Q1) Let A m×n be a given matrix with m>n. If the time taken to compute the determinant of a square matrix of size j is j^3, find upper bound on the

(a) Total time taken to find the rank of A using determinants.

(b) No. of additions and multiplications required to determine the rank using the elimination procedure.

Note :->

Rank in terms of determinant :->

order of non-zero minor of highest order (size)

$^nC_r = \frac{n!}{(n-r)!\, r!}$

Eg :-

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \\ 6 & 7 \end{bmatrix}_{3\times 2}$$

$$\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix}_{2\times 2}, \quad \begin{vmatrix} 1 & 2 \\ 6 & 7 \end{vmatrix}_{2\times 2} \quad \begin{vmatrix} 3 & 2 \\ 6 & 7 \end{vmatrix}_{2\times 2} = 3$$

$= 6 - 2$ $\qquad = 7 - 12$ $\qquad = 21 - 12$

$= 4 \neq 0$ $\qquad = -5 \neq 0$ $\qquad = 9 \neq 0$

$|3|_{1\times1}$ $|2|_{1\times1}$ $|1|_{1\times1}$ $|2|_{1\times1}$ $|6|_{1\times1}$ $|7|_{1\times1}$ $= 6$

Total $= 9$

$\therefore \sum_{i=1}^{2} {}^mC_i \, {}^nC_i$

$m = 3, \; n = 2$

$= \sum_{i=1}^{2} {}^3C_i \, {}^2C_i$

$= {}^3C_1 \, {}^2C_1 + {}^3C_2 \, {}^2C_2$

$= (3\times 2) + \frac{3!}{(3-2)!\, 2!}(1)$

$= 6 + \frac{3\times \cancel{2!}}{1! \times \cancel{2!}} \times (1)$

$= 6 + 3$

$= \underline{9}$

$$\therefore \sum_{i=1} ({}^{m}C_i \; {}^{n}C_i) = (9 \ldots)$$

$$\therefore \sum_{i=1}^{2} {}^{m}C_i \, {}^{n}C_i = 9$$

Eg (2)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 6 & 2 \\ 7 & 6 & 9 \end{bmatrix}_{4+3} \qquad = 12$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 6 & 2 \end{vmatrix}_{3+3}, \begin{vmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 7 & 6 & 9 \end{vmatrix}_{3+3} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 6 & 2 \\ 7 & 6 & 9 \end{vmatrix}_{3+3} \begin{vmatrix} 7 & 8 & 9 \\ 4 & 6 & 2 \\ 7 & 6 & 9 \end{vmatrix}_{3+3} = 4$$

$$\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix}, \begin{vmatrix} 7 & 8 \\ 4 & 6 \end{vmatrix}, \begin{vmatrix} 8 & 9 \\ 6 & 2 \end{vmatrix} \begin{vmatrix} 7 & 9 \\ 4 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 6 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix}, \begin{vmatrix} 4 & 6 \\ 7 & 6 \end{vmatrix}, \begin{vmatrix} 6 & 2 \\ 6 & 9 \end{vmatrix} \begin{vmatrix} 4 & 2 \\ 7 & 9 \end{vmatrix} \qquad = 18$$

$$\begin{vmatrix} 7 & 8 \\ 7 & 6 \end{vmatrix}, \begin{vmatrix} 8 & 9 \\ 6 & 9 \end{vmatrix}, \begin{vmatrix} 7 & 9 \\ 7 & 9 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 7 & 6 \end{vmatrix} \begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix}$$

Total = 34

Now

$$= \sum_{i=1}^{3} {}^{m}C_i \, {}^{n}C_i \qquad m = 4, \; n = 3$$

$$= \sum_{i=1}^{3} {}^{4}C_i \, {}^{3}C_i = {}^{4}C_1 \, {}^{3}C_1 + {}^{4}C_2 \, {}^{3}C_2 + {}^{4}C_3 \, {}^{3}C_3$$

$$= (4 \times 3) + \left(\frac{4!}{(4-2)!\, 2!} \times \frac{3!}{(3-2)!\, 2!} \right) + \left(\frac{4!}{(4-3)!\, 3!} \times 1 \right)$$

$$= 12 + \left[\frac{4+3+\cancel{2!}}{\cancel{2!}+2!} \times \frac{3 \times \cancel{2!}}{1! \times \cancel{2!}} \right] + \left[\frac{4 \times \cancel{3!}}{1! \times \cancel{3!}} \times 1 \right]$$

$$= 12 + (6 \times 3) + (4 \times 1)$$

$$= 12 + 18 + 4 = 34$$

$$\therefore \sum_{i=1}^{3} {}^4C_i \, {}^nC_i = 34$$

In general

No. of determinants $= \displaystyle\sum_{i=1}^{\min(m,n)} {}^mC_i \, {}^nC_i$

where $\min(m,n)$ max. size of square matrix in A

${}^mC_i$ is the combination of i columns out of m

${}^nC_i$ is the combination of i rows out of n

Given that time to compute the determinant of a square matrix of size i is i^3

Hence total time taken to compute determinant of a square matrix is $\displaystyle\sum_{i=1}^{i=\min(m,n)} {}^mC_i \, {}^nC_i \, i^3$

