Introduction to Statistical Methods

ISM Team

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Session 3: Conditional Probability, Independent events and, Total Probability

(Session 3: 3rd/4th June 2023)

IMP Note to Self

RECAPTIULATION:

- Introduction to Probability
- Review of Set Theory
- Counting Principles
- Definition of Probability
- Axioms of Probability
- Addition Rule of Probability

Contact Session 3

Contact Session 3: Module 2(Conditional Probability & Bayes theorem)

Contact Session	List of Topic Title	Reference
CS - 3	Introduction to conditional probability,indepents events, Total probability	T1 & T2
HW	Problems on conditional probability	T1 & T2
Lab		

CONTENTS:

- Conditional Probability
- Independent Events
- Total Probability

CONDITIONAL PROBABILITY

The probabilities assigned to various events depend on what is known about the experimental situation when the assignment is made. Subsequent to the initial assignment, partial information relevant to the outcome of the experiment may become available. Such information may cause us to revise some of our probability assignments

CONDITIONAL PROBABILITY

We examine how the information "an event B has occurred" affects the probability assigned to A.

For example, A might refer to an individual having a particular disease in the presence of certain symptoms. If a blood test is performed on the individual and the result is negative , then the probability of having the disease will change (it should decrease, but not usually to zero, since blood tests are not infallible). We will use the notation to represent the conditional probability of A given that the event B has occurred. B is the "conditioning event."

CONDITIONAL PROBABILITY

Let A and B be two events in sample space. The **conditional probability** that event A occurs given that event B has occurred and it is denoted by

$$P\left(\frac{A}{B}\right) = \frac{P(A \text{ and } B)}{P(B)} \quad \text{OR} \quad P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$$

It can also written as $P(A \cap B) = P(B)\, P(A/B) \qquad P(B) \neq 0$

$= P(A)\, P(B/A) \qquad P(A) \neq 0$

Let A,B and C be three events in a sample space S,

then $P(A \cap B \cap C) = P(A)\, P(B/A)\, P(C/A \cap B)$

and it is called Multiplication Rule

CONDITIONAL PROBABILITY

Multiplication Rule

In general, $A_1, A_2, \ldots\ldots. A_n$ are events in S, then

$$P(A_1 \cap A_2 \cap \ldots. \cap A_n) = P(A_1)\, P(A_2/A_1) P(A_3/A_1 \cap A_2)$$

$$\ldots\ldots. P(A_n/A_1 \cap A_2 \cap \ldots\ldots \cap A_{n-1})$$

Examples:1

In a certain college, 25% of the students failed Maths, 15% of the students failed chemistry and 10% of the students failed both maths and chemistry. A student is selected at random, find

a. If he failed chemistry, what is the probability that he failed Maths?

$$P(M) = 0.25, \quad P(C) = 0.15, \; P(M \cap C) = 0.1$$

$$P\left(\frac{M}{C}\right) = \frac{P(M \cap C)}{P(C)} = \frac{0.1}{0.15} = 0.6667$$

b. If the failed maths, what is the probability he failed chemistry?

$$P\left(\frac{C}{M}\right) = \frac{P(C \cap M)}{P(M)} = \frac{0.1}{0.25} = 0.4$$

c. What is the probability that the student failed in Maths or chemistry?

Examples:2

In a housing colony, 70% of the houses are well planned and 60% of the houses are well planned and well built. Find the probability that an arbitrarily chosen house in this colony is well built given that it is well planned.

Solution:

Let A be the event that the house is well planned B be the event that the house is well built therefore

$$P(A) = 0.70, \qquad P(A \cap B) = 0.60$$

$$P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)} = \frac{0.60}{0.70} = 0.8571$$

Examples: 3

The probabilities of a regularly scheduled flight departs on time is 0.83, arrives on time is 0.82 & it departs and arrives on time is 0.78. Find the probability that a plane

(i) arrives on time given that it departed on time,

(ii) departed on time given that it has arrived on time and

(iii) find $\boldsymbol{P}\left(\frac{\boldsymbol{A}}{\overline{\boldsymbol{D}}}\right)$

Ans:

Let D and A be the events that the flight departs and arrives on time respectively. Then, $P\ (D) = 0.83, P(A) = 0.82$ and $P\ (D \cap A) = 0.78$

