x \to 0} \dfrac{\sin(5x)}{\sin(4x)}$

Ans: $\frac{5}{4}$

8 $\lim\limits_{x \to 0} \dfrac{x}{\tan(3x)}$

Ans: $\frac{1}{3}$

9 $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ if $f(x) = \cos(x)$

Ans: $-\sin(x)$

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1.8 Continuity

f is continuous at $x = a$ if:

1. $f(a)$ is defined.
2. $\lim_{x\to a} f(x)$ exists $\Leftrightarrow \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$.
3. $\lim_{x\to a} f(x) = f(a) = L$.

1 Determine whether the function $f(x) = \dfrac{4x+12}{x^2-9}$ is continuous at $x = 0$.

Ans: continuous

2 Determine whether the function $f(x) = \dfrac{4x+1}{x^2-9}$ is continuous at $x = -3$.

Ans: discontinuous

3 Determine whether the following function is continuous at $x = 3$.

$$f(x) = \begin{cases} 2x^2 - 5 & x < 3 \\ 4x + 1 & x \geq 3 \end{cases}$$

Ans: continuous

4 Find the value(s) of k if f is continuous at $x = 1$.

$$f(x) = \begin{cases} x^2 - 2 & x < 1 \\ kx - 4 & x \geq 1 \end{cases}$$

Ans: 3

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2 Differentiation

2.1 Definition of Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Find the derivative of f using the definition of derivative.

1 $f(x) = x + 4$

Ans: 1

2 $f(x) = x^2 + 4$

Ans: $2x$

$\boxed{\mathbf{3}}$ $f(x) = \sqrt{x+4}$

Ans: $\frac{1}{2\sqrt{x+4}}$

$\boxed{\mathbf{4}}$ $f(x) = \dfrac{1}{x+4}$

Ans: $-\frac{1}{(x+4)^2}$

$\boxed{\mathbf{5}}$ $f(x) = \sin(x)$

Ans: $\cos(x)$

$\boxed{\mathbf{6}}$ $f(x) = \cos(x)$

Ans: $-\sin(x)$

Scan for guides

2.2 Techniques of Differentiation

1. Power Rule: $(x^n)' = nx^{n-1}; n \in \Re$
2. Product Rule: $[u(x)\,v(x)]' = v(x)u'(x) + u(x)v'(x)$
3. Quotient Rule: $\left[\dfrac{u(x)}{v(x)}\right]' = \dfrac{v(x)u'(x) - u(x)v'(x)}{v(x)^2}$

Find the derivative $f'(x)$.

1 $f(x) = 3x^8 - 2x^5 + 6x + 1$

Ans: $24x^7 - 10x^4 + 6$

2 $f(x) = \left(4x^2 - 1\right)\left(7x^3 + x\right)$

Ans: $140x^4 - 9x^2 - 1$

3 $f(x) = \dfrac{x^2 - 1}{x^4 + 1}$

Ans: $\dfrac{-2x^5 + 4x^3 + 2x}{(x^4 + 1)^2}$

4 $f(x) = x^2 \tan x$

Ans: $(\tan x)(2x) + x^2 \sec^2 x$

1. Derivative of logarithmic function: $\left[\log_b(u(x))\right]' = \dfrac{u'(x)}{u(x)\ln b}$
2. Derivative of natural logarithmic function: $[\ln(u(x))]' = \dfrac{u'(x)}{u(x)}$
3. Derivative of exponential function: $\left[b^{u(x)}\right]' = b^{u(x)}(\ln b)\,u'(x)$
4. Derivative of exponential function: $\left[e^{u(x)}\right]' = b^{u(x)}\,u'(x)$

$\boxed{\mathbf{5}}$ $f(x) = \log_2(x^2+1)$

Ans: $\dfrac{2x}{(x^2+1)\ln 2}$

$\boxed{\mathbf{6}}$ $f(x) = \ln(x^2+1)$

Ans: $\dfrac{2x}{x^2+1}$

$\boxed{\mathbf{7}}$ $f(x) = 2^{\sin(x)}$

Ans: $2^{\sin x}(\cos(x))\ln 2$

$\boxed{\mathbf{8}}$ $f(x) = e^{\cos x}$

Ans: $-e^{\cos x}\sin x$

2.3 The Chain Rule

Chain Rule: $(f \circ g)'(x) = f'(g(x)) \times g'(x)$

1 $f(x) = (x^3 + 2x - 3)^4$

Ans: $4(x^3+2x-3)^3(3x^2+2)$

2 $f(x) = 4\sin(x^3)$

Ans: $12x^2\cos(x^3)$

3 $f(x) = \dfrac{1}{x^3 + 2x - 3}$

Ans: $-\dfrac{3x^2+2}{(x^3+2x-3)^2}$

4 $f(x) = \sqrt{x^3 + 2x - 3}$

Ans: $\dfrac{3x^2+2}{2\sqrt{x^3+2x-3}}$

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2.4 Implicit Differentiation

1 $\dfrac{d}{dx}\left(x^2+3x+4\right)$

Ans: $2x+3$

2 $\dfrac{d}{dx}\left[\left(x^2+3x+4\right)^5\right]$

Ans: $5\left(x^2+3x+4\right)^4(2x+3)$

3 Find $\dfrac{dy}{dx}$ for $y=f(x)$.

Ans: $\dfrac{df}{dx}$

4 Find $\dfrac{dy}{dx}$ for $y=[f(x)]^5$.

Ans: $5[f(x)]^4\dfrac{df}{dx}$

5 Find $\dfrac{dy}{dx}$ for $y^3 = [f(x)]^5$.

Ans: $\dfrac{5[f(x)]^4 \frac{df}{dx}}{3y^2}$

6 Find $\dfrac{dy}{dx}$ for $y^3 = \left(x^2 + 3x + 4\right)^5$.

Ans: $\dfrac{5\left(x^2 + 3x + 4\right)^4 (2x + 3)}{3y^2}$

7 Find $\dfrac{dy}{dx}$ for $5y^2 + \sin y = x^2$.

Ans: $\dfrac{2x}{10y + \cos y}$

8 Find $\dfrac{dy}{dx}$ for $x^3 + y^3 = 3xy$ (Folium of Descartes).

Ans: $\dfrac{y - x^2}{y^2 - x}$

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2.5 Equation of Tangent Line

$$y - y_1 = m\left(x - x_1\right)$$

1 Find the equation of tangent line to $f(x) = x^2 + 1$ at $x = 2$.

Ans: $y = 4x - 3$

2 Find the equation of tangent line to $f(x) = 2x^2$ at the point $(3, 18)$.

Ans: $y = 12x - 18$

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2.6 Linear Approximations and Differentials

1. Local Linear Approximation of f at x_0

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

2. Differentials: For the function y = f (x), we define the following:

   (a) dx, called the differential of x, given by the relation $dx = \delta x$

   (b) dy, called the differential of y, given by the relation $dy = f'(x)dx$

$\boxed{\mathbf{1}}$ Find the local linear approximation of $f(x) = \sqrt{x}$ at $x_0 = 4$.

