Jeffrey R. Chasnov

Fibonacci Numbers and the Golden Ratio

bookboon
The eBook company

JEFFREY R. CHASNOV

FIBONACCI NUMBERS AND THE GOLDEN RATIO

Fibonacci Numbers and the Golden Ratio
1st edition

ISBN 978-87-403-1559-2
Peer review Prof. Shingyu Leung, The Hong Kong University of Science & Technology

CONTENTS

	Preface	**vi**
1	**The Fibonacci sequence**	**1**
2	**The Fibonacci sequence redux**	**3**
3	**The golden ratio**	**5**
4	**Fibonacci numbers and the golden ratio**	**7**
5	**Binet's formula**	**9**
6	**The Fibonacci Q-matrix**	**13**
7	**Cassini's identity**	**17**
8	**The Fibonacci bamboozlement**	**19**
9	**Sum of Fibonacci numbers**	**23**

10 **Sum of Fibonacci numbers squared** **25**

11 **The golden rectangle** **27**

12 **Spiraling squares** **29**

13 **The golden spiral** **33**

14 **An inner golden rectangle** **37**

15 **The Fibonacci spiral** **41**

16 **Fibonacci numbers in nature** **43**

17 **Continued fractions** **45**

18 **The golden angle** **49**

19 **The growth of a sunflower** **53**

A **Mathematical induction** **55**

B **Matrix algebra** **57**
B.1 Addition and Multiplication 57
B.2 Determinants 58

C **Problem solutions** **61**

Preface

View an Introduction on YouTube

This book forms the lecture notes for my Massive Open Online Course (MOOC) on the web platform Coursera. These lecture notes are divided into chapters called Lectures, and each Lecture corresponds to a video on Coursera. I have also uploaded the Coursera videos to YouTube, and links are placed at the top of each Lecture.

Most of the Lectures also contain problems for students to solve. Less experienced students may find some of these problems difficult. Do not despair! The Lectures can be read and watched, and the material understood and enjoyed without actually solving any problems. But mathematicians do like to solve problems and I have selected those that I found to be interesting. Try some of them, but if you get stuck, full solutions can be read in the Appendix.

My aim in writing these lecture notes was to place the mathematics at the level of an advanced high school student. Proof by mathematical induction and matrices, however, may be unfamiliar to a typical high school student and I have provided a short and hopefully readable discussion of these topics in the Appendix. Although all the material presented here can be considered elementary, I suspect that some, if not most, of the material may be unfamiliar to even professional mathematicians since Fibonacci numbers and the golden ratio are topics not usually covered in a University course. So I welcome both young and old, novice and experienced mathematicians to peruse these lecture notes, watch my lecture videos, solve some problems, and enjoy the wonders of the Fibonacci sequence and the golden ratio.

Jeffrey R. Chasnov
Hong Kong
Oct. 2016

Lecture 1

The Fibonacci sequence

View this lecture on YouTube

Fibonacci published in the year 1202 his now famous rabbit puzzle:

> A man put a male-female pair of newly born rabbits in a field. Rabbits take a month to mature before mating. One month after mating, females give birth to one male-female pair and then mate again. No rabbits die. How many rabbit pairs are there after one year?

To solve, we construct Table 1.1. At the start of each month, the number of juvenile pairs, adult pairs, and total number of pairs are shown. At the start of January, one pair of juvenile rabbits is introduced into the population. At the start of February, this pair of rabbits has matured. At the start of March, this pair has given birth to a new pair of juvenile rabbits. And so on.

month	J	F	M	A	M	J	J	A	S	O	N	D	J
juvenile	1	0	1	1	2	3	5	8	13	21	34	55	89
adult	0	1	1	2	3	5	8	13	21	34	55	89	144
total	1	1	2	3	5	8	13	21	34	55	89	144	233

Table 1.1: Fibonacci's rabbit population.

We define the Fibonacci numbers F_n to be the total number of rabbit pairs at the start of the nth month. The number of rabbits pairs at the start of the 13th month, $F_{13} = 233$, can be taken as the solution to Fibonacci's puzzle.

Further examination of the Fibonacci numbers listed in Table 1.1, reveals that these numbers satisfy the recursion relation

$$F_{n+1} = F_n + F_{n-1}. \tag{1.1}$$

This recursion relation gives the next Fibonacci number as the sum of the preceeding two numbers. To start the recursion, we need to specify F_1 and F_2. In Fibonacci's rabbit problem, the initial month starts with only one rabbit pair so that $F_1 = 1$. And this initial rabbit pair is newborn and takes one month to mature before mating so $F_2 = 1$.

The first few Fibonacci numbers, read from the table, are given by

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

and has become one of the most famous sequences in mathematics.

Problems for Lecture 1

1. The Fibonacci numbers can be extended to zero and negative indices using the relation $F_n = F_{n+2} - F_{n+1}$. Determine F_0 and find a general formula for F_{-n} in terms of F_n. Prove your result using mathematical induction.

2. The Lucas numbers are closely related to the Fibonacci numbers and satisfy the same recursion relation $L_{n+1} = L_n + L_{n-1}$, but with starting values $L_1 = 1$ and $L_2 = 3$. Determine the first 12 Lucas numbers.

3. The generalized Fibonacci sequence satisfies $f_{n+1} = f_n + f_{n-1}$ with starting values $f_1 = p$ and $f_2 = q$. Using mathematical induction, prove that

$$f_{n+2} = F_n p + F_{n+1} q. \tag{1.2}$$

4. Prove that

$$L_n = F_{n-1} + F_{n+1}. \tag{1.3}$$

5. Prove that

$$F_n = \frac{1}{5}\left(L_{n-1} + L_{n+1}\right).$$

6. The generating function for the Fibonacci sequence is given by the power series

$$f(x) = \sum_{n=1}^{\infty} F_n x^n.$$

Assuming the power series converges, prove that

$$f(x) = \frac{x}{1 - x - x^2}.$$

Solutions to the Problems

Lecture 2

The Fibonacci sequence redux

View this lecture on YouTube

We can solve another puzzle that also leads to the Fibonacci sequence:

> How many ways can one climb a staircase with n steps, taking one or two steps at a time?

Any single climb can be represented by a string of ones and twos which sum to n. We define a_n as the number of different strings that sum to n. In Table 1, we list the possible strings for the first five values of n. It appears that the a_n's form the beginning of the Fibonacci sequence.

To derive a relationship between a_n and the Fibonacci numbers, consider the set of strings that sum to n. This set may be divided into two nonoverlapping subsets: those strings that start with one and those strings that start with two. For the subset of strings that start with one, the remaining part of the string must sum to $n-1$; for the subset of strings that start with two, the remaining part of the string must sum to $n-2$. Therefore, the number of strings that sum to n is equal to the number of strings that sum to $n-1$ plus the number of strings that sum to $n-2$. The number of strings that sum to $n-1$ is given by a_{n-1} and the number of strings that sum to $n-2$ is given by a_{n-2}, so that

$$a_n = a_{n-1} + a_{n-2}.$$

And from the table we have $a_1 = 1 = F_2$ and $a_2 = 2 = F_3$, so that $a_n = F_{n+1}$ for all positive integers n.

n	strings	a_n
1	1	1
2	11, 2	2
3	111, 12, 21	3
4	1111, 112, 121, 211, 22	5
5	11111, 1112, 1121, 1211, 2111, 122, 212, 221	8

Table 2.1: Strings of ones and twos that add up to n.

Problems for Lecture 2

1. Consider a string consisting of the first n natural numbers, $123\ldots n$. For each number in the string, allow it to either stay fixed or change places with one of its neighbors. Define a_n to be the number of different strings that can be formed. Examples for the first four values of n are shown in Table 2.2. Prove that $a_n = F_{n+1}$.

n	strings	a_n
1	1	1
2	12, 21	2
3	123, 132, 213	3
4	1234, 1243, 1324, 2134, 2143	5

Table 2.2: Strings of natural numbers obtained by allowing a number to stay fixed or change places with its neighbor.

2. Consider a problem similar to that above, but now allow the first 1 to change places with the last n, as if the string lies on a circle. Suppose $n \geq 3$, and define b_n as the number of different strings that can be formed. Show that $b_n = L_n$, where L_n is the nth Lucas number.

Solutions to the Problems

Lecture 3

The golden ratio

View this lecture on YouTube

Figure 3.1: *The golden ratio satisfies* $x/y = (x + y)/x$.

We now present the classical definition of the golden ratio. Referring to Fig. 3.1, two positive numbers x and y, with $x > y$ are said to be in the golden ratio if the ratio between the larger number and the smaller number is the same as the ratio between their sum and the larger number, that is,

$$\frac{x}{y} = \frac{x+y}{x}. \tag{3.1}$$

Denoting $\Phi = x/y$ to be the golden ratio, (Φ is the capital Greek letter Phi), the relation (3.1) becomes

$$\Phi = 1 + \frac{1}{\Phi}, \tag{3.2}$$

or equivalently Φ is the positive root of the quadratic equation

$$\Phi^2 - \Phi - 1 = 0. \tag{3.3}$$

Straightforward application of the quadratic formula results in

$$\Phi = \frac{\sqrt{5}+1}{2} \approx 1.618.$$

The negative of the negative root of the quadratic equation (3.3) is what we will call the golden ratio conjugate ϕ, (the small Greek letter phi), and is equal to

$$\phi = \frac{\sqrt{5}-1}{2} \approx 0.618.$$

The relationship between the golden ratio conjugate ϕ and the golden ratio Φ, is given by

$$\phi = \Phi - 1,$$

or using (3.2),

$$\phi = \frac{1}{\Phi}.$$

Problems for Lecture 3

1. The golden ratio Φ and the golden ratio conjugate ϕ can be defined as

$$\Phi = \frac{\sqrt{5}+1}{2}, \qquad \phi = \frac{\sqrt{5}-1}{2}.$$

From these definitions, prove the following identities by direct calculation:

(a) $\phi = \Phi - 1$,

(b) $\phi = 1/\Phi$,

(c) $\Phi^2 = \Phi + 1$,

(d) $\phi^2 = -\phi + 1$,

2. Prove that the golden ratio satisfies the Fibonacci-like relationship

$$\Phi^{n+1} = \Phi^n + \Phi^{n-1}.$$

3. Prove that the golden ratio conjugate satisfies

$$\phi^{n-1} = \phi^n + \phi^{n+1}.$$

Solutions to the Problems

Lecture 4

Fibonacci numbers and the golden ratio

View this lecture on YouTube

The recursion relation for the Fibonacci numbers is given by

$$F_{n+1} = F_n + F_{n-1}.$$

Dividing by F_n yields

$$\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}. \tag{4.1}$$

We assume that the ratio of two consecutive Fibonacci numbers approaches a limit as $n \to \infty$. Define $\lim_{n\to\infty} F_{n+1}/F_n = \alpha$ so that $\lim_{n\to\infty} F_{n-1}/F_n = 1/\alpha$. Taking the limit, (4.1) becomes $\alpha = 1 + 1/\alpha$, the same identity satisfied by the golden ratio. Therefore, if the limit exists, the ratio of two consecutive Fibonacci numbers must approach the golden ratio for large n, that is,

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \Phi.$$

The ratio of consecutive Fibonacci numbers and this ratio minus the golden ratio is shown in Table 4.1. The last column appears to be approaching zero.

n	F_{n+1}/F_n	value	$F_{n+1}/F_n - \Phi$
1	1/1	1.0000	−0.6180
2	2/1	2.0000	0.3820
3	3/2	1.5000	−0.1180
4	5/3	1.6667	0.0486
5	8/5	1.6000	−0.0180
6	13/8	1.6250	0.0070
7	21/13	1.6154	−0.0026
8	34/21	1.6190	0.0010
9	55/34	1.6176	−0.0004
10	89/55	1.6182	0.0001

Table 4.1: Ratio of consecutive Fibonacci numbers approaches Φ.

FIBONACCI NUMBERS AND THE GOLDEN RATIO

Problems for Lecture 4

1. Assuming $\lim_{n\to\infty} F_{n+1}/F_n = \Phi$, prove that

$$\lim_{k\to\infty} \frac{F_{k+n}}{F_k} = \Phi^n.$$

2. Using $\Phi^2 = \Phi + 1$, prove by mathematical induction the following linearization of powers of the golden ratio:

$$\Phi^n = F_n\Phi + F_{n-1}, \tag{4.2}$$

where n is a positive integer and $F_0 = 0$.

3. Using $\phi^2 = -\phi + 1$, prove by mathematical induction the following linearization of powers of the golden ratio conjugate:

$$(-\phi)^n = -F_n\phi + F_{n-1}, \tag{4.3}$$

where n is a positive integer and $F_0 = 0$.

Solutions to the Problems

Lecture 5

Binet's formula

View this lecture on YouTube

The Fibonacci numbers are uniquely determined from their recursion relation,

$$F_{n+1} = F_n + F_{n-1}, \tag{5.1}$$

and the initial values, $F_1 = F_2 = 1$. An explicit formula for the Fibonacci numbers can be found, and is called Binet's Formula.

