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## On the Growth Rate of the Minimal Goldbach Prime

### A Computational Study of $p_{\min}(N)$ for Even Integers up to $10^9$

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#### Abstract

For any even integer  $N > 2$ , Goldbach's conjecture asserts the existence of primes  $p$  and  $q$  such that  $N = p + q$ . We define  $p_{\min}(N)$  as the smallest such prime  $p$ . Through exhaustive computation of  $p_{\min}(N)$  for all even integers up to one billion, we present empirical evidence that the maximum value of  $p_{\min}$  grows as  $O(\ln(N)^3)$ . We provide a heuristic derivation of this cubic logarithmic growth using extreme value theory and the Prime Number Theorem.

**Keywords:** Goldbach conjecture, prime pairs, computational number theory, extreme value statistics

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#### 1. Introduction

Goldbach's conjecture (1742) states that every even integer greater than 2 can be expressed as the sum of two primes. Despite nearly three centuries of effort, the conjecture remains unproven, though it has been verified computationally up to  $4 \times 10^{18}$  (Oliveira e Silva et al., 2014).

Rather than asking whether such a decomposition exists, we investigate a quantitative question: **How small can the smaller prime be?**

**Definition 1.1.** For an even integer  $N > 2$ , we define

$$p_{\min}(N) = \min\{ p : p \text{ prime}, N - p \text{ prime} \}$$

For example: -  $p_{\min}(12) = 5$ , since  $12 = 5 + 7$  -  $p_{\min}(30) = 7$ , since  $30 = 7 + 23$  -  $p_{\min}(98) = 19$ , since smaller odd primes 3, 5, 7, 11, 13, 17 all fail

The function  $\text{pmin}(N)$  measures the “difficulty” of finding a Goldbach decomposition. If  $\text{pmin}$  is bounded by a slow-growing function of  $N$ , then Goldbach decompositions are computationally inexpensive to find.

**Scope and Limitations.** This paper presents computational findings, not mathematical proofs. Our formulas are empirical fits to data up to  $10^9$ . Whether these patterns persist to infinity remains an open question.

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## 2. Computational Method

We computed  $\text{pmin}(N)$  for all even integers from 6 to 1,000,000,000 using a straightforward algorithm:

1. **Prime Sieve:** Generate a boolean array marking all primes up to  $10^9$  using the Sieve of Eratosthenes.
2. **pmin Search:** For each even  $N$ , test candidates  $p = 3, 5, 7, \dots$  until both  $p$  and  $N - p$  are prime.

**Implementation Details:** - Language: C with 64-bit integers - Compiler: GCC 13.3.0 with `-O3` optimization - Memory: 1 GB (one byte per integer for primality lookup) - Total computations: 500 million  $\text{pmin}$  evaluations

**Benchmark Environment:** - CPU: Intel Core i7-13700H (14 cores, 20 threads, up to 5.0 GHz) - L3 Cache: 24 MB - RAM: 16 GB DDR5 - OS: Ubuntu 24.04 (Linux 6.14)

**Measured Runtime (single-threaded):** | Phase | Time | | Sieve of Eratosthenes ( $10^9$ ) | ~8 seconds | |  $\text{pmin}$  computation (500M even numbers) | ~10 seconds | | **Total** | **~18 seconds** |

Note: The fast runtime is achieved because (1) the sieve provides  $O(1)$  primality lookup, and (2) most even numbers have very small  $\text{pmin}$  (95% have  $\text{pmin} \leq 103$ ), so the inner loop terminates quickly. The worst-case  $\text{pmin} = 1,789$  requires testing only 282 primes, and such cases are rare (33 records out of 500 million).

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## 3. Results

### 3.1 Distribution of $\text{pmin}$ Values

Our first finding concerns the distribution of  $\text{pmin}$  across all even integers.

Percentile	$\text{pmin} \leq$	Odd primes to test
95%	103	26
99%	191	42
99.9%	331	66
99.999%	631	114

**Finding 1.** 99.999% of even integers up to  $10^9$  have  $\text{pmin} \leq 631$ .

This means virtually all Goldbach decompositions can be found by testing only the first 114 odd primes. The remaining 0.001% (about 5,000 cases out of 500 million) require testing more primes, with the worst case needing 282 primes.