(i) Probability that the plane arrives on time given that it departed on time is

$$P\left(\frac{A}{D}\right) = \frac{P(D \cap A)}{P(D)} = \frac{0.78}{0.83} = 0.9398$$

(ii) Probability that the plane departed on time given that it has arrived on time is $P\left(\frac{D}{A}\right) = \frac{P(D \cap A)}{P(A)} = \frac{0.78}{0.82} = 0.9512$

(iii) $P\ \left(\frac{A}{\bar{D}}\right) = \frac{P(A \cap \bar{D})}{P(\bar{D})} = \frac{0.82 - 0.78}{1 - 0.83} = 0.24$

This is the probability that the flight arrives on time given that it did not depart on time

Example: 4

In a housing colony, 70% of the houses are well planned and 60% of the houses are well planned and well built. Find the probability that an arbitrarily chosen house in this colony is well built given that it is well planned.

Solution: Let A be the event that the house is well planned

B be the event that the house is well built

therefore $P(A) = 0.70$, $P(A \cap B) = 0.60$

$$P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)} = \frac{0.60}{0.70} = 0.8571$$

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Example: 5

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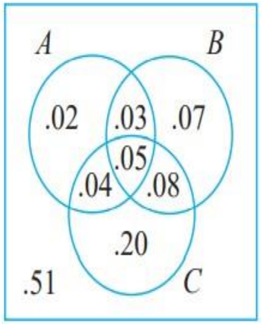
A news magazine publishes three columns entitled "Art" (A), "Books" (B), and "Cinema" (C). Reading habits of a randomly selected reader with respect to these columns are

Read regularly	A	B	C	$A \cap B$	$A \cap C$	$B \cap C$	$A \cap B \cap C$
Probability	.14	.23	.37	.08	.09	.13	.05



Find

A). P(A/B)

B). P(A/BUC)

C). P(A/ reads at least one)

Solution

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We thus have

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.08}{.23} = .348$$

$$P(A|B \cup C) = \frac{P(A \cap (B \cup C))}{P(B \cup C)} = \frac{.04 + .05 + .03}{.47} = \frac{.12}{.47} = .255$$

$$P(A|\text{reads at least one}) = P(A|A \cup B \cup C) = \frac{P(A \cap (A \cup B \cup C))}{P(A \cup B \cup C)}$$

$$= \frac{P(A)}{P(A \cup B \cup C)} = \frac{.14}{.49} = .286$$

and

$$P(A \cup B|C) = \frac{P((A \cup B) \cap C)}{P(C)} = \frac{.04 + .05 + .08}{.37} = .459$$

INDEPENDENT EVENTS

We can deduce an important result from the conditional probability:

If B has no effect on A, then, $\boldsymbol{P\left(\frac{A}{B}\right) = P(A)}$ Also $\boldsymbol{P\left(\frac{B}{A}\right) = P(B)}$

and we say the events are independent.

i.e., The probability of A does not depend on B.

so, $$\boldsymbol{P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}}$$

becomes, $$\boldsymbol{P(A) = \frac{P(A \cap B)}{P(B)}}$$

or $$\boxed{P(A \cap B) = P(A) \times P(B)}$$

Example: 1

A problem in statistics is given to 3 students A,B,C. Their chances of solving it are 1/2,1/3/,1/4. Find the probability that the problem is solved.

Solution: Problem can be solved by either A or B or C

Therefore we have to use

P(AUBUC) = P(A) + P(B) + P(C) - P(A∩B) - P(B∩C) - P(A∩C) + P(A ∩ B ∩ C)

Using complement of the event i.e the problem is not solved . Therefore P(problem solved is)=1-P(not solved)

$$= 1 - P(\overline{A} \cap \overline{B} \cap \overline{C})$$

$$= 1 - P(\overline{A})\, P(\overline{B})\, P(\overline{C})$$

$$= 1 - \left[1 - \frac{1}{2}\right]\left[1 - \frac{1}{3}\right]\left[1 - \frac{1}{4}\right]$$

$$= \frac{1}{4}$$

Note: If A, B,C are independent then $\overline{A}, \overline{B}, \overline{C}$ are also independent.

Example: 2

A box contains 20 fuses of which 5 are defective. If two fuses are choosen at random one after the other. What is probability that both the fuses are defective if

(i) the first fuse is replaced,

(ii) (ii) the first fuse is not replaced.

