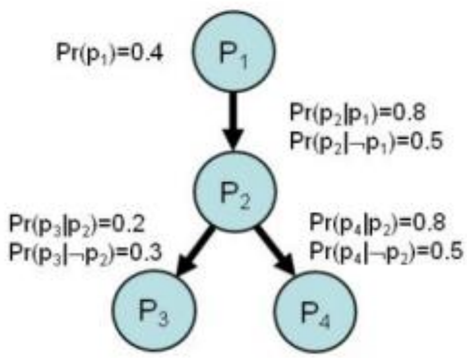
Probability basics problem and with solution

Exercise 4. *Given the network below, calculate marginal and conditional probabilities* $Pr(\neg p_3)$, $Pr(p_2|\neg p_3)$, $Pr(p_1|p_2, \neg p_3)$ *a* $Pr(p_1|\neg p_3, p_4)$. *Apply the method of* **inference by enumeration**.



Inference by enumeration sums the joint probabilities of atomic events. They are calculated from the network model: $Pr(P_1, \ldots, P_n) = Pr(P_1|parents(P_1)) \times \cdots \times Pr(P_n|parents(P_n))$. The method does not take advantage of conditional independence to further simplify inference. It is a routine and easily formalized algorithm, but computationally expensive. Its complexity is exponential in the number of variables.

$$
\begin{aligned}
Pr(\neg p_3) &= \sum_{P_1,P_2,P_4} Pr(P_1,P_2,\neg p_3,P_4) = \sum_{P_1,P_2,P_4} Pr(P_1)Pr(P_2|P_1)Pr(\neg p_3|P_2)Pr(P_4|P_2) = \\
&= Pr(p_1)Pr(p_2|p_1)Pr(\neg p_3|p_2)Pr(p_4|p_2) + Pr(p_1)Pr(p_2|p_1)Pr(\neg p_3|p_2)Pr(\neg p_4|p_2) + \\
&+ Pr(p_1)Pr(\neg p_2|p_1)Pr(\neg p_3|\neg p_2)Pr(p_4|\neg p_2) + Pr(p_1)Pr(\neg p_2|p_1)Pr(\neg p_3|\neg p_2)Pr(\neg p_4|\neg p_2) + \\
&+ Pr(\neg p_1)Pr(p_2|\neg p_1)Pr(\neg p_3|p_2)Pr(p_4|p_2) + Pr(\neg p_1)Pr(p_2|\neg p_1)Pr(\neg p_3|p_2)Pr(\neg p_4|p_2) + \\
&+ Pr(\neg p_1)Pr(\neg p_2|\neg p_1)Pr(\neg p_3|\neg p_2)Pr(p_4|\neg p_2) + Pr(\neg p_1)Pr(\neg p_2|\neg p_1)Pr(\neg p_3|\neg p_2)Pr(\neg p_4|\neg p_2) = \\
&= .4 \times .8 \times .8 \times .8 + .4 \times .8 \times .8 \times .2 + .4 \times .2 \times .7 \times .5 + .4 \times .2 \times .7 \times .5 + \\
&+ .6 \times .5 \times .8 \times .8 + .6 \times .5 \times .8 \times .2 + .6 \times .5 \times .7 \times .5 + .6 \times .5 \times .7 \times .5 = \\
&= .2048 + .0512 + .028 + .028 + .192 + .048 + .105 + .105 = \mathbf{.762}
\end{aligned}
$$

$$
\begin{aligned}
Pr(p_2|\neg p_3) &= \frac{Pr(p_2,\neg p_3)}{Pr(\neg p_3)} = \frac{.496}{.762} = \mathbf{.6509} \\
Pr(p_2,\neg p_3) &= \sum_{P_1,P_4} Pr(P_1,p_2,\neg p_3,P_4) = \sum_{P_1,P_4} Pr(P_1)Pr(p_2|P_1)Pr(\neg p_3|p_2)Pr(P_4|p_2) = \\
&= Pr(p_1)Pr(p_2|p_1)Pr(\neg p_3|p_2)Pr(p_4|p_2) + Pr(p_1)Pr(p_2|p_1)Pr(\neg p_3|p_2)Pr(\neg p_4|p_2) + \\
&+ Pr(\neg p_1)Pr(p_2|\neg p_1)Pr(\neg p_3|p_2)Pr(p_4|p_2) + Pr(\neg p_1)Pr(p_2|\neg p_1)Pr(\neg p_3|p_2)Pr(\neg p_4|p_2) = \\
&= .4 \times .8 \times .8 \times .8 + .4 \times .8 \times .8 \times .2 + .6 \times .5 \times .8 \times .8 + .6 \times .5 \times .8 \times .2 = \\
&= .2048 + .0512 + .192 + .048 = \mathbf{.496}
\end{aligned}
$$