Ans: $\frac{x+4}{4}$

2 Use differentials to approximate $\sqrt{3.98}$.

Ans: $1\frac{199}{200}$

3 Use differentials to approximate $\sqrt{4.02} - \dfrac{1}{\sqrt{4.02}}$.

Ans: $1\frac{81}{160}$

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3 | Applications of Differentiation

3.1 Related Rates

Find the rate at which some quantity is changing by relating the quantity to other quantities whose rates of change is known.

1. Given $y = f(x)$ and $x = g(t)$ and a constant rate of change $\dfrac{dx}{dt}$.
2. Find the changes of y in time (or rate of change or how fast is the changing) when $x = \text{‘}something'$.
3. $\dfrac{dy}{dt} = \left.\dfrac{df}{dx}\right|_{x=\text{‘}something'} \times \dfrac{dx}{dt}$.

$\boxed{1}$ The value of x is increasing at a constant rate of 4. How fast is $y = 3x^2 + 2$ changing at the instant $x = 2$.

Ans: $\dfrac{dy}{dt} = 48$

$\boxed{2}$ The value of y is decreasing at a constant rate of 1. $x^2 + y^2 = 625$. How fast is x changing at the instant $x = 7$.

Ans: $\dfrac{dx}{dt} = \dfrac{7}{24}$

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3.2 Critical Points

$x = c$ is a critical point of $f(x)$ if:

1. $f'(c) = 0$, or
2. $f'(c)$ does not exists

Determine all the critical point(s) for the following functions:

1 $x^2 - 4x + 3$

Ans: $x = 2$

2 $x^3 - 3x + 2$

Ans: $x = \{-1, 1\}$

3 $\dfrac{x-1}{x+2}$

Ans: $x = -2$

Scan for guides

3.3 Intervals of Increasing and Decreasing Functions

Let f be a function that is continuous on a closed interval $[a,b]$ and is differentiable on the open interval (a,b).

	x on interval (a,b)		
$f'(x)$	$+$	$-$	0
$f(x)$	increasing, $f\uparrow$	decreasing $\downarrow$	constant

Determine the intervals where the following functions are decreasing or decreasing.

1 x^2-4x+3

Ans: decreasing: $\{(-\infty,2)\}$; increasing: $\{(2,+\infty)\}$

2 x^3-3x+2

Ans: decreasing: $\{(-1,1)\}$; increasing: $\{(-\infty,-1),(1,+\infty)\}$

3 $\dfrac{x-1}{x+2}$

Ans: decreasing: none; increasing: $\{(-\infty,-2),(-2,+\infty)\}$

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3.4 Concavity and Inflection Points

Let f be twice differentiable on the open interval (a, b).

	x on interval (a, b)	
$f''(x)$	$+$	$-$
$f(x)$	concave up, $f \cup$	concave down $\cap$

$x = c$ is an **inflection point** of $f(x)$ if the concavity changes at $x = c$.

1 $x^2 - 4x + 3$

Ans: concave up: $\{(-\infty, +\infty)\}$; concave down: none; no inflection point

2 $x^3 - 3x + 2$

Ans: concave up: $\{(0, +\infty)\}$; concave down: $\{(-\infty, 0)\}$; inflection point: $x = 0$

3 $\dfrac{x-1}{x+2}$

Ans: concave up: $\{(-\infty, -3)\}$; concave down: $\{(-2, +\infty)\}$; no inflection point

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3.5 Asymptotes

1. **vertical asymptotes:** vertical lines which correspond to the zeroes of the denominator of rational function.
2. **horizontal asymptotes:** $\lim\limits_{x \to \pm\infty} f(x)$

$\boxed{\mathbf{1}}$ $f(x) = \dfrac{x-1}{x+2}$

Ans: vertical asymptotes: $x = -2$; horizontal asymptotes: $y = 1$

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3.6 Curve Sketching: Even Polynomial Function

Steps:

1. $y-$intercepts
2. $x-$intercepts
3. Intervals of decrease and increase
4. Concavity and inflection points
5. Relative extrema
6. Sketch graph

Sketch the graph of the following function.

1 $f(x) = x^2 - 4x + 3$

Scan for guides

3.7 Curve Sketching: Odd Polynomial Function

Steps:

1. $y-$intercepts
2. $x-$intercepts
3. Intervals of decrease and increase
4. Concavity and inflection points
5. Relative extrema
6. Sketch graph

Sketch the graph of the following function.

1 $f(x) = x^3 - 3x + 2$

Scan for guides

3.8 Curve Sketching: Rational Function

Steps:

1. $y-$intercepts
2. $x-$intercepts
3. Intervals of decrease and increase
4. Concavity and inflection points
5. Relative extrema
6. **Asymptotes**
7. Sketch graph

Sketch the graph of the following function.

$\boxed{\mathbf{1}}$ $f(x) = \dfrac{x-1}{x+2}$

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3.9 Rolle's Theorem; Mean-Value Theorem

Mean Value Theorem: f is differentiable on (a,b) and continuous on $[a,b]$. Then, there is at least one number c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$\boxed{1}$ Determine all the numbers c which satisfy the conclusions of the Mean Value Theorem for $f(x) = x^3 + 1$ on $(0,2)$.

Ans: $x = +1.1547$

Rolle's Theorems: f is differentiable on (a,b).

$$f(a) = f(b) = 0$$

Then, there is at least one number c in (a,b) such that

$$f'(c) = 0$$

$\boxed{2}$ Determine all the numbers c which satisfy the conclusions of the Rolle's Theorem for $f(x) = x^3 - x$ on $[0,1]$.