To solve (5.1) for the Fibonacci numbers, we first look at the equation

$$x_{n+1} = x_n + x_{n-1}. \tag{5.2}$$

This equation is called a second-order, linear, homogeneous difference equation with constant coefficients, and its method of solution closely follows that of the analogous differential equation. The idea is to guess the general form of a solution, find two such solutions, and then multiply these solutions by unknown constants and add them. This results in a general solution to (5.2), and one can then solve (5.1) by satisfying the specified initial values.

To begin, we guess the form of the solution to (5.2) as

$$x_n = \lambda^n, \tag{5.3}$$

where λ is an unknown constant. Substitution of this guess into (5.2) results in

$$\lambda^{n+1} = \lambda^n + \lambda^{n-1},$$

or upon division by λ^{n-1} and rearrangement of terms,

$$\lambda^2 - \lambda - 1 = 0.$$

Use of the quadratic formula yields two roots, both of which are already familiar. We have

$$\lambda_1 = \frac{1+\sqrt{5}}{2} = \Phi, \quad \lambda_2 = \frac{1-\sqrt{5}}{2} = -\phi,$$

where Φ is the golden ratio and ϕ is the golden ratio conjugate.

We have thus found two independent solutions to (5.2) of the form (5.3), and we can now use these two solutions to find a solution to (5.1). Multiplying the solutions by constants and adding them, we obtain

$$F_n = c_1\Phi^n + c_2(-\phi)^n, \tag{5.4}$$

which must satisfy the initial values $F_1 = 1$ and $F_2 = 1$. The algebra for finding the unknown constants can be made simpler, however, if instead of F_2, we use the value $F_0 = F_2 - F_1 = 0$.

Application of the values for F_0 and F_1 results in the system of equations given by

$$\begin{aligned} c_1 + c_2 &= 0, \\ c_1\Phi - c_2\phi &= 1. \end{aligned}$$

We use the first equation to write $c_2 = -c_1$, and substitute into the second equation to get

$$c_1(\Phi + \phi) = 1.$$

Since $\Phi + \phi = \sqrt{5}$, we can solve for c_1 and c_2 to obtain

$$c_1 = 1/\sqrt{5}, \quad c_2 = -1/\sqrt{5}. \tag{5.5}$$

Using (5.5) in (5.4) then derives the surprising formula

$$F_n = \frac{\Phi^n - (-\phi)^n}{\sqrt{5}}, \tag{5.6}$$

known as Binet's formula.

Problems for Lecture 5

1. Prove Binet's formula (5.6) by mathematical induction.

2. Use Binet's formula to prove the limit

$$\lim_{n\to\infty} F_{n+1}/F_n = \Phi.$$

3. Use the linearization formulas

$$\Phi^n = F_n\Phi + F_{n-1} \tag{5.7}$$
$$(-\phi)^n = -F_n\phi + F_{n-1} \tag{5.8}$$

to derive Binet's formula.

4. Use the generating function for the Fibonacci sequence

$$\sum_{n=1}^{\infty} F_n x^n = \frac{x}{1-x-x^2}$$

to derive Binet's formula.

5. Determine the analogue to Binet's formula for the Lucas numbers, defined as

$$L_{n+1} = L_n + L_{n-1}$$

with the initial values $L_1 = 1$ and $L_2 = 3$. Again it will be simpler to define the value of L_0 and use it and L_1 as the initial values.

6. Use Binet's formula for F_n and the analogous formula for L_n to show that

$$\Phi^n = \frac{L_n + \sqrt{5}F_n}{2}.$$

Solutions to the Problems

Lecture 6

The Fibonacci Q-matrix

View this lecture on YouTube

month	J	F	M	A	M	J	J	A	S	O	N	D	J
juvenile	1	0	1	1	2	3	5	8	13	21	34	55	89
adult	0	1	1	2	3	5	8	13	21	34	55	89	144

Table 6.1: Fibonacci's rabbit population consists of juveniles and adults.

Consider again Fibonacci's growing rabbit population of juvenile and adult rabbit pairs shown in Table 6.1. Let a_n denote the number of adult rabbit pairs at the start of month n, and let b_n denote the number of juvenile rabbit pairs. The number of adult pairs at the start of month $n+1$ is just the sum of the number of adult and juvenile pairs at the start of month n. The number of juvenile pairs at the start of month $n+1$ is just the number of adult pairs at the start of month n. This can be written as a system of recursion relations given by

$$\begin{aligned} a_{n+1} &= a_n + b_n, \\ b_{n+1} &= a_n; \end{aligned}$$

or in matrix form as

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}. \tag{6.1}$$

The matrix in (6.1) is called the Fibonacci Q-matrix, defined as

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.2}$$

Repeated multiplication by Q advances the population additional months. For example, advancing k months is achieved by

$$\begin{pmatrix} a_{n+k} \\ b_{n+k} \end{pmatrix} = Q^k \begin{pmatrix} a_n \\ b_n \end{pmatrix}.$$

Powers of the Q-matrix are related to the Fibonacci sequence. Observe what happens when we multiply an arbitrary matrix by Q. We have

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ a & b \end{pmatrix}.$$

Multiplication of a matrix by Q replaces the first row of the matrix by the sum of the first and second rows, and the second row of the matrix by the first row.

If we rewrite Q itself in terms of the Fibonacci numbers as

$$Q = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix},$$

and then make use of the Fibonacci recursion relation, we find

$$Q^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} F_3 & F_2 \\ F_2 & F_1 \end{pmatrix}.$$

In a similar fashion, Q^3 is given by

$$Q^3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_3 & F_2 \\ F_2 & F_1 \end{pmatrix} = \begin{pmatrix} F_4 & F_3 \\ F_3 & F_2 \end{pmatrix},$$

and so on. The self-evident pattern can be seen to be

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}. \tag{6.3}$$

Problems for Lecture 6

1. Prove (6.3) by mathematical induction.

2. Using the relation $Q^nQ^m = Q^{n+m}$, prove the Fibonacci addition formula

$$F_{n+m} = F_{n-1}F_m + F_nF_{m+1}. \tag{6.4}$$

3. Use the Fibonacci addition formula to prove the Fibonacci double angle formulas

$$F_{2n-1} = F_{n-1}^2 + F_n^2, \qquad F_{2n} = F_n\left(F_{n-1} + F_{n+1}\right). \tag{6.5}$$

4. Show that

$$F_{2n} = L_nF_n.$$

Solutions to the Problems

Lecture 7

Cassini's identity

View this lecture on YouTube

Last lecture's result for the Fibonacci Q-matrix is given by

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \tag{7.1}$$

with

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7.2}$$

From the theory of matrices and determinants (see Appendix B), we know that

$$\det AB = \det A \det B.$$

Repeated application of this result yields

$$\det Q^n = (\det Q)^n. \tag{7.3}$$

Applying (7.3) to (7.1) and (7.2) results directly in Cassini's identity (1680),

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n. \tag{7.4}$$

Examples of this equality can be obtained from the first few numbers of the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$. We have

$$\begin{aligned} 2 \times 5 - 3^2 &= 1, \\ 3 \times 8 - 5^2 &= -1, \\ 5 \times 13 - 8^2 &= 1, \\ 8 \times 21 - 13^2 &= -1 \\ 13 \times 34 - 21^2 &= 1. \end{aligned}$$

Cassini's identity is the basis of an amusing dissection fallacy, called the Fibonacci bamboozlement, discussed in the next lecture.

Problems for Lecture 7

1. Prove Cassini's identity by mathematical induction.

2. Using the Cassini's identity (7.4) and the Fibonacci addition formula (6.4), prove Catalan's identity

$$F_n^2 - F_{n-r}F_{n+r} = (-1)^{n-r}F_r^2. \tag{7.5}$$

Solutions to the Problems

Lecture 8

The Fibonacci bamboozlement

View this lecture on YouTube

Cassini's identity

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

can be interpreted geometrically: $F_{n+1}F_{n-1}$ is the area of a rectangle of side lengths F_{n+1} and F_{n-1}, and F_n^2 is the area of a square of side length F_n. Cassini's identity states that the absolute difference in area between the rectangle and the square is only one unit of area. As n becomes large, this one unit of area difference becomes small relative to the areas of the square and the rectangle, and Cassini's identity becomes the basis of an amusing dissection fallacy, called the Fibonacci bamboozlement.

To perform the Fibonacci bamboozlement, one dissects a square with side length F_n in such a way that by rearranging the pieces, one appears able to construct a rectangle with side lengths F_{n-1} and F_{n+1}, with either one unit of area larger or smaller than the original square.

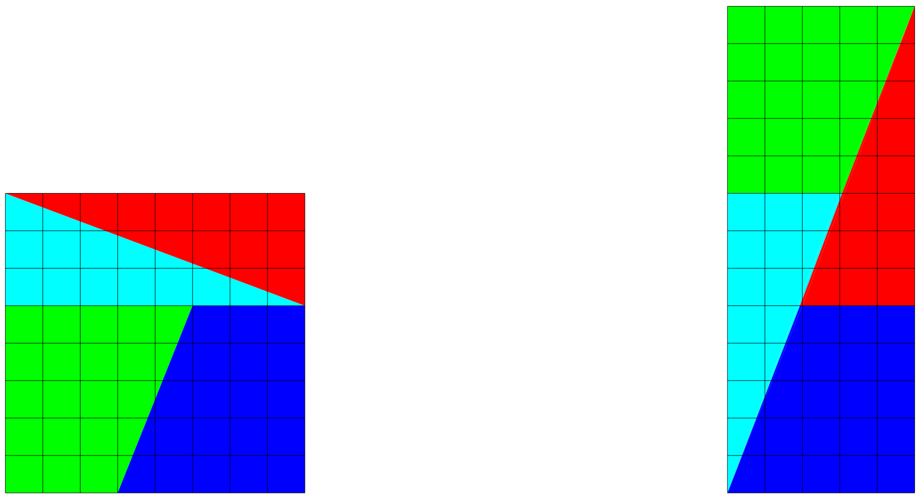
We illustrate this bamboozlement in Fig. 8.1 using a square of area $8 \times 8 = 64$, and a rectangle of area $5 \times 13 = 65$, corresponding to $n = 6$ in Cassini's identity. We begin by dissecting a square into two rectangles, the bottom rectangle of dimension 8-by-5 and the top rectangle of dimension 8-by-3. Here, 5 and 3 are the two Fibonacci numbers immediately preceeding 8.

The bottom 8-by-5 rectangle is then further dissected into two equal trapezoids. The bases of the two trapezoids are the Fibonacci numbers 5 and 3, and the heights are 5. The top rectangle is further dissected into two equal right triangles. The bases of the triangles are the Fibonacci number 8 and the heights of the triangles are the Fibonacci number 3. The dissected square is shown in Fig. 8.1a.

To construct a rectangle of dimension 5-by-13, one of the trapezoids fits into the bottom of the rectangle and the other trapezoid fits into the top of the rectangle. The two triangles then fill the remaining spaces. The resulting rectangle is shown in Fig. 8.1b, and it superficially appears that the the square has been recombined to form a rectangle of a larger area.

The honestly reconstructed rectangle, however, is shown in Fig. 8.2, where the missing unit area is seen to be almost evenly distributed along the diagonal of the rectangle.

Why is the missing (or extra) unit area distributed along the diagonal of the rectangle? In our example, the side slope of the trapezoids is given by $F_5/F_3 = 5/2 = 2.5$ while the slope of the triangle's hypotenuse is given by $F_6/F_4 = 8/3 \approx 2.67$. This slight mismatch in slopes results in a steady increase or decrease in distance between the trapezoid and the triangle when they are aligned. Here, the gap between the trapezoid and the triangle can be easily hidden by splitting the difference between the aligned pieces, as done in the dishonestly constructed rectangle.



(a) Square of dimension 8-by-8 and area 64.

(b) Rectangle of dimension 13-by-5 and area 65.

Figure 8.1: The Fibonacci Bamboozlement.

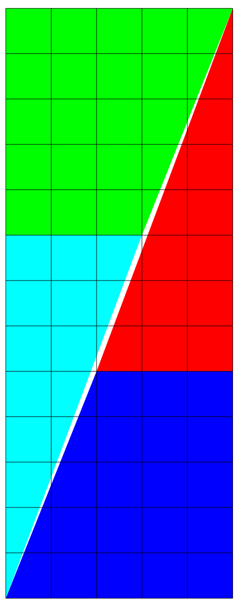


Figure 8.2: *The honest rectangle. The white space shows the missing unit area.*

Problems for Lecture 8

1. Cut an 8×8 chess board, or ruled paper, into two trapezoids and two triangles, and fool your friends by using the pieces to reconstruct a 5×13 rectangle.

Lecture 9

Sum of Fibonacci numbers

View this lecture on YouTube

In this lecture, I derive the summation identity

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1. \tag{9.1}$$

For example, consider the first eight Fibonacci numbers, $1, 1, 2, 3, 5, 8, 13, 21$. With $n = 6$ in (9.1), we have

$$\sum_{i=1}^{n} F_i = 1 + 1 + 2 + 3 + 5 + 8 = 20,$$

and

$$F_{n+2} - 1 = 21 - 1 = 20.$$

One can use mathematical induction to prove (9.1), but a direct derivation uses the relation $F_n = F_{n+2} - F_{n+1}$. Constructing a list of identities, we have

$$\begin{aligned}
F_n &= F_{n+2} - F_{n+1} \\
F_{n-1} &= F_{n+1} - F_n \\
F_{n-2} &= F_n - F_{n-1} \\
\vdots\ &\ \vdots \\
F_2 &= F_4 - F_3 \\
F_1 &= F_3 - F_2.
\end{aligned}$$

Adding all the left hand sides yields the sum over the first n Fibonacci numbers, and adding all the right-hand-sides results in the cancellation of all terms except the first and the last. Using $F_2 = 1$ results in (9.1).

