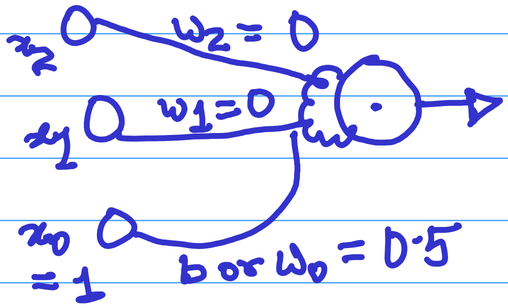
What did we try?



$y = \sigma(a) = a$ unit function

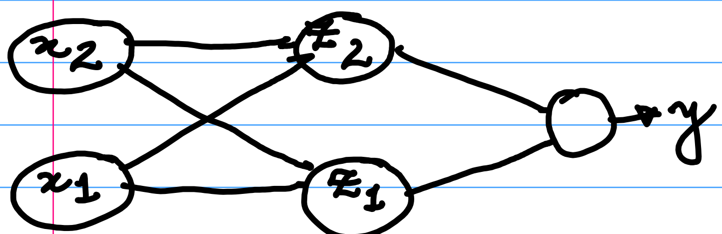
$y = \underline{w}^T \underline{x} + b$
(non-homogeneous)

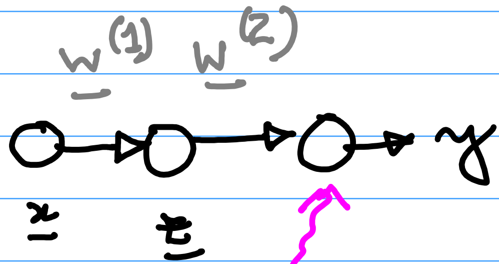
$= w_2 x_2 + w_1 x_1 + b$
$= 0.5$ always.

Some points: →

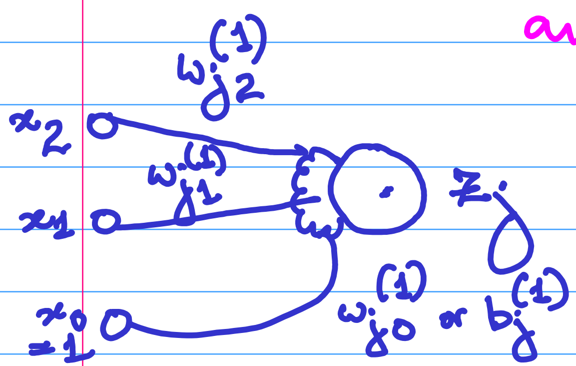
- This is a simple example with regression
  (though the problem is actually one of classification)
- This is one of the myriad possible solutions
- Unnecessarily ~~trying~~ to build a classifier out of a regressor (that too, for binary inputs)
- Handcrafted

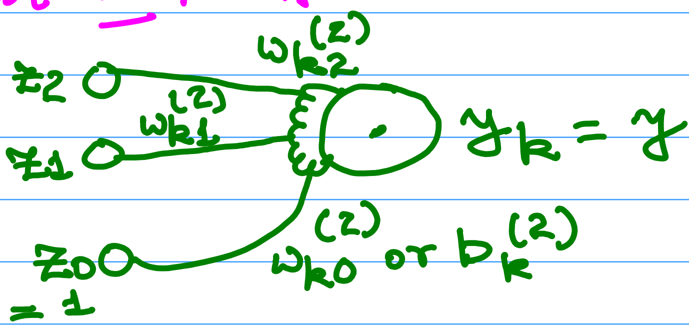
ONE possible solution: –





regressor applied to $\underline{z} = \underline{\phi}(\underline{x})$, and not $\underline{x}$ itself





$z_j = h(a_j)$

choose $h(\cdot)$ to be a ReLU

$$a_j = \underbrace{\underline{w}_j^{(1)T} \underline{x}}_{\text{homogeneous}} = \underbrace{\underline{w}_j^{(1)T} \underline{z} + b_j^{(1)}}_{\text{non-homogeneous}}$$

$b_j^{(1)}$ or $w_{j0}^{(1)}$

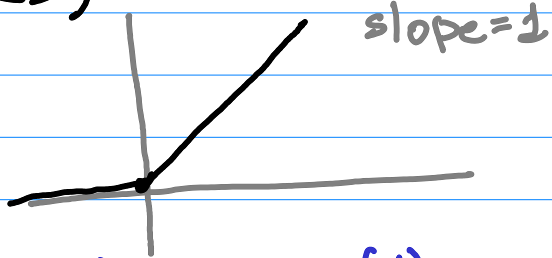
$y_k = y = \sigma(a_k)$

choose σ to be the unit function

$$y_k = y = \underbrace{\underline{w}_k^{(2)T} \underline{z} + b_k^{(2)}}_{\text{non-homogeneous}}$$

