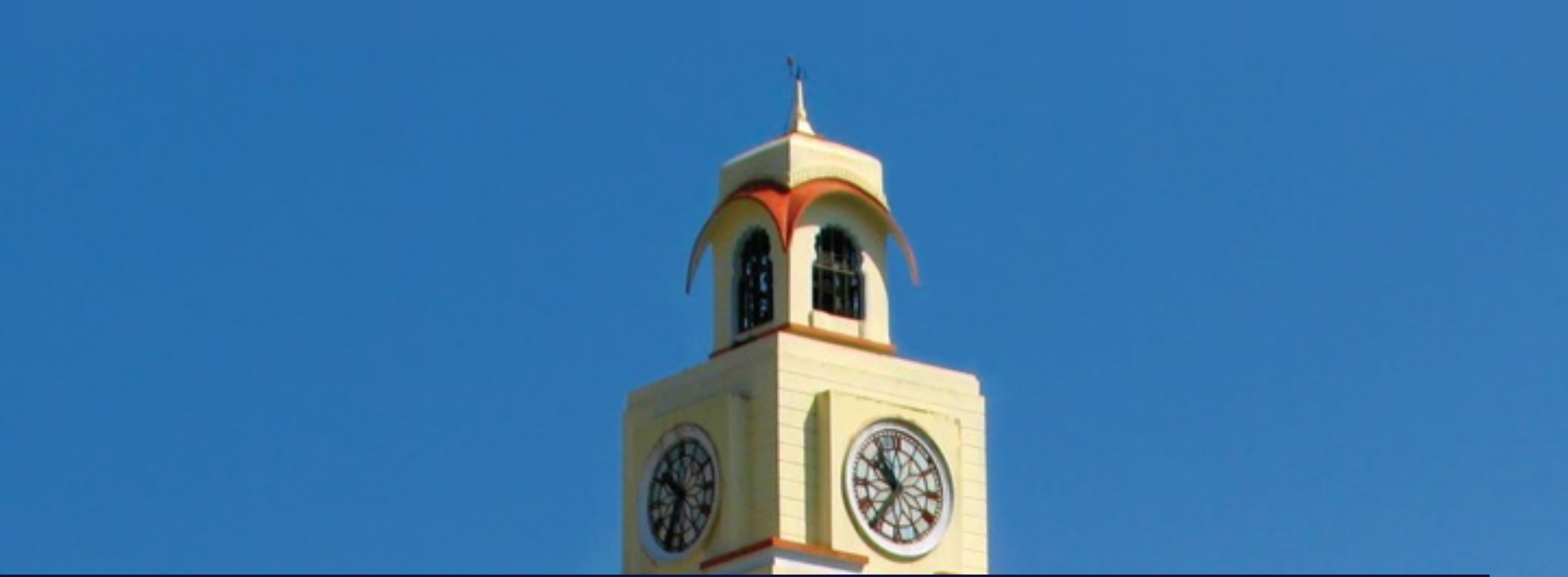

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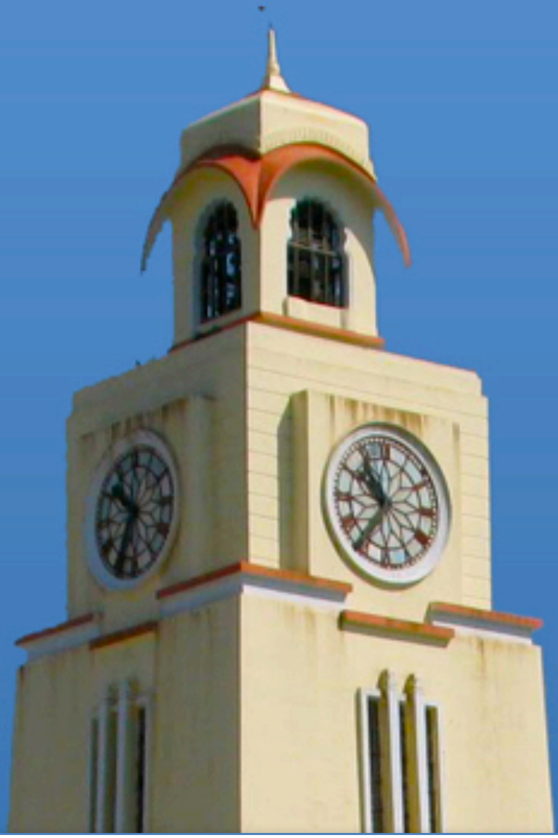

