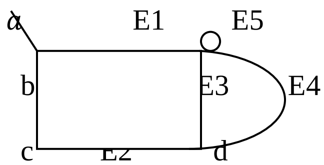
GRAPH THEORY:::

Definition: A graph is an ordered triple (V,E, g) where V is a non empty set of vertices, E is a set of edges and g is a function from E to set of unordered pair of vertices.

Note: Mostly we write a graph as G = (V, E) omitting the function g or simply G. V and E are called the vertex set and edge set of G respectively, V and E are also written as V(G) and E(G).

Example:



We represent a graph by a diagram, points in the diagram represents vertices and line segment or arc represents an edge E who end points are the vertices of the unordered pair associated with the edge E.

If two edges are associated with the same unordered pair of vertices they are called parallel or multiple edges (ex: E4 and E3).

An edge associated with (a, a) for some vertex a is called a loop (ex: E5)

A graph is called simple if it does not have parallel edges or loops.

Let G=(V,E) be a graph and {u,v} be the unordered pair associated with the edge e, we say (i) u and v the end vertices of edge e (ii) the edge e is between u and v (iii) e joins u and v (iv) e is incident with u and v.

Let G= (V,E) be a graph with v in V. the degree of v denoted by deg(v) of d(v) is number of edges incident with v. example: deg(c) = 2 (i.e. number of edges incident on vertex c of above figure); deg(d) = 3 and deg(loop) is always 2 as it is counted clockwise and anti clock wise. In this graph the vertex a is called pendent vertex (since a vertex of degree 1 is called pendent vertex). Any vertex with degree zero is called isolated vertex. A vertex with degree odd is said to be odd vertex and a vertex with even degree is said to be even vertex, for example the vertex c is even vertex and vertex d is said to be odd vertex.

In the given graph G the vertices a and b are said to be adjacent to each other as there is an edge between them. Similarly the vertices c and d are adjacent to each other. But the vertices a and c are not adjacent to each other.

Sum of degree's theorem: The sum of degrees of all vertices of graph G=(V,E) is twice the number of edges. i.e., $\sum_{v\in V} \deg(v) = 2|E|$.

Proof: Let G = (V,E) be a graph, where V is set of vertices and E is set of all edges of graph G. deg(v) represent of degree of vertices v ∈ V. since every edge is incident on minimum two vertices, so every edge contributes 2 to sum of degree of those vertices.

E
Example: u----------v here E is an edge incident of vertices u and v, where deg(u) is one and deg(v) is one so edge E contribute 2 to sum of degrees of u and v. Similarly if there are e number of edges in a graph then sum of the degrees of all vertices is 2 (e). Hence we have ., $\sum_{v\in V} \deg(v) = 2|E|$.

Result: Let G= (V,E) be a graph then the number of vertices of odd degree is even.

Proof: Let us partition the vertex set V of G as (V1 and V2) where V1 is set of vertices of odd degree and V2 is set of vertices of even degree respectively. then we have .,

$\sum_{v\in V} \deg(v) = \sum_{v\in V1} \deg(v) + \sum_{v\in V2} \deg(v)$; since by sum of degree theorem (an edge e between the vertices u and v is considered once while calculating d(u) and for the second time while calculating d(v), so each edge is considered twice in calculating $\sum_{v\in V} \deg(v)$).

Therefore $\sum_{v\in V} \deg(v) = 2|E| = \sum_{v\in V1} \deg(v) + \sum_{v\in V2} \deg(v)$; since sum of degree of even vertices is even and the LHS of equation is even , then sum of degrees of odd vertices must be even. i.e, $\sum_{v\in V1} \deg(v)$ is even. Hence the number of odd vertices in G is even.

Edge sequence, walk, path and circuits: Let G be a graph then (i) An edge sequence of length n in G is a finite alternating sequence u e1 u1 e2 u2…..e_n v of vertices and edges, starting and ending in vertices such that the edge e_i joins the preceding vertex u_{i-1} and the succeeding vertex u_i.