Note :-

Eg (1) $\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$

$\boxed{\rho(A) = 1}$

$\begin{vmatrix} 2 & 2 \\ 4 & 4 \end{vmatrix} = 8 - 8 = 0$

minor of $2 = |4|_{1\times1}$

-"- of $2 = |4|_{1\times1}$

-"- of $4 = |2|_{1\times1}$

-"- of $4 = |2|_{1\times1}$

Eg $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 10 \end{bmatrix}$ $R_2 \to R_2 + 3R_1$

minor of $1 = 4$
-"- of $2 = -3$
-"- of $-3 = 2$
-"- of $4 = 1$

$\rho(A) = 2$

Rank in terms of determinants

order of non-zero minor of highest order (size)

Eg $\begin{bmatrix} 2 & 3 & 1 \\ 4 & 5 & 7 \end{bmatrix}_{2\times 3} \Rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$\boxed{\therefore \rho(A) = 2}$

$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$ $\begin{vmatrix} 2 & 1 \\ 4 & 7 \end{vmatrix}$ $\begin{vmatrix} 3 & 1 \\ 5 & 7 \end{vmatrix}$

$= 10 - 12$ $\quad 14 - 4$ $\quad 21 - 5$

$= -2 \neq 0$ $\quad = 10 \neq 0$ $\quad = 16 \neq 0$ non-zero minors

Total time $= 3 \times 2^3 + 6 \times 1^3$

|||ly $\begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 4 & 5 \end{bmatrix}_{3\times 2}$

$\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$ $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$ $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$

$= 4 - 3$ $\quad = 10 - 12$ $\quad = 5 - 8$

$= 1 \neq 0$ $\quad = -2 \neq 0$ $\quad = -3 \neq 0$

Total time $= \boxed{3 \times 2^3 + 6 \times 1^3}$

Max. time taken

$$\sum_{i=1}^{i=\min(m,n)} \left({}^mC_i \; {}^nC_i \right) i^3$$

(6)

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}_{m \times n} \rightarrow nm \quad nA$$

LE, 1M 1A, 1M 1A, nm nA

REF

$$R_2' \longrightarrow R_2 - \left(\frac{a_{21}}{a_{11}}\right) R_1$$

1M, 1A

I^st row operation :-

$$\text{multi} :\to \underbrace{n+n+n+\cdots+n}_{(m-1)\text{ rows}}$$

$$= n(m-1) \text{ times}$$

$$A \approx \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}' & \cdots & a_{2n}' \\ 0 & a_{32}' & \cdots & a_{3n}' \\ \vdots & \vdots & & \vdots \\ 0 & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

LE, 1M 1A

II^nd row operation $\quad R_3' \longrightarrow R_3 - \left(\frac{a_{32}'}{a_{22}'}\right) R_2$

$$\text{multi} :- \underbrace{(n-1)+(n-1)+(n-1)\cdots+(n-1)}_{(m-2)\text{ rows}}$$

$$= (n-1)(m-2) \text{ times}$$

III^{ly} Add $= (n-1)(m-2)$

The no of multiplications required to make all the entries below a_{11} to zeros is $n(m-1)$

The no of multiplications required to make all the entries below a_{22} to zeros is $(n-1)(m-2)$

⋮

The no of multiplications required all the entries to make all the entries a_{kk} to zeros is

$(n-k+1)(m-k)$

∴ total no of multiplication required is

$$= n(m-1)+(n-1)(m-2)+ \cdots$$

$$= \sum_{i=1}^{n} (m-k)(n-k+1)$$

|||ly total no of addition required

$$= \sum_{i=1}^{n} (m-k)(n-k+1)$$

Hence the total no of addition and multiplication required $= 2\sum_{i=1}^{n} (m-k)(m-k+1)$

(4)

Q.4) Compute the total number of division, multiplication and additions required to perform the forward elimination and back substitution in solving a system of linear equations $A_{n\times n}X = b$ using the Gauss elimination method