S2 -22_DSECLZC416, MFDS

Webinar 2

Agenda

Discussion on

- **Solutions of Homework Problems**
- **Gram-Schmidt process with example**
- **Example of SVD**

Homework Problem

Qus-1: Let $B = (b_1, b_2, \ldots, b_{r-1}, b_r, b_{r+1}, \ldots, b_n)$ be a non-singular matrix. If column b_r is replaced by a and that the resulting matrix is called B_a along with $a = \sum_{i=1}^{n} y_i b_i$, then state the necessary and sufficient conditions for B_a to be non-singular.

Sol: Given

$$B = \begin{array}{c} \begin{array}{cccccccc} b_1 & b_2 & \cdots & b_{r-1} & b_r & b_{r+1} & \cdots & b_n \end{array} \\ \begin{bmatrix} * & * & \cdots & * & * & * & \cdots & * \\ * & * & \cdots & * & * & * & \cdots & * \\ * & * & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \end{bmatrix} \end{array}$$

Linearly independent columns

is non-singular.

Homework Problem

$$\begin{array}{c} \begin{array}{cccccccc} b_1 & b_2 & \dots & b_{r-1} & b_r & b_{r+1} & \dots & b_n \end{array} \\ B = \begin{bmatrix} * & * & \cdots & * & * & * & \cdots & * \\ * & * & \cdots & * & * & * & \cdots & * \\ * & * & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \end{bmatrix} \end{array}$$

Linearly independent columns

- This means no column of B can be written as a linear combination of other columns.
- Replacing column b_r by a, we get

$$\begin{array}{c} \begin{array}{cccccccc} b_1 & b_2 & \dots & b_{r-1} & a & b_{r+1} & \dots & b_n \end{array} \\ B_a = \begin{bmatrix} * & * & \cdots & * & * & * & \cdots & * \\ * & * & \cdots & * & * & * & \cdots & * \\ * & * & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \end{bmatrix} \end{array}$$

Homework Problem

Looking at, $a = \sum_{i=1}^{n} y_i b_i$, we see that

$$\begin{array}{c} \begin{array}{cccccccc} b_1 & b_2 & \dots & b_{r-1} & a & b_{r+1} & \dots & b_n \end{array} \\ \begin{bmatrix} * & * & \dots & * & * & * & \dots & * \\ * & * & \dots & * & * & * & \dots & * \\ * & * & \dots & * & * & * & \dots & * \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \end{bmatrix} \end{array}$$

B_a

- a is a linear combination of $b_1, b_2, \dots, b_n$.
- Suppose $y_r = 0$, i.e.

$$a = y_1 b_1 + y_2 b_2 + \dots + y_{r-1} b_{r-1} + y_{r+1} b_{r+1} + \dots + y_n b_n \, .$$

This means a is a linear combination of other columns of B_a. Hence B_a is singular.

Homework Problem

- Suppose $y_r \neq 0$, then

$$a = y_r b_r + y_1 b_1 + y_2 b_2 + \ldots + y_{r-1} b_{r-1} + y_{r+1} b_{r+1} + \ldots + y_n b_n .$$

Not a linear combination of columns $b_1, \ldots, b_{r-1}, b_{r+1}, \ldots, b_n$ of B_a

$$\begin{array}{c} \begin{array}{cccccccc} b_1 & b_2 & \ldots & b_{r-1} & a & b_{r+1} & \ldots & b_n \end{array} \\ \begin{bmatrix} * & * & \cdots & * & * & * & \cdots & * \\ * & * & \cdots & * & * & * & \cdots & * \\ * & * & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \end{bmatrix} \end{array}$$

- Thus, a is not a linear combination of other columns of B_a. B_a

- Hence, B_a is non-singular, iff $y_r \neq 0$.

