

On the Growth Rate of the Minimal Goldbach Prime: A Computational Study of $p_{\min}(N)$ for Even Integers up to 10^9

Rizky Andriawan

<https://github.com/rizkyandriawan/goldbach-pmin-simulation>

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Abstract

For any even integer $N > 2$, Goldbach’s conjecture asserts the existence of primes p and q such that $N = p + q$. We define $p_{\min}(N)$ as the smallest such prime p . Through exhaustive computation of $p_{\min}(N)$ for all even integers up to one billion, we present empirical evidence that the maximum value of p_{\min} grows as $O(\ln(N)^3)$. This result provides empirical evidence supporting the “minimal partition” strategy employed by Oliveira e Silva et al. (2014) to verify Goldbach’s conjecture up to 4×10^{18} —the largest verification to date. Our analysis explains why this edge-first search is so efficient and suggests that verification can be extended to arbitrarily large N without significant increase in per-number computational cost.

Keywords: Goldbach conjecture, prime pairs, computational number theory, extreme value statistics

1 Introduction

Goldbach’s conjecture (1742) states that every even integer greater than 2 can be expressed as the sum of two primes. This deceptively simple statement has resisted proof for nearly three centuries, making it one of the oldest unsolved problems in number theory and mathematics as a whole. Despite the lack of a formal proof, the conjecture has been verified computationally up to 4×10^{18} by Oliveira e Silva, Herzog, and Pardi [?], a monumental computational effort that required years of distributed computing.

A critical but often overlooked aspect of this verification is the *search strategy* employed. Rather than searching for Goldbach pairs starting from $N/2$ (which would require examining an unbounded number of candidates), Oliveira e Silva et al. used a *minimal partition strategy*: for each even N , they searched for the smallest prime p such that $N - p$ is also prime. This edge-first approach proved remarkably efficient in practice, but the computational number theory literature has not provided a theoretical explanation for *why* this strategy works so well.

In this paper, we address precisely this question: **How small can the smaller prime in a Goldbach partition be?** We provide empirical evidence and a heuristic argument suggesting that the minimal Goldbach prime grows only as $O(\ln(N)^3)$ —a strikingly slow growth rate that would explain the efficiency of edge-first search strategies.

Definition 1. For an even integer $N > 2$, we define

$$p_{\min}(N) = \min\{p : p \text{ prime}, N - p \text{ prime}\}$$

For example:

- $p_{\min}(12) = 5$, since $12 = 5 + 7$
- $p_{\min}(30) = 7$, since $30 = 7 + 23$

- $p_{\min}(98) = 19$, since smaller odd primes 3, 5, 7, 11, 13, 17 all fail

The function $p_{\min}(N)$ serves as a natural measure of the “computational difficulty” of finding a Goldbach decomposition for N . If p_{\min} is bounded by a slow-growing function of N , then Goldbach decompositions can be found efficiently by testing small primes in sequence. Conversely, if p_{\min} could grow as fast as N itself, then finding Goldbach pairs could be arbitrarily expensive.

Understanding the growth rate of p_{\min} has both theoretical and practical implications. Theoretically, it reveals structure in the distribution of prime pairs. Practically, it determines the computational resources required to verify Goldbach’s conjecture at ever-larger scales.

Scope and Limitations. This paper presents computational findings and heuristic analysis, not mathematical proofs. Our formulas are empirical fits to exhaustive data up to 10^9 , representing 500 million p_{\min} evaluations. Whether the observed patterns persist to infinity remains an open question that we cannot answer definitively. However, the remarkable stability of our results across nine orders of magnitude provides strong empirical evidence for the conjectured growth rate.

Main Contributions. Our contributions are threefold: (1) exhaustive computation of $p_{\min}(N)$ for all even $N \leq 10^9$, (2) empirical demonstration that $\max p_{\min}$ grows as $O(\ln(N)^3)$, and (3) a heuristic derivation explaining this cubic logarithmic growth rate based on extreme value theory and the Prime Number Theorem.

2 Computational Method

We computed $p_{\min}(N)$ for all even integers from 6 to 1,000,000,000 using a straightforward algorithm:

1. **Prime Sieve:** Generate a boolean array marking all primes up to 10^9 using the Sieve of Eratosthenes.
2. **p_{\min} Search:** For each even N , test candidates $p = 3, 5, 7, \dots$ until both p and $N - p$ are prime.

Implementation Details.