Ans: $x = +0.5774$

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3.10 Maximum and Minimum Values of a Function

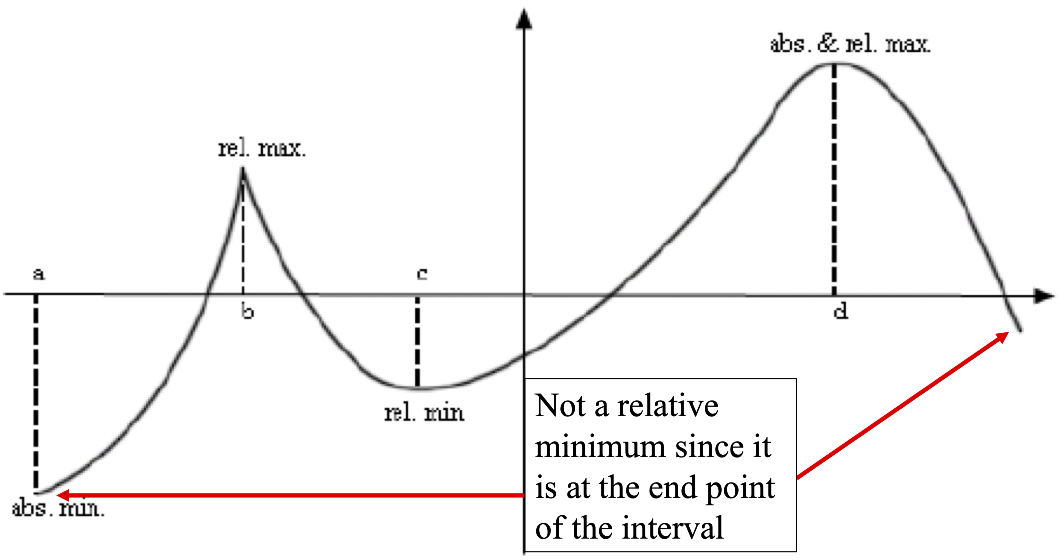

Figure 3.1: Maximum and minimum values.

1. **First Derivative test:** Suppose that f is continuous at a critical number x_0.
   (a) **Relative maximum** at x_0: $f'(x < x_0) > 0$ and $f'(x > x_0) < 0$
   (b) **Relative minimum** at x_0: $f'(x < x_0) < 0$ and $f'(x > x_0) > 0$
   (c) **No relative extremum** at x_0: No changes in sign of $f'(x_0)$
2. **Second Derivative test:** Suppose that f is twice differentiable x_0 and $f'(x_0) = 0$.
   (a) **Relative minimum** at x_0: $f'' > 0$
   (b) **Relative maximum** at x_0: $f'' < 0$
   (c) **Inconclusive:** $f'' = 0$

1 Find the relative and absolute extremum values for $f(x) = 2x^3 - 15x^2 + 36x$ on the interval $[1, 5]$.

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4 | Integration

4.1 The Indefinite Integral

$$\int b\,dx = bx + C$$

1 $\displaystyle\int 4\,dx$

Ans: $4x + C$

2 $\displaystyle\int 0.4\,dy$

Ans: $0.4y + C$

3 $\displaystyle\int \frac{1}{3}\,dx$

Ans: $\frac{x}{3} + C$

4 $\displaystyle\int e\,d\theta$

Ans: $e\theta + C$

5 $\displaystyle\int \sqrt{2}\,dx$

Ans: $\sqrt{2}x + C$

6 $\displaystyle\int (3+\sqrt{2})\,dz$

Ans: $(3+\sqrt{2})z + C$

7 $\displaystyle\int \pi\,dx$

Ans: $\pi x + C$

$$\int x^r \, dx = \frac{x^{r+1}}{r+1} + c; r \neq -1$$

8 $\int x \, dx$

Ans: $\frac{x^2}{2} + C$

9 $\int 4x \, dx$

Ans: $2x^2 + c$

10 $\int x^2 \, dx$

Ans: $\frac{x^3}{3} + C$

11 $\int 3x^2 \, dx$

Ans: $x^3 + c$

12 $\int x^3 \, dx$

Ans: $\frac{x^4}{4} + C$

13 $\int 0.5x^3 \, dx$

Ans: $\frac{x^4}{8} + C$

14 $\int x^{0.5} \, dx$

Ans: $\frac{x^{1.5}}{1.5} + C$

$$\int x^r \, dx = \frac{x^{r+1}}{r+1} + c; r \neq -1$$

15 $\int (x + x^2) \, dx$

Ans: $\frac{x^2}{2} + \frac{x^3}{3} + C$

16 $\int (3x^6 - 2x^2 + 7x + 1) \, dx$

Ans: $3\frac{x^7}{7} - 2\frac{x^3}{3} + 7\frac{x^2}{2} + x + C$

$$\int x^r \, dx = \frac{x^{r+1}}{r+1} + c; r \neq -1$$

17 $\int \sqrt{x} \, dx$

Ans: $\frac{2}{3} x^{\frac{3}{2}} + C$

18 $\int \sqrt[3]{x} \, dx$

Ans: $\frac{3}{4} x^{\frac{4}{3}} + C$

19 $\int \sqrt{x^3} \, dx$

Ans: $\frac{2}{5} x^{\frac{5}{2}} + C$

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + c;\ r \neq -1$$

20 $\displaystyle\int (x+2)(x-3)\,dx$

Ans: $\frac{x^3}{3} - \frac{x^2}{2} - 6x + C$

21 $\displaystyle\int \frac{x^3+2x^2}{x}\,dx$

Ans: $\frac{x^3}{3} + x^2 + C$

22 $\displaystyle\int \frac{1}{x^3}\,dx$

Ans: $-\frac{1}{2x^2} + C$

23 $\displaystyle\int \frac{3}{x^2}\,dx$

Ans: $-\frac{3}{x} + C$

1. $\displaystyle\int x^r\,dx = \frac{x^{r+1}}{r+1} + c; r \neq -1$
2. $\displaystyle\int \frac{1}{u}\,du = \ln u + C$
3. $\displaystyle\int e^u\,du = e^u + C$
4. $\displaystyle\int b^u\,du = \frac{b^u}{\ln b} + C$

24 $\displaystyle\int \frac{2}{x}\,dx$

Ans: $2\ln x + C$

25 $\displaystyle\int 5e^x\,dx$

Ans: $5e^x + C$

26 $\displaystyle\int 2^x\,dx$

Ans: $\dfrac{2^x}{\ln 2} + C$

27 $\displaystyle\int \pi^x\,dx$

Ans: $\dfrac{\pi^x}{\ln \pi} + C$

1. $\int \sin(x)\,dx = -\cos(x) + C$
2. $\int \cos(x)\,dx = \sin(x) + C$
3. $\int \sec^2(x)\,dx = \tan(x) + C$
4. $\int \csc^2(x)\,dx = -\cot(x) + C$
5. $\int \sec(x)\tan(x)\,dx = \sec(x) + C$
6. $\int \csc(x)\cot(x)\,dx = -\csc(x) + C$

28 $\int 2\sin(x)\,dx$

Ans: $-2\cos(x) + C$

29 $\int 10\cos(x)\,dx$

Ans: $10\sin(x) + C$

30 $\int 5\sec^2(x)\,dx$

Ans: $5\tan(x) + C$

31 $\int 2\csc^2(x)\,dx$

Ans: $-2\cot(x) + C$

32 $\int 10\sec(x)\tan(x)\,dx$

Ans: $10\sec(x) + C$

33 $\int 2\csc(x)\cot(x)\,dx$

Ans: $-2\csc(x) + C$

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4.2 Integration by Substitution