Solution: Let A be the event that the first fuse is defective and B be the event that the second fuse is defective

(i) When the first fuse is replaced, the events are independent hence

$$P(A \cap B) = P(A) \times P(B) = \frac{5C_1}{20C_1} \times \frac{5C_1}{20C_1} = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

(ii) When first fuse is not replaced, the events are not independent then

$$P(B \cap A) = \mathrm{P(A)} \times P\left(\frac{B}{A}\right) = \frac{5C_1}{20C_1} \times \frac{4C_1}{19C_1} = \frac{1}{19}$$

Miscellaneous examples

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1. A box A contains 2 white balls and 4 black balls. Another box contains 5 white balls and 7 black balls. A ball is transferred fro box A to box B, than a ball is drawn from box B. What is th probability that it is white.

2. A person X can hit a target three times in five shots, a second person twice in five shots and a third person Z thrice in four shots. They a fire once simultaneously. Find the probability that (i) atleast two sh hit, (ii) exactly two shots hit.

Continued

3. If A and B are two events with P(A)= ½, P(B)= 1/3 and $P(A \cap B) = 1/4$
Find

$$P\left(\frac{A}{B}\right), P\left(\frac{B}{A}\right), \mathrm{P}\left(\frac{\overline{\mathrm{A}}}{\overline{B}}\right), \mathrm{P}\left(\frac{\overline{\mathrm{B}}}{\overline{A}}\right), \mathrm{P}\left(\frac{\mathrm{A}}{\overline{B}}\right)$$

4.Three students X,Y,Z write an examination. Their chances of passing are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ respectively. Find the probability that

(i) all of them pass,

(ii) at least one of them passes,

(iii) at least two of them pass.

Continued

5. The probability that Ram will hit a target is 0.56 and the probability that shyam will hit the target is 0.45. The probability that both will hit the target is 0.25. find the probability that
   (i) at least one will hit the target
   (ii) exactly one will hit the target.

6. If A and B are two events such that

$$P(A \cap B) = 0.05, P(B) = 0.7, P(A \cap \overline{B}) = 0.02$$

Find (i) $P(A \cup B)$, (ii) $P(A \cup \overline{B})$

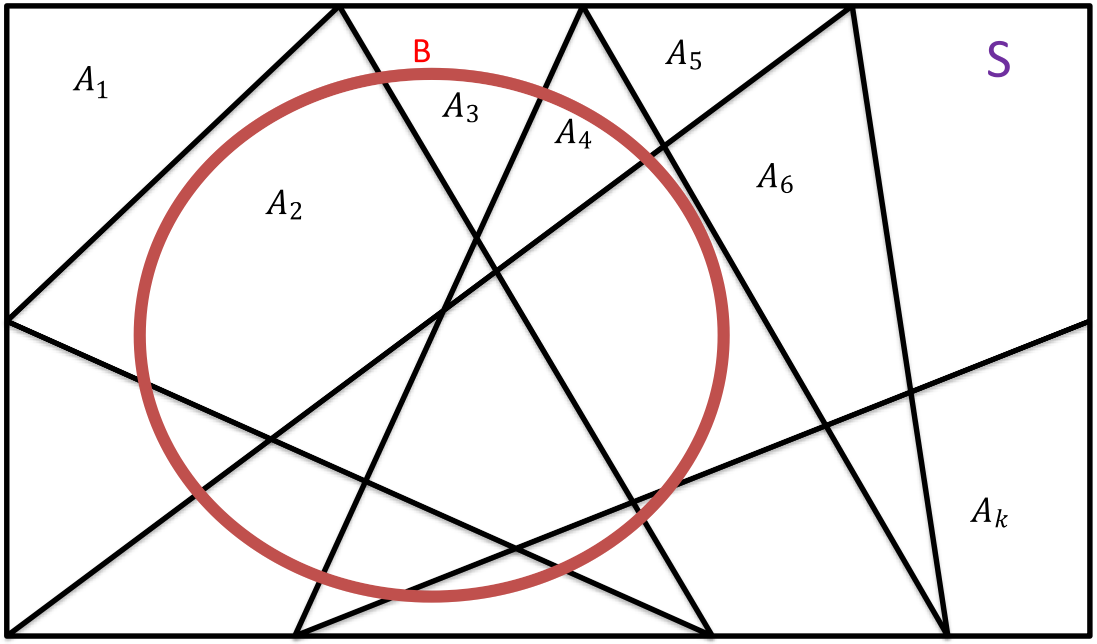
Total Probability

The Law of Total Probability

Let $A_1, \ldots, A_k$ be mutually exclusive and exhaustive events. Then for any other event B,