Problems for Lecture 9

1. Prove by mathematical induction that the sum over the first n Fibonacci numbers is given by

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1. \tag{9.2}$$

2. Prove by construction that the sum over the first n Lucas numbers is given by

$$\sum_{i=1}^{n} L_i = L_{n+2} - 3. \tag{9.3}$$

3. Prove by construction that the sums over the first n odd or n even Fibonacci numbers are given by

$$\sum_{i=1}^{n} F_{2i-1} = F_{2n}, \qquad \sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1.$$

Solutions to the Problems

Lecture 10

Sum of Fibonacci numbers squared

View this lecture on YouTube

In this lecture, I derive a combinatorial identity obtained by summing over the squares of the Fibonacci numbers:

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}. \tag{10.1}$$

For example, consider the first seven Fibonacci numbers, $1, 1, 2, 3, 5, 8, 13$. With $n = 6$ in (10.1), we have

$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 = 8 \times 13,$$

where by doing the arithmetic one finds that both sides are equal to 104.

To prove (10.1), we work with the right-hand side, using the Fibonacci recursion relation. We have

$$\begin{aligned} F_n F_{n+1} &= F_n(F_n + F_{n-1}) \\ &= F_n^2 + F_{n-1}F_n \\ &= F_n^2 + F_{n-1}(F_{n-1} + F_{n-2}) \\ &= F_n^2 + F_{n-1}^2 + F_{n-2}F_{n-1} \\ &= \dots \\ &= F_n^2 + F_{n-1}^2 + \dots + F_2^2 + F_1F_2. \end{aligned}$$

Because $F_2 = F_1$, the identity (10.1) is proved.

We will revisit this combinatorial identity in a later lecture.

Problems for Lecture 10

1. Prove (10.1) by mathematical induction

2. Prove by construction that the sum over the first n Lucas numbers squared is given by

$$\sum_{i=1}^{n} L_i^2 = L_n L_{n+1} - 2. \tag{10.2}$$

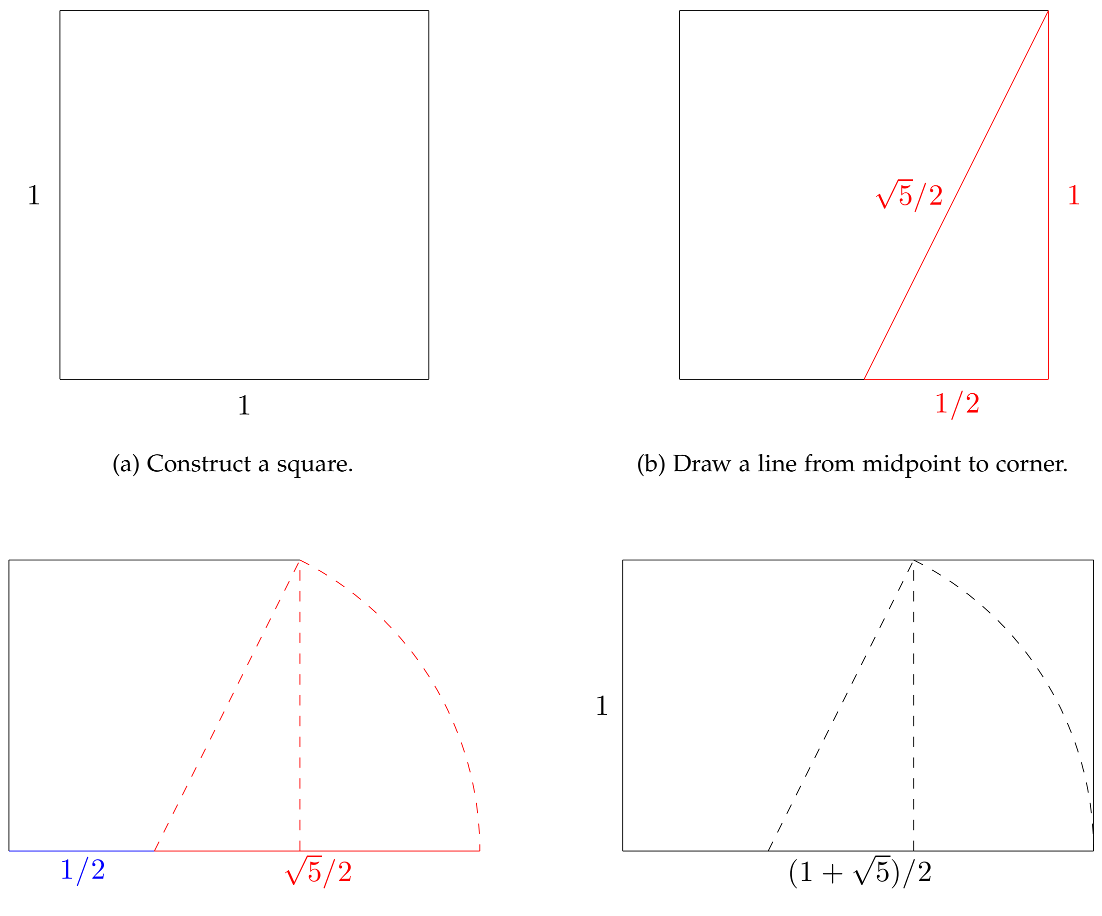
Solutions to the Problems

Lecture 11

The golden rectangle

View this lecture on YouTube

A golden rectangle is a rectangle whose side lengths are in the golden ratio. In a classical construction, first one draws a square. Second, one draws a line from the midpoint of one side to a corner of the opposite side. Third, one draws an arc from the corner to an extension of the side with the midpoint. Fourth, one completes the rectangle. The procedure is illustrated in Fig. 11.1.



(a) Construct a square. (b) Draw a line from midpoint to corner.

(c) Draw an arc using the internal line as radius. (d) Complete the golden rectangle.

Figure 11.1: Classical construction of the golden rectangle.

Problems for Lecture 11

1. Use the online software GeoGebra to construct a golden rectangle. Or try it with a real straightedge and a compass!

Lecture 12

Spiraling squares

View this lecture on YouTube

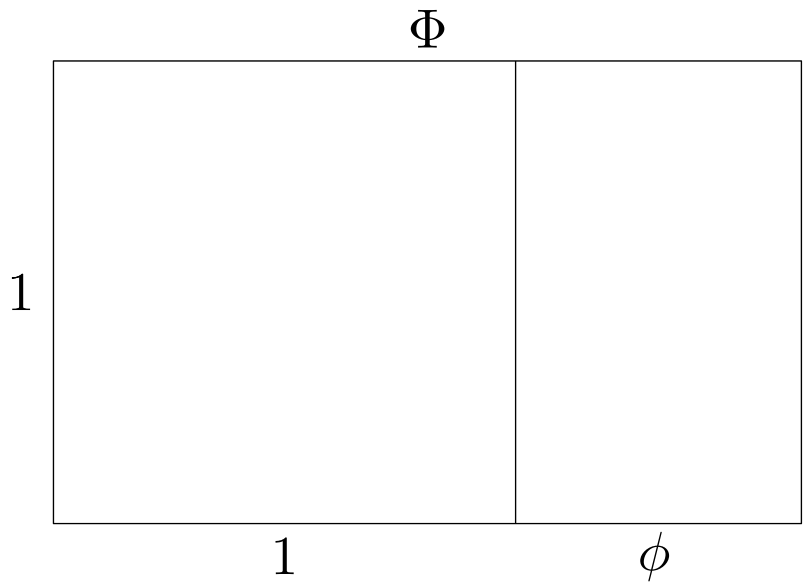


Figure 12.1: Two golden rectangles. The full rectangle and the rectangle next to the square are both golden rectangles. Here, $\phi = \Phi - 1 = 1/\Phi$.

To construct a golden rectangle of length $L = \Phi$ and width $W = 1$, a smaller rectangle was attached to a unit square as illustrated in Fig. 12.1. The smaller rectangle has vertical length $L = 1$ and horizontal width $W = \Phi - 1$, but since $\Phi - 1 = 1/\Phi$, the smaller rectangle satisfies $L/W = \Phi$, and so it too is a golden rectangle.

This smaller golden rectangle can again be subdivided into a still smaller square and golden rectangle, and this process can be continued ad infinitum. At each subdivision, the length of the square is reduced by a factor of $\phi = 1/\Phi$.

The subdivisions can be done in either a clockwise or counterclockwise fashion. For clockwise, the square is positioned first on the left, then top, then right, and then bottom of the rectangle, and so on. Eventually, we obtain Fig. 12.2, where the side lengths of some of the squares are written in their centers as powers of ϕ.

Notice that each golden rectangle in Fig. 12.2 is a reduced-scale copy of the whole. Objects containing reduced-scale copies of themselves are called self-similar.

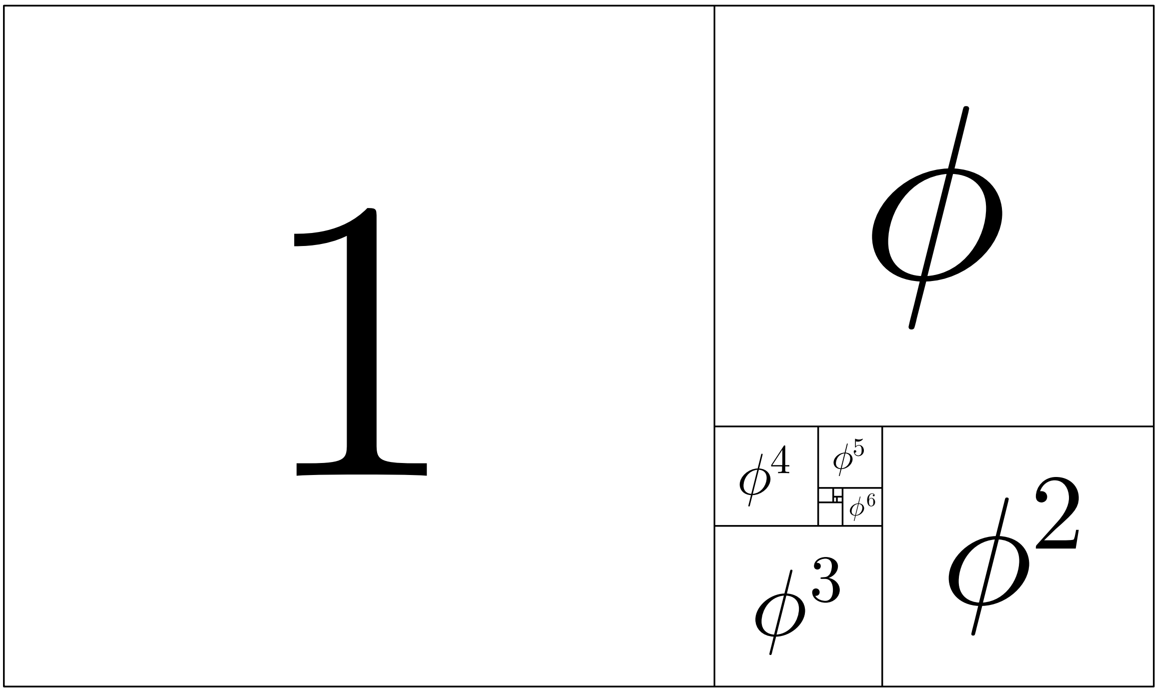


Figure 12.2: Spiraling squares. The side lengths of the squares are the numbers in their centers.

Problems for Lecture 12

1. Prove that

$$\sum_{i=0}^{\infty} \phi^{2i} = \Phi.$$

A visual proof of this identity is given by Fig. 12.2, where the left-hand-side is the sum of the areas of all the imbedded squares and the right-hand-side is the area of the big rectangle.

Solutions to the Problems

Lecture 13

The golden spiral

View this lecture on YouTube

The celebrated golden spiral is a special case of the more general logarithmic spiral whose radius r is given by

$$r = ae^{b\theta}, \tag{13.1}$$

where θ is the usual polar angle, and a and b are constants. Jacob Bernoulli (1655-1705) studied this spiral in depth and gave it the name *spira mirabilis*, or miraculous spiral, asking that it be engraved on his tombstone with the enscription *"Eadem mutata resurgo"*, roughly translated as "Although changed, I arise the same." A spiral was engraved at the bottom of his tombstone, but sadly it was not his beloved logarithmic spiral.