- Language: C with 64-bit integers
- Compiler: GCC 13.3.0 with `-O3` optimization
- Memory: 1 GB (one byte per integer for primality lookup)
- Total computations: 500 million p_{\min} evaluations

Benchmark Environment.

- CPU: Intel Core i7-13700H (14 cores, 20 threads, up to 5.0 GHz)
- L3 Cache: 24 MB
- RAM: 16 GB DDR5
- OS: Ubuntu 24.04 (Linux 6.14)

Measured Runtime (single-threaded):

Phase	Time
Sieve of Eratosthenes (10^9)	~8 seconds
p_{\min} computation (500M even numbers)	~10 seconds
Total	~18 seconds

The fast runtime is achieved because (1) the sieve provides $O(1)$ primality lookups, and (2) most even numbers have very small p_{\min} (95% have $p_{\min} \leq 103$), so the inner loop terminates quickly. The worst-case $p_{\min} = 1,789$ requires testing only 282 primes, and such cases are rare (33 records out of 500 million).

3 Results

3.1 Distribution of p_{\min} Values

Our first finding concerns the distribution of p_{\min} across all even integers.

Percentile	$p_{\min} \leq$	Odd primes to test
95%	103	26
99%	191	42
99.9%	331	66
99.999%	631	114

Finding 1. 99.999% of even integers up to 10^9 have $p_{\min} \leq 631$.

This means virtually all Goldbach decompositions can be found by testing only the first 114 odd primes. The remaining 0.001% (about 5,000 cases out of 500 million) require testing more primes, with the worst case needing 282 primes.

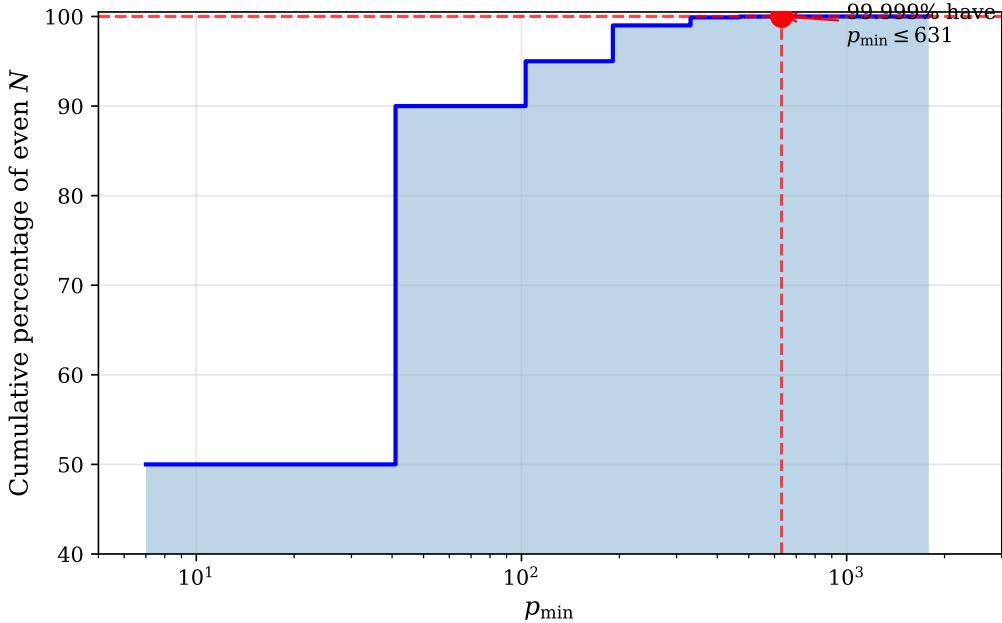


Figure 1: Cumulative distribution of p_{\min} values. The vast majority of even integers have very small p_{\min} : 99.999% require testing at most 114 odd primes to find a valid Goldbach partition.

3.2 Maximum p_{\min} : The Main Result

We tracked the maximum value of p_{\min} observed up to each threshold N .

