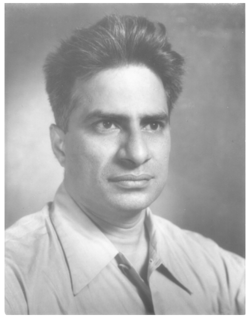
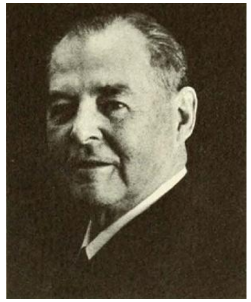
The ‘KLT’: Introduction


D. D. Kosambi
[1907-1966]
1943

K. Karhunen
[1915-1992]
1945


M. Loève
[1907-1979]
1948


H. Hotelling
[1895-1973]

https://upload.wikimedia.org/wikipedia/commons/0/0f/Kosambi-dd.jpg

https://upload.wikimedia.org/wikipedia/commons/d/d0/Michel_Lo%C3%A8ve.jpg

https://upload.wikimedia.org/wikipedia/en/4/49/Harold_Hotelling.jpg

People call it names!

- Karhunen-Loeve Transform
- Hotelling Transform
- Principal Component Analysis
- Eigenvalue-Eigenvector Transform

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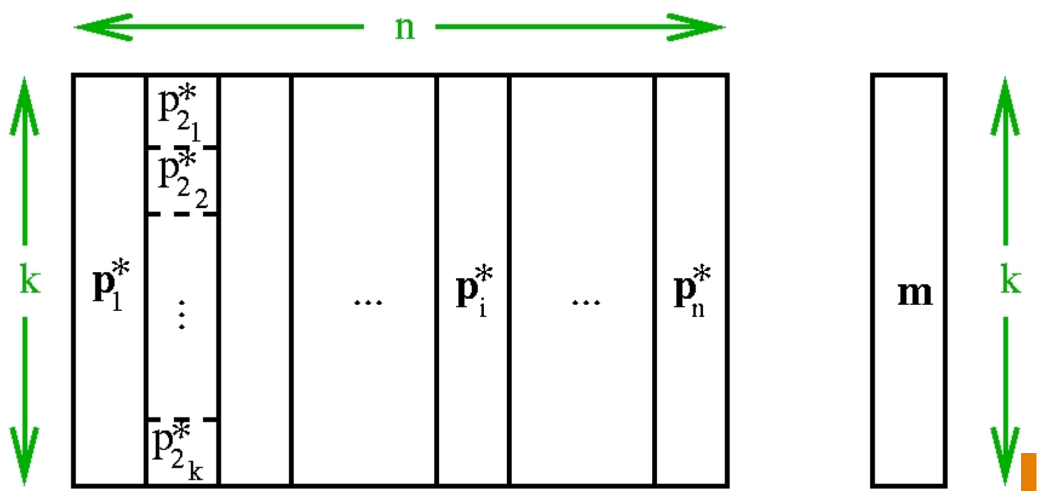
Pattern Recognition Terms

- A ‘pattern’ is a $k \times 1$ column vector - a 1-D signal can be represented as a ‘pattern’. A $k_1 \times k_2$ 2-D signal (an image) can be represented as a ‘pattern’ by taking all pixels in raster scan order (row major order) to form a $k \times 1$ ‘pattern’, $k = k_1 \cdot k_2$
- k-dimensional ‘patterns’ $\mathbf{p}_i^*$, $1 \leq i \leq n$
- Stack them up together (in any order) to form a $k \times n$ Pattern Matrix $\mathbf{P}^*$



- Normalise each pattern: $\mathbf{p}_i \triangleq \mathbf{p}_i^*$ - $\mathbf{m}$
- $\mathbf{A} \triangleq \frac{1}{n} \mathbf{P}\ \mathbf{P}^T$: The Covariance Matrix
- Stack together EigenVectors $\mathbf{u}_i$ of $\mathbf{A}$ in decreasing order of the corresponding EigenValues to get the $k \times k$ matrix $\mathbf{U}$

Linear Algebra Fundamentals

- Phys significance of Eigenvalues & Eigenvectors
- Similar Matrices
- Diagonalisation of a $k \times k$ matrix
- Gram-Schmidt Orthogonalisation
- Eigenvalues of a symmetric real matrix are real
- Eigenvecs of a symmetric matrix: orthonormality

Phys Sig of E'values, E'vectors

- For a $k \times k$ matrix $\mathbf{B}$, if $\mathbf{B}\mathbf{u_i} = \lambda_i \mathbf{u_i}$, λ_i are the eigenvalues, and $\mathbf{u_i}$, the corresponding eigenvectors
- Phys sig: matrix $\times$ vector $\equiv$ scaling it!
- Computing eigenvalues: $\mathbf{B}\mathbf{u} - \lambda \mathbf{u} = \mathbf{0} \Longrightarrow (\mathbf{B} - \lambda \mathbf{I})\mathbf{u} = \mathbf{0} \Longrightarrow$ non-trival solution: $|\mathbf{B} - \lambda \mathbf{I}| = 0$
- E'vecs: not unique! scaled versions also e'vecs

