BITS Pilani
Pilani Campus

M.Tech. (AIML)

Session-5 (Random Variables)

Team ISM

Session-5 Agenda

Random variables –

Discrete & continuous Expectation of a random variable,

mean and variance of a random variable –

Single random random variable &

Joint distributions

Contact Session 5: Module 3: Probability Distributions

Contact Session	List of Topic Title	Reference
CS - 5	Random variables - Discrete & continuous Expectaion of a random variable,mean and variance of a random variable – Sinlge random random variable & Joint distributions	T1 & T2
HW	Problems on random variables	T1 & T2
Lab	Probability Distributions & Sampling	Lab 3

Random Variables

- A **random variable** is a variable that assumes numerical values associated with the random outcome of an experiment, where one (and only one) numerical value is assigned to each sample point.
- In mathematical language, a random variable is a function whose domain is the sample space and whose range is the set of real numbers.

Random Variables

- A random variable can be classified as being either discrete or continuous depending on the numerical values it assumes.
- A discrete random variable may assume either finite or countably infinite number of values
- A continuous random variable may assume any numerical value in an interval or collection of intervals.
- Continuous random variables are generated in experiments where things are "measured' as opposed to "counted".
- Experimental outcomes based on measurement of time, distance, weight, volume etc. generate continuous RV.

Types of random Variables

- A **discrete random variable** can assume a countable number of values.
  - Number of steps to the top of the Eiffel Tower*

- A **continuous random variable** can assume any value along a given interval of a number line.
  - The time a tourist stays at the top once s/he gets there

Two Types of Random Variables

- **Discrete random variables**
  - Number of sales
  - Number of calls
  - Shares of stock
  - People in line
  - Mistakes per page

- **Continuous random variables**
  - Length
  - Depth
  - Volume
  - Time
  - Weight

Discrete Probability Distributions

- The probability distribution for a random variable describes how probabilities are distributed over the values of the random variable.
- The probability distribution is defined by a probability function, denoted by $f(x)$, which provides the probability for each value of the random variable.

The required conditions for a discrete probability function are:

$$f(x) \geq 0$$

$$\Sigma f(x) = 1$$

- We can describe a discrete probability distribution with a table, graph, or equation.
- Advantage: once the probability distribution is known, it is relatively easy to determine the probability of a variety of events that may be of interest to the decision maker.

Probability Distributions for Discrete Random Variables

- Say a random variable x follows this pattern: $p(x) = (.3)(.7)^{x-1}$ for $x > 0$.
  - This table gives the probabilities (rounded to two digits) for x between 1 and 10.

x	$P(x)$
1	.30
2	.21
3	.15
4	.11
5	.07
6	.05
7	.04
8	.02
9	.02
10	.01

Expected Value and Variance

- The expected value, or mean, of a random variable is a measure of its central location.
  - The mean or Expected value of a discrete random variable:

$$E(x) = \mu = \Sigma x f(x)$$

- The variance summarizes the variability in the values of a random variable.
  - Variance of a discrete random variable:

$$\mathrm{Var}(x) = \sigma^2 = \Sigma(x - \mu)^2 f(x)$$

- The standard deviation, σ, is defined as the positive square root of the variance.

Rules of Expected Value

- Multiplying RV by a constant a, $E(aX) = a.E(X)$
- Adding a constant b, $E(X+b) = E(X) + b$
- Therefore, $E(aX + b) = ?$

Variability of Discrete Random Variables

- The **variance** of a **discrete random variable** *x* is

$$\sigma^2 = E[(x-\mu)^2] = \sum (x-\mu)^2 p(x).$$

- The **standard deviation** of a **discrete random variable** *x* is

$$\sqrt{\sigma^2} = \sqrt{E[(x-\mu)^2]} = \sqrt{\sum (x-\mu)^2 p(x)}.$$

Rules of variability

- Multiplying RV by a constant a, $V(aX) = a^2 . V(X)$
- Adding a constant b, $V(X+b) = V(X)$
- $\sigma_{aX} = |a| \cdot \sigma_X, \quad \sigma_{X+b} = \sigma_X$

Example

At a shooting range, a shooter is able to hit a target in either 1, 2 or 3 shots. Let x be a random variable indicating the number of shots fired to hit the target. The following probability function was proposed.

$$f(x) = x/6$$

Is this probability function valid?