### 3.2 Maximum pmin: The Main Result

We tracked the maximum value of pmin observed up to each threshold N.

log10(N)	N	max pmin	$0.2 \times \ln(N)^3$	Ratio
2	100	19	20	0.195
3	1,000	73	66	0.221
4	10,000	173	156	0.221
5	100,000	293	305	0.192
6	1,000,000	523	527	0.198
7	10,000,000	751	837	0.179
8	100,000,000	1,093	1,250	0.175
9	1,000,000,000	1,789	1,780	0.201

**Finding 2.** The maximum pmin up to N is well-approximated by

$$\max \text{pmin}(N) \sim 0.2 \times \ln(N)^3$$

To verify that  $\ln(N)^3$  is the correct growth rate (rather than  $\ln(N)^2$  or  $\ln(N)^4$ ), we examined the stability of various ratios:

Ratio tested	Average ( $N \leq 10^5$ )	Average ( $N > 10^5$ )	Change
pmin / $\ln(N)$	20.4	59.8	+193%
pmin / $\ln(N)^2$	1.97	3.45	+75%
pmin / $\ln(N)^3$	0.201	0.202	+0.6%

Only the cubic ratio remains stable across the entire range, confirming  $\ln(N)^3$  as the correct functional form.

**Note on statistical methodology:** The ratio stability analysis above constitutes our primary evidence for the  $\ln(N)^3$  growth rate. Traditional confidence intervals are not applicable here, as our dataset is exhaustive (all 500 million even integers up to  $10^9$ ), not a statistical sample. The constant 0.2 is a descriptive fit to complete data, not an estimate with sampling error. The relevant question is not “how confident are we in 0.2?” but rather “does this pattern persist beyond  $10^9$ ?”—which remains open.

### 3.3 Theoretical Basis for Cubic Growth

The  $\ln(N)^3$  growth rate is not coincidental. We provide a heuristic derivation based on extreme value theory.

**Finding 3.** The cubic growth  $\max \text{pmin} \sim \ln(N)^3$  arises from three multiplicative factors.

**Step 1: Prime Density.** By the Prime Number Theorem, the probability that a random integer near N is prime is approximately  $1/\ln(N)$ .

**Step 2: Pair Probability.** For  $N = p + q$  to be a valid Goldbach decomposition, both  $p$  and  $N - p$  must be prime. Treating these as approximately independent events:

$$P(\text{valid pair}) \sim (1/\ln(N)) \times (1/\ln(N)) = 1/\ln(N)^2$$

Note: This independence assumption is a simplification. The actual probability involves correction factors (the Hardy-Littlewood singular series) that depend on  $N$ 's divisibility by small primes. However, these corrections affect the constant factor, not the growth exponent.

**Step 3: Expected Search Depth.** If each candidate  $p$  has probability  $\sim 1/\ln(N)^2$  of success, then the expected number of trials until success is  $\sim \ln(N)^2$ . This gives the typical value of  $p_{\min}$ :

$$\text{typical } p_{\min} \sim \ln(N)^2$$

**Step 4: Maximum vs. Typical.** We seek not the typical  $p_{\min}$ , but the maximum across  $N/2$  even integers. This is an extreme value problem.

Consider an analogy: if we flip a coin until we get heads, the expected number of flips is 2. But if we repeat this experiment one million times, the longest streak will be much larger than 2.

For geometric distributions, the maximum of  $n$  independent samples grows as  $\log(n)$  times the mean. Applying this:

$$\max p_{\min} \sim \log(N/2) \times \ln(N)^2 \sim \ln(N) \times \ln(N)^2 = \ln(N)^3$$

This explains the cubic growth rate. The constant 0.2 is determined empirically.

### 3.4 Comparison with Prime Gaps

For context, we compare the growth of  $\max p_{\min}$  with the growth of maximum prime gaps.

Quantity	Empirical Growth	Theoretical Basis
Max prime gap	$\sim 0.5 \times \ln(N)^2$	Cramér's conjecture
Max $p_{\min}$	$\sim 0.2 \times \ln(N)^3$	This paper

The extra factor of  $\ln(N)$  in  $p_{\min}$  growth reflects the additional constraint: finding a Goldbach pair requires **both**  $p$  and  $N-p$  to be prime, whereas a prime gap only concerns the distance to the **next** prime.

## 4. Implications for Goldbach's Conjecture

### 4.1 Computational Efficiency

A naive approach to finding Goldbach pairs might start from the middle: test whether  $N/2$  is prime, then try  $(N/2 - 1, N/2 + 1)$ , and so on. This is inefficient because: - Most integers near  $N/2$  are composite - The search space is unbounded in the worst case

Our findings demonstrate that **searching from the small end is far more efficient**. By testing  $p = 3, 5, 7, 11, \dots$  in sequence: - 99.999% of even  $N$  find a valid pair within the first 114 odd primes ( $p \leq 631$ ) - The worst case up to  $10^9$  requires only 282 odd primes ( $p \leq 1,789$ ) - The search empirically terminates quickly

This transforms Goldbach verification from a potentially expensive search into a bounded, predictable computation. For any  $N$  up to one billion, we are guaranteed to find a valid pair within a short reach of the prime sequence.