Evaluate the following integral:

1 $\displaystyle\int (x^2+1)^{50} 2x\,dx$

Ans: $\dfrac{(x^2+1)^{51}}{51}+C$

2 $\displaystyle\int (x^2+1)^{50} x\,dx$

Ans: $\dfrac{(x^2+1)^{51}}{102}+C$

3 $\displaystyle\int (x-8)^5\,dx$

Ans: $\dfrac{(x-8)^6}{6}+C$

4 $\displaystyle\int \frac{1}{\left(\frac{x}{3}-8\right)^5}\,dx$

Ans: $-\dfrac{3}{4\left(\frac{x}{3}-8\right)^4}+C$

5 $\displaystyle\int \sin(x+9)\,dx$

Ans: $-\cos(x+9)+C$

6 $\displaystyle\int \cos(5x)\,dx$

Ans: $\dfrac{\sin(5x)}{5}+C$

7 $\displaystyle\int \left(\frac{1}{x^2} + \sec^2(\pi x)\right) dx$

Ans: $-\dfrac{1}{x} + \dfrac{\tan(\pi x)}{\pi} + C$

8 $\displaystyle\int \sin^2(x)\cos(x)\, dx$

Ans: $\dfrac{\sin^3(x)}{3} + C$

9 $\displaystyle\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\, dx$

Ans: $2\sin(\sqrt{x}) + C$

10 $\displaystyle\int t^4 \sqrt[3]{3-5t^5}\, dt$

Ans: $-\dfrac{3}{100}\left(3-5t^5\right)^{\frac{4}{3}} + C$

$\boxed{\textbf{11}}$ $\displaystyle\int x^2\sqrt{x-1}\,dx$

Ans: $\frac{2}{7}(x-1)^{\frac{7}{2}}+\frac{4}{5}(x+1)^{\frac{5}{2}}+\frac{2}{3}(x-1)^{\frac{3}{2}}+C$

$\boxed{\textbf{12}}$ $\displaystyle\int \frac{3x^2}{\left(x^3-1\right)^5}\,dx$

Ans: $-\frac{\left(x^3-1\right)^{-4}}{4}+C$

$\boxed{\textbf{13}}$ $\displaystyle\int \cos^3(x)\,dx$

Ans: $\sin(x)-\frac{\sin^3(x)}{3}+C$

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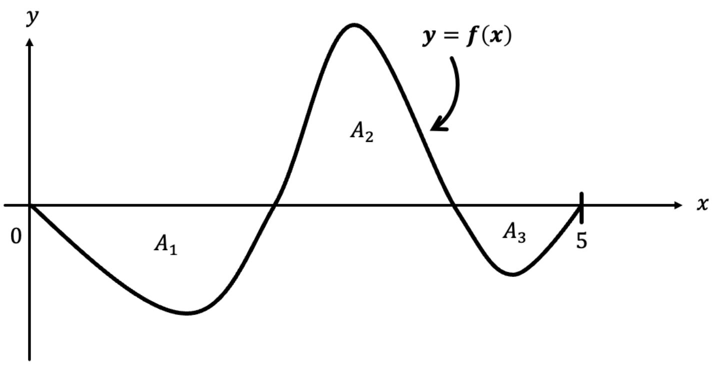
4.3 The Definite Integral

1. **First Fundamental Theorem of Calculus:** If f is continuous on $[a,b]$ and F is antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

2. **Area** =
The sum of the areas **above the x-axis** and **under** the graph −
The sum of the areas **under the x-axis** and **above** the graph
$= A_2 - A_1 - A_3$
$= \int_0^5 f(x)\,dx$



1 $\int_{-1}^{4} \frac{1}{x^2}\,dx$

Ans: undefined

2 $\int_{1}^{2} x\,dx$

Ans: $\frac{3}{2}$

3 $\displaystyle\int_0^3 \left(9 - x^2\right) dx$

Ans: 18

4 $\displaystyle\int_0^{\frac{\pi}{3}} \sec^2(x)\, dx$

Ans: $\sqrt{3}$

5 $\displaystyle\int_1^1 x^2\, dx$

Ans: 0

6 $\displaystyle\int_4^0 x\, dx$

Ans: -8

$\boxed{\mathbf{7}}$ Evaluate the integral $\int_0^6 f(x)\,dx$ if $f(x) = \begin{cases} x^2 & x < 2 \\ 3x-2 & x \geq 2 \end{cases}$.

Ans: $\frac{128}{3}$

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4.4 Evaluating Definite Integrals by Substitution

1 $\displaystyle\int_0^2 x\left(x^2+1\right)^3\,dx$

Ans: 78

2 $\displaystyle\int_0^{\frac{\pi}{8}} \sin^5(2x)\cos(2x)\,dx$

Ans: $\frac{1}{96}$

3 $\displaystyle\int_2^5 (2x-5)(x-3)^9\,dx$

Ans: $\frac{52233}{110}$

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4.5 The Second Fundamental Theorem of Calculus

If f is continuous on interval I, A is any number in I, F is an antiderivative of F on I.

1. $F(x) = \int_a^x f(t)\,dt \implies F'(x) = \dfrac{d}{dx}\int_a^x f(t)\,dt = f(x)$
2. $F(x) = \int_a^{g(x)} f(t)\,dt \implies F'(x) = \dfrac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))g'(x)$

1 $\dfrac{d}{dx}\int_1^x t^3\,dt$

Ans: x^3

2 $\dfrac{d}{dx}\int_1^x \dfrac{\sin(t)}{t}\,dt$

Ans: $\dfrac{\sin(x)}{x}$

3 $\dfrac{d}{dx}\int_2^{3x^2} 4u\,du$

Ans: $72x^3$

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4.6 Mean-Value Theorem for Integrals

If f is continuous on $[a,b]$, then there is at least one number c in $[a,b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

1 Find the value of c in $[1,4]$, if $f(x) = x^2$ that satisfy the Mean-Value theorem for integrals.

Ans: $c = +\sqrt{7}$

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5 | Applications of Integration

5.1 Area Between Two Curves

Area between two curves $y = f(x)$ (upper function) and $y = g(x)$ (lower function) on the interval $[a, b]$ is given by

$$A = \int_a^b \big(f(x) - g(x)\big)\, dx$$

1 Find the area of the region
bounded above by $y = x + 6$,
bounded below by $y = x^2$, and
bounded of the sides by the lines $x = 0$ and $x = 2$.
Ans: $\frac{34}{3}$