ReLU as an activation function

(often the choice in most deep feedforward networks)



$h(a) = \max\{0, a\}$

piecewise linear

$$a_j = \underline{w}_j^{(1)} \underline{x} = b_j^{(1)} \text{ or } w_{j0}^{(1)}$$

$$z_j = h(a_j) = \max\{0, a_j\} = \max\{0, \underline{w}_j^{(1)T} \underline{x} + b_j^{(1)}\}$$

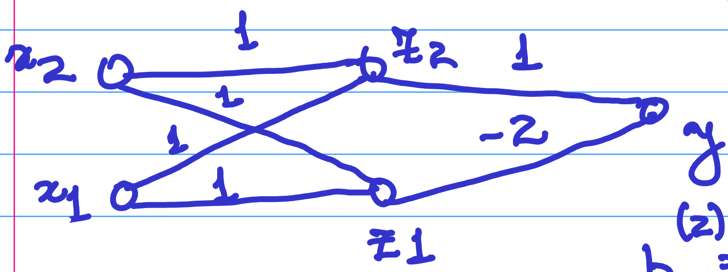
Handcrafted example:– (all these values are given!)

$$\underline{w}_j^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \underline{b}^{(1)} = \begin{bmatrix} b_2^{(1)} \\ b_1^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\underline{w}^{(2)} = \begin{bmatrix} w_2^{(2)} \\ w_1^{(2)} \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}; \quad b^{(2)} = 0$$

X = matrix of all possible inputs

$$\underline{x} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \underset{\underline{x}_{(1)}}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}, \underset{\underline{x}_{(2)}}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}, \underset{\underline{x}_{(3)}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \underset{\underline{x}_{(4)}}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$



$b^{(2)} = 0$

$$\underline{b}^{(1)} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

(given)

$$a_j = \underline{w}_j^{(1)T} \underline{x} + b_j^{(1)} \quad \text{or } w_{j0}^{(1)}$$

$$a_2 = \underline{w}_2^{(1)T} \underline{x} + b_2^{(1)}$$

$$a_1 = \underline{w}_1^{(1)T} \underline{x} + b_1^{(1)}$$

$$\underline{a}_{(n)} = \begin{bmatrix} \underline{w}_2^{(1)T} \\ \underline{w}_1^{(1)T} \end{bmatrix} \underline{x}_{(n)} + \begin{bmatrix} b_2^{(1)} \\ b_1^{(1)} \end{bmatrix}$$

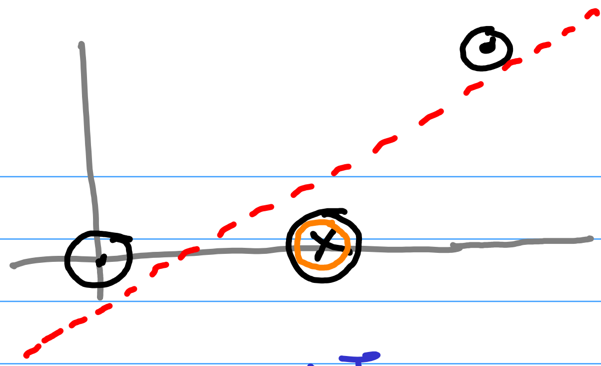
$$\Rightarrow \underline{a}_{(1)} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad \underline{z}_{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underline{a}_{(2)} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \underline{z}_{(2)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\underline{a}_{(3)} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \underline{z}_{(3)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\underline{a}_{(4)} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \underline{z}_{(4)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$z_j = h(a_j) \Rightarrow \underline{z} = \begin{bmatrix} z_2 \\ z_1 \end{bmatrix} = \begin{bmatrix} h(a_2^{(1)}) \\ h(a_1^{(1)}) \end{bmatrix} = \begin{bmatrix} \max(a_2, 0) \\ \max(a_1, 0) \end{bmatrix}$$



linear decision boundary

$$a_k^{(2)} = \underbrace{\underline{w}_k^{(2)T} \underline{z}}_{\text{homogeneous}} = \underbrace{\underline{w}_k^{(2)T} \underline{z} + b_k^{(2)}}_{\text{non-homogeneous}}$$