A walk is an edge sequence in which no edge is repeated.

Note: The edge sequence u e1 u1 e2 u2…..e_n v is edge sequence between the vertices u and v or a u-v edge sequence. We can define a u- v walk similarly.

Definition: Let G be a graph. Then (i) A u- v walk is open if $u \neq v$ (ii) A u- v walk is closed if u = v. (iii) A u- v path is an open u – v walk in which no vertex is repeated. (iv) A circuit is a closed walk in which no vertex except the first and last vertex is repeated.

Definition: A graph G is connected if there is a path between every pair of distinct vertices. A graph which is not connected is called disconnected.

Note: A graph consisting of a single isolated vertex is connected.

Directed graphs: We define a graph as G = (V,E, g) where g is a function from the set E of edges to the set of un ordered pairs of vertices. When g is a function from E to set Vx V of ordered pair of vertices we get a directed graph or digraph. We denote digraph by D. If the ordered pair of vertices (u, v) is associated with the edge e then u is called the initial vertex of e and v is called the terminal vertex of e. the edges in the digraph are called directed edges.

Note: If we ignore the direction of an edge, we get edges of usual graph.

Definition: let v be a vertex in a digraph D. The indegree of v is the number of edges that have v as their terminal vertex. The out degree of v is the number of edges that have v as their initial vertex. The indegree and out degree of v are denoted by $d^-(v)$ and $d^+(v)$ respectively.

Types of graphs:

1) K- regular graph: The minimum of all the degrees of the vertices of the graph G is denoted by $\delta(G)$ and maximum of all the degrees of the vertices of G is denoted by $\Delta(G)$. If $\delta(G) = \Delta(G)$ then G is said to be K-regular graph. i.e. If every vertex of G has degree k, then G is K-regular graph.
2) Simple graph: A graph with out parallel edges and loops is called simple graph.
   Note: A simple graph of order k cannot have a vertex of degree k.
3) Subgraph: If G =(V,E) and H= (V, E) are graphs then H is a subgraph of G iff V(H) is a subset of V(G) and E(G) is a subset of E(G).
4) Spanning subgraph: A subgraph H of G is called a spanning subgraph of G iff V(H) = V(G).
5) Subgraph induced by subset: If W is any subset of V(G) then the subgraph induced by W is the subgraph H of G obtaining by taking V(H) = W and E(H) to be those edges of G that join pairs of vertices in W.
6) Vertex removal: If v is a vertex of G, we use notation G-v to denoted the graph obtained by removing the vertex v from G to with all edges incident on v. More generally, we write G – {v1,v2,…vk}for the graph obtained by deleting the vertices v1,v2,…..vk and all the edges incident on any of them.
7) Edge removal: If e is an edge of given graph G, then G- e denotes the graph obtained from G by deleting the edge e.
8) Complete graph: A simple non directed graph with n mutually adjacent vertices is called a complete graph on n vertices. It is denoted by K_n.
   A complete graph with n vertices has $\frac{n(n-1)}{2}$ number of edges and each of its vertices has degree (n-1).
9) Complement: If H is a subgraph of G then complement of H in G is denoted by $\overline{H(G)}$, is the subgraph G- E(H). i.e. the edges of H are deleted from those of G.
   Note: If H is a simple graph with n vertices the complement $\overline{H}$ of H is the complement of H in K_n
   In complement of subgraph, $V(\overline{H}) = V(H)$ and any two vertices are adjacent in $\overline{H}$ iff they are not adjacent in H. The degree of a vertex in $\overline{H}$ plus its degree in H is (n-1) where n = |V(H)|.
10) Intersection and Union of graphs: Let G and G' be two graphs then G∩G' is a graph whose vertex set is V(G) ∩ V(G') and edge set is E(G) ∩ E(G'). similarly union of graphs G∪ G' is a graph whose vertex set V(G) ∪ V(G') and edge set is E(G) ∪ E(G').
11) Cycle graph: A cycle graph of order n is a connected graph whose edges form a cycle of length n. It is denoted by C_n.
12) Wheel graph: A wheel graph of order n is a graph obtained by joining a single new vertex(" the hub") to each vertex of the cycle graph of order(n- 1). It is denoted by W_n.
13) Path graph: A path graph of order n is obtained by removing an edge from a cycle graph C_n. It is denoted by P_n.
14) Null graph: A Null graph of order n is a graph with n vertices and no edges. It is denoted by N_n.
15) Note: Null graph is contrast to the empty graph, which has no vertices and no edges.
16) Bipartitie graph: A bipartite graph is a non directed graph whose set of vertices can be partitioned into two sets M and N in such a way that each edge joins a vertex in M to a vertex in N.

