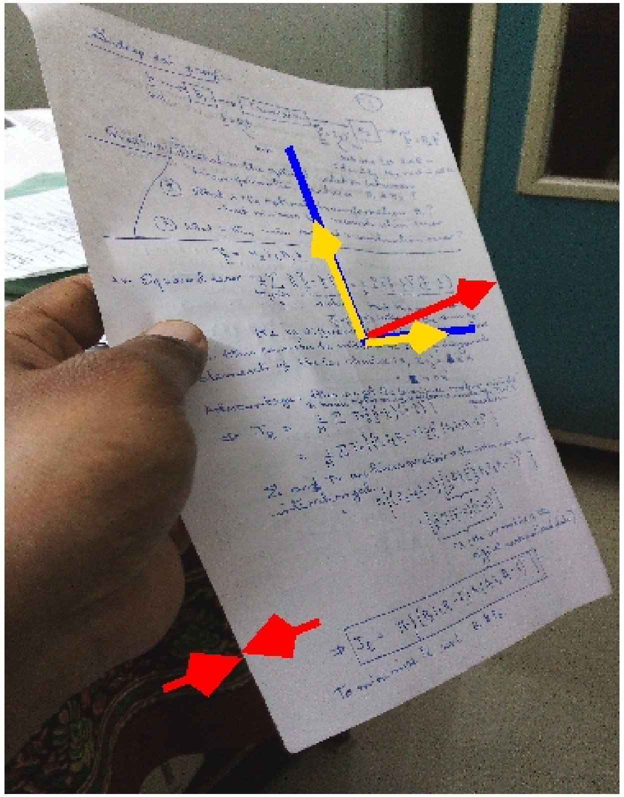
Illustration: Dim Reduction

2-D sheet of paper in 3-D space



Home Page
Title Page
Contents
Page 50 of 55
Go Back
Full Screen
Close
Quit

The Singular Value Decomp

- Generalises the KLT $\forall$ cases, including $k > n$
- A construction: always exists!
- KL-Transform: square matrices, SVD: rectangular
- Can't diag any square matrix, SVD always exists!
- Key observation: $\mathbf{P}_{k\times n}$ not amenable for eigendecomposition, but not $\mathbf{PP^T}_{k\times k}$ & $\mathbf{P^TP}_{n\times n}$
- $\mathbf{PP^T}_{k\times k}$ & $\mathbf{P^TP}_{n\times n}$: Positive Semi-Definite: non-negative eigenvalues! How?

$(\mathbf{PP^T})_{k\times k}$ PSD: $\mathbf{x}^T_{1\times k}(\mathbf{PP^T})_{k\times k}\mathbf{x}_{k\times 1} = (\mathbf{x^TP})(\mathbf{P^Tx})$
$= (\mathbf{x^TP})_{1\times n}(\mathbf{x^TP})^T_{n\times 1} = \mathbf{y^Ty} = \sum y_i^2 \geq 0$

$(\mathbf{P^TP})_{n\times n}$ PSD: $\mathbf{x}^T_{1\times n}(\mathbf{P^TP})_{n\times n}\mathbf{x}_{n\times 1} = (\mathbf{x^TP^T})(\mathbf{Px})$
$= (\mathbf{x^TP^T})_{1\times k}(\mathbf{Px})_{k\times 1} = \mathbf{y^Ty} = \sum y_i^2 \geq 0$

Eigenvalues of $\mathbf{PP^T}$: $\mathbf{PP^Tu} = \lambda\mathbf{u}$: $\mathbf{u^TPP^Tu} = \lambda\mathbf{u^Tu}$
LHS ≥ 0, $\mathbf{u^Tu} \geq 0$ (e'vecs) $\implies \lambda \geq 0$. ||ly, $\mathbf{P^TP}$