The golden spiral is a logarithmic spiral whose radius either increases or decreases by a factor of the golden ratio Φ with each one-quarter turn, that is, when θ increases by $\pi/2$. The golden spiral therefore satisfies the equation

$$r = a\Phi^{2\theta/\pi}. \tag{13.2}$$

In our figure of the spiraling squares within the golden rectangle, the dimension of each succeeding square decreases by a factor of Φ, with four squares composing each full turn of the spiral. It should then be possible to inscribe a golden spiral within our figure of spiralling squares. We place the central point of the spiral at the accumulation point of all the squares, and fit the parameter a so that the golden spiral passes through opposite corners of the squares. The resulting beautiful golden spiral is shown in Fig. 13.1.

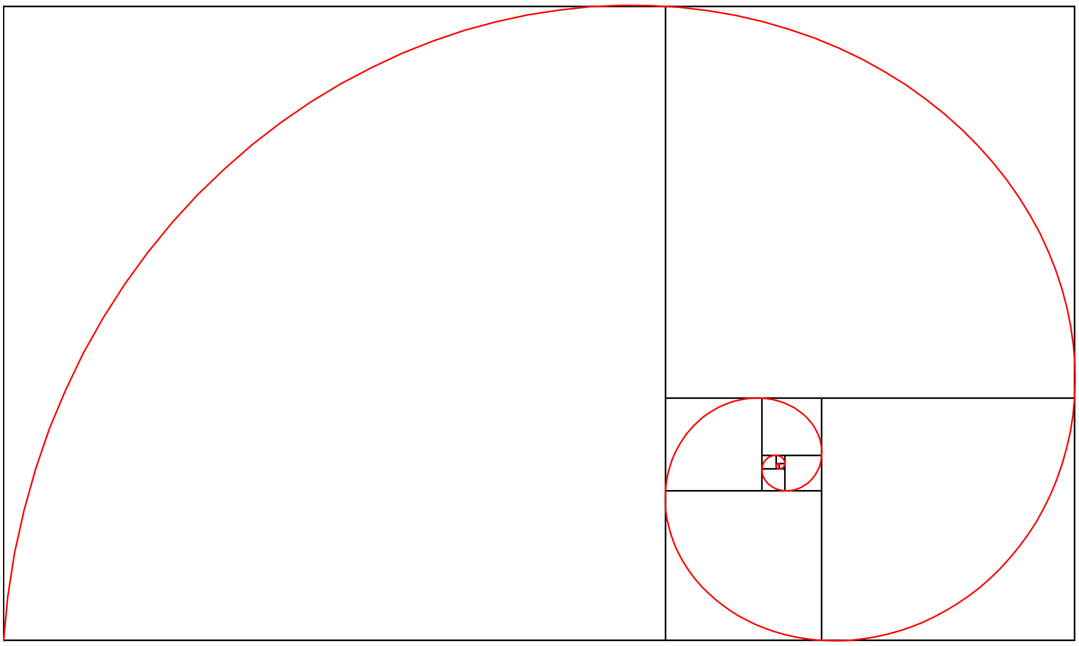
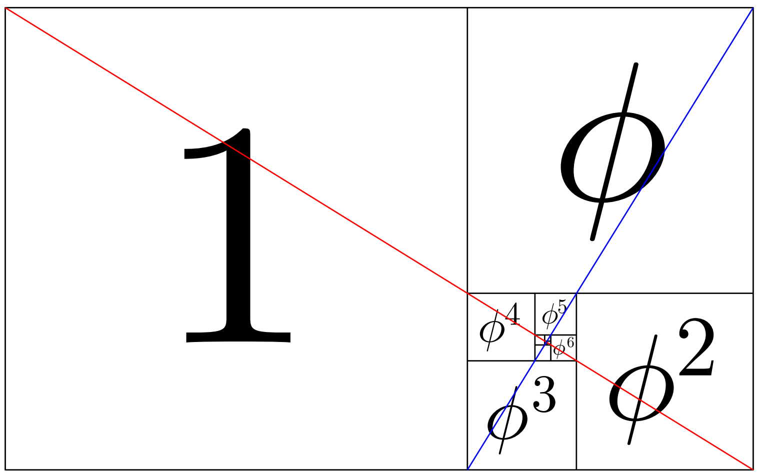


Figure 13.1: The golden spiral. The central point is where the squares accumulate.

Problems for Lecture 13

1. Prove that the accumulation point of all the spiralling squares is the intersection point of the diagonal lines of the two largest golden rectangles, as illustrated below. Find the coordinates of this intersection point.



Solutions to the Problems

Lecture 14

An inner golden rectangle

View this lecture on YouTube

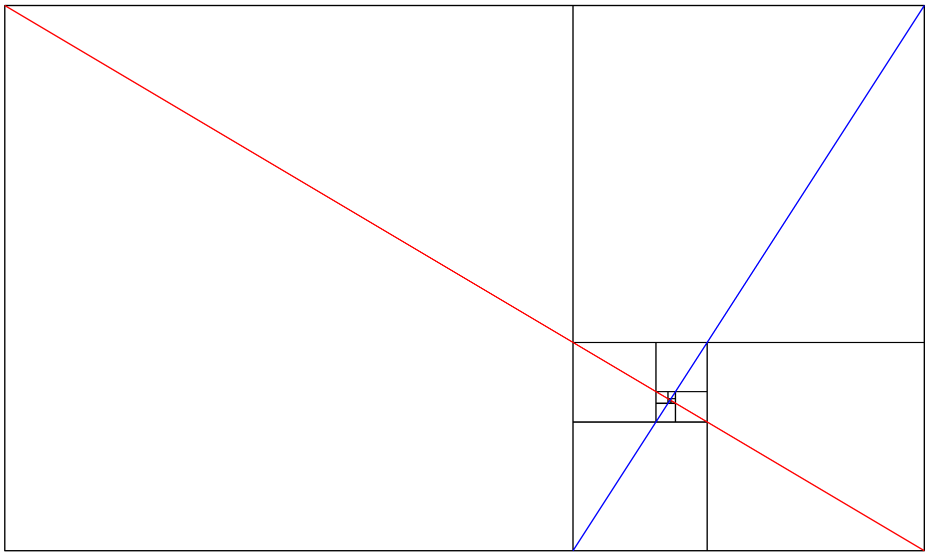


Figure 14.1: Spiral center. The intersection of the red and blue diagonal lines marks the accumulation point of all the golden rectangles, and locates the center of the golden spiral.

Consider again the spiralling squares shown in Fig. 14.1. As shown in the problems, if the diagonals of the two largest golden rectangles are drawn, their intersection point marks the center of a golden spiral.

By symmetry, we should be able to mark four possible spiral centers. These four centers are located in Fig. 14.2 as the intersection of the red and blue diagonals. If we then draw the rectangle with vertices at the four spiral centers, we obtain yet another golden rectangle, with horizontal length $L = \Phi/\sqrt{5}$ and vertical width $W = 1/\sqrt{5}$.

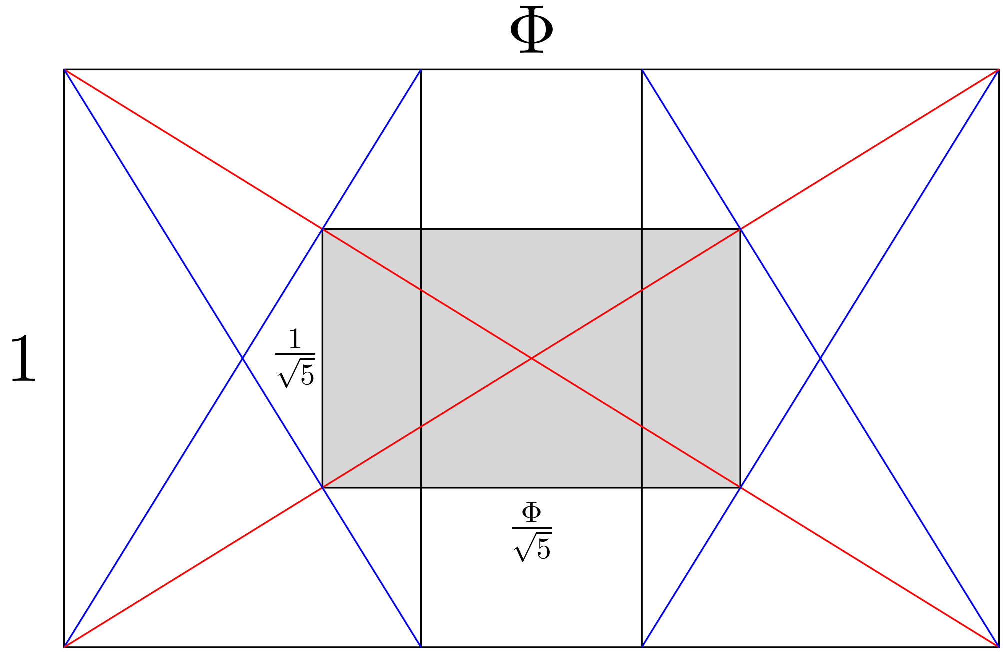


Figure 14.2: An inner golden rectangle. The four spiral centers in a golden rectangle are indicated by the intersections of the red and blue diagonals. These four points form the vertices of another golden rectangle, whose sides are reduced from that of the original by the factor $\sqrt{5}$.

Problems for Lecture 14

1. Show that the inner golden rectangle with corners at the centers of the four possible golden spirals is reduced in scale from the outer golden rectangle by the factor $\sqrt{5}$.

Solutions to the Problems

Lecture 15

The Fibonacci spiral

View this lecture on YouTube

Consider again the sum of the Fibonacci numbers squared:

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}. \tag{15.1}$$

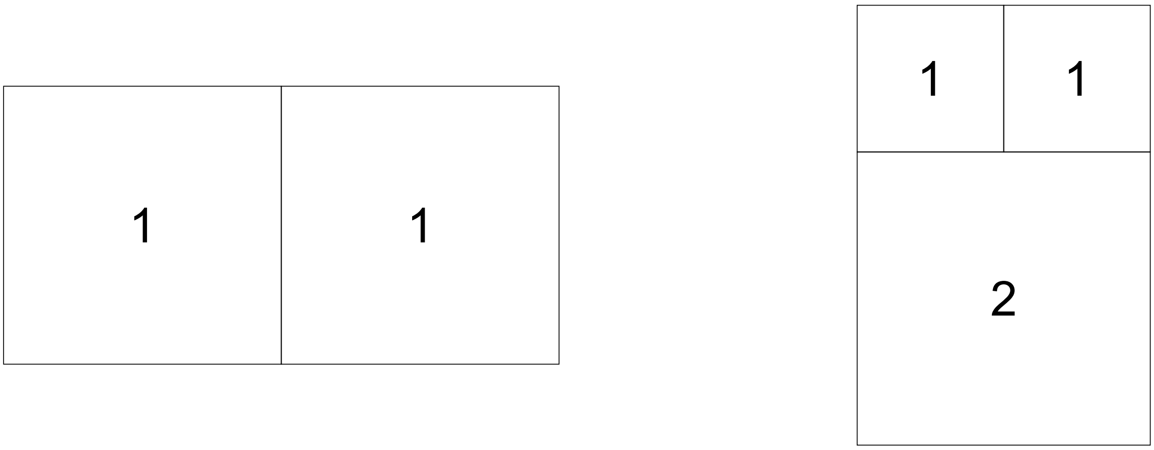
This identity can be interpreted as an area formula. The left-hand-side is the total area of squares with sides given by the first n Fibonacci numbers; the right-hand-side is the area of a rectangle with sides F_n and F_{n+1}.

For example, consider $n = 2$. The identity (15.1) states that the area of two unit squares is equal to the area of a rectangle constructed by placing the two unit squares side-by-side, as illustrated in Fig. 15.1a.

For $n = 3$, we can position another square of side length two directly underneath the first two unit squares. Now, the sum of the areas of the three squares is equal to the area of a 2-by-3 rectangle, as illustrated in Fig. 15.1b. The identity (15.1) for larger n is made self-evident by continuing to tile the plane with squares of side lengths given by consecutive Fibonacci numbers.

The most beautiful tiling occurs if we keep adding squares in a clockwise, or counterclockwise, fashion. Fig. 15.2 shows the iconic result obtained from squares using the first six Fibonacci numbers, where quarter circles are drawn within each square thereby reproducing the Fibonacci spiral.

Consider the close similarity between the golden spiral in Fig. 13.1 and the Fibonacci spiral in Fig. 15.2. Both figures contain spiralling squares, but in Fig. 15.2 the squares spiral outward, and in Fig. 13.1 the squares spiral inward. Because the ratio of two consecutive Fibonacci numbers approaches the golden ratio, the Fibonacci spiral, as it spirals out, will eventually converge to the golden spiral.



(a) $n = 2$: $1^2 + 1^2 = 1 \times 2$.

(b) $n = 3$: $1^2 + 1^2 + 2^2 = 2 \times 3$.

Figure 15.1: Illustrating the sum of the Fibonacci numbers squared. The center numbers represent the side lengths of the squares.