Sol^n Gauss elimination method :-

consider the system of equation

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}\right\} \quad AX = b$$

The augmented matrix is given by

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{3\times 1}$

The Augmented matrix is given by [Forward Elimination]

$$[A:B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ a_{21} & a_{22} & a_{23} & : & b_2 \\ a_{31} & a_{32} & a_{33} & : & b_3 \end{bmatrix}$$

REF

$R_2' \longrightarrow R_2 - \dfrac{a_{21}}{a_{11}} R_1$ $\qquad$ $D \to 1,\ M \to 3,\ A \text{ (or) } S \to 3$

$R_3' \longrightarrow R_3 - \dfrac{a_{31}}{a_{11}} R_1$ $\qquad$ $D \to 1,\ M \to 3,\ A \text{ (or) } S \to 3$

$$[A:B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & : b_1 \\ 0 & a'_{22} & a'_{23} & : b'_2 \\ 0 & a'_{32} & a'_{33} & : b'_3 \end{bmatrix}$$

$$R'_3 \longrightarrow R_3 - \frac{a'_{31}}{a'_{22}} R_2$$

$D \to 1$, $M = 2$; A (or) $S \to 2$

$$[A:B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & : b_1 \\ 0 & a'_{22} & a'_{23} & : b'_2 \\ 0 & 0 & a''_{33} & : b''_3 \end{bmatrix}$$

Total:-

$D \to 3$; $M = 8$, A (or) $S \to 8$

Total = 19

Solving to find x, y and z [Backward substitution]

we have

$$a_{11}x + a_{12}y + a_{13}z = b_1 \longrightarrow (1)$$

$$a'_{22}y + a'_{23}z = b'_2 \longrightarrow (2)$$

$$a''_{33}z = b''_3 \longrightarrow (3)$$

from (3)

$$a''_{33}z = b''_3$$

$$\boxed{z = \frac{b''_3}{a''_{33}}}$$

$D \to 1$

from (2) $a'_{22}y + a'_{23}z = b'_2$

$$a'_{22}y = b'_2 - a'_{23}z$$

$$\boxed{y = \frac{b'_2 - a'_{23}z}{a_{22}}}$$

$D \to 1$, $M = 1$, A (or) $S = 1$

From (1) $a_{11}x + a_{12}y + a_{13}z = b_1$

$$a_{11}x = b_1 - a_{12}y - a_{13}z$$

$$\boxed{x = \frac{b_1 - a_{12}y - a_{13}z}{a_{11}}}$$

D → 1, M → 2 A (or) S → 2

Total :- D → 3 M → 3, A (or) S → 3

Total = 9

Total of Forward eliminatation and Back substitution

	D	M	A (or) S	
FE →	3	8	8	= 19
BS →	3	3	3	= 9
Total →	6	11	11	= 28

Ikhi Chendrule $\quad A_{n\times n}x = b$

Forward Elimination:-

$$[A:B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & : & b_2 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & & b_n \end{bmatrix}$$

Reduced into REF [Gauss elimination]

$$R_2 \rightarrow R_2 - \frac{a_{21}}{a_{11}} R_1 \qquad \begin{matrix} D \rightarrow 1 \\ A \rightarrow 1 \\ M \rightarrow 1 \end{matrix}$$

Stage-I:-

$M = n + n + \cdots + n = (n-1)$ times

$A = n + n + n + \cdots + n = (n-1)$ times

$D = 1 + 1 + \cdots + 1 = (n-1)$ times

IInd Stage:-

$m = (n-1) + (n-1) + \cdots + (n-1) = (n-2)$ times

$A = (n-1) + (n-1) + \cdots + (n-1) = (n-2)$ times

$D = 1 + 1 + \cdots + 1 = (n-2)$ times

IIIrd Stage:-

$m = n(n-1) + (n-1)(n-2) + \cdots + 1$

$A = n(n-1) + (n-1)(n-2) \cdots + 1$

$D = (n-1) + (n-2) + \cdots + 1$

$= 1 + 2 + \cdots + (n-1)$

$= \sum_{K=1}^{n} (n-K)$

Back substitution :-

we have

$$\begin{bmatrix} 2 & 1 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \\ 12 \end{bmatrix}$$