**Extrapolation:** If the formula  $\max p_{\min} \sim 0.2 \ln(N)^3$  continues to hold: - At  $N = 10^{12}$ :  $\max p_{\min} \sim 4,200$  (testing  $\sim 600$  primes) - At  $N = 10^{18}$ :  $\max p_{\min} \sim 14,000$  (testing  $\sim 1,700$  primes)

Note: Goldbach's conjecture has been verified up to  $4 \times 10^{18}$  (Oliveira e Silva, 2014). Our formula predicts  $\max p_{\min} \sim 14,000$  at that scale, meaning even the hardest cases would require testing fewer than 2,000 small primes.

## 4.2 What Would It Take for Goldbach to Fail?

We emphasize that **this paper does not prove Goldbach's conjecture**. Computational verification, no matter how extensive, cannot prove a statement about all integers.

However, our findings reveal what a counterexample would require. A Goldbach counterexample is an even  $N$  such that  $p_{\min}(N)$  does not exist—equivalently,  $p_{\min}(N) > N/2$  (since we cannot have  $p > N/2$  in a valid decomposition).

For Goldbach to fail, the orderly growth pattern  $p_{\min} \sim \ln(N)^3$  would need to catastrophically break down. Some unprecedented arithmetic chaos would need to occur, causing  $p_{\min}$  to jump from  $O(\ln(N)^3)$  to  $O(N)$ .

To illustrate the magnitude of this jump: - At  $N = 10^9$ : observed  $\max p_{\min} = 1,789$ , while  $N/2 = 500,000,000$  - The ratio is approximately  $1 : 280,000$

For a counterexample to exist,  $p_{\min}$  would need to increase by a factor of 280,000 beyond its expected value. Our data shows no hint of such behavior—the ratio  $p_{\min}/\ln(N)^3$  remains remarkably stable at  $\sim 0.2$  across nine orders of magnitude.

This does not constitute a proof, but it quantifies precisely how dramatic a deviation from established patterns would be required for Goldbach to fail.

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## 5. Directions for Further Investigation

1. **Extended computation:** Verify the  $\ln(N)^3$  formula up to  $10^{12}$  or  $10^{15}$  using distributed computing.
  2. **Refined constants:** Determine whether the constant 0.2 has a closed-form expression involving known mathematical constants.
  3. **Secondary terms:** Investigate whether  $\max p_{\min} = A \ln(N)^3 + B \ln(N)^2 \ln(\ln(N)) + \dots$  provides a better fit.
  4. **Rigorous bounds:** Attempt to prove upper bounds on  $p_{\min}$  using sieve methods or other analytic techniques.
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## 6. Summary of Findings

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#	Finding	Formula/Result
1	Most N are light	99.999% have $p_{\min} \leq 631$ (114 odd primes)
2	Maximum growth	$\max p_{\min} = O(\ln(N)^3)$
3	Why cubic	Extreme value of $\ln(N)^2$ typical values
4	Comparison	$p_{\min}$ grows as $\ln^3$ , prime gaps as $\ln^2$

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## Appendix: Record-Breaking Values

All 33 pairs (N,  $p_{\min}$ ) where  $p_{\min}$  exceeded all previous values:

N	$p_{\min}$	$\log_{10}(N)$
6	3	0.78
12	5	1.08
30	7	1.48
98	19	1.99
220	23	2.34
308	31	2.49
556	47	2.75
992	73	3.00
2,642	103	3.42
5,372	139	3.73
7,426	173	3.87
43,532	211	4.64
54,244	233	4.73
63,274	293	4.80
113,672	313	5.06
128,168	331	5.11
194,428	359	5.29
194,470	383	5.29
413,572	389	5.62
503,222	523	5.70
1,077,422	601	6.03
3,526,958	727	6.55
3,807,404	751	6.58
10,759,922	829	7.03
24,106,882	929	7.38
27,789,878	997	7.44
37,998,938	1039	7.58
60,119,912	1093	7.78
113,632,822	1163	8.06
187,852,862	1321	8.27
335,070,838	1427	8.53
419,911,924	1583	8.62
721,013,438	1789	8.86

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*Computed December 2025. Source code available at: <https://github.com/rizkyandriawan/goldbach-pmin-simulation>*