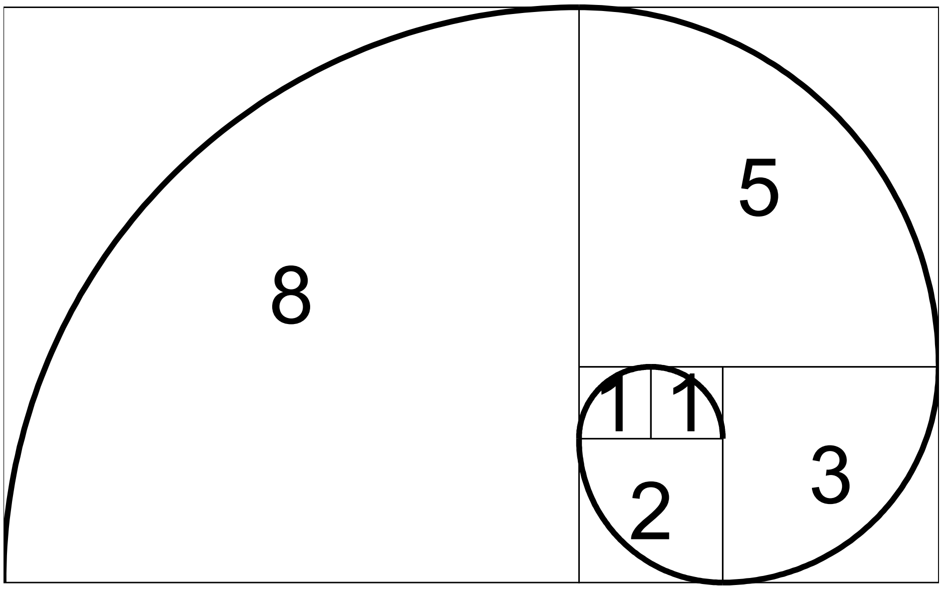


Figure 15.2: The sum of Fibonacci numbers squared for $n = 6$. The Fibonacci spiral is drawn.

Lecture 16

Fibonacci numbers in nature

View this lecture on YouTube

Figure 16.1: The flowering head of a sunflower.

Consider the photo of a sunflower shown in Fig. 16.1, and notice the apparent spirals in the florets radiating out from the center to the edge. These spirals appear to rotate both clockwise and counterclockwise. By counting them, one finds 21 clockwise spirals and 34 counterclockwise spirals. Surprisingly, the numbers 21 and 34 are consecutive Fibonacci numbers. In the following lectures, we will try to explain why this might not be a coincidence.

Lecture 17

Continued fractions

View this lecture on YouTube

The appearance of consecutive Fibonacci numbers in some sunflower heads can be related to a very special property of the golden ratio. To reveal that property requires first a short lesson on continued fractions.

Recall that a rational number is any number that can be expressed as the quotient of two integers, and an irrational number is any number that is not rational. Rational numbers have finite continued fractions; irrational numbers have infinite continued fractions.

A finite continued fraction represents a rational number x as

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}, \tag{17.1}$$

where a_1, a_2, $\dots$, a_n are positive integers and a_0 is any integer. The convenient shorthand form of (17.1) is

$$x = [a_0; a_1, a_2, \dots, a_n].$$

If x is irrational, then $n \to \infty$.

Now for some examples. To construct the continued fraction of the rational number $x = 3/5$, we can write

$$\begin{aligned} 3/5 &= \frac{1}{5/3} = \frac{1}{1+2/3} \\ &= \cfrac{1}{1+\cfrac{1}{3/2}} = \cfrac{1}{1+\cfrac{1}{1+1/2}}, \end{aligned}$$

which is of the form (17.1), so that $3/5 = [0; 1, 1, 2]$.

To construct the continued fraction of an irrational number, say π, we can write

$$\begin{aligned}\pi &= 3 + 0.14159\dots \\ &= 3 + \frac{1}{7.06251\dots} \\ &= 3 + \cfrac{1}{7 + \cfrac{1}{15.99659\dots}},\end{aligned}$$

and so on, yielding the beginning sequence $\pi = [3; 7, 15, \dots]$. The historically important first-order approximation is given by $\pi \approx [3; 7] = 22/7 = 3.142857\dots$, which was already known by Archimedes in ancient times.

Finally, to determine the continued fraction for the golden ratio Φ, we can use a trick and write

$$\Phi = 1 + \frac{1}{\Phi},$$

which is a recursive definition that can be continued as

$$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{\Phi}},$$

and so on, yielding the remarkably simple form

$$\Phi = [1; \overline{1}],$$

where the bar indicates an infinite repetition.

Because the trailing a_i's are all equal to one, the continued fraction for the golden ratio (and other related numbers with trailing ones) converges especially slowly. Furthermore, the successive rational approximations to the golden ratio are just the ratios of consecutive Fibonacci numbers, that is, $1/1$, $2/1$, $3/2$, $5/3$, $8/5$, and so on. Because of the very slow convergence of this sequence, we say that the golden ratio is the most difficult number to approximate by a rational number. More poetically, the golden ratio has been called the most irrational of the irrational numbers.

Problems for Lecture 17

1. Starting with $\sqrt{2} = 1 + (\sqrt{2} - 1)$, find a recursive definition for $\sqrt{2}$ and use it to derive its continued fraction.

2. Use the same trick of Problem 1 to find the continued fraction for $\sqrt{3}$.

3. Show that $e = [2; 1, 2, 1, 1, 4, \dots]$

4. Define x_n to be the nth rational approximation to x obtained from its continued fraction, where, for example, $x_0 = [a_0;]$, $x_1 = [a_0; a_1]$, and $x_2 = [a_0; a_1, a_2]$. Using $\Phi = [1; \overline{1}]$, verify that Φ_0, Φ_1, Φ_2, and Φ_3 are just the ratios of consecutive Fibonacci numbers.

5. Prove by induction that

$$\Phi_n = \frac{F_{n+2}}{F_{n+1}}. \tag{17.2}$$

Solutions to the Problems

Lecture 18

The golden angle

View this lecture on YouTube

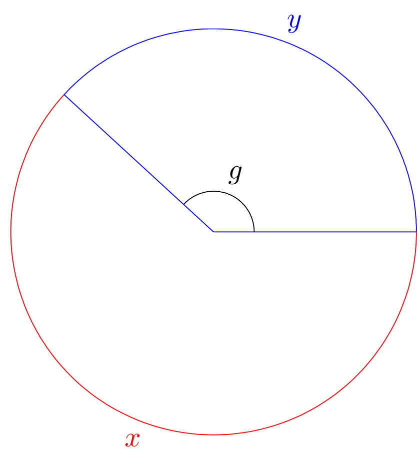


Figure 18.1: *The golden angle g is determined from requiring $x/y = \Phi$.*

Our model of the sunflower will make use of the golden angle. The golden angle is defined as the acute angle g that divides the circumference of a circle into two arcs with lengths in the golden ratio (see Fig. 18.1).

The golden ratio Φ and the golden ratio conjugate ϕ satisfy

$$\Phi = \frac{x}{y}, \quad \phi = \frac{y}{x},$$

with $\Phi = 1 + \phi$. We can determine the golden angle by writing

$$\begin{aligned} \frac{g}{2\pi} &= \frac{y}{x+y} = \frac{\phi}{1+\phi} \\ &= \frac{\phi}{\Phi} = \phi^2; \end{aligned}$$

and since $\phi^2 = 1 - \phi$, we obtain

$$g = 2\pi(1-\phi).$$

Expressed in degrees, this is $g \approx 137.5°$.

To determine the continued fraction for $g/2\pi$, we write

$$\begin{aligned}\frac{g}{2\pi} = \frac{y}{x+y} &= \frac{1}{1+\Phi} \\ &= \cfrac{1}{1+1+\cfrac{1}{1+\cfrac{1}{1+\dots}}},\end{aligned}$$

which yields $g/2\pi = [0;2,\overline{1}]$. The trailing ones in the continued fraction ensure that $g/2\pi$ is difficult to represent as a rational number. Indeed, the successive rational approximations to $g/2\pi$ can be computed to be $1/2$, $1/3$, $2/5$, $3/8$, $5/13$, and so on, which is just the sequence given by the ratio F_n/F_{n+2}.

Problems for Lecture 18

1. Define x_n to be the nth rational approximation to x obtained from its continued fraction, where, for example, $x_0 = [a_0;]$, $x_1 = [a_0; a_1]$, and $x_2 = [a_0; a_1, a_2]$. Using $g/2\pi = [0; 2, \bar{1}]$, determine $g_0/2\pi$, $g_1/2\pi$, $g_2/2\pi$, and $g_3/2\pi$.

2. Prove by induction that

$$\frac{g_n}{2\pi} = \frac{F_n}{F_{n+2}}. \tag{18.1}$$

Solutions to the Problems

Lecture 19

A simple model for the growth of a sunflower

View this lecture on YouTube

We can now understand a simple model for the growth of a sunflower head, and why the Fibonacci numbers might appear. Suppose that during development, florets are created close to the center of the head and subsequently move radially outward with constant speed during growth. Also suppose that as each new floret is created at the center, it is rotated through a constant angle before moving radially. Our goal is to derive an angle of rotation that in some sense is optimum: the resulting sunflower head consists of well-spaced florets.

Let us denote the rotation angle by $2\pi\alpha$. We first consider the possibility that α is a rational number, say n/m, where n and m are positive integers with no common factors, and $n < m$. Since after m rotations florets will return to the radial line on which they started, the resulting sunflower head consists of florets lying along m straight lines. A simulation of such a sunflower head for $\alpha = 1/7$ is shown in Fig. 19.1a, where one observes seven straight lines. Evidently, rational values for α do not result in well-spaced florets.

What about irrational values? For α irrational, no number of rotations will return the florets to their first radial line. Nevertheless, the resulting sunflower head may still not have well-spaced florets. For example, if $\alpha = \pi - 3$, then the resulting sunflower head looks like Fig. 19.1b. There, one can see seven counterclockwise spirals. Recall that a good rational approximation to π is $22/7$, which is slightly larger than π. On every seventh counterclockwise rotation, new florets fall just short of the radial line of florets created seven rotations ago.

The irrational numbers that are most likely to construct a sunflower head with well-spaced florets are those that can not be well-approximated by rational numbers. Here, we choose the golden angle, taking $\alpha = 1 - \phi$. The rational approximations to $1 - \phi$ are given by F_n/F_{n+2}, so that the number of spirals observed will correspond to the Fibonacci numbers.

Two simulations of the sunflower head with $\alpha = 1 - \phi$ are shown in Fig. 19.2. These simulations differ only by the choice of radial velocity, v_0. In Fig. 19.2a, one counts 13 clockwise spirals and 21 counterclockwise spirals; in Fig. 19.2b, one counts 21 counter clockwise spirals and 34 clockwise spirals, the same as the sunflower head shown in Fig. 16.1.

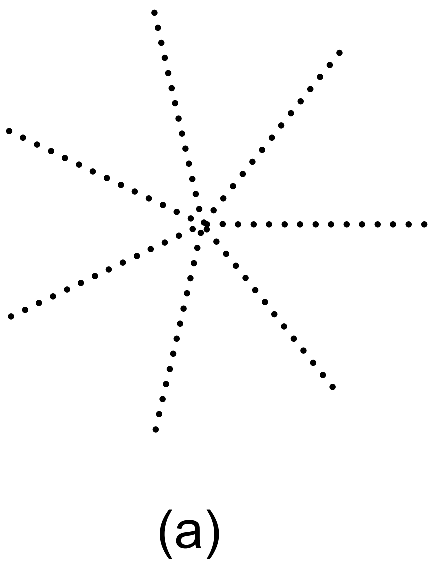


Figure 19.1: Simulation of the sunflower model for (a) $\alpha = 1/7$; (b) $\alpha = \pi - 3$ and counterclockwise rotation.

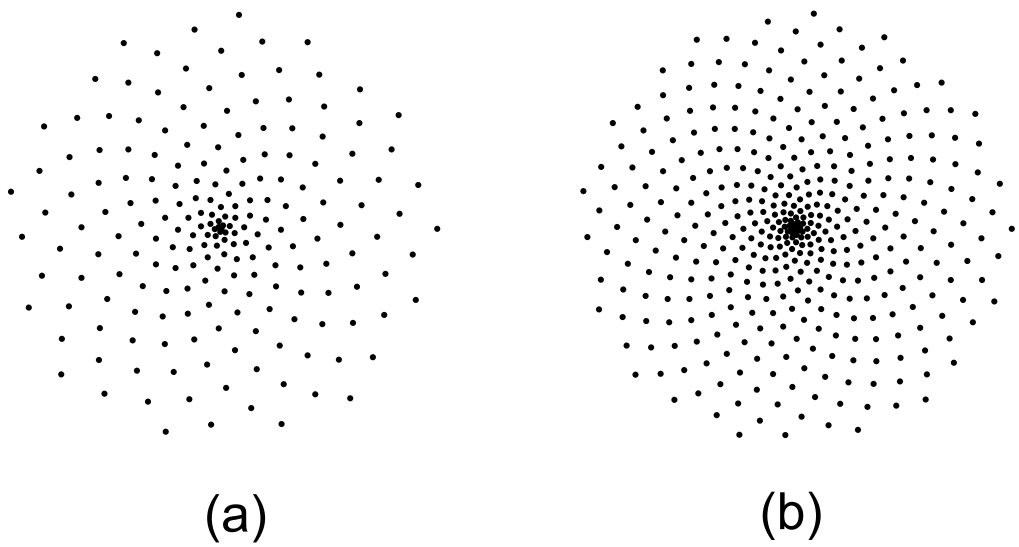


Figure 19.2: Simulation of the sunflower model for $\alpha = 1 - \phi$ and clockwise rotation. (a) $v0 = 1/2$; (b) $v0 = 1/4$.