17) Complete bipartite graph: A complete bipartite graph is a bipartite graph in which every vertex of M is adjacent to every vertex of N. It is denoted by $K_{m,n}$; where |M| =m and |N| =n.
18) Star graph: Any graph that is $K_{1,n}$ is called a star graph.

Problem: Let G be a simple graph all of whose vertices have degree 3 and |E| = 2|V| - 3. What can be said about G?

Solution: Given G is a simple graph. Every vertex have degree 3 (i.e. every vertex has same degree 3) so it is also a 3- regular graph. Given |E| = 2|V| - 3. By sum of degree theorem we have $\sum_{v\in V} \deg(v) = 2|E|$ and since every vertex has degree 3 then we have 3|V| = 2|E| , thus 3|V| = 2[2|V| - 3] = 4|V| - 6, which implies |V| = 6. Hence G a has 6 vertices. So we can conclude G is isomorphic to $K_{3,3}$.

Problem: Is there a graph with degree sequence(1,3,3,3,5,6,6)?

Solution: No, because no.of odd vertices is not even.

Problem: Is there a simple graph with degree sequence (1,1,3,3,3,4,6,7)?

Solution: No, there is no such simple graph because a simple graph of order k can not have a vertex of degree k, i.e. there are 7 vertices in given graph with a vertex of degree 7, so cannot have a simple graph, as there exits loops in graph with degree sequence (1,1,3,3,3,4,6,7). Suppose there is a simple graph with the given degree sequence (1,1,3,3,3,4,6,7) then the vertex of degree 7 is adjacent to all other vertices, so in particular it must be adjacent to vertex of degree 1. Hence a vertex of degree 6 cann't be adjacent to either of the two vertices of degree 1. Thus there must be loop which is contradiction. So there is no simple graph.

Problem: How many vertices will the following graphs have if they contain (i) 16 edges and all vertices of degree 2 (ii) 21 edges with 3 vertices of degree 4 and other vertices of degree 3 (iii) 24 edges and all vertices of same degree.

Solution: (i) given there are 16 edges and all vertices of degree 2, then $k|V| \leq \sum_{v\in V} \deg(v) = 2|E|$, which implies $2|V| \leq 2(16) = 32$, then $|V| \leq 16$.

(ii) there are 3 vertices of degree 4 and other vertices of degree 3, then 3(4) + 3(v) = 2|E| 12+3(v) = 2x 21 = 42, which implies 3(v) = 30 then (v) = 10. There are 10 + 3 = 13 vertices.

(iii) given every vertex of same degree, let us take degree = k then k|V| = 2|E| = 2x 24 = 48, Then |V| = 48/k i.e. divisors of 48.

Isomorphism of Graphs:

Definition: Two graphs G and G' are isomorphic if there is a function f: V(G) -→ V(G') from the vertices of G to the vertices of G' such that (i) f is 1-1 (ii) f is onto and (iii) for each pair of vertices u and v of G {u,v } ∈ E(G) iff { f(u), f(v) } ∈ E(G'). A function f with these three properties is called an isomorphism from G to G' , then the graphs G and G' are said to be isomorphic to each other. The condition (iii) is that vertices u and v are adjacent in G iff f(u) and f(v) are adjacent in G'. In other words we say that the function f preserving adjacency.