Identify the r.v to be discrete or continuous?

EXPERIMENT	Random Variable (x)
Audit 50 tax returns	Number of returns that contains error
Operate a restaurant for one day	Number of customers
Observe an employee’s work	No. of productive hours in an 8-hour workday

Discrete Uniform Probability Distribution

- The discrete uniform **probability** distribution is the simplest example of a discrete probability distribution given by a formula.

- The discrete uniform probability function is

$$f(x) = 1/n$$

where:

n = the number of values the random variable may assume

Note that the values of the random variable are equally likely.

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Random Variables

Discrete

$X = \{1, 2, 3, 4\}$

→ No of students

→ No of bits transmitted

Continuous

$X \in (1, 3)$

→ pressure

→ Temperature

→ Time

→ Voltage

Random Variables

Discrete

$P(x)$

Validation :-

(1) $0 \leq P(x) \leq 1$

2) $\sum P(x) = 1$

probability distribution function

continuous

$f(x)$

1) $0 \leq f(x) \leq 1$

2) $\int f(x)\,dx = 1$

probability density function

Example:

x	1	2	4	8	16
$P(x)$	0.05	0.10	0.35	0.40	0.10

compute the following

a) $E(x)$

b) $V(x)$ directly from the definition

c) standard deviation of X

d) $V(x)$ using the shortcut formula

Example

Let x be a random variable with PDF given by

$$f(x) = \begin{cases} cx^2, & -1 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

a) Find constant 'c'.

b) Find $E(x)$ and $V(x)$

c) Find $P(x \geq 1/2)$

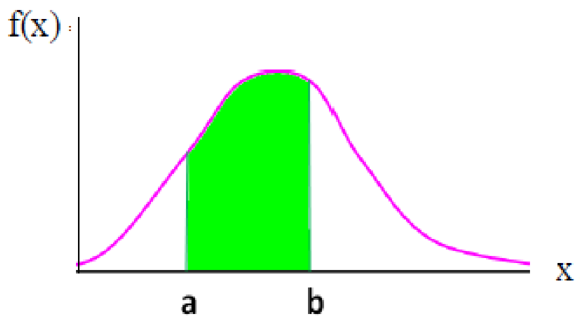
Continuous Probability Distributions

- A <u>continuous random variable</u> can assume any value in an interval on the real line or in a collection of intervals.
- It is not possible to talk about the probability of the random variable assuming a particular value.
- Instead, we talk about the probability of the random variable assuming a value within a given interval.
- The probability of the random variable assuming a value within some given interval from x_1 to x_2 is defined to be the <u>area under the graph</u> of the <u>probability density function</u> between x_1 and x_2.

Continuous Random Variables

A continuous random variable can assume any value in an interval on the real line or in a collection of intervals.

The probability of the random variable assuming a value within some given interval from x1 to x2 is defined to be the area under the graph of the **probability density function** between x1 and x2



Example:

- Height of students in a class
- Amount of ice tea in a glass
- Change in temperature throughout a day
- Price of a car in next year

Continuous Random Variables

Probability Density Function

For a continuous random variable X, a **probability density function** is a function such that

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(3) $P(a \leq X \leq b) = \int_{a}^{b} f(x)\,dx$ = area under $f(x)$ from a to b for any a and b (4.1)

Continuous Random Variables

Cumulative Distribution Function

The **cumulative distribution function** of a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\,du$$

for $-\infty < x < \infty$.

Probability Density Function from the Cumulative Distribution Function

Given $F(x)$,

$$f(x) = \frac{dF(x)}{dx}$$

as long as the derivative exists.

