- Definiteness of a symmetric matrix depends on the sign of its eigenvalues. Quadratic Form: for $\mathbf{B}_{k \times k}$ the scalar $\mathbf{x}^T \mathbf{B} \mathbf{x}$ is a quadratic form = $\sum_{i=1}^{k} \sum_{j=1}^{k} b_{ij} x_i x_j$. $(\mathbf{x}^T \mathbf{B} \mathbf{x})^T = \mathbf{x}^T \mathbf{B}^T \mathbf{x}$: $\mathbf{B}$ is symmetric!

  Quadratic form: defined only for symmetric $\mathbf{B}_{k \times k}$

  Positive Definite: $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0, \ \forall \mathbf{x}$

  Positive Semi-Definite: $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0, \ \forall \mathbf{x}$

  $(*)$ PSD matrix has non-negative eigenvalues: Let λ be an eigenvalue of $\mathbf{A}_{k \times k}$ with eigenvector $\mathbf{u}$. $\mathbf{A}\mathbf{u} = \lambda \mathbf{u}$: $\mathbf{u}^T \mathbf{A} \mathbf{u} = \lambda \mathbf{u}^T \mathbf{u}$. Since $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0 \ \forall \mathbf{x}$, $\mathbf{u}^T \mathbf{A} \mathbf{u} \geq 0$. $\mathbf{u}^T \mathbf{u} \geq 0 \implies \lambda \geq 0$

  $(*)$ Non-negative eigenvalues $\implies$ PSD: Symmetric matrix $\mathbf{A} = \mathbf{U} \mathbf{\Lambda} \mathbf{U}^T$. $\mathbf{x}^T \mathbf{A} \mathbf{x} = \mathbf{x}^T \mathbf{U} \mathbf{\Lambda} \mathbf{U}^T \mathbf{x} = \mathbf{y}^T \mathbf{\Lambda} \mathbf{y} = \sum_{i=1}^{k} \lambda_i y_i^2$. If all $\lambda_i \geq 0$, $\mathbf{x}^T \mathbf{A} \mathbf{x}$ is PSD

Home Page

Title Page

Contents

Page 34 of 55

Go Back

Full Screen

Close

Quit

- The Quadratic form for Constrained Optimisation of a Symmetric Matrix: $\max/\min_{\mathbf{x}} \mathbf{x}^T\mathbf{A}\mathbf{x},\ ||\mathbf{x}||^2 = 1$

  Take $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} + \lambda(\mathbf{x}^T\mathbf{x} - 1)$ (or $(1 - \mathbf{x}^T\mathbf{x})$)

  $\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = 0:\ 2\mathbf{A}\mathbf{x} + \lambda 2\mathbf{x} = \mathbf{0}$: $(\mathbf{A} + \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$

  For this, take $\mu = -\lambda$ (λ suffices for $(1 - \mathbf{x}^T\mathbf{x})$)

  $(\mathbf{A} - \mu\mathbf{I})\mathbf{x} = \mathbf{0}$: Soln: eigenvec $\mathbf{u}$ of $\mathbf{A} \equiv$ eigenval μ

  To optimise: $\mathbf{x}^T\mathbf{A}\mathbf{x}$. At the opt: $\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{u}^T\mathbf{A}\mathbf{u} = \lambda\mathbf{u}^T\mathbf{u} = \lambda \implies \max = \lambda_{\max},\ \min = \lambda_{\min}$

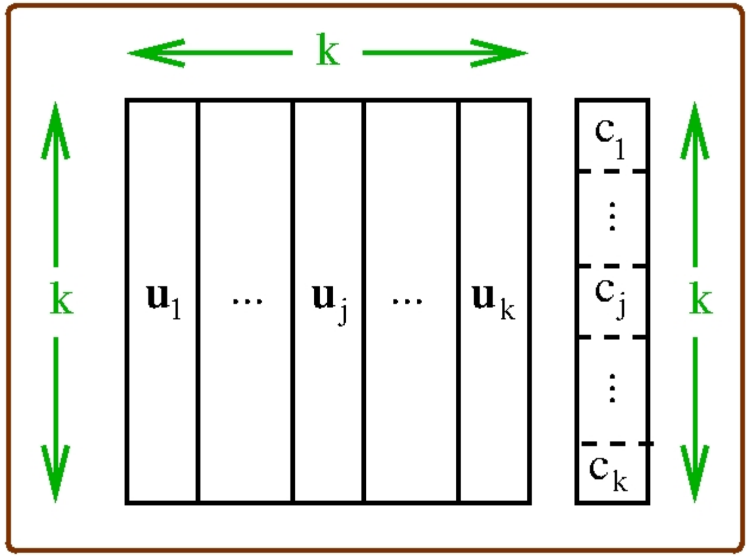
Similar Matrices

- For a $k \times k$ matrix $\mathbf{B}$ and any invertible $k \times k$ matrix $\mathbf{E}$, $\mathbf{EBE}^{-1}$ and $\mathbf{B}$ are Similar Matrices
- $\mathbf{Bu} = \lambda \mathbf{u}, \implies \mathbf{EBu} = \lambda \mathbf{Eu}, \implies \mathbf{EB}\ \mathbf{E}^{-1}\mathbf{E}\ \mathbf{u} = \lambda \mathbf{Eu}$ $\implies \mathbf{EBE}^{-1}\ \mathbf{Eu} = \lambda\ \mathbf{Eu}, \implies \mathbf{EBE}^{-1}\ \mathbf{v} = \lambda\ \mathbf{v}$

Diagonalisation of a $k \times k$ matrix $\mathbf{B}$

- $\mathbf{B}$: eigenvectors $\mathbf{u_i}$, stacked to get $\mathbf{U}$. $\mathbf{u_i}$'s not necessarily orthonormal, assume lin indep (non-repeated e'vals) $\implies$ basis of $k-$dim space
- What if accidentally end up with repeated eigen-values? SVD always works. Applications: use eigenvectors as an orthonormal basis. Extend with extra orthonormal vectors to span the space
- Any k-dimensional pattern $\mathbf{p_i}$ can be written as a linear combination of these basis vectors
- $\mathbf{p_i} = \sum_{j=1}^{k} c_j \mathbf{u_j}$

Home Page
Title Page
Contents
Page 36 of 55
Go Back
Full Screen
Close
Quit



- $\mathbf{p_i} = \sum_{j=1}^{k} c_j \mathbf{u_j} = \mathbf{Uc}$
- $\mathbf{Bp_i} = \sum_{j=1}^{k} c_j \mathbf{Bu_j}$
  $= \sum_{j=1}^{k} c_j \lambda_j \mathbf{u_j} = \mathbf{U\Lambda c}$
- $\mathbf{BUc} = \mathbf{U\Lambda c} \implies$
  $\mathbf{BU} = \mathbf{U\Lambda} \implies$
- $\mathbf{B} = \mathbf{U\Lambda U}^{-1} \iff$
- $\mathbf{U}^{-1}\mathbf{BU} = \mathbf{\Lambda}$

If $\mathbf{U}$ is additionally orthonormal, $\mathbf{U}^{-1} = \mathbf{U}^{T}$