Homework Problem

Qus-2: Let V be a finite dimensional vector space over $\mathbb{R}$. If S is a set of elements in V such that span$(S) = V$, what is the relationship between S and the basis of V?

Sol: Given

$$\text{span}(S) = V$$

- Every vector in V can be expressed as a linear combination of elements of S.
- S is a generating set of V.
- Since basis is a minimal generating set, we have

$$\text{Number of elements in a basis of } V \leq \text{Number of elements in } S$$

Homework Problem

Qus-3:

a) Let P be a real square matrix satisfying $P = P^T$and $P^2 = P$.

1. Can the matrix P have complex eigenvalues? If so, construct an example, else, justify y answer.
2. What are the eigenvalues of P?

Sol: 1. $P = P^T \implies P$ is symmetric.

Spectral theorem: If $A \in \mathbb{R}^{n \times n}$ is symmetric there exists an orthonormal basis of the corresponding vector space V consisting of the eigenvectors of A, and each eigenvalue is real.

Therefore, by the Spectral theorem, P cannot have complex eigenvalues.

Homework Problem

Sol: 2. Suppose λ is an eigenvalue of P and let v be the corresponding eigenvector. Then

$$
\begin{aligned}
& Pv = \lambda v \\
\implies\ & P^2 v = \lambda P v = \lambda^2 v \\
\implies\ & Pv = \lambda^2 v \\
\implies\ & \lambda v = \lambda^2 v \qquad\qquad (\because P^2 = P) \\
\implies\ & (\lambda^2 - \lambda) v = \mathbf{0}
\end{aligned}
$$

Since $v \neq \mathbf{0}$, we have

$$\lambda^2 - \lambda = 0 \implies \lambda(\lambda - 1) = 0 \qquad \implies \lambda = 0 \text{ or } \lambda = 1$$

Hence, the eigenvalues of P are 0 and 1.

Homework Problem

b) Given the following matrix

$$A = \begin{bmatrix} 1 & 2 & r \\ c & 1 & 7 \\ c & 1 & 7 \end{bmatrix}$$

where c and r are arbitrary real numbers and $5.5 < r \leq 6.5$, and the fact that $\lambda 1 = 3$ is one of the eigenvalues, is it possible to determine the other two eigenvalues? If so, compute them and give reasons for your answer.

Sol: (ii) Since A has two identical rows

$$\det(A) = 0$$

i.e. 0 is an eigenvalue of A.

Homework Problem

Next, we recall that

Trace of a matrix = sum of its eigenvalues

$$A = \begin{bmatrix} 1 & 2 & r \\ c & 1 & 7 \\ c & 1 & 7 \end{bmatrix}$$

- Let λ_3 be the third eigenvalue of A.

- Since trace$(A) = 9$, we have

$$3 + 0 + \lambda_3 = 9 \implies \lambda_3 = 6.$$

- Hence, the other two eigenvalues of A are 0 and 6.

Homework Problem

Qus-4: The Fibonacci sequence is defined by

$$V_n = V_{n-1} + V_{n-2}, \qquad \text{for } n \geq 2$$

with starting values $V_0 = 0$ and $V_1 = 1$. Observe that the calculation of V_k requires the calculation of $V_2, V_3, \ldots, V_{k-1}$. To avoid this, could this problem be written as an eigenvalue problem and solved for V_n directly? If so, find the explicit formula for V_n.