Continuous Random Variables

Mean and Variance

Suppose that X is a continuous random variable with probability density function $f(x)$. The **mean** or **expected value** of X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

The **variance** of X, denoted as $V(X)$ or σ^2, is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$$

The **standard deviation** of X is $\sigma = \sqrt{\sigma^2}$.

Integration Formulas

$$\int kf(u)du = k\int f(u)du$$

$$\int u^n du = \frac{u^{n+1}}{n+1}$$

$$\int e^u du = e^u$$

$$\int \sin u \, du = -\cos u$$

$$\int \cos u \, du = \sin u$$

$$\int [f(u) \pm g(u)]du = \int f(u)du \pm \int g(u)du$$

$$\int udv = uv - \int vdu$$

Continuous Random Variables

EXAMPLE 1 **Calculating probabilities from the probability density function**

If a random variable has the probability density

$$f(x) = \begin{cases} 2e^{-2x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

find the probabilities that it will take on a value

(a) between 1 and 3;

(b) greater than 0.5.

Solution Evaluating the necessary integrals, we get

(a)
$$\int_1^3 2e^{-2x}\,dx = e^{-2} - e^{-6} = 0.133$$

(b)
$$\int_{0.5}^{\infty} 2e^{-2x}\,dx = e^{-1} = 0.368$$

With reference to the preceding example, find the distribution function and use it to determine the probability that the random variable will take on a value less than or equal to 1.

Performing the necessary integrations, we get

$$F(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ \int_0^x 2e^{-2t}\,dt = 1 - e^{-2x} & \text{for } x > 0 \end{cases}$$

and substitution of $x = 1$ yields

$$F(1) = 1 - e^{-2} = 0.865$$

Determining the mean and variance using the probability density function

With reference to Example 1, find the mean and the variance of the given probability density.

Performing the necessary integrations, using integrations by parts, we get

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{0}^{\infty} x \cdot 2 e^{-2x}\, dx = \frac{1}{2}$$

Alternatively, the expectation of x is $E(X) = 0.5$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = \int_{0}^{\infty} \left(x - \frac{1}{2}\right)^2 \cdot 2 e^{-2x}\, dx = \frac{1}{4}$$

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A probability density function assigns probability one to $(-\infty, \infty)$

Find k so that the following can serve as the probability density of a random variable:

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ kxe^{-4x^2} & \text{for } x > 0 \end{cases}$$

Solution

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} kxe^{-4x^2}\,dx = \int_{0}^{\infty} \frac{k}{8} \cdot e^{-u}\,du = \frac{k}{8} = 1$$

so that $k = 8$. ∎

Continuous Random Variables

5.4 If the probability density of a random variable is given by

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2 - x & \text{for } 1 \le x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

find the probabilities that a random variable having this probability density will take on a value

(a) between 0.2 and 0.8; (b) between 0.6 and 1.2.

5.14 Find μ and σ^2 for the probability density of Exercise 5.4.

Continuous Random Variables

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2 - x & \text{for } 1 \leq x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find μ and σ^2 for the probability density

Continuous Random Variables

5.6 Given the probability density $f(x) = \dfrac{k}{1+x^2}$ for $-\infty < x < \infty$, find k.

Continuous Random Variables

5.10 The length of satisfactory service (years) provided by a certain model of laptop computer is a random variable having the probability density

$$f(x) = \begin{cases} \dfrac{1}{4.5} e^{-x/4.5} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

Find the probabilities that one of these laptops will pro vide satisfactory service for

(a) at most 2.5 years; (b) anywhere from 4 to 6 years; (c) at least 6.75 years.

Continuous Random Variables

$$f(x) = \begin{cases} \frac{1}{4.5} e^{-x/4.5} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

(a) at most 2.5 years;

(b) anywhere from 4 to 6 years;

(c) at least 6.75 years.