Home Page

Title Page

Contents

Page 51 of 55

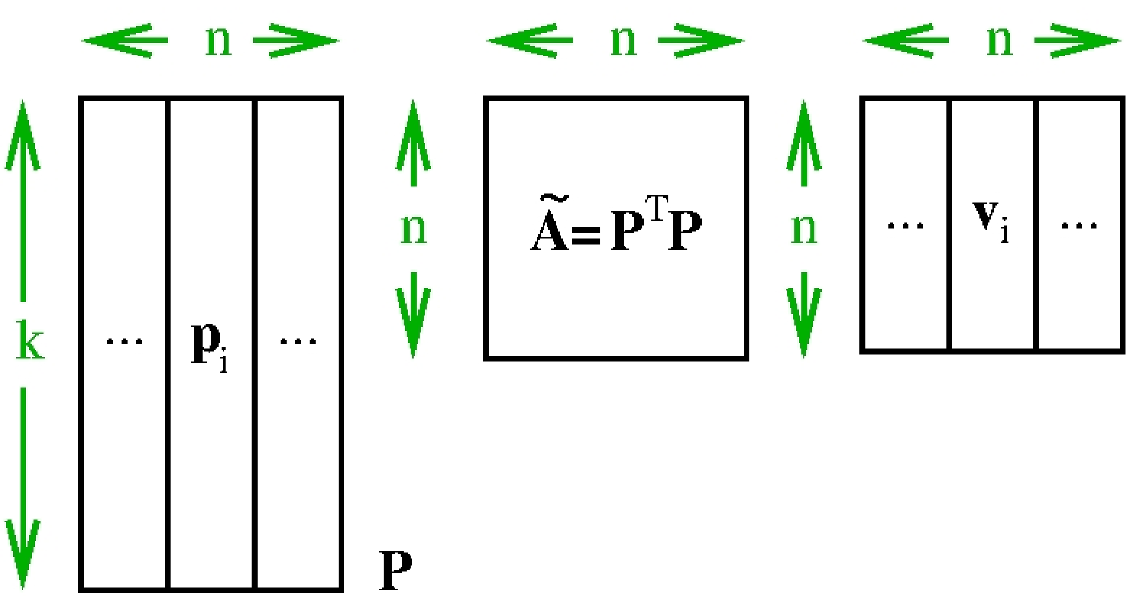
Go Back

Full Screen

Close

Quit

- $k > n$: difficult to numerically calculate the eigenvalues and eigenvectors of $\mathbf{A}_{k \times k} = \frac{1}{n}\mathbf{P}\mathbf{P}^T$. Trick: consider the pseudo-covariance matrix $\tilde{\mathbf{A}} = \mathbf{P}^T\mathbf{P}$:
- n $k-$dim vectors $\equiv$ k $n-$dim vectors
- Let $rank(\mathbf{P}) = r$, where r is at most $\min(k, n)$.
- Let $\tilde{\mathbf{A}}$ have r non-zero eigenvalues $\lambda_1, \ldots \lambda_r$ and corresponding eigenvectors $\mathbf{v}_1, \ldots \mathbf{v}_r$ (each $n \times 1$). Stack these together to form a $n \times r$ matrix
- Append $n - r$ orthonormal vectors to get $\mathbf{V}_{n \times n}$ (all Euclidean bases: related by $\mathbf{R}, \mathbf{t}$)
- Definiton $\sigma_i \triangleq \sqrt{\lambda_i}$. $\boxed{(\mathbf{u_i})_{k \times 1} \triangleq \frac{1}{\sigma_i}\mathbf{P}_{k \times n}(\mathbf{v_i})_{n \times 1}}$, $i \in \{1, r\}$ Stack these together to form a $k \times r$ matrix
- Append $k - r$ orthonormal vectors to get $\mathbf{U}_{k \times k}$