Sol: We can write

$$V_{n+1} = V_n + V_{n-1}$$

$$V_n = 1.V_n$$

Or

$$\begin{bmatrix} V_{n+1} \\ V_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} V_n \\ V_{n-1} \end{bmatrix}$$

Homework Problem

Define

$$F_n = \begin{bmatrix} V_{n+1} \\ V_n \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Then, we have

$$F_n = A \begin{bmatrix} V_n \\ V_{n-1} \end{bmatrix} = AF_{n-1}$$

- $F_1 = AF_0$
- $F_2 = AF_1 = A^2F_0$
- $F_3 = AF_2 = A^3F_0, \ \ldots$

Homework Problem

Continuing like this, we get

$$F_n = A^n F_0$$

where $F_0 = \begin{bmatrix} V_1 \\ V_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

$$F_n = \begin{bmatrix} V_{n+1} \\ V_n \end{bmatrix}$$

- If we find A^n, then from F_n, we can find the value of V_n.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

- Since A is a symmetric matrix, it is diagonalisable, i.e.

$$A = PDP^{-1}$$

where columns of P consists of eigenvectors of A and D is the diagonal matrix with eigenvalues of A as its diagonal entries.

Homework Problem

- Note that

$$
\begin{aligned}
A^n &= (PDP^{-1})(PDP^{-1})\ldots(PDP^{-1}) && (n \text{ times}) \\
&= (PD^2P^{-1})(PDP^{-1})\ldots(PDP^{-1}) && (n-1 \text{ times}) \\
&\vdots \\
&= PD^nP^{-1}
\end{aligned}
$$

- Therefore, to find A^n, it is enough to find P and D.

- To find P and D, we need to find the eigenvalues and eigenvectors of A.

Homework Problem

Suppose λ is an eigenvalue of A. Then

$$\det(A-\lambda I)=0 \implies \begin{vmatrix} 1-\lambda & 1 \\ 1 & \lambda \end{vmatrix} = 0$$

$$\implies -\lambda(1-\lambda)-1=0$$

$$\implies \lambda^2-\lambda-1=0$$

$$\implies \lambda = \frac{1\pm\sqrt{5}}{2}$$

Thus, the eigenvalues of A are $\dfrac{1\pm\sqrt{5}}{2}$ and hence $D = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}$.

Homework Problem

Suppose $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be an eigenvector of A corresponding to the eigenvalue $\dfrac{1+\sqrt{5}}{2}$.

Then

$$Av = \frac{1+\sqrt{5}}{2}v \qquad \Longrightarrow \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{1+\sqrt{5}}{2} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Longrightarrow v_1 + v_2 = \frac{1+\sqrt{5}}{2}v_1 \quad \text{and } v_1 = \frac{1+\sqrt{5}}{2}v_2$$

- Solving these two equations, we see that v_2 is an independent variable.
- Choosing $v_2 = 1$, we get $v = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$

Homework Problem

- Similarly, we can see $\begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\dfrac{1-\sqrt{5}}{2}$.

- Hence, $P = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}$.

- It is easy to compute that, $P^{-1} = \dfrac{1}{\sqrt{5}} \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & 1+\frac{\sqrt{5}}{2} \end{bmatrix}$.

Homework Problem

Using $A^n = PD^nP^{-1}$, we compute

$$A^n = \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & 1+\frac{\sqrt{5}}{2} \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} & \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \\ * & * \end{bmatrix} \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & 1+\frac{\sqrt{5}}{2} \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} & * \\ * & * \end{bmatrix}.$$

Homework Problem

- Using

$$F_n = A^n F_0, \qquad F_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad F_n = \begin{bmatrix} V_{n+1} \\ V_n \end{bmatrix}$$

we get

$$V_{n+1} = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right]$$

or

$$V_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n} - \left(\frac{1-\sqrt{5}}{2} \right)^{n} \right].$$

Homework Problem

Qus-5: Prove that if A is a square diagonalisable matrix of size $n \times n$, then $A^k \to 0$ as $k \to \infty$ if and only if $|\lambda_i| < 1, \forall i$.

Sol:

Since A is diagonalisable, we can write

$$A = PDP^{-1}$$

where D is the diagonal matrix with eigenvalues of A as its diagonal entries.

