On the Growth Rate of the Minimal Goldbach Prime: A Computational Study of $p_{\min}(N)$ for Even Integers up to 10^9

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https://github.com/rizkyandriawan/goldbach-pmin-simulation

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Abstract

For any even integer $N > 2$, Goldbach's conjecture asserts the existence of primes p and q such that $N = p+q$. We define $p_{\min}(N)$ as the smallest such prime p. Through exhaustive computation of $p_{\min}(N)$ for all even integers up to one billion, we present empirical evidence that the maximum value of $p_{\min}$ grows as $O(\ln(N)^3)$. We provide a heuristic derivation of this cubic logarithmic growth using extreme value theory and the Prime Number Theorem.

Keywords: Goldbach conjecture, prime pairs, computational number theory, extreme value statistics

1 Introduction

Goldbach's conjecture (1742) states that every even integer greater than 2 can be expressed as the sum of two primes. Despite nearly three centuries of effort, the conjecture remains unproven, though it has been verified computationally up to 4×10^{18} [2].

Notably, Oliveira e Silva et al. achieved this verification using a *minimal partition strategy*: for each even N, they searched for the smallest prime p such that $N - p$ is also prime. This approach is far more efficient than searching from $N/2$ downward, but no theoretical analysis has explained *why* it works so well.

In this paper, we investigate precisely this question: **How small can the smaller prime be?**

Definition 1. *For an even integer $N > 2$, we define*

$$p_{\min}(N) = \min\{p : p \text{ prime}, N - p \text{ prime}\}$$

For example:

- $p_{\min}(12) = 5$, since $12 = 5 + 7$
- $p_{\min}(30) = 7$, since $30 = 7 + 23$
- $p_{\min}(98) = 19$, since smaller odd primes $3, 5, 7, 11, 13, 17$ all fail

The function $p_{\min}(N)$ measures the "difficulty" of finding a Goldbach decomposition. If $p_{\min}$ is bounded by a slow-growing function of N, then Goldbach decompositions are computationally inexpensive to find.

Scope and Limitations. This paper presents computational findings, not mathematical proofs. Our formulas are empirical fits to data up to 10^9. Whether these patterns persist to infinity remains an open question.

2 Computational Method

We computed $p_{\min}(N)$ for all even integers from 6 to 1,000,000,000 using a straightforward algorithm:

1. **Prime Sieve:** Generate a boolean array marking all primes up to 10^9 using the Sieve of Eratosthenes.
2. $p_{\min}$ **Search:** For each even N, test candidates $p = 3, 5, 7, \ldots$ until both p and $N - p$ are prime.

Implementation Details.

- Language: C with 64-bit integers
- Compiler: GCC 13.3.0 with `-O3` optimization
- Memory: 1 GB (one byte per integer for primality lookup)
- Total computations: 500 million $p_{\min}$ evaluations

Benchmark Environment.

- CPU: Intel Core i7-13700H (14 cores, 20 threads, up to 5.0 GHz)
- L3 Cache: 24 MB
- RAM: 16 GB DDR5
- OS: Ubuntu 24.04 (Linux 6.14)

Measured Runtime (single-threaded):

Phase	Time
Sieve of Eratosthenes (10^9)	$\sim$8 seconds
$p_{\min}$ computation (500M even numbers)	$\sim$10 seconds
Total	**$\sim$18 seconds**

The fast runtime is achieved because (1) the sieve provides $O(1)$ primality lookup, and (2) most even numbers have very small $p_{\min}$ (95% have $p_{\min} \leq 103$), so the inner loop terminates quickly. The worst-case $p_{\min} = 1{,}789$ requires testing only 282 primes, and such cases are rare (33 records out of 500 million).

3 Results

3.1 Distribution of $p_{\min}$ Values

Our first finding concerns the distribution of $p_{\min}$ across all even integers.

Percentile	$p_{\min} \leq$	Odd primes to test
95%	103	26
99%	191	42
99.9%	331	66
99.999%	631	114

Finding 1. *99.999% of even integers up to 10^9 have $p_{\min} \leq 631$.*

This means virtually all Goldbach decompositions can be found by testing only the first 114 odd primes. The remaining 0.001% (about 5,000 cases out of 500 million) require testing more primes, with the worst case needing 282 primes.

3.2 Maximum $p_{\min}$: The Main Result

We tracked the maximum value of $p_{\min}$ observed up to each threshold N.