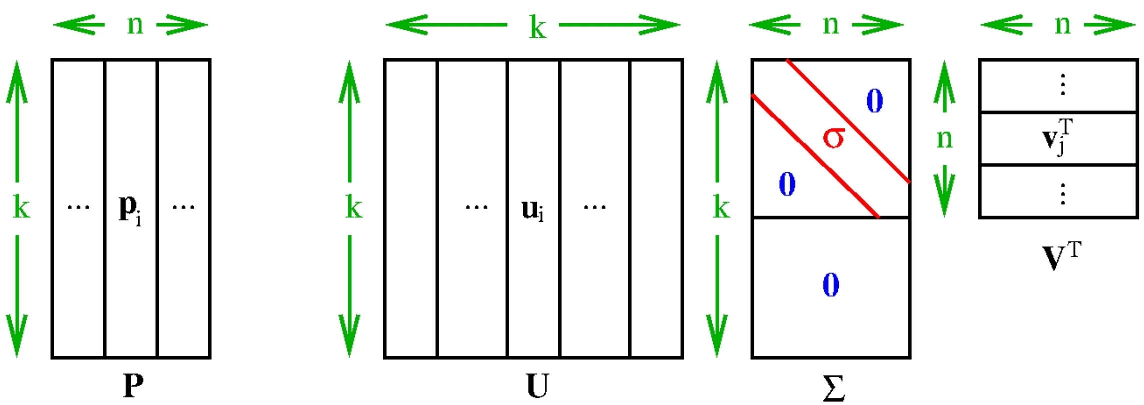


- $\mathbf{u}_i$ orthonormal; $\mathbf{U}_{k\times k}$: orthonormal basis matrix

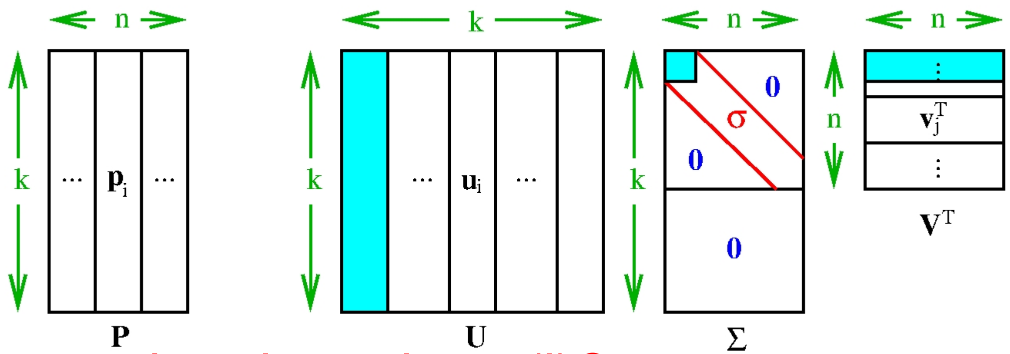
  $\mathbf{u}_i^T\mathbf{u}_j = \frac{1}{\sigma_i}\mathbf{v}_i^T\mathbf{P}^T\frac{1}{\sigma_j}\mathbf{P}\mathbf{v}_j = \frac{1}{\sigma_i\sigma_j}\mathbf{v}_i^T(\mathbf{P}^T\mathbf{P})\mathbf{v}_j = \frac{\lambda_j}{\sigma_i\sigma_j}\mathbf{v}_i^T\mathbf{v}_j$

  $\mathbf{v}_i$ orthonormal. $i = j$, RHS = 1; $i \neq j$, RHS = 0.

- $\mathbf{u}_i^T\mathbf{P}\mathbf{v}_j = \frac{1}{\sigma_i}\mathbf{v}_i^T\mathbf{P}^T\mathbf{P}\mathbf{v}_j = \frac{1}{\sigma_i}\mathbf{v}_i^T\lambda_j\mathbf{v}_j$. $\mathbf{v}_i$ orthonormal $i = j$: value = σ_i; $i \neq j$: value = 0 $\boxed{\mathbf{U}^T\mathbf{P}\mathbf{V} = \mathbf{\Sigma};\ \mathbf{P} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T}$