Multiplying A by P^{-1} on the left and by P on the right, we get

$$D = P^{-1}AP$$

$$\implies D^k = P^{-1}A^kP$$

Homework Problem

- Suppose $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. Then

$$D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \implies D^k = \begin{bmatrix} \lambda_1^k & 0 & \ldots & 0 \\ 0 & \lambda_2^k & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n^k \end{bmatrix}$$

- Since $D^k = P^{-1}A^kP$,

$$A^k \to 0 \text{ as } k \to \infty$$

$$\iff D^k \to 0 \text{ as } k \to \infty$$

$$\iff (\lambda_i)^k \to 0 \text{ as } k \to \infty, \ \forall i$$

$$\iff |\lambda_i| < 1, \forall i$$

Homework Problem

Qus-6: Construct examples of matrices for which the defect is positive, negative and zero wherever possible.

Sol: Suppose λ is an eigenvalue of a matrix A. Then

Algebraic multiplicity of λ → Multiplicity of λ as a root of the characteristic polynomial of A

Geometric multiplicity of λ → the maximum number of linearly independent eigenvectors associated with λ

→ **Defect of λ** = Algebraic multiplicity − Geometric multiplicity

Homework Problem

- Since

$$\text{Algebraic multiplicity} \geq \text{Geometric multiplicity}$$

The defect cannot be negative.

- **Example of matrix with zero defect.**

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

A is a diagonal matrix with eigenvalues 2 and 3. Since all the eigenvalues are distinct, the defect of each eigenvalue is 0.

It is also easy to compute the eigenvectors corresponding to each eigenvalue and check algebraic and geometric multiplicity is same for each eigenvalue.

Homework Problem

Example of matrix with positive defect.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The characteristic polynomial of A is

$$\lambda^2 - 2\lambda + 1 = 0$$

Therefore, 1 is the only eigenvalue of A and has multiplicity 2.

$\Longrightarrow$ Algebraic multiplicity of $1 \;= 2.$

Homework Problem

Suppose $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be an eigenvector of A corresponding to the eigenvalue 1. Then

$$Av = v \qquad \Longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Longrightarrow v_1 + v_2 = v_1$$

$\Longrightarrow v_2 = 0$ and v_1 is a free variable

$$\Longrightarrow v = v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\Longrightarrow$ Geometric multiplicity of $1 \; = 1.$

Hence, Defect of eigenvalue 1 $= 2 - 1 = 1 > 0.$

Gram Schmidt process

- The Gram–Schmidt process is a method for orthonormalizing a set of vectors in an inner product space

- The Gram–Schmidt process takes a finite, linearly independent set of vectors

$$S = \{v_1, \ldots, v_k\} \quad \text{for } k \leq n$$

and generates an orthogonal set

$$S' = \{u_1, \ldots, u_k\}$$

- The Gram–Schmidt process works as follows:

$$u_1 = v_1$$

$$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1$$

$$u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2$$

Gram Schmidt process (contd.)

$$u_4 = v_4 - \frac{\langle v_4, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_4, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 - \frac{\langle v_4, u_3 \rangle}{\langle u_3, u_3 \rangle} u_3$$

$$\vdots$$

$$\vdots$$

$$u_k = v_k - \sum_{j=1}^{k-1} \frac{\langle v_k, u_j \rangle}{\langle u_j, u_j \rangle} u_j$$

- The sequence $u_1, \ldots, u_k$ is the required system of orthogonal vectors.
- The normalized vectors

$$e_1 = \frac{u_1}{||u_1||}, e_2 = \frac{u_2}{||u_2||} \ldots, e_k = \frac{u_k}{||u_k||}$$

form an orthonormal set

Example of Gram Schmidt process

Qus: Obtain an orthogonal basis for the subspace of $\mathbb{R}^4$ spanned by

$$x_1 = (1,0,1,0), x_2 = (1,1,1,1), x_3 = (-1,2,0,1)$$

Sol: Following the Gram-Schmidt process, we set

$$v_1 = x_1 = (1,0,1,0)\,.$$

Next, we have

$$v_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{||v_1||^2} v_1 = (1,1,1,1) - \frac{2}{2}(1,0,1,0) = (0,1,0,1)$$

and

Example of Gram Schmidt process (contd.)