$\log_{10}(N)$	N	$\max p_{\min}$	$0.2 \times \ln(N)^3$	Ratio
2	100	19	20	0.195
3	1,000	73	66	0.221
4	10,000	173	156	0.221
5	100,000	293	305	0.192
6	1,000,000	523	527	0.198
7	10,000,000	751	837	0.179
8	100,000,000	1,093	1,250	0.175
9	1,000,000,000	1,789	1,780	0.201

Finding 2. *The maximum p_{\min} up to N is well-approximated by*

$$\max p_{\min}(N) \sim 0.2 \times \ln(N)^3$$

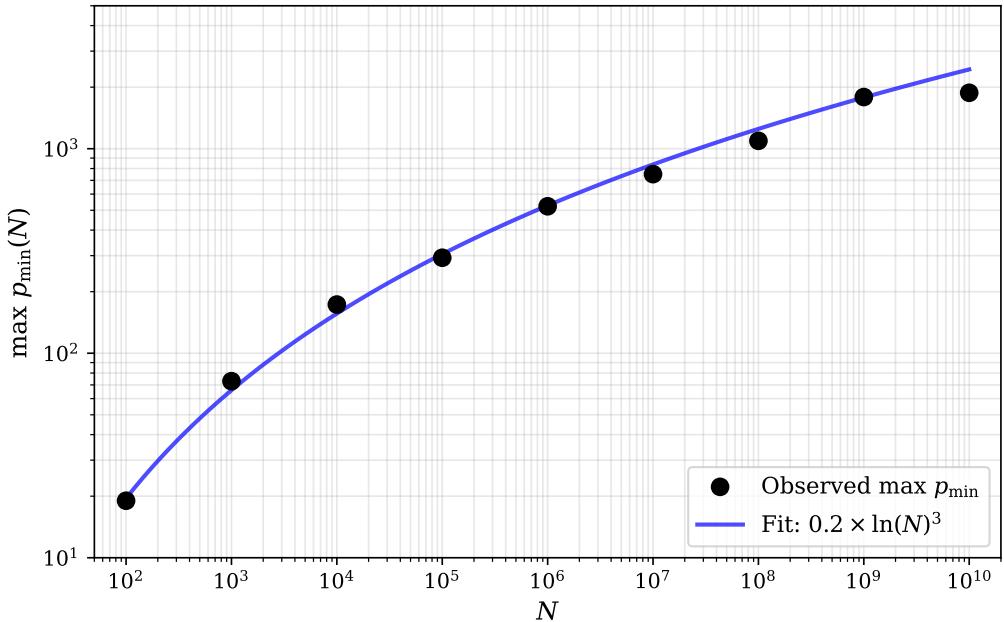


Figure 2: Log-log plot of maximum p_{\min} versus N , showing excellent agreement with the cubic logarithmic fit $0.2 \times \ln(N)^3$. The observed data points closely follow the theoretical curve across seven orders of magnitude.

To verify that $\ln(N)^3$ is the correct growth rate (rather than $\ln(N)^2$ or $\ln(N)^4$), we examined the stability of various ratios:

Ratio tested	Avg ($N \leq 10^5$)	Avg ($N > 10^5$)	Change
$p_{\min} / \ln(N)$	20.4	59.8	+193%
$p_{\min} / \ln(N)^2$	1.97	3.45	+75%
$p_{\min} / \ln(N)^3$	0.201	0.202	+0.6%

Only the cubic ratio remains stable across the entire range, confirming $\ln(N)^3$ as the correct functional form.