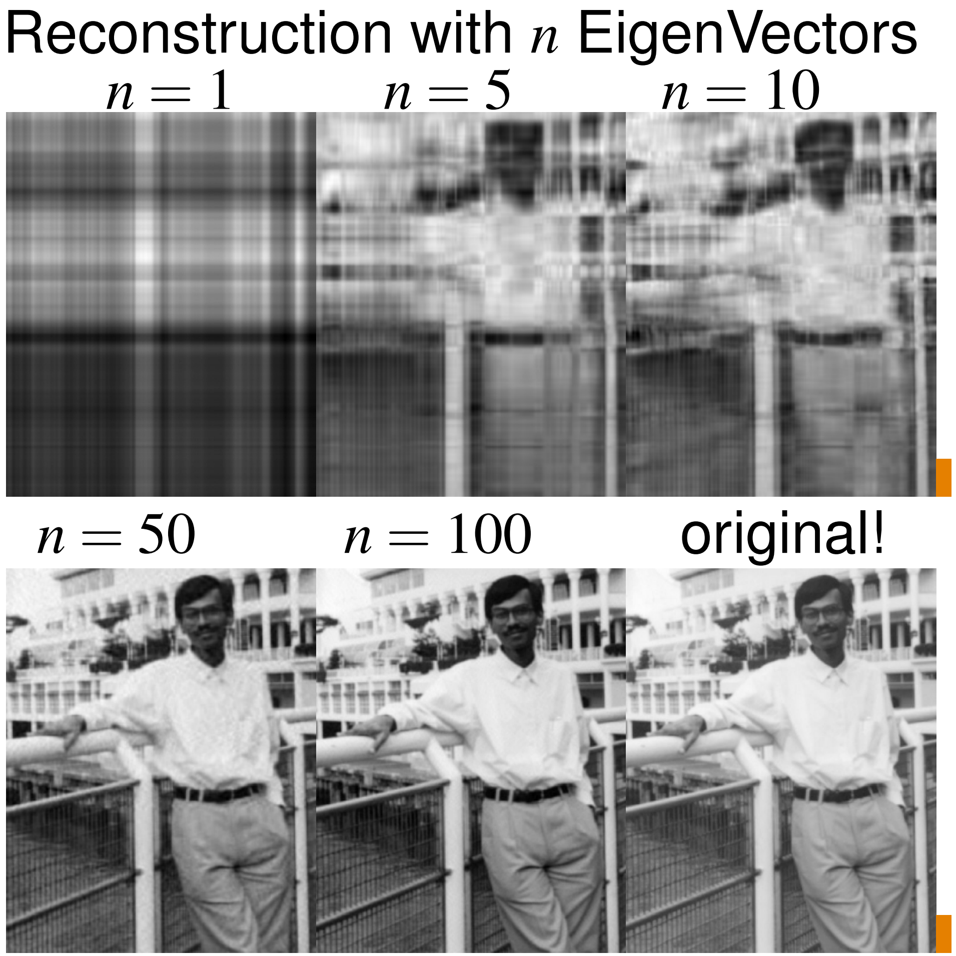
Dimensionality Reduction: Take $l < n < k$ orthonormal basis vectors $\mathbf{u}_i$: $\mathbf{P}_{k\times n} \approx \mathbf{U}_{k\times l}\boldsymbol{\Sigma}_{l\times l}\mathbf{V}^T_{l\times n}$



How many singular values (l)?

e.g., min to make up 95% energy. $\min l : \frac{\sum_{i=1}^{l} \sigma_i}{\sum_{i=1}^{k} \sigma_i} \geq 0.95$

Reconstruction



References

- B. P. Lathi.
  Communication Systems.
  B. S. Publications, 2001.
- R. C. Gonzalez and R. E. Woods.
  Digital Image Processing, Second Ed.
  Pearson Education Inc., 2002.
- A. K. Jain.
  Fundamentals of Digital Image Processing.
  Prentice-Hall International, Inc., 1989.