$\log_{10}(N)$	N	max $p_{\min}$	$0.2 \times \ln(N)^3$	Ratio
2	100	19	20	0.195
3	1,000	73	66	0.221
4	10,000	173	156	0.221
5	100,000	293	305	0.192
6	1,000,000	523	527	0.198
7	10,000,000	751	837	0.179
8	100,000,000	1,093	1,250	0.175
9	1,000,000,000	1,789	1,780	0.201

Finding 2. *The maximum $p_{\min}$ up to N is well-approximated by*

$$\max p_{\min}(N) \sim 0.2 \times \ln(N)^3$$

To verify that $\ln(N)^3$ is the correct growth rate (rather than $\ln(N)^2$ or $\ln(N)^4$), we examined the stability of various ratios:

Ratio tested	Avg ($N \leq 10^5$)	Avg ($N > 10^5$)	Change
$p_{\min}/\ln(N)$	20.4	59.8	+193%
$p_{\min}/\ln(N)^2$	1.97	3.45	+75%
$p_{\min}/\ln(N)^3$	0.201	0.202	+0.6%

Only the cubic ratio remains stable across the entire range, confirming $\ln(N)^3$ as the correct functional form.

Note on statistical methodology. The ratio stability analysis above constitutes our primary evidence for the $\ln(N)^3$ growth rate. Traditional confidence intervals are not applicable here, as our dataset is exhaustive (all 500 million even integers up to 10^9), not a statistical sample. The constant 0.2 is a descriptive fit to complete data, not an estimate with sampling error. The relevant question is not "how confident are we in 0.2?" but rather "does this pattern persist beyond 10^9?"—which remains open.

3.3 Theoretical Basis for Cubic Growth

The $\ln(N)^3$ growth rate is not coincidental. We provide a heuristic derivation based on extreme value theory.

Finding 3. *The cubic growth* $\max p_{\min} \sim \ln(N)^3$ *arises from three multiplicative factors.*

Step 1: Prime Density. By the Prime Number Theorem, the probability that a random integer near N is prime is approximately $1/\ln(N)$.

Step 2: Pair Probability. For $N = p + q$ to be a valid Goldbach decomposition, both p and $N - p$ must be prime. Treating these as approximately independent events:

$$P(\text{valid pair}) \sim \frac{1}{\ln(N)} \times \frac{1}{\ln(N)} = \frac{1}{\ln(N)^2}$$

Note: This independence assumption is a simplification. The actual probability involves correction factors (the Hardy-Littlewood singular series) that depend on N's divisibility by small primes. While the independence assumption is known to be imperfect due to arithmetic correlations, it is sufficient for explaining the observed growth exponent rather than the precise constant.

Step 3: Expected Search Depth. If each candidate p has probability $\sim 1/\ln(N)^2$ of success, then the expected number of trials until success is $\sim \ln(N)^2$. This gives the typical value of $p_{\min}$:

$$\text{typical } p_{\min} \sim \ln(N)^2$$

Step 4: Maximum vs. Typical. We seek not the typical $p_{\min}$, but the maximum across $N/2$ even integers. This is an extreme value problem.

Consider an analogy: if we flip a coin until we get heads, the expected number of flips is 2. But if we repeat this experiment one million times, the longest streak will be much larger than 2.

For geometric distributions, the maximum of n independent samples grows as $\log(n)$ times the mean. Applying this:

$$\max p_{\min} \sim \log(N/2) \times \ln(N)^2 \sim \ln(N) \times \ln(N)^2 = \ln(N)^3$$

This explains the cubic growth rate. The constant 0.2 is determined empirically.

3.4 Comparison with Prime Gaps

For context, we compare the growth of max $p_{\min}$ with the growth of maximum prime gaps.

Quantity	Empirical Growth	Theoretical Basis
Max prime gap	$\sim 0.5 \times \ln(N)^2$	Cramér's conjecture
Max $p_{\min}$	$\sim 0.2 \times \ln(N)^3$	This paper

The extra factor of $\ln(N)$ in $p_{\min}$ growth reflects the additional constraint: finding a Goldbach pair requires **both** p and $N-p$ to be prime, whereas a prime gap only concerns the distance to the **next** prime.

4 Implications for Goldbach's Conjecture

4.1 Computational Efficiency and Connection to Prior Work

A naive approach to finding Goldbach pairs might start from the middle: test whether $N/2$ is prime, then try $(N/2-1, N/2+1)$, and so on. This is inefficient because:

- Most integers near $N/2$ are composite
- The search space is unbounded in the worst case

Our findings demonstrate that **searching from the small end is far more efficient**. By testing $p = 3, 5, 7, 11, \ldots$ in sequence:

- 99.999% of even N find a valid pair within the first 114 odd primes ($p \leq 631$)
- The worst case up to 10^9 requires only 282 odd primes ($p \leq 1{,}789$)
- The search empirically terminates quickly

This is precisely the strategy employed by Oliveira e Silva et al. [2] to verify Goldbach's conjecture up to 4×10^{18}. Their implementation searched for the "minimal Goldbach partition"—exactly what we call $p_{\min}(N)$—using highly optimized segmented sieves. Our analysis provides a theoretical framework explaining *why* this approach is so efficient: because $p_{\min}$ grows only as $O(\ln(N)^3)$, the search terminates after testing a vanishingly small fraction of candidates.