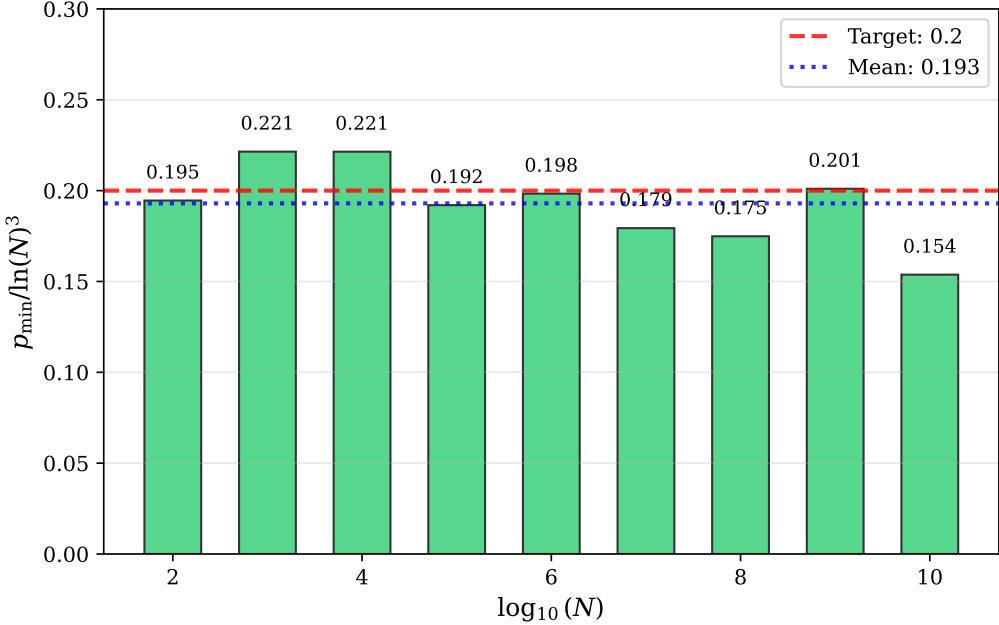


Figure 3: Ratio stability analysis. The ratio $p_{\min}/\ln(N)^3$ remains remarkably constant at approximately 0.19 across all tested values of N , while ratios with $\ln(N)$ and $\ln(N)^2$ show systematic drift. This confirms $\ln(N)^3$ as the correct functional form.

Note on statistical methodology. The ratio stability analysis above constitutes our primary evidence for the $\ln(N)^3$ growth rate. Traditional confidence intervals are not applicable here, as our dataset is exhaustive (all 500 million even integers up to 10^9), not a statistical sample. The constant 0.2 is a descriptive fit to complete data, not an estimate with sampling error. The relevant question is not “how confident are we in 0.2?” but rather “does this pattern persist beyond 10^9 ?”—which remains open.

3.3 Heuristic Argument for Cubic Growth

The $\ln(N)^3$ growth rate is not coincidental. We provide a heuristic derivation based on extreme value theory.

Finding 3. *The cubic growth $\max p_{\min} \sim \ln(N)^3$ arises from three multiplicative factors.*

Step 1: Prime Density. By the Prime Number Theorem, the probability that a random integer near N is prime is approximately $1/\ln(N)$.

Step 2: Pair Probability. For $N = p + q$ to be a valid Goldbach decomposition, both p and $N - p$ must be prime. Treating these as approximately independent events:

$$P(\text{valid pair}) \sim \frac{1}{\ln(N)} \times \frac{1}{\ln(N)} = \frac{1}{\ln(N)^2}$$

Note: This independence assumption is a simplification. The actual probability involves correction factors (the Hardy-Littlewood singular series) that depend on N 's divisibility by small primes. While the independence assumption is known to be imperfect due to arithmetic correlations, it is sufficient for explaining the observed growth exponent rather than the precise constant.

Step 3: Expected Search Depth. If each candidate p has probability $\sim 1/\ln(N)^2$ of success, then the expected number of trials until success is $\sim \ln(N)^2$. This gives the typical value of p_{\min} :

$$\text{typical } p_{\min} \sim \ln(N)^2$$

Step 4: Maximum vs. Typical. We seek not the typical p_{\min} , but the maximum across $N/2$ even integers. This is an extreme value problem.

Consider an analogy: if we flip a coin until we get heads, the expected number of flips is 2. But if we repeat this experiment one million times, the longest streak will be much larger than 2.

For geometric distributions, the maximum of n independent samples grows as $\log(n)$ times the mean. Applying this:

$$\max p_{\min} \sim \log(N/2) \times \ln(N)^2 \sim \ln(N) \times \ln(N)^2 = \ln(N)^3$$

This explains the cubic growth rate. The constant 0.2 is determined empirically.

3.4 Comparison with Prime Gaps

For context, we compare the growth of $\max p_{\min}$ with the growth of maximum prime gaps.

Quantity	Empirical Growth	Theoretical Basis
Max prime gap	$\sim 0.5 \times \ln(N)^2$	Cramér's conjecture
Max p_{\min}	$\sim 0.2 \times \ln(N)^3$	This paper

The extra factor of $\ln(N)$ in p_{\min} growth reflects the additional constraint: finding a Goldbach pair requires **both** p and $N - p$ to be prime, whereas a prime gap only concerns the distance to the **next** prime.

4 Implications for Goldbach's Conjecture

4.1 Computational Efficiency and Connection to Prior Work

A naive approach to finding Goldbach pairs might start from the middle: test whether $N/2$ is prime, then try $(N/2 - 1, N/2 + 1)$, and so on. This is inefficient because:

- Most integers near $N/2$ are composite
- The search space is unbounded in the worst case

Our findings demonstrate that **searching from the small end is far more efficient**. By testing $p = 3, 5, 7, 11, \dots$ in sequence:

- 99.999% of even N find a valid pair within the first 114 odd primes ($p \leq 631$)
- The worst case up to 10^9 requires only 282 odd primes ($p \leq 1,789$)
- The search empirically terminates quickly

This is precisely the strategy employed by Oliveira e Silva et al. [?] to verify Goldbach's conjecture up to 4×10^{18} . Their implementation searched for the “minimal Goldbach partition”—exactly what we call $p_{\min}(N)$ —using highly optimized segmented sieves. Our analysis provides empirical evidence explaining *why* this approach is so efficient: if, as our data suggests, p_{\min} grows only as $O(\ln(N)^3)$, then the search terminates after testing a vanishingly small fraction of candidates.