Appendix A

Mathematical induction

View this lecture on YouTube

Mathematical induction is a method of proof used to establish a statement about the natural numbers. The standard example used to illustrate this form of proof is

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}, \tag{A.1}$$

valid for n a positive integer. We will suppose that this formula is given, and that our task is to prove it using mathematical induction.

The first step in the proof is to establish what is called the base case. Here, we prove that the given statement is true for the smallest integer for which it is claimed to be valid:

Base case: For $n = 1$, the left-hand side of (A.1) equals one, and the right-hand side of (A.1) equals $(1 \times 2)/2 = 1$, so that (A.1) is true for $n = 1$.

The next step is called the induction step. Here one assumes that the statement is true for $n = k$, and then proves it is true for $n = k + 1$. Once the induction step is proved, then the statement is declared true for all integers greater than or equal to the base case.

Why is the proof then complete? Well, if the statement is true for $n = 1$ as shown in the base case, then the induction step shows it is also true for $n = 2$. And if the statement is true for $n = 2$, then the induction step shows it is true for $n = 3$. And so on, leading to the conclusion that the statement is true for all the counting numbers starting from one.

So let us proceed to prove the induction step. The method of proof usually writes the left-hand side of the statement when $n = k + 1$, and then makes use of the induction hypothesis—the assumption that the statement is true for $n = k$—and some additional information to obtain the right-hand side of the statement.

Induction step: Now, suppose (A.1) is true for $n = k$. Then

$$\begin{aligned}
1+2+3+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) && \text{(from induction hypothesis)}\\
&= \frac{k(k+1)+2(k+1)}{2} && \text{(from combining fractions)}\\
&= \frac{(k+1)(k+2)}{2}, && \text{(from factoring)}
\end{aligned}$$

and since $(k+2) = (k+1)+1$, we have shown that A.1 is true for $n = k+1$. By the principle of induction, A.1 is therefore true for all positive integers n.

Because statements about the Fibonacci numbers F_n are typically statements true for $n = 1, 2, 3, \ldots$, proofs can often make use of mathematical induction. Oftentimes, the induction step requires the assumption that the statement is true for both $n = k - 1$ and $n = k$. When this happens, the base case must verify the truth of the statement for both $n = 1$ and $n = 2$. Then if the statement is true for $n = 1$ and $n = 2$, it must be true for $n = 3$. And if the statement is true for $n = 2$ and $n = 3$, it must be true for $n = 4$. And so on, leading to the conclusion that the statement is true for all positive integers n.

Appendix B

Matrix algebra

For those readers who have never studied matrices or linear algebra, it will be helpful to understand a few basic concepts. A matrix with n rows and m columns is called an n-by-m matrix. Here, we need only consider the simple case of two-by-two matrices.

A two-by-two matrix A, with two rows and two columns, can be written as

$$\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The first row has elements a and b, the second row has elements c and d. The first column has elements a and c; the second column has elements b and d.

B.1 Addition and Multiplication

View this lecture on YouTube

Matrices can be added and multiplied. Matrices can be added if they have the same dimension, and addition proceeds element by element, following

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}.$$

Matrices can be multiplied if the number of columns of the left matrix equals the number of rows of the right matrix. A particular element in the resulting product matrix, say in row k and column l, is obtained by multiplying and summing the elements in row k of the left matrix with the elements in column l of the right matrix. For example, a two-by-two matrix can multiply a two-by-one column vector as follows

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}.$$

The first row of the left matrix is multiplied against and summed with the first (and only) column of the right matrix to obtain the element in the first row and first column of the product matrix, and so on for the element in the second row and first column. The product of two two-by-two matrices is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}.$$

B.2 Determinants

View this lecture on YouTube

Consider the system of equations given by

$$\begin{aligned} ax + by &= 0, \\ cx + dy &= 0, \end{aligned} \tag{B.1}$$

which can be written in matrix form as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When does there exist a nontrivial (not identically zero) solution for x and y?

To answer this question, we solve directly the system of equations given by (B.1). Multiplying the first equation by d and the second by b, and subtracting the second equation from the first, results in

$$(ad - bc)x = 0.$$

Similarly, multiplying the first equation by c and the second by a, and subtracting the first equation from the second, results in

$$(ad - bc)y = 0.$$

Therefore, a nontrivial solution of (B.1) exists only if $ad - bc = 0$. The quantity $ad - bc$ defines the determinant of the two-by-two matrix, that is

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc. \tag{B.2}$$

The determinants of larger square matrices can be found similarly. Just for fun, I can show you the determinant of a three-by-three matrix:

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg).$$

We will need the following result to prove Cassini's identity:

$$\det AB = \det A \det B.$$

Although this is a general result for all n-by-n square matrices, we need only the result for the 2-by-2 case, which can be easily proved by an explicit calculation.

Let

$$\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \mathrm{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Then

$$\mathrm{AB} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix},$$

and

$$\begin{aligned}
\det AB &= (ae+bg)(cf+dh)-(af+bh)(ce+dg)\\
&= (acef+adeh+bcfg+bdgh)-(acef+adfg+bceh+bdgh)\\
&= (adeh+bcfg)-(adfg+bceh)\\
&= ad(eh-fg)-bc(eh-fg)\\
&= (ad-bc)(eh-fg)\\
&= \det A \det B.
\end{aligned}$$

Appendix C

Problem solutions

Solutions to the Problems for Lecture 1

1. We calculate the first few terms.

$F_0 = F_2 - F_1 = 0,$
$F_{-1} = F_1 - F_0 = 1,$
$F_{-2} = F_0 - F_{-1} = -1,$
$F_{-3} = F_{-1} - F_{-2} = 2,$
$F_{-4} = F_{-2} - F_{-3} = -3,$
$F_{-5} = F_{-3} - F_{-4} = 5,$
$F_{-6} = F_{-4} - F_{-5} = -8.$

The correct relation appears to be

$$F_{-n} = (-1)^{n+1} F_n. \tag{C.1}$$

We now prove (C.1) by mathematical induction.

Base case: Our calculation above already shows that (C.1) is true for $n = 1$ and $n = 2$, that is, $F_{-1} = F_1$ and $F_{-2} = -F_2$.

Induction step: Suppose that (C.1) is true for positive integers $n = k - 1$ and $n = k$. Then we have

$$\begin{aligned}
F_{-(k+1)} &= F_{-(k-1)} - F_{-k} && \text{(from definition)}\\
&= (-1)^k F_{k-1} - (-1)^{k+1} F_k && \text{(from induction hypothesis)}\\
&= (-1)^{k+2} (F_{k-1} + F_k)\\
&= (-1)^{k+2} F_{k+1}, && \text{(from recursion relation)}
\end{aligned}$$

so that (C.1) is true for $n = k + 1$. By the principle of induction, (C.1) is therefore true for all positive integers.

2. 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322

3. We now prove (1.2) by mathematical induction.

Base case: To prove that (1.2) is true for $n = 1$, we write $F_1 p + F_2 q = p + q = f_3$. To prove that (1.2) is true for $n = 2$, we write $F_2 p + F_3 q = p + 2q = f_3 + f_2 = f_4$.

Induction step: Suppose that (1.2) is true for positive integers $n = k - 1$ and $n = k$. Then we have

$$\begin{aligned}
f_{k+3} &= f_{k+2} + f_{k+1} \\
&= (F_k p + F_{k+1} q) + (F_{k-1} p + F_k q) && \text{(from induction hypothesis)} \\
&= (F_k + F_{k-1})\, p + (F_{k+1} + F_k)\, q \\
&= F_{k+1} p + F_{k+2} q, && \text{(from recursion relation)}
\end{aligned}$$

so that (1.2) is true for $n = k + 1$. By the principle of induction, (1.2) is therefore true for all positive integers.

4. To generate the Lucas sequence, we take $p = 1$ and $q = 3$. Therefore, we have

$$\begin{aligned}
L_n &= F_{n-2} + 3F_{n-1} && \text{(from (1.2))} \\
&= 2F_{n-1} + (F_{n-1} + F_{n-2}) \\
&= F_{n-1} + (F_{n-1} + F_n) && \text{(from recursion relation)} \\
&= F_{n-1} + F_{n+1}. && \text{(from recursion relation)}
\end{aligned}$$

5. We have

$$\begin{aligned}
\frac{1}{5}\left(L_{n-1} + L_{n+1}\right) &= \frac{1}{5}\left((F_{n-2} + F_n) + (F_n + F_{n+2})\right) && \text{(from (1.3))} \\
&= \frac{1}{5}\left(F_{n-2} + 2F_n + F_n + F_{n+1}\right) && \text{(from recursion relation)} \\
&= \frac{1}{5}\left(F_{n-2} + 3F_n + F_n + F_{n-1}\right) && \text{(from recursion relation)} \\
&= F_n. && \text{(from recursion relation)}
\end{aligned}$$

6. We write

$$\begin{aligned}
f(x) &= F_1 x + F_2 x^2 + F_3 x^3 + F_4 x^4 + \ldots \\
x f(x) &= \qquad\quad F_1 x^2 + F_2 x^3 + F_3 x^4 + \ldots \\
x^2 f(x) &= \qquad\qquad\qquad F_1 x^3 + F_2 x^4 + \ldots.
\end{aligned}$$

We then subtract the second and third equation from the first, and use $F_1 = F_2 = 1$ and $F_{n+1} - F_n - F_{n-1} = 0$ to obtain

$$(1 - x - x^2) f(x) = x,$$

resulting in

$$f(x) = \frac{x}{1 - x - x^2}.$$

Solutions to the Problems for Lecture 2

1. Consider the set of different possible strings. This set may be divided into two nonoverlapping subsets: those strings that start with one and those strings for which one and two are interchanged. For the former, the remaining $n-1$ numbers can form a_{n-1} different strings. For the latter, the remaining $n-2$ numbers may can form a_{n-2} different strings. The total number of different strings is therefore given by the Fibonacci recursion relation

$$a_n = a_{n-1} + a_{n-2}.$$

Together with $a_1 = 1 = F_2$ and $a_2 = 2 = F_3$, we obtain $a_n = F_{n+1}$.

2. Again consider the set of different possible strings. This set may be divided into two nonoverlapping subsets: those strings for which the one and n are not interchanged, and those strings for which they are interchanged. For the former, the number of different strings is given by $a_n = F_{n+1}$. For the latter, the number of different strings is given by $a_{n-2} = F_{n-1}$. We therefore have

$$b_n = F_{n+1} + F_{n-1}.$$

From (1.3), the relation satisfied by b_n is the same as that satisfied by the nth Lucas number, so that $b_n = L_n$.

Solutions to the Problems for Lecture 3

1.

(a)

$$\begin{aligned}\Phi - 1 &= \frac{\sqrt{5}+1}{2} - 1 \\ &= \frac{\sqrt{5}-1}{2} \\ &= \phi,\end{aligned}$$

(b)

$$\begin{aligned}\frac{1}{\Phi} &= \frac{2}{1+\sqrt{5}} \times \frac{1-\sqrt{5}}{1-\sqrt{5}} \\ &= \frac{2\left(1-\sqrt{5}\right)}{-4} \\ &= \frac{\sqrt{5}-1}{2} \\ &= \phi.\end{aligned}$$

(c)

$$\begin{aligned}\Phi^2 &= \left(\frac{\sqrt{5}+1}{2}\right)^2 \\ &= \frac{5+2\sqrt{5}+1}{4} \\ &= \frac{\sqrt{5}+3}{2} \\ &= \Phi + 1.\end{aligned}$$

(d)

$$\begin{aligned}\phi^2 &= \left(\frac{\sqrt{5}-1}{2}\right)^2 \\ &= \frac{5-2\sqrt{5}+1}{4} \\ &= \frac{-\sqrt{5}+3}{2} \\ &= -\phi + 1.\end{aligned}$$

2. We multiply

$$\frac{1}{\Phi} = \Phi - 1$$

by Φ and rearrange to obtain

$$\Phi^2 = \Phi + 1.$$

Multiplying both sides by Φ^{n-1} yields the desired result:

$$\Phi^{n+1} = \Phi^n + \Phi^{n-1}.$$

3. We substitute $\phi = 1/\Phi$ into

$$\frac{1}{\Phi} = \Phi - 1$$

to obtain

$$\phi = \frac{1}{\phi} - 1.$$

Multiplying both sides by ϕ^n and rearranging terms yields the desired result:

$$\phi^{n-1} = \phi^n + \phi^{n+1}.$$

Solutions to the Problems for Lecture 4

1. Write

$$\frac{F_{k+n}}{F_k} = \frac{F_{k+n}}{F_{k+n-1}} \times \frac{F_{k+n-1}}{F_{k+n-2}} \times \cdots \times \frac{F_{k+1}}{F_k}.$$

Then taking $\lim_{k\to\infty}$, and using

$$\lim_{j\to\infty} \frac{F_j}{F_{j-1}} = \Phi,$$

one obtains directly

$$\lim_{k\to\infty} \frac{F_{k+n}}{F_k} = \Phi^n.$$

2. We prove (4.2) by mathematical induction.

Base case: For $n = 1$, the relation (4.2) becomes $\Phi = \Phi$, which is true.