The cumulative distribution function of the random variable Y is

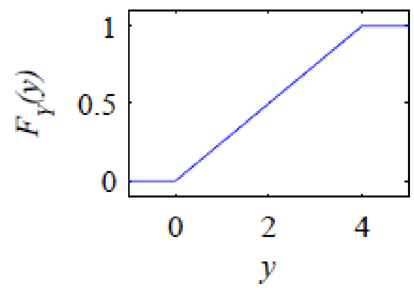
$$F_Y(y) = \begin{cases} 0 & y < 0, \\ y/4 & 0 \le y \le 4, \\ 1 & y > 4. \end{cases}$$

Sketch the CDF of Y and calculate the following probabilities:

(1) $P[Y \le -1]$ (2) $P[Y \le 1]$

(3) $P[2 < Y \le 3]$ (4) $P[Y > 1.5]$

The CDF of Y is



From the CDF $F_Y(y)$, we can calculate the probabilities:

(1) $P[Y \le -1] = F_Y(-1) = 0$

(2) $P[Y \le 1] = F_Y(1) = 1/4$

(3) $P[2 < Y \le 3] = F_Y(3) - F_Y(2) = 3/4 - 2/4 = 1/4$

(4) $P[Y > 1.5] = 1 - P[Y \le 1.5] = 1 - F_Y(1.5) = 1 - (1.5)/4 = 5/8$

The probability density function of the random variable Y is

$$f_Y(y) = \begin{cases} 3y^2/2 & -1 \le y \le 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Sketch the PDF and find the following:

(1) the expected value $E[Y]$

(2) the second moment $E[Y^2]$

(3) the variance $\mathrm{Var}[Y]$

(4) the standard deviation σ_Y

Recall - Continuous Random Variables

Properties:

$$f(x) \geq 0$$

$$\int_a^b f(x)\,dx = 1$$

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx = F(b) - F(a)$$

$$f(x) = P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

$$f(c) = P(X = c) = P(c \leq X \leq c) = \int_c^c f(x)\,dx = 0$$

$$E(x) = \int_a^b x f(x)\,dx$$

$$E(x^2) = \int_a^b x^2 f(x)\,dx$$

$$V(X) = E(X^2) - ((E(X))^2$$

Exercise - Continuous Random Variables

5.7 If the distribution function of a random variable is given by

$$F(x) = \begin{cases} 1 - \dfrac{4}{x^2} & \text{for } x > 2 \\ 0 & \text{for } x \le 2 \end{cases}$$

find the probabilities that this random variable will take on a value

(a) less than 3; (b) between 4 and 5.

5.9 Let the phase error in a tracking device have probability density

$$f(x) = \begin{cases} \cos x & 0 < x < \pi/2 \\ 0 & \text{elsewhere} \end{cases}$$

Find the probability that the phase error is

(a) between 0 and $\pi/4$; (b) greater than $\pi/3$.

Find μ and σ for the distribution of the phase error

4.3.1 The random variable X has probability density function

$$f_X(x) = \begin{cases} cx & 0 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Use the PDF to find

(a) the constant c,

(b) $P[0 \le X \le 1]$,

(c) $P[-1/2 \le X \le 1/2]$,

(d) the CDF $F_X(x)$.

4.4.4 The probability density function of random variable Y is

$$f_Y(y) = \begin{cases} y/2 & 0 \le y < 2, \\ 0 & \text{otherwise.} \end{cases}$$

What are $E[Y]$ and $\text{Var}[Y]$?

4.4.5 The cumulative distribution function of the random variable Y is

$$F_Y(y) = \begin{cases} 0 & y < -1, \\ (y+1)/2 & -1 \le y \le 1, \\ 1 & y > 1. \end{cases}$$

What are $E[Y]$ and $\text{Var}[Y]$?