$6z = 12 \Rightarrow \boxed{z = \frac{12}{6} = 2}$ $\quad D = 1$

$3y + 5z = 10$

$\Rightarrow 3y = 10 - 5(2)$

$\Rightarrow 3y = 10 - 5 \times 2$

$\boxed{y = \frac{10 - 5 \times 2}{3}}$

$D = 1, \ M = 1, \ A = 1$

$2x + y + 4z = 11$

$2x = 11 - y - 4z$

$x = \frac{11 - y - 4z}{2}$

$\boxed{x = \frac{11 - y - 4z}{2}}$

$D = 1, \ M = \cancel{2}, \ A = \cancel{2}$

$M = 2 \quad A = 2$

$\therefore \ M = 1 + 2 = 3$

$A = 1 + 2 = 3$

$D = 1 + 1 + 1 = 3$

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

· Number of divisions required in Gauss elimination $\Big\} = \sum_{k=1}^{n} (n-k)$

Number of multiplication and addition in Gauss elimination $\Big\} = 2\sum_{k=1}^{n-1} (n-k)(n-k+1)$

$\Big\}$ Forward Elimination Elimination

Number of operations required back substitution $\Big\} = 2\sum_{k=1}^{n} (n-k) + n \Big\}$ Back substitution

Now

Let $f(n)$ be forward elimination

$$\therefore f(n) = \sum_{k=1}^{k=n-1} (n-k) + 2\sum_{k=1}^{k=(n-1)} (n-k)(n-k+1) \longrightarrow (1)$$

Now $\boxed{n-k = s}$ | $k=1$ | $k = n-1$

$\boxed{n-s=k}$ | $n-s=1$ | $\cancel{n}-s = \cancel{n}-1$

| $\boxed{s=n-1}$ | $\boxed{s=1}$

Note:- $\sum_{k=1}^{n} k = 1+2+3+ \dots + n = \dfrac{n(n+1)}{2}$

$$\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

Eqn (1) becomes

$$= \sum_{s=1}^{s=n-1} s + 2\sum_{s=1}^{s=n-1} s(s+1)$$

$$\Rightarrow = \sum_{s=1}^{s=n-1} s + 2\left[\sum_{s=1}^{n-1} s^2 + \sum_{s=1}^{n-1} s\right]$$

$$= \cancel{n} \frac{(n-1)n}{2} + 2\left[\frac{(n-1)n(2n-1)}{6} + \frac{(n-1)n}{2}\right]$$

$$= \frac{n^2-n}{2} + \cancel{2}\left[\frac{(n^2-n)(2n-1) + 3n^2-3n}{\cancel{6}\,3}\right]$$

$$= \frac{3n^2-3n+2(2n^3-2n^2-n^2+n+3n^2-3n)}{6}$$

$$= \frac{3n^2-3n+2(2n^3\cancel{-3n^2}+n\cancel{+3n^2}-3n)}{6}$$

$$= \frac{3n^2-3n+4n^3+2n-6n}{6}$$

$$= \frac{4n^3}{6} + \frac{3n^2}{6} - \frac{7n}{6}$$

$$\boxed{f(n) = \frac{2n^3}{3} + \frac{n^2}{2} - \frac{7n}{6}} \Rightarrow O(n^3) = f(n)$$

Let $b(n)$ be back substitution

$$b(n) = 2\sum_{k=1}^{n}(n-k) + n$$

$$= 2\left[n\sum_{k=1}^{n}1 - \sum_{k=1}^{n}k\right] + n$$

$$= 2\left[n \cdot n - \frac{n(n+1)}{2}\right] + n$$

$$= 2\left[n^2 - \frac{n^2}{2} - \frac{n}{2}\right] + n$$

$$= 2\left[\frac{n^2}{2} - \frac{n}{2}\right] + n$$

$$= n^2 - \cancel{n} + \cancel{n}$$

$$\boxed{b_n = n^2} \Rightarrow \boxed{O(n^2) = b(n)}$$

Theorem - 1 :- product of any lower triangular matrices is a lower matrices

Eg :-

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 5 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 6 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1\times2)+(0\times3)+(0\times2) & (1\times0)+(0\times1)+(0\times6) & (1\times0)+(0\times0)+(0\times1) \\ (2\times2)+(2\times3)+(0\times2) & (2\times0)+(2\times1)+(0\times6) & (2\times0)+(2\times0)+(0\times1) \\ (5\times2)+(2\times3)+(1\times2) & (5\times0)+(2\times1)+(1\times6) & (5\times0)+(2\times0)+(1\times1) \end{bmatrix}$$

$$= \begin{bmatrix} 2+0+0 & 0+0+0 & 0+0+0 \\ 4+6+0 & 0+2+0 & 0+0+0 \\ 10+6+2 & 0+2+6 & 0+0+1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 0 & 0 \\ 10 & 2 & 0 \\ 18 & 8 & 1 \end{bmatrix}$$