given

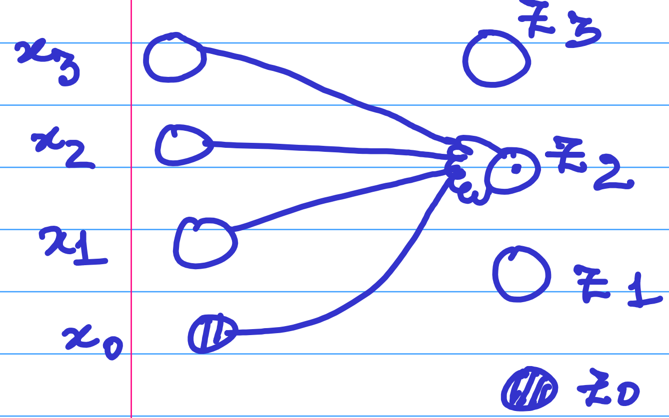
$$a_{(n)}^{(2)} = \underbrace{[1\ \ -2]}\, \underline{z}_{(n)} + b^{(2)} \rightarrow = 0$$

$$\underline{w}_k^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad b_k^{(2)} = 0$$

$$\left.\begin{aligned}
a_{(1)}^{(2)} &= [1\ -2]\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \\
a_{(2)}^{(2)} &= [1\ -2]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \\
a_{(3)}^{(2)} &= [1\ -2]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \\
a_{(4)}^{(2)} &= [1\ -2]\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0
\end{aligned}\right\}$$

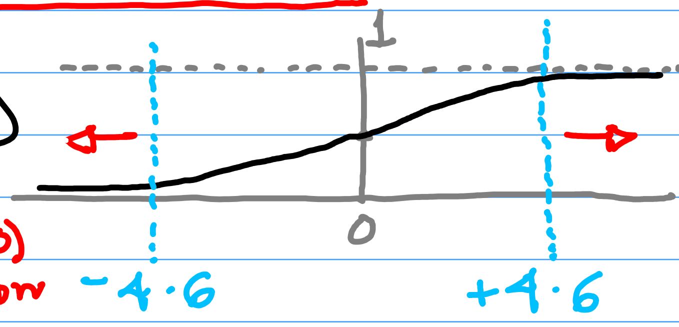
this is what we wanted as the XOR outputs

Notation (Vector-Matrix Notation)



SOME HANDCRAFTED NN EXAMPLES

$$\boxed{h(a) = \frac{1}{1+e^{-a}}} \quad \text{(sigmoid)}$$

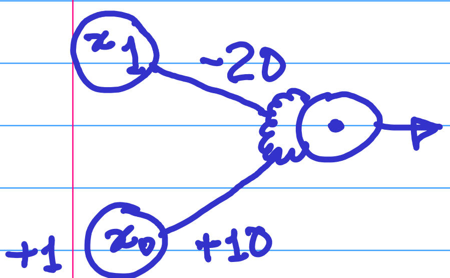


$e^{4.6} \approx 100$

when $a = -4.6$, $h(a) = \dfrac{1}{1+e^{4.6}} \approx \dfrac{1}{1+100} \approx 0.01$

when $a = +4.6$, $h(a) = \dfrac{1}{1+e^{-4.6}} \approx \dfrac{1}{1+1/100} = \dfrac{100}{101} \approx 0.99$

Example 1 : NOT



$a = \underline{w}^T \underline{x} + b$

$= -20x_1 + 10$

x_1	$\overline{x_1}$
0	1
1	0

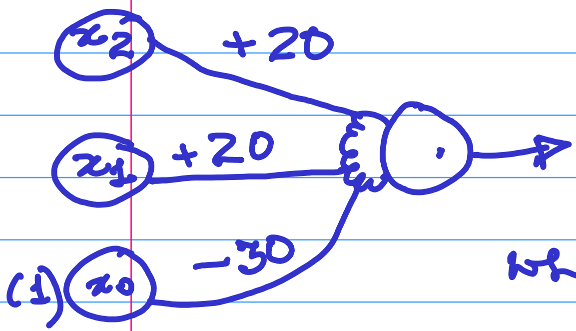
$x_1 = 0$, $a = 10$, $h(+10) \approx 1$

$x_1 = 1$, $a = -10$, $h(-10) \approx 0$

Example 2 : AND

x_2	x_1	$x_2 \cdot x_1$
0	0	0
0	1	0
1	0	0
1	1	1

$a = \underline{w}^T \underline{x} + b$



$a = 20x_2 + 20x_1 - 30$

when $x_2 = 0$, $x_1 = 0$: $a = -30$

$h(-30) \approx 0$

when $x_2 = 0$, $x_1 = 1$, $a = 20(0) + 20(1) - 30 = -10$, $h(-10) \approx 0$