MFML Assignment | 05-07-2023
2022ac05668

Q1

$Lx = b$, $b \neq 0$, where L is a $10^5 \times 10^5$ lower triangular matrix & 'b' is a $10^5 \times 1$ vector

Solve this system with the constraints
- store only 1 row at a time
- As few writes as possible.

→ Forward substitution is a technique used to solve system of equations when the matrix is lower triangular.

$$\rightarrow L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ l_{31} & l_{32} & \cdots & 0 \\ l_{m1} & l_{m2} & \cdots & l_{mm} \end{bmatrix}$$

For $Lx = b$

$$l_{11}x_1 + \cdots = b_1$$
$$l_{21}x_2 + l_{22}x_2 \cdots = b_2$$
$$l_{m1}x_1 + l_{m2}x_2 + \cdots = b_m$$

$$\therefore x_1 = b_1 / l_{11}$$

$$x_2 = \frac{b_2 - l_{21}x_1}{l_{22}}$$

$$\cancel{x_m = \frac{b_2 + l_{21}x_1}{l_{22}}}$$

$$\boxed{x_m = \frac{b_m - \sum_{i=1}^{m-1} l_{mi}x_i}{l_{mm}}}$$

→ 1. Initialize Empty vector x, size (00000 x)
2. Start a loop for $i = 0$ to 10^5
3. Read the i_{th} row of the matrix L from memory.
4. Calculate dot product of x with current row L, excluding the ith element.
5. Subtract the dot product from corresponding element b vector.
6. Divide by the ith element of current row L, of x vector
7. Store this value in the ith element
8. Repeat steps (3 - 7) for all rows of L.
9. End of the loop, The resultant x vector will contain solution $Lx = b$.

In the above, we are not storing intermediary matrix, but reads only one row at a time and writes directly to x vector instead of storing intermediary results. without storing intermediary results.

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Q2 $E_{ij_{50\times 50}}$ → Elimination matrix
ith, jth row below main diagonal is non-zero, all other off diagonal elements are zero.
For $i=5$, $j=2$, find set of elimination matrices E_{pq} where $E_{pq} E_{ij} = E_{ij} E_{pq}$

→ An elimination matrix is a matrix used in Gaussian elimination or row reduction methods to transform a matrix into row echelon form (REF) or RREF.
It performs row operations to eliminate certain entities in a matrix.
→ $E_{ij_{50\times 50}}$, ith row, jth colum $\neq 0$
→ If $i=5$, $j=2$, $E_{pq} = ?$ where $E_{pq} E_{52} = E_{52} E_{pq}$
→ Commutating will hold for some pair of elimination matrices. Commutativity means that order of multiplication does not matter.
→ Therefore if E_{ij} (E_{52}) has non-zero in 5, 2, it will affect 5 row, 2 column of matrix.
→ E_{pq} must perform row operations that do not affect 5th row & 2nd column of E_{52}.
→ ∴ For any p & q, where $(p,q) \neq (5,2)$ the entry in the pth row & qth column of E_{pq} should be 0.
i.e for $E_{pq} E_{52}$, non-zero element in E_{pq} should be a row other than $p=5$ & column $q=2$ & they should match entry in E_{ij}.
→ If the elimination matrix affect different elements independent & their then they can be performed order don't matter.

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Question 3

Find $A_{2\times2}$ matrix, Eigen Value $= \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$

$\lambda_1\ \lambda_2 \longrightarrow$ complex numbers

$\therefore$ for A^n for $0<n<49$

But Real for $n=50$ i.e. A^{50}

$|\lambda_1|, |\lambda_2| = 1$

We chose a rotation matrix, as its Eigen values have modulus of 1 Eigen values of a rotation matrix are complex numbers that lie on unit circle in complex plain i.e. Modulus = 1

$\rightarrow$ For $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\rightarrow$ Eigen values of A equation

$\det(A-\lambda I) = 0 \rightarrow I =$ Identity matrix

$\lambda =$ Eigen value.

$$\det\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} \cos\theta-\lambda & -\sin\theta \\ \sin\theta & \cos\theta-\lambda \end{bmatrix}\right) = 0$$

$$((\cos\theta-\lambda)^2 + \sin^2\theta) = 0.$$

$$(\cos^2\theta - 2\cos\theta\lambda + \lambda^2 + \sin^2\theta) = 0.$$

$$1 - 2\cos\theta + \lambda^2 = 0 \quad \text{—} \quad \therefore \sin^2\theta + \cos^2\theta = 1$$

Solving the above quadratic equation.

$$\lambda = \frac{(-b \pm \sqrt{b^2-4ac})}{2a}$$

$a=1$, $b=-2\cos\theta$, $c=1$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$\lambda = \cos\theta \pm \sqrt{\cos^2\theta - 1}$$

$$\lambda = \cos\theta \pm \sqrt{-\sin^2\theta} \quad - \quad \therefore \sin^2\theta + \cos^2\theta = 1$$

$$\lambda = \cos\theta \pm i\sqrt{\sin^2\theta}$$

Modulus is given by $|a+bi| = \sqrt{a^2+b^2}$

$$\therefore |\cos\theta \pm i\sqrt{\sin^2\theta}| = \sqrt{\cos^2\theta + i(\sqrt{\sin^2\theta})^2}$$

$$= \sqrt{\cos^2\theta - \sin^2\theta(-1)}$$

$$= \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

By Euler's formula

$$\boxed{\lambda = e^{i\theta} \pm i e^{i\theta}}$$

For A^{50}

As its reals for $n=50$

$\lambda^{50} = e^{50i\theta}$

$= \cos 50\theta + i\sin(50\theta)$

$\therefore \sin(50\theta) = 0 \rightarrow$ as imaginary should be 0.

$\therefore \theta = \pi/50$ satisfies the condition when the Eigen values become real for A^{50}.

$$\boxed{\therefore A = \begin{bmatrix} \cos(\pi/50) & -\sin(\pi/50) \\ \sin(\pi/50) & \cos(\pi/50) \end{bmatrix}}$$

is a suitable matrix that satisfies the given properties.

Q4 $A_{n\times n}$ matrix, n is atleast 6
$A_{n\times n}$ is upper triangular matrix
Can matrix be solved in $O(n)$ time? (for λ)

- Please note in an upper triangular matrix, all element below are 0.
- By property of upper triangular matrix, the Eigen values are the diagonal elements.
- Hence, B needs to only query the elements once to obtain the Eigen values.
- This takes $O(n)$ time, as the amount of operation for ~~each~~ each element.
- Therefore B's confidence is justified!

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Q5 MFML Assignment

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$C \in \mathbb{R}'_{n \times n}$ sush that

$C^T C = C C^T$

Using SVD, $C = U \Sigma V^T$

since $C^{T'} C = C C^T$, we see that

$V \Sigma^T U^T U \Sigma V^T = U \Sigma V^T V \Sigma U^T$

& $V \Sigma^T \Sigma V^T = U \Sigma \Sigma^T U^T$

$\therefore U = V$, & C is a symmetric matrix.

Is the above argument justified?

→ The argument suggests that since $V \Sigma^T \Sigma V^T = U \Sigma \Sigma^T U^T$, then $U = V$ & C is symmetric matrix.

→ The SVD of a matrix C is given by $C = U \Sigma V^T$, where U & V are orthogonal matrices & Σ is the diagonal matrix with singular values.

But the matrix C can be any rectangular matrix, not necessarily symmetric.

→ The derived equation holds true for any matrix C & its corresponding SVD. The equation istead is dependent on U & V being orthogonal & Σ being the diagonal matrix.

→ Hence this is not a justified argument.