$$\begin{aligned} P(B) &= P(B \mid A_1)P(A_1) + \cdots + P(B \mid A_k)P(A_k) \\ &= \sum_{i=1}^{k} P(B \mid A_i)P(A_i) \end{aligned}$$

$S = A_1 \cup A_2 \cup \ldots \cup A_n$ and $A_1 \cap A_2 \cap \ldots \cap A_n = \emptyset$



$$\therefore \boldsymbol{B} = \boldsymbol{B} \cap \boldsymbol{S} = \boldsymbol{B} \cap \{\boldsymbol{A}_1 \cup \boldsymbol{A}_2 \cup \boldsymbol{A}_3, \; . \quad . \quad . \cup \boldsymbol{A}_n\}$$

Proof:

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We have $\boldsymbol{S} = \{A_1 \cup A_2 \cup A_3 \cup \;.\;\;.\;\;. \cup A_n\}$ and $A \subset S$

$$\therefore B = B \cap S = B \cap \{A_1 \cup A_2 \cup A_3, \;.\;\;.\;\;. \cup A_n\}$$

Using distributive law in the R.H.S, we get

Since B ∩ A_i (i = 1 to n) are mutually exclusive, we have by applying addition rule of probability,

$$P(B) \;=\; P(B \cap A_1) + P(B \cap A_2) + \cdots\cdots + P(B \cap A_n)$$

$$i.e., \; P(B) \;=\; \sum_{i=1}^{i=n} P(B \cap A_i)$$

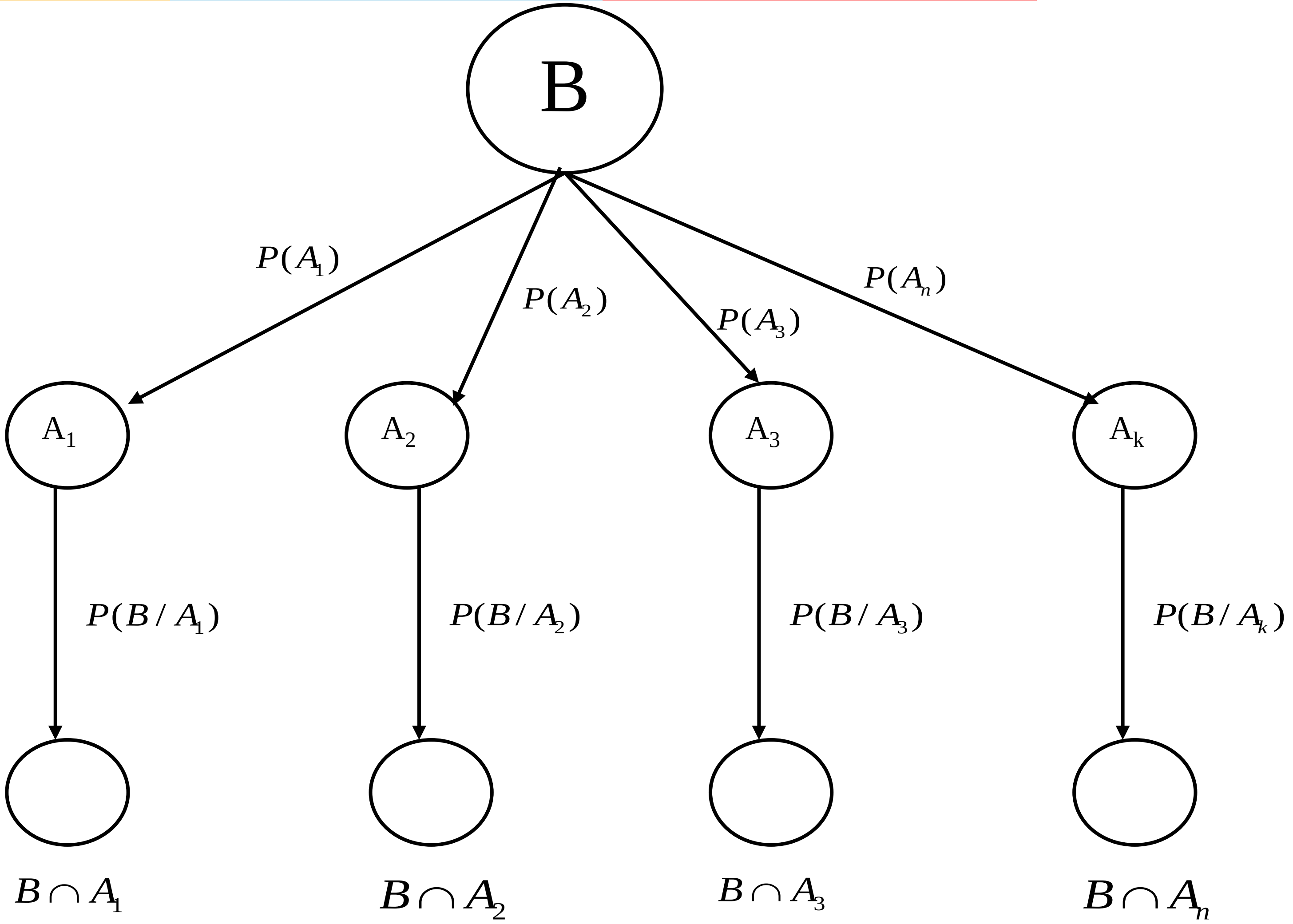
Using multiplication rule on each term on R.H.S, namely