2 Find the area of the region
that is enclosed between the curves
$y = x^2$ and $y = x + 6$.
Ans: $\frac{125}{6}$

3 Find the area of the region that is enclosed between the curves $x = y^2$ and $y = x - 2$.

Ans: $\frac{9}{2}$

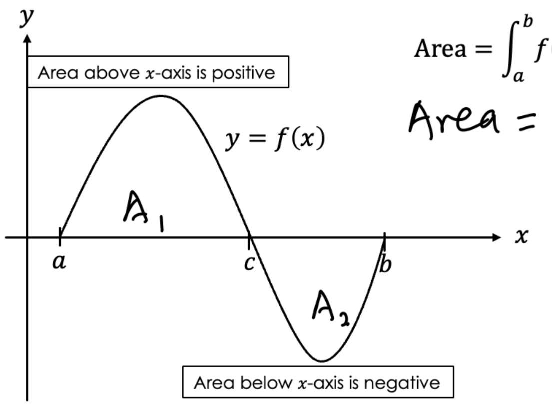
Area between two curves $x = f(y)$ (right function) and $x = g(y)$ (left function) on the interval $[c, d]$ is given by

$$A = \int_c^d \big(f(y) - g(y)\big)\, dy$$

$\boxed{\textbf{1}}$ Find the area of the region that is enclosed between the curves $x = y^2$ and $y = x - 2$.

Ans: $\frac{9}{2}$

Area between a curve and the x-axis.



$$\text{Area} = \int_a^b f(x)\,dx$$

$$\text{Area} = A_1 - A_2$$

Find the area under the curve $y = \cos(x)$ over the following intervals:

1 $\left[0, \frac{\pi}{2}\right]$

Ans: 1

2 $\left[\frac{\pi}{2}, \pi\right]$

Ans: 1

3 $[0, \pi]$

Ans: 2

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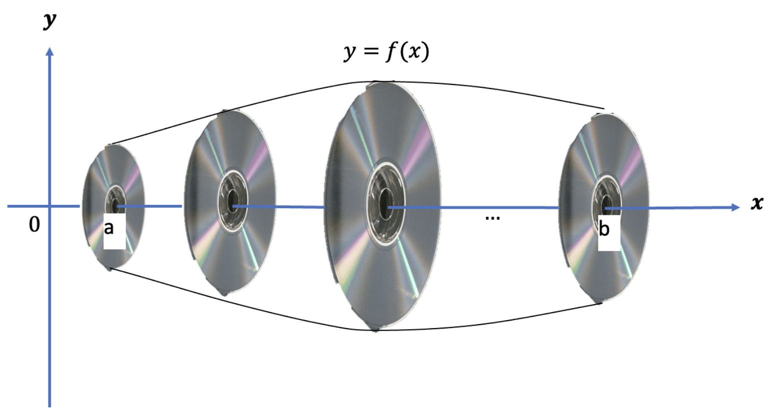
5.2 Method of Disks/Washers (Perpendicular to the x-axis)



1. Area $= \pi\left(f(x)-0\right)^2 = \pi\left(f(x)\right)^2$

2. Volume $= \pi \displaystyle\int_{x=a}^{b} \left[f(x)\right]^2\, dx$

$\boxed{1}$ Find the volume of the solid that is obtained when the region under the curve $y=\sqrt{x}$ over the interval $[1,4]$ is revolved about the x-axis.

Ans: $\frac{15}{2}\pi$

$\boxed{\mathbf{2}}$ Find the volume of the solid that is obtained when the region between the graph $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the x-axis.

Ans: $\frac{69}{10}\pi$

$\boxed{\mathbf{3}}$ Find the volume of the solid that is obtained when the region
bounded by the graph $x = y^2$ and $y = x^2$
is revolved about the x-axis.

Ans: $\frac{3}{10}\pi$

4 Find the volume of the solid that is obtained when the region bounded by the graph $y = x^2$ and $y = x^3$ is revolved about the line $y = -1$.

Ans: $\frac{47}{210}\pi$

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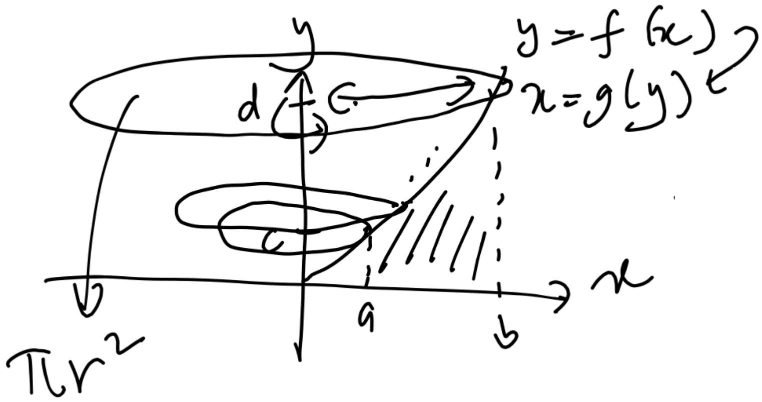
5.3 Method of Disks/Washers (Perpendicular to the y-axis)



1. Area $= \pi\left(g(y) - 0\right)^2 = \pi\left[g(y)\right]^2$
2. Volume $= \pi \int_{y=c}^{d} \left[g(y)\right]^2 \, dy$

$\boxed{1}$ Find the volume of the solid that is generated when the region enclosed by $y = \sqrt{x}$, $y = 2$ and $x = 0$ is revolved about the y-axis.

Ans: $\frac{32}{5}\pi$

2 Find the volume of the solid that is generated when the region enclosed by $x = y^2$ and $y = x^2$ is revolved about the y-axis.

Ans: $\frac{3}{10}\pi$

$\boxed{\mathbf{3}}$ Find the volume of the solid that is generated when the region bounded by $y = x^2$ and $y = x^3$
is revolved about the line $x = -1$.
Ans: $\frac{4}{15}\pi$

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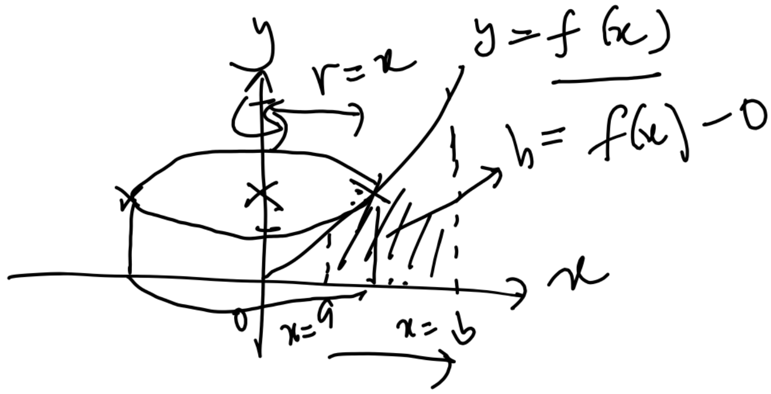
5.4 Cylindrical Shells (Revolved about the y-axis)



1. Volume of a cylinder, $V = 2\pi r h$
2. $r = x$; $h = f(x)$
3. $V = 2\pi \int_{x=a}^{b} x f(x)\, dx$

$\boxed{1}$ Use cylindrical shells to find the volume of the solid generated when the region enclosed between $y = \sqrt{x}$, $x = 1$, $x = 4$ and the x-axis is revolved about the y-axis.

