Generating functions

Definition

The *generating function* for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

Generating functions

Definition

The *generating function* for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

Remark

We can define generating functions for finite sequences of real numbers by extending a finite sequence $a_0, a_1, \ldots, a_n$ into an infinite sequence by setting $a_{n+1} = 0$, $a_{n+2} = 0$, and so on. The generating function $G(x)$ of this infinite sequence $\{a_n\}$ is a polynomial of degree n:

$$G(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

Useful facts about power series

Example

The function $f(x) = 1/(1 - ax)$ is the generating function of the sequence $1, a, a^2, a^3, \ldots,$ because

$$\frac{1}{1 - ax} = 1 + ax + a^2x^2 + \cdots$$

when $|ax| < 1$, or equivalently, for $|x| < 1/|a|$ for $a \neq 0$.

Useful facts about power series

Solution

Using the extended binomial theorem, we have

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k.$$

Useful facts about power series

Solution

Using the extended binomial theorem, we have

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k.$$

There is a nice expression for these coefficients:

$$(1+x)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k.$$

Useful facts about power series

Solution

Using the extended binomial theorem, we have

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k.$$

There is a nice expression for these coefficients:

$$(1+x)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k.$$

Replacing x by $-x$, we also find that

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k.$$

Useful facts about power series

Example

Find the number of solutions of $e_1 + e_2 + e_3 = 17$, where e_1, e_2, and e_3 are nonnegative integers with $1 + i \leq e_i \leq 4 + i$.

Useful facts about power series

Example

Find the number of solutions of $e_1 + e_2 + e_3 = 17$, where e_1, e_2, and e_3 are nonnegative integers with $1 + i \leq e_i \leq 4 + i$.

The number of solutions with the indicated constraints is the coefficient of x^{17} in the expansion of

$$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7).$$

Useful facts about power series

Example

Find the number of solutions of $e_1 + e_2 + e_3 = 17$, where e_1, e_2, and e_3 are nonnegative integers with $1 + i \leq e_i \leq 4 + i$.

The number of solutions with the indicated constraints is the coefficient of x^{17} in the expansion of

$$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7).$$

Explicit expansion shows this coefficient to be 3. (Note that performing the expansion is about as much work as an explicit enumeration of the solutions.)

Useful facts about power series

Example

Use generating functions to determine the number of ways to insert tokes with \$1, \$2, and \$5 into a vending machine to pay for an item that costs r dollars in both the cases when the order in which the tokens are inserted does not matter and when the order does matter. (For example: there are two ways to pay for an item that costs \$3 when the order does not matter and three when it does.)

Useful facts about power series

Solution: Unordered

Because we can use any number of \$1, \$2, and \$5 tokens, the answer is the coefficient of x^r in the generating function

$$(1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^5 + x^{10} + \cdots).$$

For example, the number of ways to pay for an item costing \$7 is given by the coefficient of x^7, which is 6.