$$P(B \cap A_i) = P(A_i) \cdot P(B \mid A_i) \qquad \text{(A)}$$

$$P(B) \;=\; \sum_{i=1}^{i=n} P(A_i)\, P(B \mid A_i)$$

(B)-Total theorem on Probability

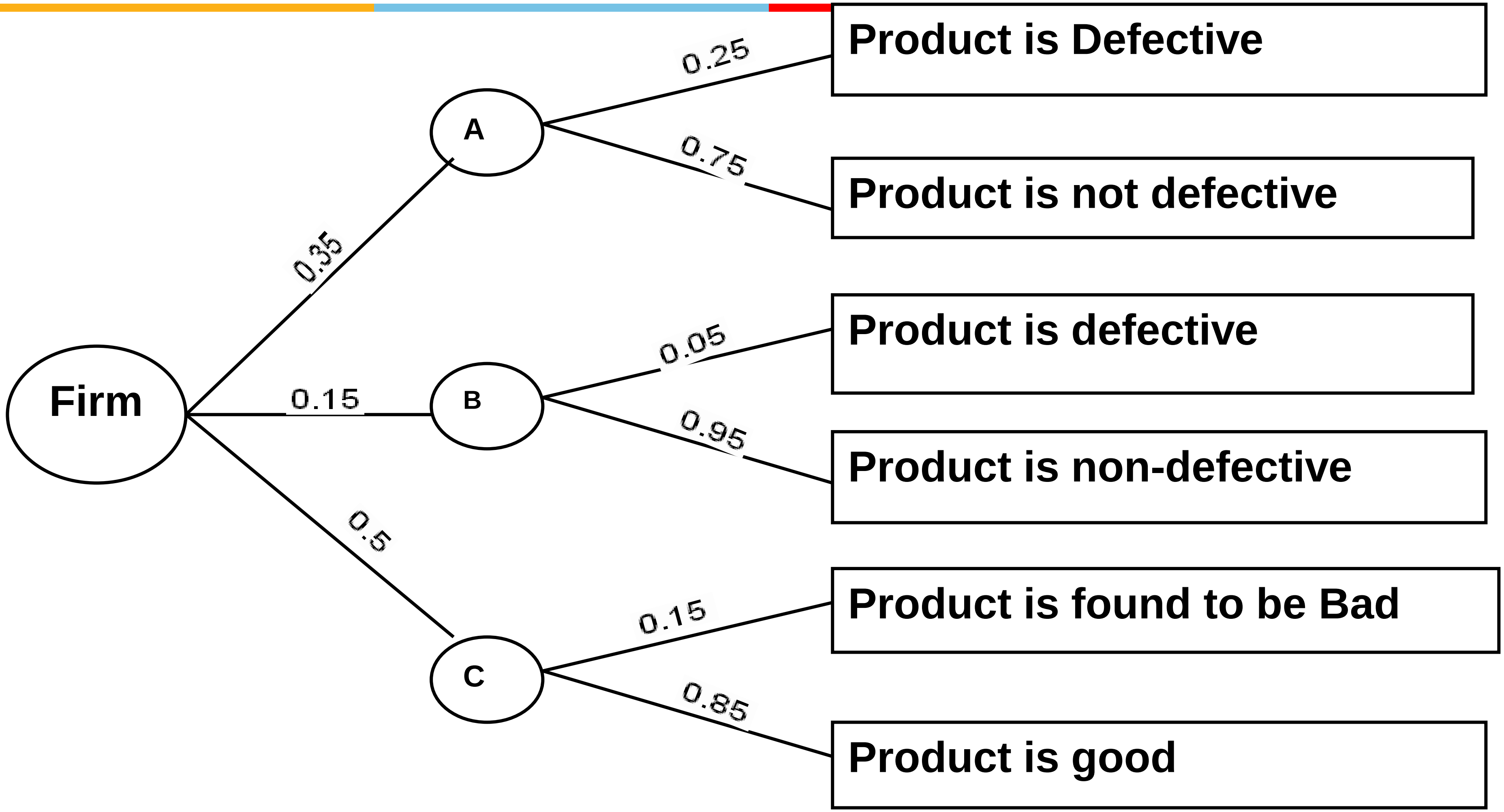
The Theorem of Total Probability
(tree diagram)