Note: If the graphs G and G' are isomorphic and f is an isomorphism of G to G', then we have

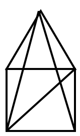
(i) |V(G)| = |V(G')|;

(ii) |E(G)| = |E(G')| ;

(iii) if v ∈ V(G) then deg(v) = deg(f(v)) and thus the degree sequence of G and G' must be same.

(iv) if {v,v} is a loop in G, then {f(v), f(v) } is a loop in G'.

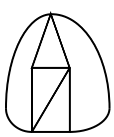
(v) if v0-v1-v2-…..v0 is a cycle of length k in G then f(v0)- f(v1) –f(v2)-……f(v0) is a cycle of length k in G'
(vi) if v1, v2,…..vn are the vertices of G, then the adjacency matrix for this ordering of the vertices of G is the nxn matrix A, where the i jth entry A(i,j) of A is 1 iff there is edge between vi and vj; otherwise A(i,j) = 0. Thus A is a symmetric matrix each of whose entries is either 0 or 1.
Note: Two simple graphs are isomorphic iff their complements are isomorphic.
Planar graphs: when drawing a graph on a piece of paper, we often find it convenient to permit edges to intersect at points other than at vertices of the graph. These points of intersection are called cross overs and the intersecting edges(or crossing edges) are said to cross over each other.


In this graph there are cross over's that is the edges meeting other than vertex points.

Definition: A graph G is said to be planar if it can be drawn on a plane without any cross over's otherwise G is said to be non planar.

Note: A graph G has been drawn with crossing edges, this does not mean that G is non planar, that means there may be another way to draw the graph without cross overs.

Example: the above graph with crossing edges can be drawn with out crossing edges as follows:


Now this graph is without crossing edges, so the graph is planar.

Problem: Suppose we have three houses and three utility outlets (electricity, gas and water) situated so that each utility outlet is connected to each house. Is it possible to connect each utility to each of the three houses without crossing lines(edges)?

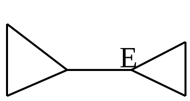
Application: Planar graphs has applications of graph theory to computer science especially in designing and building electrical circuiting.

Definition: Region::- A plane graph G can be thought of dividing the plane into regions or faces. The regions are the connected portions of the plane remaining after all the curves and points of the plane corresponding , respectively to edges and vertices of G have been deleted.

The vertices and edges of G incident with a region r make up the boundary of the region r.

Cut edge: An edge of a graph whose deletion increases the graphs number of connected components , i.e. an edge is a bridge iff it is not contained in any cycle.

Example:



An edge E is cut edge by removing it the number of connected components increases.

If G is connected then the boundary of a region r is a closed path in which each cut edge is traversed twice.

When the boundary contains no cut edges of G, then the boundary of region r is a cycle of G.

In either case the degree of region r is the length of its boundary.

Note: A cycle of G need not be the boundary of a region .

Construction of Dual graph G*: Let G be a plane graph, then we define another multi graph G* as follows: Corresponding to each region r of G, there is a vertex r* in G*; corresponding to each edge e of G there is an edge e* of G*; If two vertices r* and s* are joined by an edge e* in G* iff their corresponding regions r and s are separated by the edge e in G. In particular a loop is added at a vertex r* of G* for each cut edge of G that belongs to the boundary of the region r.

Let |E*| denotes no of edges of G* ; |R*| denotes no.of regions of G* and |V*| denotes no.of vertices of G* and |E|, |R| and |V| denotes no.of edges, regions and vertices of G respectively; then the following relations are direct consequence of the definition of G*; for the plane graph G we have |E*| = |E|; |V*| = |R| and deg(r*) = deg(r) for each vertex r* of G* corresponding region r of G, more over if G is connected it can be shown that |R*| = |V|.

Theorem: If G is plane graph, then the sum of the degree of the regions determined by G is 2|E|.