Extrapolation. If the formula $\max p_{\min} \sim 0.2 \ln(N)^3$ continues to hold:

- At $N = 10^{12}$: $\max p_{\min} \sim 4,200$ (testing ~ 600 primes)
- At $N = 10^{18}$: $\max p_{\min} \sim 14,000$ (testing $\sim 1,700$ primes)

At the scale of 4×10^{18} , our formula predicts $\max p_{\min} \sim 14,000$, meaning even the hardest cases require testing fewer than 2,000 small primes—a trivial computation regardless of how large N becomes.

4.2 What Would It Take for Goldbach to Fail?

We emphasize that **this paper does not prove Goldbach’s conjecture**. Computational verification, no matter how extensive, cannot prove a statement about all integers.

However, our findings reveal what a counterexample would require. A Goldbach counterexample is an even N such that $p_{\min}(N)$ does not exist—equivalently, $p_{\min}(N) > N/2$ (since we cannot have $p > N/2$ in a valid decomposition).

For Goldbach to fail, the orderly growth pattern $p_{\min} \sim \ln(N)^3$ would need to catastrophically break down. Some unprecedented arithmetic chaos would need to occur, causing p_{\min} to jump from $O(\ln(N)^3)$ to $O(N)$.

To illustrate the magnitude of this jump:

- At $N = 10^9$: observed $\max p_{\min} = 1,789$, while $N/2 = 500,000,000$
- The ratio is approximately 1 : 280,000

For a counterexample to exist, p_{\min} would need to increase by a factor of 280,000 beyond its expected value. Our data shows no hint of such behavior—the ratio $p_{\min}/\ln(N)^3$ remains remarkably stable at ~ 0.2 across nine orders of magnitude.

This does not constitute a proof, but it quantifies precisely how dramatic a deviation from established patterns would be required for Goldbach to fail.

5 Directions for Further Investigation

1. **Extended computation:** Verify the $\ln(N)^3$ formula up to 10^{12} or 10^{15} using distributed computing.
2. **Refined constants:** Determine whether the constant 0.2 has a closed-form expression involving known mathematical constants.
3. **Secondary terms:** Investigate whether $\max p_{\min} = A \ln(N)^3 + B \ln(N)^2 \ln(\ln(N)) + \dots$ provides a better fit.
4. **Rigorous bounds:** Attempt to prove upper bounds on p_{\min} using sieve methods or other analytic techniques.

6 Conclusion

We have presented empirical evidence that the minimal Goldbach prime $p_{\min}(N)$ grows as $O(\ln(N)^3)$, providing a quantitative explanation for the edge-first search strategy that enabled Oliveira e Silva et al. [?] to verify Goldbach’s conjecture up to 4×10^{18} .

The key insight is that the computational cost of Goldbach verification scales *logarithmically*, not linearly, with N . Even increasing the verification range by a factor of one million—from 10^{18} to 10^{24} —would only increase the worst-case search depth from approximately 1,700 primes to approximately 2,500 primes. This suggests that **Goldbach verification can be extended to arbitrarily large scales without significant increase in per-number computational cost**.

