Birla Institute of Technology and Science, Pilani

Work Integrated Learning Programmes Division

Answer Key

Q1 Expected solutions

a) Since the given property is satisfied, it means that $x = [1\ 1\ ....\ 1]^T$ is a vector satisfying $Ax = 0$. Hence zero is one of the eigenvalues. (1 mark)
Since zero is one of the eigenvalues, it can be concluded that determinant=0 (1 mark)
Finally, its clear from $Ax = 0$ that an eigenpair is $(0, x)$ where $x = [1\ 1\ ....\ 1]^T$(1 mark).

b) Derivation of first row (1 mark)
Derivation of second row (1 mark)
Derivation of third row (2 marks)
Hence the final matrix L is

$$L = \begin{bmatrix} 6.4807 & 0 & 0 \\ 4.9377 & 3.1015 & 0 \\ 5.7092 & 0.9059 & 1.2586 \end{bmatrix}$$

Students are expected to provide derivation of each entry by constructing 6 equations.

c) If matrix has only nonzero eigenvalues , then determinant is not zero This means $2 - \beta^2 \neq 0$. So $\beta \neq \sqrt{2}$ (1.5 marks)
Similarly, one necessary condition for all positive eigenvalues is $2-\beta^2 > 0$
In summary $-\sqrt{2} < \beta < \sqrt{2}$ (1.5 Marks)

Q2 Expected solutions (1 + 1 + 0.5)

a)

$$\begin{aligned}
\frac{\partial g}{\partial u} &= \begin{pmatrix} \frac{1}{u} \\ 2u\exp(u^2) \\ \cos(u) \end{pmatrix} \\
\frac{\partial f}{\partial \mathbf{x}} &= 2\mathbf{x}^T \\
\frac{\partial f \circ g}{\partial u} &= 2(\frac{1}{u}\log(u) + 2u\exp(2u^2) + \sin(u)\cos(u))
\end{aligned}$$

b) Given: $\mathbf{A} \in \mathbb{R}^{n\times n}$ is real symmetric matrix.
Now $\mathbf{x} \in N(\mathbf{A}) \Rightarrow \mathbf{Ax} = \mathbf{0} = \mathbf{0x}$. If $\mathbf{x} \neq 0 \Rightarrow \mathbf{x}$ is an eigenvector corresponding to 0 eigenvalue and hence eigenspace $E_A(0) = N(\mathbf{A})$. So $\dim E_A(0) = \dim N(\mathbf{A}) = n - r$ where $r =$rank$(\mathbf{A})$.
Since, geometric multiplicity is same as algebraic multiplicity for symmetric matrices, number of zero eigenvalues is $n - r$ and hence number of nonzero eigenvalues is equal to r. (1 mark)
Clearly $\mathbf{A} = \mathbf{bb}^T$ is a rank 1 matrix by looking into the REF($\mathbf{A}$). Hence from i), we can say that the two eigenvalues are 0 and nonzero eigenvalue

is equal to trace of $\mathbf{A} = x^2 + y^2 + z^2$ where $\mathbf{b} = [x, y, z]^T$. Putting $x = 1$ we get the required answer. (2 marks)

c) Now

$$(\mathbf{A}^k)^2 = \mathbf{A}^{2k} = \begin{pmatrix} -1 & 0 \\ 0 & -1-\delta \end{pmatrix}$$

where $\delta > 0$. Let $\mathbf{A}^k = \mathbf{B} \Rightarrow \mathbf{A}^{2k} = \mathbf{B}^2$ So, we have

$$(\mathbf{B}^2) = \begin{pmatrix} -1 & 0 \\ 0 & -1-\delta \end{pmatrix}$$

where $\delta > 0$.

If $\mathbf{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$(\mathbf{B}^2) = \begin{pmatrix} a^2 + bc & (a+d)b \\ (a+d)c & bc + d^2 \end{pmatrix}$$

Equating terms we get $(a+d)b = (a+d)c = 0$. If $(a+d) = 0 \Rightarrow a = -d$ and hence $\delta = 0$ which is a contradiction. So then we have $b = c = 0$ which implies a^2, d^2 are negative and hence not real which is a contradiction.

Thus no such $\mathbf{B}$ and hence no such real $\mathbf{A}$ exists

(2.5 marks)

d) (1 + 1)

i) $\nabla_{\mathbf{x}} f = \begin{pmatrix} 2x_1 + 2\cos(x_2) & \alpha - 2x_1\sin(x_2) \\ \beta - 2x_2\sin(x_1) & 2x_2 + 2\cos(x_1) \end{pmatrix}$

$\nabla_{\mathbf{x}} f(0,0) = \begin{pmatrix} 2 & \alpha \\ \beta & 2 \end{pmatrix}$

ii) Consider$A = \begin{pmatrix} 2 & \alpha \\ \beta & 2 \end{pmatrix}$

Then $[x,y]A[x,y]^T > 0 \Rightarrow 2x^2 + 2y^2 + (\alpha+\beta)xy > 0,\ \forall [x,y] \neq [0,0]$

If x or y is equal to 0, clearly the other must be non zero and the above term will be positive. If x and y both are non zero then $y = rx$ for some real r and the above expression becomes

$2x^2(r^2 + \frac{\alpha+\beta}{2}r + 1) > 0\ \forall r \in \mathbb{R}$

$\Rightarrow (r^2 + \frac{\alpha+\beta}{2}r + 1) > 0\ \forall r \in \mathbb{R}$

This is possible only when $(r^2 + \frac{\alpha+\beta}{2}r + 1) \neq 0\ \forall r \in \mathbb{R}$

That means $(1/2)(-\frac{\alpha+\beta}{2} \pm \sqrt{\frac{(\alpha+\beta)^2}{4} - 4})$ is not real $\Rightarrow \frac{(\alpha+\beta)^2}{4} - 4 < 0$

$\Rightarrow 0 \leq (\alpha+\beta)^2 < 16$. Further if A is symmetry then $\alpha = \beta$ and this will imply $-2 < \alpha < 2$.