Proof: let us use the notation $\sum_{r \in R(G)} \text{degree}(r)$ for the sum of the degrees of all the regions determined by G. then if G* is the dual graph of G and let $\sum_{r^* \in V(G^*)} \text{degree}(r^*)$ denotes the sum of degrees of the vertices of G*. then $\sum_{r \in R(G)} \text{degree}(r) = \sum_{r^* \in V(G^*)} \text{degree}(r^*)$ = 2|E*| (by sum of degrees theorem) = 2|E| (since |E*| = |E|) . Hence the sum of degree of the regions determined by G is 2|E|.

Euler’s Formula:

If G is a connected planar graph then any drawing of G in the plane as a plane graph will always form |R|= |E| -|V| +2 ; where |R| , |E| and |V| denotes the number of regions , edges and vertices of G respectively. regions includes the exterior regions also.

Theorem: If G is a connected plane graph then |V| -|E| + |R| = 2.

Proof: we prove this theorem by induction method. Let us prove for one region. Since G is connected plane graph there must have at least two vertices in a graph i.e., |V| = 2 then the number of edges is one less than the number of vertices i.e. |E| = 1; and an edge is always incident on two vertices and an edge with two vertices represent one region. Then |R| = 1. Hence |V| -|E| + |R| = 2 -1+1 = 2. Therefore for one region i.e. R = 1 the result is true. Let us assume that the result is true for |R| = k ; that means |V| - |E| + |R| =2 holds good for k number of regions.

Let us prove the result holds for k+1 regions. Let us consider a connected graph with k+1 number of regions. Now delete an edge common to the boundary of two separate regions. The resulting graph G’ has the same number of vertices ; one fewer edge and also one fewer region as two previous regions have been consolidated by the removal of edge. Let |E'| |V'| and |R'| denote the number of edges, vertices and regions in G' then |E'| = |E| -1 ; |R'| = |R| -1 and |V'| = |V| .

Therefore |V| - |E| + |R| = |V'| - [|E'| +1] + [|R'| +1]

$$= |V'| - |E'| + |R'| = 2 \text{ ; since the result holds good for k number of regions.}$$

Hence |V| - |E| + |R| = 2 ; thus the result holds good for K+ 1 number of regions. Therefore by induction process we conclude the result |V| - |E| + |R| =2.

Note: we shall assume the simple graph and |E| > 1 and assume that the degree of each region is greater than or equal to 3.Thus a connected plane graph is polyhedral if degree (r) ≥ 3 for each region r in R(G). and if degree (v) ≥ 3 for each vertex v in V(G).

In a plane graph G, if the degree of each region is ≥ k then by sum of degree's theorem k|R| ≤ 2|E|.

In particular we have 3|R| ≤ 2|E|.

Theorem: In a connected plane graph G with |E| > 1, then we have a (i) |E| ≤ 3|V| - 6 (ii) there is a vertex v in G such that deg(v) ≤ 5.

Proof: Let G be a connected plane graph with |E| > 1. (i) Then by Euler's formula we have |R| + |V| = |E| + 2; and G is plane graph 3|R| ≤ 2 |E|. then $|R| \leq \frac{2}{3}|E|$. Therefore $|R| + |V| \leq \frac{2}{3}|E| + |V|$; but LHS is |E| +2 thus the in equality become $|E| + 2 \leq \frac{2}{3}|E| + |V|$ which implies 3|V| - 6 ≥ |E| .

(ii) Let us suppose each vertex v of G has a degree greater than or equal to 6. Since by sum of degree theorem we have $\sum_{v \in V(G)} \deg(V) = 2|E|$ and as G is connected plane graph we have k|V| ≤ 2 |E| as degree of every vertex is 6 we have 6|V| ≤ 2|E| or $|V| \leq \frac{1}{3}|E|$. Now by Euler's formula we have |R| + |V| = |E| + 2; thus $\frac{2}{3}|E| + \frac{1}{3}|E| \geq |R| + |V| = |E| + 2$; which implies |E| ≥ |E| + 2 i.e. 0 ≥ 2 which is contradiction , thus the assumption that every vertex of G has degree 6 is false, hence there is a vertex v in G such that deg(v) ≤ 5.