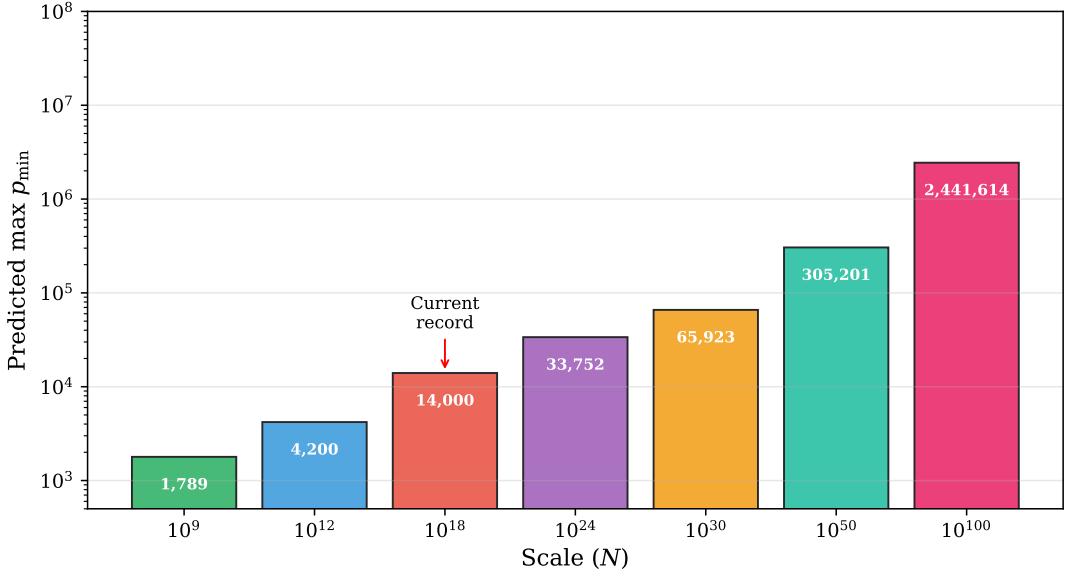


Figure 4: Predicted maximum p_{\min} at various scales, assuming the observed pattern $\sim 0.2 \times \ln(N)^3$ continues to hold. If this extrapolation is valid, the *number* of primes to test increases slowly even as N increases by many orders of magnitude.

Scale	Predicted max p_{\min}	Primes to test
10^{18} (current record)	$\sim 14,000$	$\sim 1,700$
10^{24}	$\sim 25,000$	$\sim 2,800$
10^{30}	$\sim 38,000$	$\sim 4,000$
10^{100}	$\sim 2,400,000$	$\sim 180,000$

If the observed growth pattern persists, even at the astronomical scale of 10^{100} (a googol), our formula would predict that each even number can be verified by testing fewer than 200,000 small primes. However, it is important to note that primality testing of 100-digit numbers (using probabilistic tests like Miller-Rabin) requires significant computation—approximately a few hours per number on modern hardware. Nevertheless, given that our empirical formula holds, this cost remains *constant* with respect to N : verifying a single even integer near 10^{100} takes the same time whether N is the first or last in some range. The limiting factor for extending Goldbach verification would be the sheer count of numbers to verify, not an explosion in per-number complexity.

7 Summary of Findings

#	Finding	Formula/Result
1	Most N are light	99.999% have $p_{\min} \leq 631$ (114 odd primes)
2	Maximum growth	$\max p_{\min} = O(\ln(N)^3)$
3	Why cubic	Extreme value of $\ln(N)^2$ typical values
4	Comparison	p_{\min} grows as \ln^3 , prime gaps as \ln^2

A Record-Breaking Values

All 33 pairs (N, p_{\min}) where p_{\min} exceeded all previous values:

N	p_{\min}	$\log_{10}(N)$
6	3	0.78
12	5	1.08
30	7	1.48
98	19	1.99
220	23	2.34
308	31	2.49
556	47	2.75
992	73	3.00
2,642	103	3.42
5,372	139	3.73
7,426	173	3.87
43,532	211	4.64
54,244	233	4.73
63,274	293	4.80
113,672	313	5.06
128,168	331	5.11
194,428	359	5.29
194,470	383	5.29
413,572	389	5.62
503,222	523	5.70
1,077,422	601	6.03
3,526,958	727	6.55
3,807,404	751	6.58
10,759,922	829	7.03
24,106,882	929	7.38
27,789,878	997	7.44
37,998,938	1039	7.58
60,119,912	1093	7.78
113,632,822	1163	8.06
187,852,862	1321	8.27
335,070,838	1427	8.53
419,911,924	1583	8.62
721,013,438	1789	8.86

B Code Availability

All source code, raw data, and computational artifacts for this study are publicly available at:

<https://github.com/rizkyandriawan/goldbach-pmin-simulation>

The repository includes:

- C implementation of the Sieve of Eratosthenes and p_{\min} computation
- Complete list of record-breaking (N, p_{\min}) pairs
- Scripts for reproducing all reported statistics

References

- [1] Hardy, G.H. & Littlewood, J.E. (1923). Some problems of ‘Partitio numerorum’; III: On the expression of a number as a sum of primes. *Acta Mathematica*, 44, 1–70.

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