Examples :1

A certain firm has plants A, B, C producing, respectively 35 %, 15% and 50% of the total output. The probabilities of a non – defective product are, respectively, 0.75, 0.95 and 0.85.A Customer receives a bad product, what is the Chance that product came from the plant C?

Tree Diagram



Solution

Let X: " Customer receives a defective product".

Clearly, P(X)= $P(A)P\left[\frac{X}{A}\right]+P(B)P\left[\frac{X}{B}\right]+P(C)P\left[\frac{X}{C}\right]$

=0.17

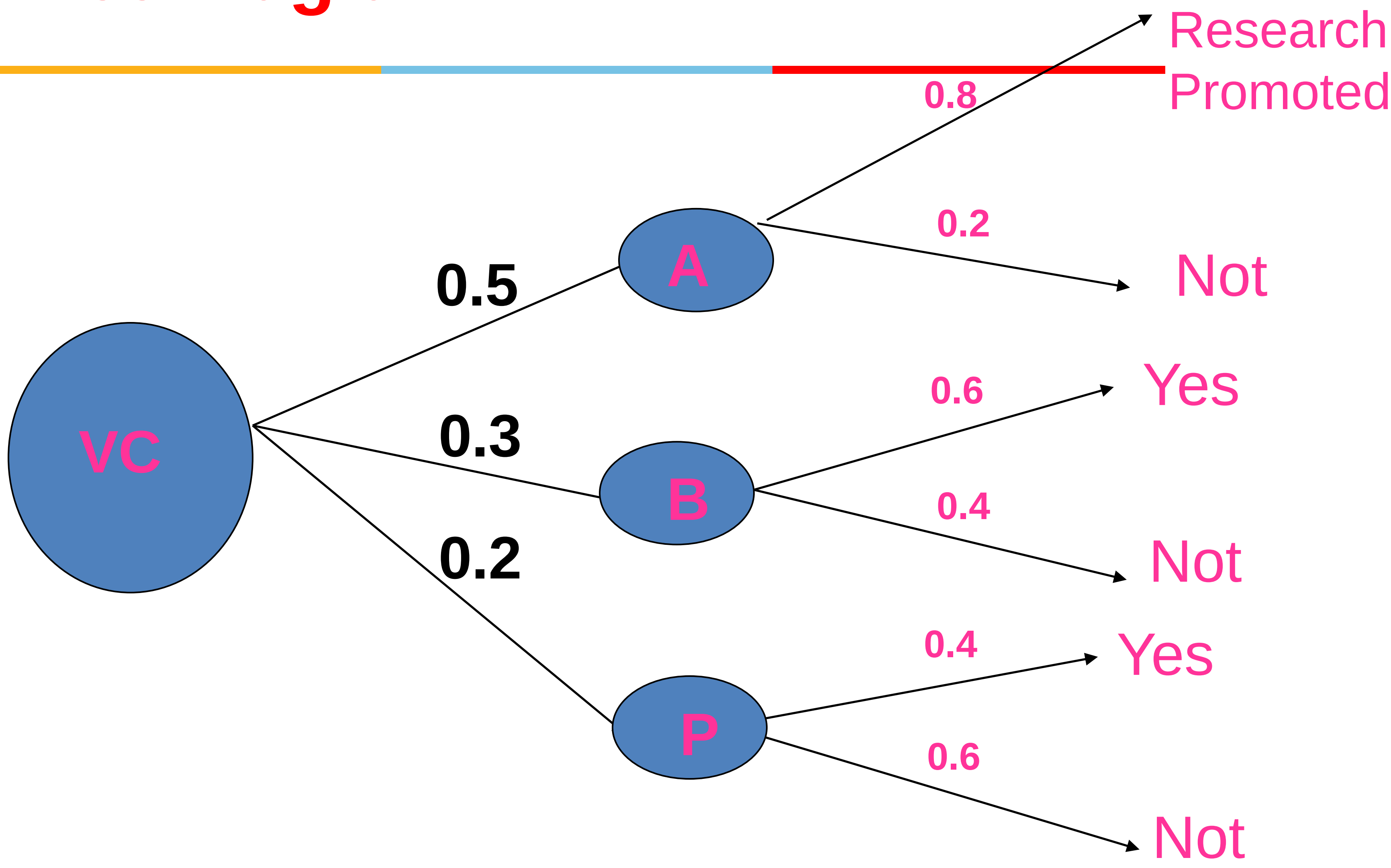
Therefore, the chance that product is manufactured by the plant C is