Extrapolation. If the formula $\max p_{\min} \sim 0.2\ln(N)^3$ continues to hold:

- At $N = 10^{12}$: max $p_{\min} \sim$ 4,200 (testing ~600 primes)
- At $N = 10^{18}$: max $p_{\min} \sim$ 14,000 (testing ~1,700 primes)

At the scale of 4×10^{18}, our formula predicts max $p_{\min} \sim$ 14,000, meaning even the hardest cases require testing fewer than 2,000 small primes—a trivial computation regardless of how large N becomes.

4.2 What Would It Take for Goldbach to Fail?

We emphasize that **this paper does not prove Goldbach's conjecture**. Computational verification, no matter how extensive, cannot prove a statement about all integers.

However, our findings reveal what a counterexample would require. A Goldbach counterexample is an even N such that $p_{\min}(N)$ does not exist—equivalently, $p_{\min}(N) > N/2$ (since we cannot have $p > N/2$ in a valid decomposition).

For Goldbach to fail, the orderly growth pattern $p_{\min} \sim \ln(N)^3$ would need to catastrophically break down. Some unprecedented arithmetic chaos would need to occur, causing $p_{\min}$ to jump from $O(\ln(N)^3)$ to $O(N)$.

To illustrate the magnitude of this jump:

- At $N = 10^9$: observed max $p_{\min} =$ 1,789, while $N/2 =$ 500,000,000
- The ratio is approximately 1 : 280,000

For a counterexample to exist, $p_{\min}$ would need to increase by a factor of 280,000 beyond its expected value. Our data shows no hint of such behavior—the ratio $p_{\min}/\ln(N)^3$ remains remarkably stable at ~0.2 across nine orders of magnitude.

This does not constitute a proof, but it quantifies precisely how dramatic a deviation from established patterns would be required for Goldbach to fail.

5 Directions for Further Investigation

1. **Extended computation:** Verify the $\ln(N)^3$ formula up to 10^{12} or 10^{15} using distributed computing.
2. **Refined constants:** Determine whether the constant 0.2 has a closed-form expression involving known mathematical constants.
3. **Secondary terms:** Investigate whether $\max p_{\min} = A\ln(N)^3 + B\ln(N)^2\ln(\ln(N)) + \cdots$ provides a better fit.
4. **Rigorous bounds:** Attempt to prove upper bounds on $p_{\min}$ using sieve methods or other analytic techniques.

6 Summary of Findings

#	Finding	Formula/Result
1	Most N are light	99.999% have $p_{\min} \leq 631$ (114 odd primes)
2	Maximum growth	$\max p_{\min} = O(\ln(N)^3)$
3	Why cubic	Extreme value of $\ln(N)^2$ typical values
4	Comparison	$p_{\min}$ grows as $\ln^3$, prime gaps as $\ln^2$

A Record-Breaking Values

All 33 pairs $(N, p_{\min})$ where $p_{\min}$ exceeded all previous values:

N	$p_{\min}$	$\log_{10}(N)$
6	3	0.78
12	5	1.08
30	7	1.48
98	19	1.99
220	23	2.34
308	31	2.49
556	47	2.75
992	73	3.00
2,642	103	3.42
5,372	139	3.73
7,426	173	3.87
43,532	211	4.64
54,244	233	4.73
63,274	293	4.80
113,672	313	5.06
128,168	331	5.11
194,428	359	5.29
194,470	383	5.29
413,572	389	5.62
503,222	523	5.70
1,077,422	601	6.03
3,526,958	727	6.55
3,807,404	751	6.58
10,759,922	829	7.03
24,106,882	929	7.38
27,789,878	997	7.44
37,998,938	1039	7.58
60,119,912	1093	7.78
113,632,822	1163	8.06
187,852,862	1321	8.27
335,070,838	1427	8.53
419,911,924	1583	8.62
721,013,438	1789	8.86

B Code Availability

All source code, raw data, and computational artifacts for this study are publicly available at:

`https://github.com/rizkyandriawan/goldbach-pmin-simulation`

The repository includes:

- C implementation of the Sieve of Eratosthenes and $p_{\min}$ computation
- Complete list of record-breaking $(N, p_{\min})$ pairs
- Scripts for reproducing all reported statistics

References

[1] Hardy, G.H. & Littlewood, J.E. (1923). Some problems of 'Partitio numerorum'; III: On the expression of a number as a sum of primes. *Acta Mathematica*, 44, 1–70.

[2] Oliveira e Silva, T., Herzog, S., & Pardi, S. (2014). Empirical verification of the even Goldbach conjecture and computation of prime gaps up to 4×10^{18}. *Mathematics of Computation*, 83(288), 2033–2060.