5.1 Joint Probability distribution

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Introduction:

- Let $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ be two discrete random variables. Then $P(x, y) = J_{ij}$ is called joint probability function of X and Y if it satisfies the conditions:

$$(i)\ J_{ij} \geq 0 \quad (ii) \sum_{i=1}^{m} \sum_{j=1}^{n} J_{ij} = 1$$

- Set of values of this joint probability function J_{ij} is called joint probability distribution of X and Y.

	y_1	y_2	...	y_n	Sum
x_1	J_{11}	J_{12}	...	J_{1n}	$f(x_1)$
x_2	J_{21}	J_{22}	...	J_{2n}	$f(x_2)$
...	...	...	...	...	...
x_m	J_{m1}	J_{m2}	...	J_{mn}	$f(x_m)$
Sum	$g(y_1)$	$g(y_2)$	...	$g(y_n)$	$Total = 1$

- So far we have been talking about the probability of a single variable, or a variable conditional on another.
- We often want to determine the joint probability of two variables, such as X and Y. Suppose we are able to determine the following information for education (X) and age (Y) for all Indian citizens based on the census.

Age (Y): Education (X)	Age : 25-35 30	Age: 35-55 45	Age: 55-85 70
None 0	.01	.02	.05
Primary 1	.03	.06	.10
Secondary 2	.18	.21	.15
College 3	.07	.08	.04

Each cell is the relative frequency (f/N).

We can define the joint probability distribution as:
$p(x, y) = \Pr(X = x \text{ and } Y = y)$

Example: what is the probability of getting a 30 year old college graduate?
$p(x,y) = \Pr(X=3 \text{ and } Y=30) = .07$

We can see that: $p(x) = \Sigma_y\, p(x,y)$
$p(x=1) = .03 + .06 + .10 = .19$

Age (Y): Education (X)		Age : 25-35 30	Age: 35-55 45	Age: 55-85 70
None	0	.01	.02	.05
Primary	1	.03	.06	.10
Secondary	2	.18	.21	.15
College	3	.07	.08	.04

Marginal Probability

- We call this the **marginal probability** because it is calculated by summing across rows or columns and is thus reported in the margins of the table.

We can calculate this for our entire table.

Age (Y): / Education (X)	30	45	70	**p(x)**
None: 0	.01	.02	.05	**.08**
Primary: 1	.03	.06	.10	**.19**
Secondary: 2	.18	.21	.15	**.54**
College: 3	.07	.08	.04	**.19**
p(y)	**.29**	**.37**	**.34**	**1**

If X and Y are discrete random variables, the joint probability distribution of X and Y is a description of the set of points (x,y) in the range of (X,Y) along with the probability of each point.

The joint probability distribution of two discrete random variables is sometimes referred to as the **bivariate probability distribution** or **bivariate distribution.**

Thus we can describe the joint probability distribution of two discrete random variables is through a **joint probability mass function**

$$f(x,y)=P(X=x, Y=y)$$

Joint Probability Mass Function

The function $f(x, y)$ is a **joint probability distribution** or **probability mass function** of the discrete random variables X and Y if

1. $f(x, y) \geq 0$ for all (x, y),
2. $\sum_{x} \sum_{y} f(x, y) = 1$,
3. $P(X = x, Y = y) = f(x, y)$.

For any region A in the xy plane, $P[(X, Y) \in A] = \sum \sum_{A} f(x, y)$.

Joint Density Function

When X and Y are continuous random variables, the **joint density function** $f(x, y)$ is a surface lying above the xy plane, and $P[(X, Y) \in A]$, where A is any region in the xy plane, is equal to the volume of the right cylinder bounded by the base A and the surface.

The function $f(x, y)$ is a **joint density function** of the continuous random variables X and Y if

1. $f(x, y) \geq 0$, for all (x, y),
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = 1$,
3. $P[(X, Y) \in A] = \int \int_A f(x, y) \, dx \, dy$, for any region A in the xy plane.

Marginal Distributions

The **marginal distributions** of X alone and of Y alone are

$$g(x) = \sum_{y} f(x,y) \quad \text{and} \quad h(y) = \sum_{x} f(x,y)$$

for the discrete case, and

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \quad \text{and} \quad h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

for the continuous case.

Consider the joint distribution of X and Y.
Compute the following probabilities:
(i) $P(X = 1, Y = 2)$ (ii) $P(X \geq 1, Y \geq 2)$
(iii) $P(X \leq 1, Y \leq 2)$ (iv) $P(X + Y \geq 2)$ (v) $P(X \geq 1, Y \leq 2)$.