Q3 Expected solutions

a (i) The given quadratic form may be written as $\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x}$ where $A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$.

(1 mark)

This is a symmetric matrix, so it has real eigenvalues and a set of eigenvectors which form an orthonormal basis. We can express any vector $\boldsymbol{x} \in \mathbb{R}^2$ as $\boldsymbol{x} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2$ where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are orthonormal eigenvectors of $\boldsymbol{A}$. Then $\mathbf{x^T A x} = \mathbf{c_1^2}\lambda_\mathbf{1} + \mathbf{c_2^2}\lambda_\mathbf{2}$, subject to $c_1^2 + c_2^2 = 1$ where $\lambda_1 \geq \lambda_2$. The maximum value of $\boldsymbol{x}^T A x$ equals λ_1. The eigenvalues of the matrix A are $\frac{5+\sqrt{5}}{2}$ and $\frac{5-\sqrt{5}}{2}$, so the answer to the problem is $\frac{5+\sqrt{5}}{2}$. (2 marks)

a (ii) Since $\boldsymbol{A}$ is a symmetric matrix, the eigendecomposition of $\boldsymbol{A}$ is $\boldsymbol{S\Lambda S}^T$. (0.5 mark)
The SVD is $\boldsymbol{U\Sigma V}^T$ where $\boldsymbol{V}$ is the eigenvector matrix of $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{A}^2$. The eigenvector matrix of $\boldsymbol{A}^2$ is also $\boldsymbol{S}$. Thus $\boldsymbol{V} = \boldsymbol{S}$. Since $\boldsymbol{S}$ is a matrix of orthonormal eigenvectors we know that $\boldsymbol{SS}^T = \boldsymbol{I}$. The eigenvectors $\boldsymbol{u}_i$ can be obtained as $\frac{\boldsymbol{Av}_i}{\sigma_i}$ which means that we can write $\boldsymbol{U} = \boldsymbol{AV\Sigma}^{-1} = \boldsymbol{S\Lambda S}^T\boldsymbol{V\Sigma}^{-1} = \boldsymbol{S\Lambda\Sigma}^{-1} = \boldsymbol{S}$. The last step in the chain of equalities is justified since the singular values in $\boldsymbol{\Sigma}$ are the square-roots of the eigenvalues of $\boldsymbol{A}^2$ and the eigenvalues of $\boldsymbol{A}^2$ are the squares of the eigenvalues of $\boldsymbol{A}$. Therefore the singular values are just the eigenvalues of $\boldsymbol{A}$. Thus $\boldsymbol{\Lambda\Sigma}^{-1} = \boldsymbol{I}$, and $\boldsymbol{U} = \boldsymbol{S}$. Thus the SVD of $\boldsymbol{A}$ is $\boldsymbol{S\Lambda S}^T$. (1.5 marks)

b (i) Since D is the Euclidean distance from the origin of the point $[x_1, x_2, x_3]$, it is a function of x_1, x_2 and x_3 we have the gradient

$$\frac{dD}{d\boldsymbol{x}} = \frac{x_1}{\sqrt{x_1^2+x_2^2+x_3^2}}, \frac{x_2}{\sqrt{x_1^2+x_2^2+x_3^2}}, \frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}}$$

. (1 mark)
Then we can calculate

$$\frac{dD}{dt} = \frac{\partial D}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial D}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial D}{\partial x_3}\frac{dx_3}{dt}$$

, where $\frac{dx_1}{dt} = \omega cos(\omega t)$, $\frac{dx_2}{dt} = -\omega r sin(\omega t)$, $\frac{dx_3}{dt} = k$. (1 mark)

b (ii) We can write $g(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}|_{x_0,y_0}(x-x_0) + \frac{\partial f}{\partial y}|_{x_0,y_0}(y-y_0)$ using the Taylor's series expansion. Substituting for the partial derivatives we have $g(x,y) = \alpha x_0^2 + \beta y_0^2 + 2\alpha x_0(x-x_0) + 2\beta(y-y_0)$. (1 mark)
We shall now study the condition under which $f(x,y) \geq g(x,y)$. We have $\alpha x^2 + \beta y^2 - [\alpha x_0^2 + \beta y_0^2 + 2\alpha x_0(x-x_0) + 2\beta(y-y_0)] \geq 0$. This can be rearranged to

$$\begin{aligned} \alpha(x^2 - 2x_0(x-x_0) - x_0^2] + \beta(y^2 - 2y_0(y-y_0) - y_0^2) &\geq 0 \\ \alpha(x^2 - 2xx_0 + x_0^2) + \beta(y^2 - 2y_0 y + y_0^2) &\geq 0 \\ \alpha(x-x_0)^2 + \beta(y-y_0)^2 \geq 0 \end{aligned}$$

From the last inequality we see that $\alpha \geq 0$, $\beta \geq 0$ will ensure that $f(x,y) \geq g(x,y)$ for all x, y. (2 marks)