$$
\begin{aligned}
Pr(p_1|p_2,\neg p_3) &= \frac{Pr(p_1,p_2,\neg p_3)}{Pr(p_2,\neg p_3)} = \frac{.256}{.496} = \mathbf{.5161} \\
Pr(p_1,p_2,\neg p_3) &= \sum_{P_4} Pr(p_1,p_2,\neg p_3,P_4) = \sum_{P_4} Pr(p_1)Pr(p_2|p_1)Pr(\neg p_3|p_2)Pr(P_4|p_2) = \\
&= Pr(p_1)Pr(p_2|p_1)Pr(\neg p_3|p_2)Pr(p_4|p_2) + Pr(p_1)Pr(p_2|p_1)Pr(\neg p_3|p_2)Pr(\neg p_4|p_2) = \\
&= .4 \times .8 \times .8 \times .8 + .4 \times .8 \times .8 \times .2 = .2048 + .0512 = \mathbf{.256} \\
Pr(p_2,\neg p_3) &= Pr(p_1,p_2,\neg p_3) + Pr(\neg p_1,p_2,\neg p_3) = .256 + .24 = \mathbf{.496} \\
Pr(\neg p_1,p_2,\neg p_3) &= \sum_{P_4} Pr(\neg p_1,p_2,\neg p_3,P_4) = \sum_{P_4} Pr(\neg p_1)Pr(p_2|\neg p_1)Pr(\neg p_3|P_2)Pr(p_4|P_2) = \\
&= Pr(\neg p_1)Pr(p_2|\neg p_1)Pr(\neg p_3|p_2)Pr(p_4|p_2) + Pr(\neg p_1)Pr(p_2|\neg p_1)Pr(\neg p_3|p_2)Pr(\neg p_4|p_2) = \\
&= .6 \times .5 \times .8 \times .8 + .6 \times .5 \times .7 \times .2 = .192 + .048 = \mathbf{.24}
\end{aligned}
$$

$$
\begin{aligned}
Pr(p_1|\neg p_3,p_4) &= \frac{Pr(p_1,\neg p_3,p_4)}{Pr(\neg p_3,p_4)} = \frac{.2328}{.5298} = \mathbf{.4394} \\
Pr(p_1,\neg p_3,p_4) &= \sum_{P_2} Pr(p_1,P_2,\neg p_3,p_4) = \sum_{P_2} Pr(p_1)Pr(P_2|p_1)Pr(\neg p_3|P_2)Pr(p_4|P_2) = \\
&= Pr(p_1)Pr(p_2|p_1)Pr(\neg p_3|p_2)Pr(p_4|p_2) + Pr(p_1)Pr(\neg p_2|p_1)Pr(\neg p_3|\neg p_2)Pr(p_4|\neg p_2) = \\
&= .4 \times .8 \times .8 \times .8 + .4 \times .2 \times .7 \times .5 = .2048 + .028 = \mathbf{.2328} \\
Pr(\neg p_3,p_4) &= Pr(p_1,\neg p_3,p_4) + Pr(\neg p_1,\neg p_3,p_4) = .2328 + .297 = \mathbf{.5298} \\
Pr(\neg p_1,\neg p_3,p_4) &= \sum_{P_2} Pr(\neg p_1,P_2,\neg p_3,p_4) = \sum_{P_2} Pr(\neg p_1)Pr(P_2|\neg p_1)Pr(\neg p_3|P_2)Pr(p_4|P_2) = \\
&= Pr(\neg p_1)Pr(p_2|\neg p_1)Pr(\neg p_3|p_2)Pr(p_4|p_2) + Pr(\neg p_1)Pr(\neg p_2|\neg p_1)Pr(\neg p_3|\neg p_2)Pr(p_4|\neg p_2) = \\
&= .6 \times .5 \times .8 \times .8 + .6 \times .5 \times .7 \times .5 = .192 + .105 = \mathbf{.297}
\end{aligned}
$$

Conclusion: $Pr(\neg p3) = 0.762$, $Pr(p2|\neg p3) = 0.6509$, $Pr(p_1|p_2, \neg p_3) = 0.5161$, $Pr(p1|\neg p3, p4) = 0.4394$.