Ans: $\frac{124}{5}\pi$

$\boxed{\mathbf{2}}$ Use cylindrical shells to find the volume of the solid generated when the region R in the first quadrant enclosed by $y = x$ and $y = x^2$ is revolved about the y-axis.

Ans: $\frac{\pi}{6}$

3 Use cylindrical shells to find the volume of the solid generated when the region R in the first quadrant enclosed by $y = x$ and $y = x^2$ is revolved about the line $x = -1$.

Ans: $\frac{\pi}{2}$

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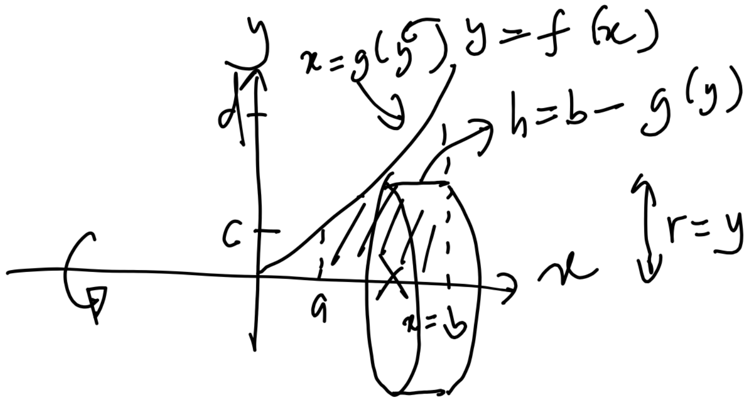
5.5 Cylindrical Shells (Revolved about the x-axis)



1. Volume of a cylinder, $V = 2\pi r h$
2. $r = y$; $h = b - g(y)$
3. $V = 2\pi \int_{y=c}^{d} y\big(b - g(y)\big)\, dy$

$\boxed{1}$ Use cylindrical shells to find the volume of the solid generated when the region R under $y = x^2$ over the interval $[0, 2]$ is revolved about the x-axis.

Ans: $\frac{32}{5}\pi$

$\boxed{\mathbf{2}}$ Use cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $x = (y-2)^2$ and $y = x$ about the line $y = -1$.

Ans: $\frac{63}{2}\pi$

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5.6 Washer vs. Cylindrical Shell

a) Find the volume of the solid generated
when the region enclosed between $y = \sqrt{x}$, $x = 1$, $x = 4$ and the x-axis
is revolved about the y-axis.

$\boxed{\textbf{1}}$ Washer

Ans: $\frac{124}{5}\pi$

$\boxed{\textbf{2}}$ Cylindrical Shell

Ans: $\frac{124}{5}\pi$

b) Find the volume of the solid generated
when the region R in the first quadrant enclosed by $y = x$ and $y = x^2$
is revolved about the x-axis.

$\boxed{\mathbf{1}}$ Washer

Ans: $\frac{2\pi}{15}$

$\boxed{\mathbf{2}}$ Cylindrical Shell

Ans: $\frac{2\pi}{15}$

c) Find the volume of the solid generated
when the region R in the first quadrant enclosed by $y = x$ and $y = x^2$
is revolved about the y-axis.

$\boxed{\mathbf{3}}$ Washer

Ans: $\frac{\pi}{6}$

$\boxed{\mathbf{4}}$ Cylindrical Shell

Ans: $\frac{\pi}{6}$

d) Use cylindrical shells to find the volume of the solid generated when the region R in the first quadrant enclosed by $y = x$ and $y = x^2$ is revolved about the line $x = -1$.

1 Washer

Ans: $\frac{\pi}{2}$

2 Cylindrical Shell

Ans: $\frac{\pi}{2}$

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A Important Formulas

A.1 Derivatives and Integrals

Derivatives:

$$\frac{d}{dx}c = 0$$
$$\frac{d}{dx}x^n = nx^{n-1}$$
$$\frac{d}{dx}e^x = e^x$$
$$\frac{d}{dx}a^x = \ln(a)\cdot a^x$$
$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$
$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$$
$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x)$$
$$\frac{d}{dx}\tan(x) = \sec^2(x)$$
$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$
$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$
$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$
$$\frac{d}{dx}e^{u(x)} = u'(x)e^{u(x)}$$
$$\frac{d}{dx}\ln|u(x)| = \frac{u'(x)}{u(x)}$$

Integrals:

$$\int 0\,dx = C$$
$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$
$$\int e^x\,dx = e^x + C$$
$$\int a^x\,dx = \frac{a^x}{\ln(a)} + C$$
$$\int \frac{1}{x}\,dx = \ln|x| + C$$
$$\int \frac{1}{x\ln(a)}\,dx = \log_a|x| + C$$
$$\int \cos(x)\,dx = \sin(x) + C$$
$$\int \sin(x)\,dx = -\cos(x) + C$$
$$\int \sec^2(x)\,dx = \tan(x) + C$$
$$\int \csc^2(x)\,dx = -\cot(x) + C$$
$$\int \sec(x)\tan(x)\,dx = \sec(x) + C$$
$$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$$
$$\int e^{u(x)}\,dx = \int e^{u(x)}u'(x)\,dx \quad \text{(use substitution)}$$
$$\int \frac{1}{u(x)}u'(x)\,dx = \ln|u(x)| + C$$

A.2 Trigonometric Identities, Index (Exponent) Rules and Logarithmic Properties

Index (Exponent) Rules:

$$a^m \cdot a^n = a^{m+n}$$
$$a^m / a^n = a^{m-n}$$
$$(a^m)^n = a^{mn}$$
$$a^0 = 1 \quad (a \neq 0)$$
$$a^{-n} = 1/a^n$$

Logarithmic Properties:

$$\log_a(xy) = \log_a(x) + \log_a(y)$$
$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$
$$\log_a(x^n) = n\log_a(x)$$
$$\log_a(a) = 1$$
$$\log_a(1) = 0$$