Some Less Important Properties

- Rank($\mathbf{B}$) = # of non-zero eigenvals
- $\sum \lambda_i$ = Trace($\mathbf{B}$) (sum of main diag), $\prod \lambda_i$ = $|\mathbf{B}|$
- A square matrix $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalues (but usually, different eigenvectors)
  $$|\mathbf{A}^T - \lambda\mathbf{I}| = |\mathbf{A}^T - \lambda\mathbf{I}^T| = |(\mathbf{A} - \lambda\mathbf{I})^T| = |\mathbf{A} - \lambda\mathbf{I}|$$
- The eigenvalues of a diagonal matrix are those! eigenvalues: $|\mathbf{B} - \lambda\mathbf{I}| = 0, \prod(b_{ii} - \lambda_i) = 0$
- $\mathbf{B}_{k\times k}$ is invertible iff 0 isn't an eigenvalue. eigenvalue 0 iff $|\mathbf{B} - 0\mathbf{I}| = 0$ iff $|\mathbf{B}| = 0$ i.e., non-invertible
- If $\mathbf{B}$ has an eigenvalue-eigenvector pair $(\lambda, \mathbf{u})$, then $\mathbf{B}^n$ $(n \in \mathcal{N})$ has the pair $(\lambda^n, \mathbf{u})$.
  $$\mathbf{B}_{k\times k}\mathbf{u}_{k\times 1} = \lambda\mathbf{u}_{k\times 1},\ \mathbf{B}\mathbf{B}\mathbf{u} = \lambda\mathbf{B}\mathbf{u},\ \mathbf{B}^2\mathbf{u} = \lambda^2\mathbf{u}, \text{ etc}$$
- If $\mathbf{B}$ has an eigenvalue-eigenvector pair $(\lambda, \mathbf{u})$, then $\mathbf{B}^{-1}$ has the pair $(\lambda^{-1}, \mathbf{u})$.
  $$\mathbf{B}_{k\times k}\mathbf{u}_{k\times 1} = \lambda\mathbf{u}_{k\times 1},\ \mathbf{B}^{-1}\mathbf{B}\mathbf{u} = \lambda\mathbf{B}^{-1}\mathbf{u},\ (1/\lambda)\mathbf{u} = \mathbf{B}^{-1}\mathbf{u}$$

- Eigenvectors of a matrix with distinct eigenvalues are linearly independent: Can form a basis
  Proof by Contradiction: Suppose not. 'Thin out' this to l indep eigenvectors $\mathbf{u}_1, \ldots \mathbf{u}_l \equiv \lambda_1, \ldots \lambda_l$
  Suppose $\mathbf{u}$ was 'thinned out' $\mathbf{u} = \sum_{j=1}^{l} c_j \mathbf{u_j}$ (1)
  1. Multiply (1) by $\mathbf{B}$: $\mathbf{Bu} = \sum c_j(\mathbf{Bu_j})$, $\lambda\mathbf{u} = \sum c_j \lambda_j \mathbf{u_j}$
  2. Multiply (1) by λ: $\lambda\mathbf{u} = \sum c_j \lambda \mathbf{u}_j$
  Subtract: $\mathbf{0} = \sum c_j(\lambda - \lambda_j)\mathbf{u}_j$. Hence, $\forall j$:
  $c_j = 0$ (no!) or $\mathbf{u}_j = \mathbf{0}$ (no, as eigenvector is a non-trival solution) or $\lambda = \lambda_j$ (no!): Contradiction!
- Eigenvalues of a symmetric real matrix are real
  $\mathbf{Au} = \lambda\mathbf{u}$ and $\mathbf{A}^*\mathbf{u}^* = \lambda^*\mathbf{u}^*$, $\mathbf{A}^* = \mathbf{A}$: real
  Pre-multiply by $\mathbf{u}^{*T}$ and $\mathbf{u}^T$, and subtract:
  $\mathbf{u}^{*T}\mathbf{Au} - \mathbf{u}^T\mathbf{Au}^* = \lambda\mathbf{u}^{*T}\mathbf{u} - \lambda^*\mathbf{u}^T\mathbf{u}^*$ LHS: Consider $(\mathbf{u}^{*T}\mathbf{Au})^T$, scalar's transpose. $= \mathbf{u}^T\mathbf{Au}^*$. LHS = 0
  RHS: $\mathbf{u}^{*T}\mathbf{u}$: sum-of-sq $\neq$ 0 unless all 0, e'vec $\neq \mathbf{0}$
  e.g., $[a - jb\ c - jd][a + jb\ c + jd]^T = a^2 + b^2 + c^2 + d^2$
  Hence $\lambda = \lambda^*$, only possible if λ is real. QED