X \ Y	0	1	2	3
0	0	1/8	1/4	1/8
1	1/8	1/4	1/8	0

Solution:

(i) $X = \{0, 1\}, Y = \{0, 1, 2, 3, 4\}$

$P(X = 1, Y = 2) = P(1, 2) = \frac{1}{8}$

(ii) If $X \geq 1, X = \{1\}$. If $Y \geq 2, Y = \{2, 3\}$

$P(X \geq 1, Y \geq 2) = P(1, 2) + P(1, 3) = \frac{1}{8} + 0 = \frac{1}{8}$

(iii) If $X \leq 1, X = \{0, 1\}$. If $Y \leq 2, Y = \{0, 1, 2\}$

$P(X \leq 1, Y \leq 2) = P(0, 0) + P(0, 1) + P(0, 2) + P(1, 0) + P(1, 1) + P(1, 2)$

$$= 0 + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

Cont.

(iv) If $X + Y \geq 2$ then

$$X + Y = 0 + 2 \; or \; 0 + 3 \; or \; 1 + 1 \; or \; 1 + 2 \; or \; 1 + 3$$

$$P(X + Y \geq 2) = P(0,2) + P(0,3) + P(1,1) + P(1,2) + P(1,3)$$

$$= \frac{1}{4} + \frac{1}{8} + \frac{1}{4} + \frac{1}{8} + 0 = \frac{3}{4}$$

(v) If $X \geq 1, X = \{1\}$. If $Y \leq 2, Y = \{0,1,2\}$

$$P(X \geq 1, Y \leq 2) = P(1,0) + P(1,1) + P(1,2)$$

$$= \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2}$$

Problem:

- Two ballpoint pens are selected at random form a box that contains blue pens, 2 red pens and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected, find the joint probability function f(x,y)
- **Solution:**

The possible pairs of values (x,y) are (0,0),(0,1),(1,0),(1,1),(0,2),(2,0)

The joint probability distribution can be represented by the formula

$$f(x,y) = \frac{\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}}{\binom{8}{2}},$$

for $x = 0, 1, 2$; $y = 0, 1, 2$; and $0 \le x + y \le 2$.

Joint distribution

f(x,y)		X			Rows Total
		0	1	2	
y	0	3/28	9/28	3/28	15/28
	1	3/14	3/14	0	3/7
	2	1/28	0	0	1/28
Columns Total		5/14	15/28	3/28	1

3. **Find the joint distribution of X and Y which are the independent random variables with the following respective distributions.**

x_i	1	2
$f(x_i)$	0.7	0.3

y_j	-2	5	8
$g(y_j)$	0.3	0.5	0.2

Solution:

Since X and Y are independent random variables,

$$J_{ij} = f(x_i)g(y_j)$$

Therefore,

$x \backslash y$	-2	5	8	$f(x)$
1	0.21	0.35	0.14	0.7
2	0.09	0.15	0.06	0.3
$g(y)$	0.3	0.5	0.2	$Total = 1$

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6. The joint probability distribution of two discrete random variables X and Y is given by $f(x, y) = k(2x + y)$ for $0 \le x \le 2$, $0 \le y \le 3$. (i) Find the value of k. (ii) The marginal distribution of X and Y (iii) Show that X and Y are dependent.