Induction step: Suppose that (4.2) is true for positive integer $n = k$. Then we have

$$\begin{aligned}
\Phi^{k+1} &= \Phi\Phi^k \\
&= \Phi\left(F_k\Phi + F_{k-1}\right) && \text{(from induction hypothesis)} \\
&= F_k\Phi^2 + F_{k-1}\Phi \\
&= F_k\left(\Phi + 1\right) + F_{k-1}\Phi && \text{(from } \Phi^2 = \Phi + 1\text{)} \\
&= \left(F_k + F_{k-1}\right)\Phi + F_k \\
&= F_{k+1}\Phi + F_k && \text{(from recursion relation)}
\end{aligned}$$

so that (4.2) is true for $n = k + 1$. By the principle of induction, (4.2) is therefore true for all positive integers.

3. We prove (4.3) by mathematical induction.

Base case: For $n = 1$, the relation (4.3) becomes $-\phi = -\phi$, which is true.

Induction step: Suppose that (4.3) is true for positive integer $n = k$. Then we have

$$\begin{aligned}
(-\phi)^{k+1} &= -\phi(-\phi)^k \\
&= -\phi\left(-F_k\phi + F_{k-1}\right) && \text{(from induction hypothesis)} \\
&= F_k\phi^2 - F_{k-1}\phi \\
&= F_k\left(-\phi + 1\right) - F_{k-1}\phi && \text{(from } \phi^2 = -\phi + 1\text{)} \\
&= -\left(F_k + F_{k-1}\right)\phi + F_k \\
&= -F_{k+1}\phi + F_k, && \text{(from recursion relation)}
\end{aligned}$$

so that (4.3) is true for $n = k + 1$. By the principle of induction, (4.3) is therefore true for all positive integers.

Solutions to the Problems for Lecture 5

1. We prove 5.6 by mathematical induction.

Base case: When $n = 1$, the left side of (5.6) is $F_1 = 1$ and the right side is $(\Phi + \phi)/\sqrt{5} = 1$, so (5.6) is true for $n = 1$. When $n = 2$, the left side of (5.6) is $F_2 = 1$ and the right side is $(\Phi^2 - \phi^2)/\sqrt{5}$. We have

$$\begin{aligned}\frac{(\Phi^2 - \phi^2)}{\sqrt{5}} &= \frac{(\sqrt{5}+1)^2 - (\sqrt{5}-1)^2}{4\sqrt{5}} \\ &= \frac{4\sqrt{5}}{4\sqrt{5}} \\ &= 1\end{aligned}$$

so (5.6) is also true for $n = 2$.

Induction step: We will need to make use of the identities

$$\Phi^{n+1} = \Phi^{n-1} + \Phi^n, \tag{C.2}$$
$$\phi^{n-1} = \phi^n + \phi^{n+1}. \tag{C.3}$$

Now, suppose (5.6) is true for $n = k-1$ and $n = k$. Then

$$\begin{aligned}\frac{\Phi^{k+1} - (-\phi)^{k+1}}{\sqrt{5}} &= \frac{(\Phi^{k-1} + \Phi^k) - (-1)^{k+1}(\phi^{k-1} - \phi^k)}{\sqrt{5}} && \text{(from (C.2) and (C.3))} \\ &= \frac{(\Phi^{k-1} - (-\phi)^{k-1}) + (\Phi^k) - (-\phi)^k}{\sqrt{5}} && \text{(from reorganizing terms)} \\ &= F_{k-1} + F_k && \text{(from induction hypothesis)} \\ &= F_{k+1}, && \text{(from recursion relation)}\end{aligned}$$

so (5.6) is true for $n = k+1$. By the principle of induction, (5.6) is therefore true for all positive integers.

2.

$$\begin{aligned}\lim_{n\to\infty} \frac{F_{n+1}}{F_n} &= \lim_{n\to\infty} \frac{\Phi^{n+1} - (-\phi)^{n+1}}{\Phi^n - (-\phi)^n} \\ &= \lim_{n\to\infty} \frac{\Phi + (-1)^n\phi(\phi/\Phi)^n}{1 + (-1)^{n+1}(\phi/\Phi)^n}. && \text{(divide numerator and denominator by } \Phi^n\text{)}\end{aligned}$$

And since

$$\lim_{n\to\infty} (\phi/\Phi)^n = 0,$$

we obtain

$$\lim_{n\to\infty} F_{n+1}/F_n = \Phi.$$

3. Subtract (5.8) from (5.7) to obtain

$$F_n\,(\Phi + \phi) = \Phi^n - (-\phi)^n.$$

With $\Phi + \phi = \sqrt{5}$, we obtain Binet's formula

$$F_n = \frac{\Phi^n - (-\phi)^n}{\sqrt{5}}.$$

4. To derive Binet's formula, we will Taylor series expand the function $f(x) = x/(1 - x - x^2)$ and equate the coefficients of the resulting power series to the Fibonacci numbers.

The roots of the quadratic equation $1 - x - x^2 = 0$ are given by ϕ and $-\Phi$ so we can write

$$\frac{x}{1 - x - x^2} = \frac{x}{(\phi - x)(\Phi + x)}.$$

A partial-fraction expansion results in

$$\begin{aligned}\frac{x}{(\phi - x)(\Phi + x)} &= \frac{1}{\sqrt{5}}\left(\frac{\phi}{\phi - x} - \frac{\Phi}{\Phi + x}\right)\\ &= \frac{1}{\sqrt{5}}\left(\frac{1}{1 - \Phi x} - \frac{1}{1 + \phi x}\right),\end{aligned}$$

where the last step uses $\phi\Phi = 1$. We now use the Taylor-series expansions

$$\begin{aligned}\frac{1}{1 - \Phi x} &= 1 + \Phi x + \Phi^2 x^2 + \Phi^3 x^3 + \ldots,\\ \frac{1}{1 - (-\phi)x} &= 1 + (-\phi)x + (-\phi)^2 x^2 + (-\phi)^3 x^3 + \ldots.\end{aligned}$$

Subtracting the second expansion from the first results in

$$\left(\frac{1}{1 - \Phi x} - \frac{1}{1 + \phi x}\right) = (\Phi - (-\phi))\, x + (\Phi^2 - (-\phi)^2)x^2 + (\Phi^3 - (-\phi)^3)x^3 + \ldots.$$

Therefore,

$$\frac{x}{1 - x - x^2} = \left(\frac{\Phi - (-\phi)}{\sqrt{5}}\right) x + \left(\frac{\Phi^2 - (-\phi)^2}{\sqrt{5}}\right) x^2 + \left(\frac{\Phi^3 - (-\phi)^3}{\sqrt{5}}\right) x^3 + \ldots,$$

and equating the coefficients of this Taylor-series expansion with the coefficients of the generating function for the Fibonacci sequence results in Binet's formula

$$F_n = \frac{\Phi^n - (-\phi)^n}{\sqrt{5}}.$$

5. The derivation follows the method we used to obtain Binet's formula, but here the initial values differ. We can use $L_0 = L_2 - L_1 = 2$, and $L_1 = 1$. The general solution to the Fibonacci recursion relation is given by

$$L_n = c_1\Phi^n + c_2(-\phi)^n.$$

Application of the two initial values that yields the Lucas sequence results in the system of equations given by

$$c_1 + c_2 = 2,$$
$$c_1 \Phi - c_2 \phi = 1.$$

Multiplying the first equation by ϕ and adding it to the second equation results in

$$c_1(\Phi + \phi) = 2\phi + 1.$$

Now $2\phi + 1 = \Phi + \phi = \sqrt{5}$. Therefore $c_1 = 1$ and $c_2 = 1$. Our solution is therefore

$$L_n = \Phi^n + (-\phi)^n.$$

6. Binet's formula and the analogous formula for the Lucas numbers are given by

$$F_n = \frac{\Phi^n - (-\phi)^n}{\sqrt{5}}, \qquad L_n = \Phi^n + (-\phi)^n.$$

Add the equation for L_n to the equation for F_n multiplied by $\sqrt{5}$, and divide by two to obtain

$$\Phi^n = \frac{L_n + \sqrt{5}F_n}{2}.$$

Solutions to the Problems for Lecture 6

1. We prove (6.3) by mathematical induction.

Base case: For $n = 1$, we obtain from (6.3)

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

which is just the definition.

Induction step: Suppose that (6.3) is true for $n = k$. Then we have

$$\begin{aligned}
Q^{k+1} &= QQ^k \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix} && \text{(from induction hypothesis)} \\
&= \begin{pmatrix} F_{k+1} + F_k & F_k + F_{k-1} \\ F_{k+1} & F_k \end{pmatrix} \\
&= \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix}, && \text{(from recursion relation)}
\end{aligned}$$

so that (6.3) is true for $n = k + 1$. By the principle of induction, (6.3) is therefore true for all positive integers.

2. We will use the formula

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

The relation $Q^n Q^m = Q^{n+m}$ is written as

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix} = \begin{pmatrix} F_{n+m+1} & F_{n+m} \\ F_{n+m} & F_{n+m-1} \end{pmatrix}.$$

Equating the first element of the resulting matrices yields

$$F_{n+1}F_{m+1} + F_n F_m = F_{n+m+1}.$$

With the substitutions $n \to n - 1$ and $m \to m$, we obtain (6.4).

3. The first result can be obtained by taking $m = n - 1$ in (6.4); the second result can be obtained by taking $m = n$.

4. Using $L_n = F_{n-1} + F_{n+1}$ in the second formula of (6.5) yields the desired result.

Solutions to the Problems for Lecture 7

1. We prove (7.4) by mathematical induction.

Base case: When $n = 1$ the left side of (7.4) is $F_2F_0 - F_1^2 = -1$ and the right side is $(-1)^1 = -1$, so (7.4) is true for $n = 1$.

Induction step: Suppose (7.4) is true for $n = k$. Then

$$\begin{aligned}
F_{k+2}F_k - F_{k+1}^2 &= (F_k + F_{k+1})\, F_k - F_{k+1}^2 && \text{(from recursion relation)}\\
&= F_k^2 + F_{k+1}\,(F_k - F_{k+1}) \\
&= F_k^2 - F_{k+1}F_{k-1} && \text{(from recursion relation)}\\
&= -(-1)^k && \text{(from induction hypothesis)}\\
&= (-1)^{k+1},
\end{aligned}$$

so (7.4) is true for $n = k + 1$. By the principle of induction, (7.4) is therefore true for all positive integers.

2. We first rewrite (7.5) using a simple substitution. Let $x = n - r$ and $y = r$. Then $n = x + y$ and (7.5) becomes

$$F_{x+y}^2 - F_xF_{x+2y} = (-1)^x F_y^2. \tag{C.4}$$

We write

$$\begin{aligned}
F_{x+y}^2 - F_xF_{x+2y} &= \left(F_{x-1}F_y + F_xF_{y+1}\right)^2 - \left(F_{x-1}F_{2y} + F_xF_{2y+1}\right) F_x && \text{(from (6.4))}\\
&= F_{x-1}^2F_y^2 + 2F_{x-1}F_xF_yF_{y+1} + F_x^2F_{y+1}^2 \\
&\qquad - F_{x-1}F_x\left(F_{y-1}F_y + F_yF_{y+1}\right) - F_x^2\left(F_y^2 + F_{y+1}^2\right) && \text{(from (6.4))}\\
&= F_{x-1}F_xF_y\left(F_{y+1} - F_{y-1}\right) + F_y^2\left(F_{x-1}^2 - F_x^2\right) \\
&= F_y^2\left(F_{x-1}\left(F_{x-1} + F_x\right) - F_x^2\right) && \text{(from recursion relation)}\\
&= F_y^2\left(F_{x-1}F_{x+1} - F_x^2\right) && \text{(from recursion relation)}\\
&= (-1)^x F_y^2, && \text{(from (7.4))}
\end{aligned}$$

which proves (C.4), and hence (7.5).

Solutions to the Problems for Lecture 9

1. We prove (9.2) by mathematical induction.

Base case: When $n = 1$ the left side of (9.2) is $F_1 = 1$ and the right side is $F_3 - 1 = 1$, so (9.2) is true for $n = 1$.

Induction step: Suppose (9.2) is true for $n = k$. Then

$$\begin{aligned}
\sum_{i=1}^{k+1} F_i &= \sum_{i=1}^{k} F_i + F_{k+1} \\
&= F_{k+2} - 1 + F_{k+1} && \text{(from induction hypothesis)} \\
&= (F_{k+1} + F_{k+2}) - 1 \\
&= F_{k+3} - 1, && \text{(from recursion relation)}
\end{aligned}$$

so (9.2) is true for $n = k + 1$. By the principle of induction, (9.2) is therefore true for all positive integers.