Trigonometric Identities:

$$\sin^2(x) + \cos^2(x) = 1$$
$$1 + \tan^2(x) = \sec^2(x)$$
$$1 + \cot^2(x) = \csc^2(x)$$
$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(2x) = \cos^2(x) - \sin^2(x)$$
$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$$
$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$
$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$
$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$
$$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$$
$$\tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$$

B | Cheat Sheets

B.1 Computing Limits

LIMIT

One-sided limit

$$\lim_{x\to 3^-}\frac{x}{x-3}=\frac{2.9}{2.9-3}=\frac{+}{-}=-\infty$$

$$\lim_{x\to 3^+}\frac{x}{x-3}=\frac{3.1}{3.1-3}=\frac{+}{+}=+\infty$$

$$\lim_{x\to 3}\frac{x}{x-3}=\frac{3}{0}=\text{undefined}$$

Continuity

$$f(x)=\begin{cases} x-3 & ; \quad x<1 \\ 2x & ; \quad 1\le x<2 \\ 4 & ; \quad x\ge 2 \end{cases}$$

$$\lim_{x\to 3}f(x)=\lim_{x\to 3}2x=2(3)=6$$

$f(1)=2(1)=2$

$$\lim_{x\to 1^-}f(x)=\lim_{x\to 1}(x-3)=1-3=-2$$

$$\lim_{x\to 1^+}f(x)=\lim_{x\to 1}2x=2(1)=2$$

$$\lim_{x\to 1}f(x)=\text{undefined}$$

$\therefore$ f is discontinuous at $x=1$

$f(2)=4$

$$\lim_{x\to 2^-}f(x)=\lim_{x\to 2}2x=2(2)=4$$

$$\lim_{x\to 2^+}f(x)=\lim_{x\to 2}4=4$$

$$\lim_{x\to 2}f(x)=4$$

Since $f(2)=\lim_{x\to 2}f(x)=4$,

$\therefore$ f is continuous at $x=2$

Two-sided limit

$$\lim_{x\to 2}\left(x^2+3x-3\right)=2^2+3(2)-3=7$$

$$\lim_{x\to 2}\frac{x+3}{x-4}=\frac{2+3}{2-4}=\frac{5}{-2}$$

$$\lim_{x\to 2}\sqrt{x+2}=\sqrt{2+2}=2$$

$$\lim_{x\to 2}\frac{x+3}{x-2}=\frac{5}{0}=\text{undefined}$$

$$\lim_{x\to 2}\sqrt{x-5}=\sqrt{2-5}=\sqrt{-3}=\text{undefined}$$

$$\lim_{x\to 2}\frac{x^2+x-6}{x-2}=\lim_{x\to 2}\frac{(x+3)(x-2)}{x-2}=\lim_{x\to 2}(x+3)=2+3=5$$

$$\lim_{x\to 0}\frac{x}{\sqrt{x+4}-2}=\lim_{x\to 0}\frac{x}{\sqrt{x+4}-2}\times\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$$

$$=\lim_{x\to 0}\frac{x\left(\sqrt{x+4}+2\right)}{x+4-4}=\lim_{x\to 0}\frac{x\left(\sqrt{x+4}+2\right)}{x}$$

$$=\lim_{x\to 0}\left(\sqrt{x+4}+2\right)$$

$$=\sqrt{0+4}+2=2+2=4$$

Limit at infinity

$$\lim_{x\to+\infty}2x^3+4x-3=\lim_{x\to+\infty}2x^3=+\infty$$

$$\lim_{x\to-\infty}2x^3+4x-3=\lim_{x\to-\infty}2x^3=-\infty$$

$$\lim_{x\to+\infty}-2x^3=\lim_{x\to+\infty}-2x^3=-\infty$$

$$\lim_{x\to-\infty}-2x^3=\lim_{x\to-\infty}-2x^3=+\infty$$

$$\lim_{x\to+\infty}\frac{3x+4}{7x+5}=\lim_{x\to+\infty}\frac{(3x+4)/x}{(7x+5)/x}=\lim_{x\to+\infty}\frac{3+4/x}{7+5/x}=\frac{3}{7}$$

$$\lim_{x\to+\infty}\sqrt[3]{\frac{3x+4}{7x+5}}=\lim_{x\to+\infty}\sqrt[3]{\frac{(3x+4)/x}{(7x+5)/x}}=\lim_{x\to+\infty}\sqrt[3]{\frac{3+4/x}{7+5/x}}=\sqrt[3]{\frac{3}{7}}$$

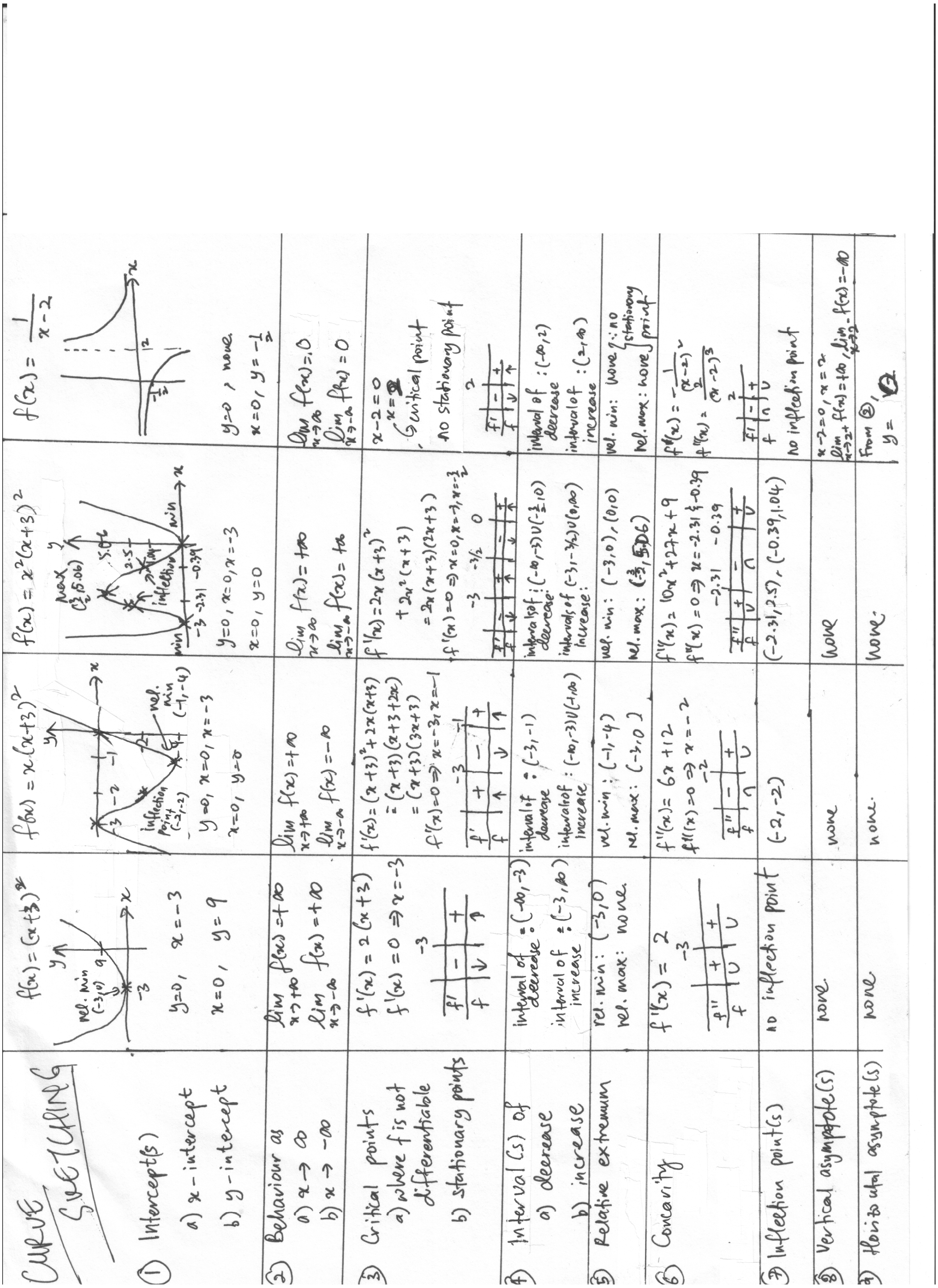
$$\lim_{x\to+\infty}\frac{\sqrt{x^2+3}}{2x-4}=\lim_{x\to+\infty}\frac{\sqrt{x^2+3}/\sqrt{x^2}}{(2x-4)/x}=\lim_{x\to+\infty}\frac{\sqrt{x^2/x^2+3/x^2}}{2x/x-4/x}$$