$$\boldsymbol{P(C \mid X)=\frac{P(C\cap X)}{P(X)}} = \boldsymbol{\frac{0.5\cdot 0.15}{0.17}} = \boldsymbol{0.4412}$$

Example 2

The chances that an academician, a business man and a politician becoming Vice Chancellor of an university are 0.5, 0.3 and 0.2 respectively. The probability that research work will be promoted in the university by these 3 gentlemen are respectively are 0.8, 0.6 and 0.4. It is found Research work has been promoted by the university. What is the chance that an academician has become the VC?

Tree Diagram



Solution

Let **X: " Research work is promoted"**

Clearly, $P(X) = 0.5 \times 0.8 + 0.3 \times 0.6 + 0.2 \times 0.4 = 0.66$

Now to find P[" An Academician is VC" / "Research work is promoted i.e. event X"] $= \frac{0.5 \times 0.8}{0.66} = 0.6061$

Practice problems

Example 1.

An office has 4 secretaries handling respectively 20%, 60%,15% and 5% of the files of all government reports. The probability that they misfile such reports are respectively 0.05, 0.1, 0.1 and 0.05. Find the probability that the misfiled report can be blamed on the first secretary.

Example 2.

In a class 70% are boys and 30% are girls. 5% of boys and 3% of girls are irregular to the classes. What is the probability of a student selected at random is irregular to the classes and what is the probability that the irregular student is a girl?

Continued

Example 3.

Three machines A, B and C produce respectively 60%, 30% and 10% of the total number of items of a factory. The percentage of defective outputs of these machines are 2%, 3% and 4%. An item is selected at random and is found to be defective.

(i) Find the probability that the item was produced by machine C?

(ii) What is the probability that the item was produced by machine C or B?

Example: 4

A card is randomly drawn from an incomplete deck of cards from which the ace of diamonds is missing.

1. What is the probability that the card is “clubs”?
2. What is the probability that the card is a “queen”?
3. Are the events “clubs” and “queen” independent

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Example: 5

In a group of children from primary school there are 18 girls and 15 boys. Of the girls, 9 have had measles. Of the boys, 6 have had measles.

1. What is the probability that a randomly chosen child from this group has had measles?

2. If we randomly choose one person from the group of 18 girls, what is the probability that this girl has had measles?

3. Are the events “boy” and “measles” in this example independent?

IMP Note to Self

Thank you