Q.9 A candy company distributed boxes of chocolates with a mixture of creams, toffees, and nuts coated in both light and dark chocolate. For a randomly selected box, let X and Y, respectively, be the proportions of the light and dark chocolates that are creams and suppose that the joint density function is

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y), & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

a) Verify whether

b) Find P[(X,Y) ∈ A], where A is the region {(x,y) | 0<x<½,¼<y<½}

c) Find g(x) and h(y) for the joint density function.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,\ y)dxdy = 1$$

Example 9 – Solution

a)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1\int_0^1 \frac{2}{5}(2x+3y)\,dx\,dy$$

$$= \int_0^1 \left.\frac{2x^2}{5}+\frac{6xy}{5}\right|_{x=0}^{x=1} dy$$

$$= \int_0^1 \left(\frac{2}{5}+\frac{6y}{5}\right)dy = \left.\frac{2y}{5}+\frac{3y^2}{5}\right|_0^1$$

$$= \frac{2}{5}+\frac{3}{5} = 1$$

Example – Solution

b)

$$P[(X,Y)\in A] = P(0 < X < \tfrac{1}{2}, \tfrac{1}{4} < Y < \tfrac{1}{2})$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}}\int_{0}^{\frac{1}{2}} \frac{2}{5}(2x+3y)\,dx\,dy$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} \left.\frac{2x^2}{5} + \frac{6xy}{5}\right|_{x=0}^{x=\frac{1}{2}} dy$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{1}{10} + \frac{3y}{5}\right) dy = \left.\frac{y}{10} + \frac{3y^2}{10}\right|_{\frac{1}{4}}^{\frac{1}{2}}$$

$$= \frac{1}{10}\left[\left(\frac{1}{2} + \frac{3}{4}\right) - \left(\frac{1}{4} + \frac{3}{16}\right)\right] = \frac{13}{160}$$

Example – Solution

By definition,

$$g(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_{0}^{1} \frac{2}{5}(2x + 3y)dy = \frac{4xy}{5} + \frac{6y^2}{10}\Big|_{y=0}^{y=1} = \frac{4x+3}{5}$$

For 0≤x ≤1, and g(x)=0 elsewhere.

Similarly, $$h(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_{0}^{1} \frac{2}{5}(2x + 3y)dx = \frac{4(1+3y)}{5}$$

For 0 ≤y ≤1, and h(y)=0 elsewhere.

Problem:

Find 'K' if the joint probability density function of a bivariate random variable (X,Y) is given by

$$f(x,y)=\begin{cases} K(1-x)(1-y) & if\, 0<x<4 \\ 0 & otherwise \end{cases}$$

7. The joint probability distribution of X and Y is given by $f(x,y) = c(x^2 + y^2)$ for $x = -1, 0, 1, 3.$ and $y = -1, 2, 3.$ (i) Find the value of c. (ii) $P(x = 0, y \leq 2)$ (iii) $P(x \leq 1, y > 2)$ (iv) $P(x \geq 2 - y)$

Solution:

By data, $X = \{-1, 0, 1, 3\}$ and $Y = \{-1, 2, 3\}$

$f(x,y) = c(x^2 + y^2)$

The joint probability distribution of X and Y:

X \ Y	-1	2	3	$f(X)$
-1	2c	$5c$	$10c$	$17c$
0	c	$4c$	$9c$	$14c$
1	$2c$	$5c$	$10c$	$17c$
3	$10c$	$13c$	$18c$	$41c$
g(Y)	$15c$	$27c$	$47c$	89c

(i) **Find c:** $1 = \Sigma f(x,y) = 89c$

$c = \frac{1}{89}$

(ii) $x = 0, y = \{-1, 2\}$

$P(x = 0, y \leq 2)$

$= P(0,-1) + P(0,2)$

$= c + 4c = 5c$

$= 5/89$

(iii) $x = \{-1, 0, 1\},\ y = \{3\}$

$P(x \leq 1, y > 2)$

$= P(-1,3) + P(0,3) + P(1,3)$

$= 10c + 9c + 10c$

$= 29c = 29/89$

Cont.

By data, $X = \{-1, 0, 1, 3\}$ and $Y = \{-1, 2, 3\}$

(iv) $P(x \geq 2 - y) = P(x + y \geq 2)$

$$= P(-1,3) + P(0,2) + P(0,3) + P(1,2) + P(1,3) + P(3,-1) + P(3,2) + P(3,3)$$

$$= 10c + 4c + 9c + 5c + 10c + 10c + 13c + 18c$$

$$= 79c = 79/89$$

Thank You !