$$=\lim_{x\to+\infty}\frac{\sqrt{1+3/x^2}}{2-4/x}=\frac{\sqrt{1}}{2}=\frac{1}{2}$$

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+3}}{2x-4}=\lim_{x\to-\infty}\frac{\sqrt{x^2+3}/\sqrt{x^2}}{(2x-4)/(-x)}=\lim_{x\to-\infty}\frac{\sqrt{x^2/x^2+3/x^2}}{2x/(-x)-4/(-x)}$$

$$=\lim_{x\to-\infty}\frac{\sqrt{1+3/x^2}}{-2+4/x}=\frac{\sqrt{1}}{-2}=\frac{1}{-2}$$

B.2 Curve Sketching



CURVE SKETCHING	$f(x)=(x+3)^2$	$f(x)=x(x+3)^2$	$f(x)=x^2(x+3)^2$	$f(x)=\frac{1}{x-2}$
① Intercept(s) a) x-intercept b) y-intercept	rel. min (-3,0); $y=0,\ x=-3$; $x=0,\ y=9$	inflection point (-2,-2); rel. min (-1,-4); $y=0,\ x=0,\ x=-3$; $x=0,\ y=0$	max $(-\frac{3}{2},5.06)$; inflection; min; $y=0,\ x=0,\ x=-3$; $x=0,\ y=0$	$y=0$, none; $x=0,\ y=-\frac{1}{2}$
② Behaviour as a) $x\to\infty$ b) $x\to-\infty$	$\lim_{x\to+\infty} f(x)=+\infty$; $\lim_{x\to-\infty} f(x)=+\infty$	$\lim_{x\to+\infty} f(x)=+\infty$; $\lim_{x\to-\infty} f(x)=-\infty$	$\lim_{x\to\infty} f(x)=+\infty$; $\lim_{x\to-\infty} f(x)=+\infty$	$\lim_{x\to\infty} f(x)=0$; $\lim_{x\to-\infty} f(x)=0$
③ Critical points a) where f is not differentiable b) stationary points	$f'(x)=2(x+3)$; $f'(x)=0\Rightarrow x=-3$; f′: − (−3) +; f: ↓ ↑	$f'(x)=(x+3)^2+2x(x+3)=(x+3)(x+3+2x)=(x+3)(3x+3)$; $f'(x)=0\Rightarrow x=-3,\ x=-1$; f′: + (−3) − (−1) +; f: ↑ ↓ ↑	$f'(x)=2x(x+3)^2+2x^2(x+3)=2x(x+3)(2x+3)$; $f'(x)=0\Rightarrow x=0,\ x=-3,\ x=-\frac{3}{2}$; f′: − (−3) + (−3/2) − (0) +; f: ↓ ↑ ↓ ↑	$x-2=0$, $x=2$ critical point; no stationary point; f′: − (2) +; f: ↓ ↑
④ Interval(s) of a) decrease b) increase	interval of decrease: $(-\infty,-3)$; interval of increase: $(-3,\infty)$	interval of decrease: $(-3,-1)$; interval of increase: $(-\infty,-3)\cup(-1,\infty)$	intervals of decrease: $(-\infty,-3)\cup(-\frac{3}{2},0)$; intervals of increase: $(-3,-\frac{3}{2})\cup(0,\infty)$	interval of decrease: $(-\infty,2)$; interval of increase: $(2,\infty)$
⑤ Relative extremum	rel. min: (−3,0); rel. max: none	rel. min: (−1,−4); rel. max: (−3,0)	rel. min: (−3,0), (0,0); rel. max: $(-\frac{3}{2},5.06)$	rel. min: none; rel. max: none ∵ no stationary point
⑥ Concavity	$f''(x)=2$; f″: + (−3) +; f: ∪ ∪	$f''(x)=6x+12$; $f''(x)=0\Rightarrow x=-2$; f″: − (−2) +; f: ∩ ∪	$f''(x)=10x^2+27x+9$; $f''(x)=0\Rightarrow x=-2.31$ & -0.39; f″: + (−2.31) − (−0.39) +; f: ∪ ∩ ∪	$f'(x)=-\frac{1}{(x-2)^2}$; $f''(x)=\frac{2}{(x-2)^3}$; f″: − (2) +; f: ∩ ∪
⑦ Inflection point(s)	no inflection point	(−2,−2)	(−2.31,2.5), (−0.39,1.04)	no inflection point
⑧ Vertical asymptote(s)	none	none	none	$x-2=0,\ x=2$; $\lim_{x\to2^+} f(x)=+\infty$, $\lim_{x\to2^-} f(x)=-\infty$
⑨ Horizontal asymptote(s)	none	none	none	From ②, $y=0$