$$v_3 = x_3 - \frac{\langle x_3, v_1 \rangle}{||v_1||^2} v_1 - \frac{\langle x_3, v_2 \rangle}{||v_2||^2} v_2$$

$$= (-1,2,0,1) + \frac{1}{2}(1,0,1,0) - \frac{3}{2}(0,1,0,1)$$

$$= (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}).$$

The orthogonal basis so obtained is

$$\{(1,0,1,0), (0,1,0,1), (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\}$$

We can normalise the vectors to get

$$\{\frac{1}{\sqrt{2}}(1,0,1,0), \frac{1}{\sqrt{2}}(0,1,0,1), (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\}$$

Example of SVD

Qus: Find Singular value decomposition of the matrix

$$A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$$

Sol: **Step 1**: Compute $A^T A$

$$A^T A = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$$

Step 2: Find the eigenvalues of $A^T A$.

The characteristic polynomial of $A^T A$ is:

$$\lambda^2 - trace(A^T A)\lambda + det(A^T A) = 0$$

$$\implies \lambda^2 - 90\lambda = 0 \qquad (\because det(A^T A) = 0)$$

Example of SVD (contd.)

$$\implies \lambda(\lambda - 90) = 0$$

$\implies \lambda_1 = 90$ and $\lambda_2 = 0$ are the two eigenvalues of $A^T A$.

Step 3: Find the singular values of A.

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{90} \text{ and } \sigma_2 = \sqrt{\lambda_2} = 0.$$

Step 4: Construct the diagonal matrix of same size as A with singular values as its diagonal entries in decreasing order.

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example of SVD (contd.)

<u>Step 5</u>: Find the eigenvectors (called right singular vectors) corresponding to each eigenvalue of A^TA.

Suppose $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ be an eigenvector of A corresponding to the eigenvalue λ_2. Then

$$A^TAz = \lambda_2 z \implies \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies 81z_1 - 27z_2 = 0 \text{ and } -27z_1 + 9z_2 = 0$$

$$\implies z_2 = 3z_1$$

$$\implies z = z_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Choose $v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ as an eigenvector corresponding to λ_2.

Example of SVD (contd.)

Similarly, we can find $v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ as an eigenvector corresponding to λ_1.

Step 6: Write the orthogonal matrix consisting of the normalised eigenvectors of $A^T A$.

$$V = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Step 7: Find the left eigenvectors (called left singular vectors) $\hat{u}_i$ using

$$\hat{u}_i = \frac{Av_i}{\sigma_i}.$$

For $i = 1$, we have.

$$\hat{u}_1 = \frac{Av_1}{\sigma_1} = \frac{1}{\sqrt{90}} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} -\frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix} = \frac{1}{\sqrt{900}} \begin{bmatrix} 10 \\ -20 \\ -20 \end{bmatrix}.$$

Since $\sigma_2 = 0$, to find the other vector we need to use the orthogonality condition

Let $w = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the next vector. Then

$$w \cdot u_1 = 0 \implies 10x - 20y - 20z = 0 \implies x = 2y + 2z$$

$$\implies w = y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Let

$w_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ And $w_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Hence the vectors orthogonal to u_1 are

$$u_2 = w_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \implies \hat{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and}$$

$$u_3 = w_2 - \frac{\langle w_2, u_2 \rangle}{||u_2||^2} u_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{5} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix} \implies \hat{u}_3 = \frac{1}{\sqrt{45}} \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}$$

Hence, $U = \begin{bmatrix} \frac{10}{\sqrt{900}} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{45}} \\ \frac{-20}{\sqrt{900}} & \frac{1}{\sqrt{5}} & -\frac{4}{\sqrt{45}} \\ \frac{-20}{\sqrt{900}} & 0 & \frac{5}{\sqrt{45}} \end{bmatrix}.$

Step 8: The singular value decomposition of A is

$$A = U\Sigma V^T = \begin{bmatrix} \frac{10}{\sqrt{900}} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{45}} \\ \frac{-20}{\sqrt{900}} & \frac{1}{\sqrt{5}} & -\frac{4}{\sqrt{45}} \\ \frac{-20}{\sqrt{900}} & 0 & \frac{5}{\sqrt{45}} \end{bmatrix} \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$$

Thank You