2. We use the relation $L_n = L_{n+2} - L_{n+1}$. Constructing a list of identities, we obtain

$$\begin{aligned}
L_n &= L_{n+2} - L_{n+1} \\
L_{n-1} &= L_{n+1} - L_n \\
L_{n-2} &= L_n - L_{n-1} \\
&\vdots \quad \vdots \\
L_2 &= L_4 - L_3 \\
L_1 &= L_3 - L_2.
\end{aligned}$$

Adding all the left hand sides yields the sum over the first n Lucas numbers, and adding all the right-hand-sides results in the cancellation of all terms except the first and the last. Using $L_2 = 3$ results in (9.3).

3. Here, we use the relation $F_{n+1} = F_{n+2} - F_n$. The first list of identities is

$$\begin{aligned}
F_{2n-1} &= F_{2n} - F_{2n-2} \\
F_{2n-3} &= F_{2n-2} - F_{2n-4} \\
F_{2n-5} &= F_{2n-4} - F_{2n-6} \\
&\vdots \quad \vdots \\
F_3 &= F_4 - F_2 \\
F_1 &= F_2 - F_0.
\end{aligned}$$

Adding the equations yields $\sum_{i=1}^{n} F_{2i-1} = F_{2n} - F_0$, and since $F_0 = 0$ the result for odd Fibonacci numbers is obtained.

The second list of identities is

$$\begin{aligned}
F_{2n} &= F_{2n+1} - F_{2n-1} \\
F_{2n-2} &= F_{2n-1} - F_{2n-3} \\
F_{2n-4} &= F_{2n-3} - F_{2n-5} \\
\vdots &\quad \vdots \\
F_4 &= F_5 - F_3 \\
F_2 &= F_3 - F_1.
\end{aligned}$$

Adding the equations yields $\sum_{i=1}^{n} F_{2i} = F_{2n+1} - F_1$, and since $F_1 = 1$ the result for even Fibonacci numbers is obtained.

Solutions to the Problems for Lecture 10

1. We prove (10.1) by mathematical induction.

Base case: When $n = 1$ the left side of (10.1) is $F_1^2 = 1$ and the right side is $F_1 F_2 = 1$, so (10.1) is true for $n = 1$.

Induction step: Suppose (10.1) is true for $n = k$. Then

$$\begin{aligned}
\sum_{i=1}^{k+1} F_i^2 &= \sum_{i=1}^{k} F_i^2 + F_{k+1}^2 \\
&= F_k F_{k+1} + F_{k+1}^2 && \text{(from induction hypothesis)} \\
&= F_{k+1}\left(F_k + F_{k+1}\right) \\
&= F_{k+1} F_{k+2}, && \text{(from recursion relation)}
\end{aligned}$$

so (10.1) is true for $n = k + 1$. By the principle of induction, (10.1) is therefore true for all positive integers.

2. Write

$$\begin{aligned}
L_n L_{n+1} &= L_n(L_n + L_{n-1}) \\
&= L_n^2 + L_{n-1} L_n \\
&= L_n^2 + L_{n-1}(L_{n-1} + L_{n-2}) \\
&= L_n^2 + L_{n-1}^2 + L_{n-2} L_{n-1} \\
&= \dots \\
&= L_n^2 + L_{n-1}^2 + \dots + L_2^2 + L_1 L_2
\end{aligned}$$

Because $L_1 = 1$ and $L_2 = 3$, we have $L_1 L_2 = L_1^2 + 2$, and bringing the two to the left-hand-side proves the identity (10.2).

Solutions to the Problems for Lecture 12

1. Let $x = \sum_{i=0}^{\infty} \phi^{2i}$. Then

$$\begin{aligned} x &= 1 + \phi^2 + \phi^4 + \phi^6 + \dots, \\ \phi^2 x &= \phantom{1 + {}} \phi^2 + \phi^4 + \phi^6 + \dots. \end{aligned}$$

Subtracting equations, one obtains $(1 - \phi^2)x = 1$, or

$$\begin{aligned} x &= \frac{1}{1-\phi^2} \\ &= \frac{\Phi^2}{\Phi^2 - 1} && (\text{from } \phi = 1/\Phi) \\ &= \frac{\Phi^2}{\Phi} && (\text{from } \Phi^2 - \Phi - 1 = 0) \\ &= \Phi. \end{aligned}$$

Solutions to the Problems for Lecture 13

1. We first obtain the equations for the two diagonal lines. Recall that $\Phi = 1 + \phi = 1/\phi$. With the origin of the coordinate system at the lower left-hand side of the largest golden rectangle, the longer diagonal passes through the boundary points $(0, 1)$ and $(\Phi, 0)$, and the shorter diagonal passes through the boundary points $(1, 0)$ and $(\Phi, 1)$. The two diagonal lines can then be determined to be

$$y = -\phi x + 1 \quad \text{(largest diagonal)}, \tag{C.5}$$
$$y = \Phi x - \Phi \quad \text{(smallest diagonal)}. \tag{C.6}$$

Now, the figure of the spiralling squares is self-similar, and the first unrotated copy of the whole can be seen to have the attached square of side length ϕ^4. If we can show that the drawn diagonal lines pass through the same boundary points of the reduced-size copy, then these two lines continue to pass through all smaller copies and must eventually intersect at the accumulation point of all the spiralling squares.

The longer diagonal of the reduced-size copy must pass through its boundary points $(1, \phi^2)$ and $(1 + \phi^3, \phi^3)$, and we need to show that these points satisfy (C.5). We will need to make use of the following relationship proved earlier:

$$\phi^2 = -\phi + 1.$$

We proceed by substituting in the x values into the equation for the diagonal line to show that we obtain the correct y values. For $x = 1$, $(y = \phi^2)$, we have

$$\begin{aligned} y &= -\phi + 1, \\ &= \phi^2, \end{aligned}$$

and for $x = 1 + \phi^3$, $(y = \phi^3)$, we have

$$\begin{aligned} y &= -\phi(1 + \phi^3) + 1 \\ &= 1 - \phi - \phi^4 \\ &= 1 - \phi - (1 - \phi)^2 \\ &= 1 - \phi - 1 + 2\phi - \phi^2 \\ &= \phi(1 - \phi) \\ &= \phi^3. \end{aligned}$$

Similarly, the shorter diagonal of the reduced-size copy must pass through its boundary points $(1 + \phi^3, \phi^2)$ and $(1 + \phi^4, \phi^3)$, and we need to show that these points satisfy (C.6). For $1 + \phi^3$, $(y = \phi^2)$, we have

$$\begin{aligned} y &= \Phi(1 + \phi^3) - \Phi, \\ &= \phi^2, \end{aligned}$$

and for $x = 1 + \phi^4$, $(y = \phi^3)$, we have

$$\begin{aligned} y &= \Phi(1 + \phi^4) - \Phi \\ &= \phi^3, \end{aligned}$$

completing our proof.

The accumation point of the squares is found from the intesection of (C.5) and (C.6). Equating the values of y gives us the equation

$$-\phi x + 1 = \Phi x - \Phi,$$

whose explicit solution can be found to be $x = (5 + 3\sqrt{5})/10$. The value of y can now be found from (C.5) and is given by $y = (5 - \sqrt{5})/10$. The approximate numerical values for the coordinates are $(x, y) \approx (1.1708, 0.2764)$.

Solutions to the Problems for Lecture 14

1. We have already found that the coordinates of one of the centers of a golden spiral is given by

$$(x, y) = \left(\frac{5+3\sqrt{5}}{10}, \frac{5-\sqrt{5}}{10} \right). \tag{C.7}$$

Note that the origin of the largest golden rectangle is taken to be at the bottom-left corner.

The centers of the four possible golden spirals are symmetric about the mid-point of the largest golden rectangle, the midpoint having coordinates $(\Phi/2, 1/2)$. The four vertices can then be determined from (C.7) to have coordinates

$$\left(\frac{\Phi}{2} + \frac{5+\sqrt{5}}{20}, \frac{1}{2} - \frac{\sqrt{5}}{10} \right), \left(\frac{\Phi}{2} - \frac{5+\sqrt{5}}{20}, \frac{1}{2} - \frac{\sqrt{5}}{10} \right),$$
$$\left(\frac{\Phi}{2} - \frac{5+\sqrt{5}}{20}, \frac{1}{2} + \frac{\sqrt{5}}{10} \right), \left(\frac{\Phi}{2} + \frac{5+\sqrt{5}}{20}, \frac{1}{2} + \frac{\sqrt{5}}{10} \right).$$

The length of the two sides of the rectangle can then be calculated to be

$$L = \frac{1+\sqrt{5}}{2\sqrt{5}}, \quad W = \frac{1}{\sqrt{5}},$$

which is just a golden rectangle reduced in dimensions by the factor of $\sqrt{5}$.

Solutions to the Problems for Lecture 17

1. We have

$$\begin{aligned}\sqrt{2} &= 1 + (\sqrt{2} - 1)\\ &= 1 + \frac{1}{1+\sqrt{2}},\end{aligned}$$

which is a recursive definition that can be iterated as follows:

$$\begin{aligned}\sqrt{2} &= 1 + \frac{1}{1+\sqrt{2}}\\ &= 1 + \cfrac{1}{2 + \cfrac{1}{1+\sqrt{2}}}\\ &= 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1+\sqrt{2}}}},\end{aligned}$$

and so on, so that $\sqrt{2} = [1; \overline{2}]$.

2. We have

$$\begin{aligned}\sqrt{3} &= 1 + (\sqrt{3} - 1)\\ &= 1 + \frac{2}{1+\sqrt{3}},\end{aligned}$$

which is a recursive definition that can be iterated as follows:

$$\begin{aligned}\sqrt{3} &= 1 + \frac{2}{1+\sqrt{3}}\\ &= 1 + \cfrac{2}{2 + \cfrac{2}{1+\sqrt{3}}}\\ &= 1 + \cfrac{1}{1 + \cfrac{1}{1+\sqrt{3}}}\\ &= 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{1+\sqrt{3}}}},\end{aligned}$$

and so on, so that $\sqrt{3} = [1; \overline{1, 2}]$.

3. We have

$$
\begin{aligned}
e &= 2 + 0.718281\ldots \\
&= 2 + \cfrac{1}{1.392211\ldots} \\
&= 2 + \cfrac{1}{1 + \cfrac{1}{2.549646\ldots}} \\
&= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1.819350\ldots}}} \\
&= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1.220479\ldots}}}} \\
&= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4.535573\ldots}}}}},
\end{aligned}
$$

giving us the beginning of the expansion $e = [2; 1, 2, 1, 1, 4, \ldots]$. Remarkably, this expansion continues in a regular fashion as

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots]$$

and is sometimes called Euler's continued fraction.

4. We have

$$\Phi_0 = [1;] = 1 = \frac{1}{1} = \frac{F_2}{F_1},$$

$$\Phi_1 = [1;1] = 1 + \frac{1}{1} = 2 = \frac{2}{1} = \frac{F_3}{F_2},$$

$$\Phi_2 = [1;1,1] = 1 + \cfrac{1}{1 + \cfrac{1}{1}} = \frac{3}{2} = \frac{F_4}{F_3},$$

$$\Phi_3 = [1;1,1,1] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}} = \frac{5}{3} = \frac{F_5}{F_4}.$$

5. We prove (17.2) by mathematical induction.

Base case: Our previous calculation already shows that (17.2) is true for $n = 0, 1, 2$, and 3.

Induction step: Suppose that (17.2) is true for positive integers $n = k$. Then we write

$$\begin{aligned}
\frac{F_{k+3}}{F_{k+2}} &= \frac{F_{k+1} + F_{k+2}}{F_{k+2}} && \text{(from recursion relation)} \\
&= 1 + \frac{F_{k+1}}{F_{k+2}} \\
&= 1 + \frac{1}{\Phi_k} && \text{(from induction hypothesis)} \\
&= \Phi_{k+1} && \text{(from inspection of the continued fraction)}
\end{aligned}$$

so that (17.2) is true for $n = k + 1$. By the principle of induction, (17.2) is therefore true for all non-negative integers.

Solutions to the Problems for Lecture 18

1. We have

$$g_0/2\pi = [0;] = 0,$$

$$g_1/2\pi = [0;2] = \frac{1}{2} = \frac{F_1}{F_3},$$

$$g_2/2\pi = [0;2,1] = \cfrac{1}{2+\cfrac{1}{1}} = \frac{1}{3} = \frac{F_2}{F_4},$$

$$g_3/2\pi = [0;2,1,1] = \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1}}} = \frac{2}{5} = \frac{F_3}{F_5}.$$

2. We prove (18.1) by mathematical induction.

Base case: Our previous calculation already shows that (18.1) is true for $n = 1$, 2, and 3.

Induction step: Suppose that (18.1) is true for positive integers $n = k$. Then we write

$$\begin{aligned}
\frac{F_{k+1}}{F_{k+3}} &= \frac{F_{k+1}}{F_{k+1}+F_{k+2}} && \text{(from recursion relation)}\\
&= \cfrac{1}{1+\cfrac{F_{k+2}}{F_{k+1}}}\\
&= \frac{1}{1+\Phi_k} && \text{(from } \Phi_k = F_{k+2}/F_{k+1}\text{)}\\
&= \frac{g_{k+1}}{2\pi} && \text{(from inspection of the continued fraction)}
\end{aligned}$$

so that (18.1) is true for $n = k+1$. By the principle of induction, (18.1) is therefore true for all positive integers.