Theorem: A complete graph K_n is planar iff n ≤ 4.

Proof: It is easy to show that K_n is planar for n = 1,2, 3 and 4. Thus we have to show that K_n is non planar if n≥ 5. And for this it is sufficient to show that k_5 is non planar. Let us assume that K_5 is planar. Then by Euler's formula |R| = |E| - |V| + 2 = 10 -5 + 2 = 7. Therefore the number of regions = 7. But since K_n is simple connected we have 3|R| ≤ 2 |E|. Hence 3. 7 ≤ 2. 10 ; that is 21 ≤ 20 which is contradiction. Therefore our assumption K_5 planar is false, hence K_5 is non planar.

Theorem: A complete bipartite graph $K_{m,n}$ is planar if m ≤ 2 and n≤ 2.

Proof: It is easy to see that $K_{m,n}$ is planar for m= n = 1 and m = n =2 . now let m ≥ 3 and n≥ 3. To prove that $K_{m,n}$ is non planar it is sufficient to prove that $K_{3,3}$ is non planar. Since $K_{3,3}$ has six vertices and nine edges and also assume $K_{3,3}$ is planar. Then by Euler's formula we have |R| = |E| -|V| +2 = 9-6+2 = 5. Thus there are 5 regions in $K_{3,3}$. Since $K_{3,3}$ is complete bipartite there are no cycles of odd length. Hence every cycle in $K_{3,3}$ has length $\geq$ 4 and thus the degree of each region would have to be greater than or equal to 4. Since the graph is simple, we have $k|R| \leq 2|E|$ where $k = 4$, therefore $4|R| \leq 2|E|$ which implies $4.\ 5 \leq 2.\ 9$ that is $20 \leq 18$ which is contradiction. Therefore $K_{3,3}$ is non planar.

Euler graph:

Euler path: An Euler path in a multi graph is a path that includes each edge of the multi graph exactly once and intersects each vertex of the multi graph at least once.

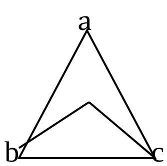
A multi graph is said to be traversable if it has an Euler path.

Euler circuit: An Euler circuit is an Euler path whose end points are identical

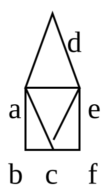
A multi graph is said to be an Eulerian multi graph if it has an Euler circuit.

A non directed multi graph has Euler circuit iff it is connected and all of its vertices are of even degree.

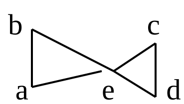
Example:



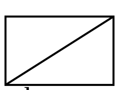
the Euler path is(c-a-b-d-c-d) in which every edge taken exactly once and every vertex atleast once., but we cann't form Euler circuit in this graph as end points in Euler path are not identical and even if we go from d to c then edge repetition takes palce.



the Euler circuit is a-d-e-a-c-e-f-c-b-a. Hence this graph is Euler graph.



in this graph the Euler circuit is (a-e-c-d-e-b-a) as it includes every edge exactly once.



Since there are two vertex of odd degree thus, their no Euler circuit, so it is not Euler graph.

Hamiltonian graph: A graph G is said to be Hamiltonian if there exists a cycle containing every vertex of G such a cycle is called a Hamiltonian cycle. Thus a Hamiltonian graph is a graph containing a Hamiltonian cycle.

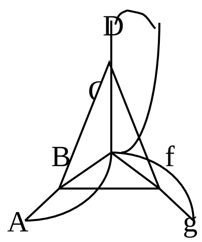
Hamiltonian path: A simple path that contains all vertices of G but where the end points may be distinct.

A Hamiltonian cycle traverse every vertex exactly once. A Hamiltonian cycle always provides a Hamiltonian path up on deletion of any edge, but Hamiltonian path may not lead to Hamiltonian cycle.

Some Basic rules for constructing Hamiltonian paths and cycles: (i) If G has n vertices then a Hamiltonian path must contain exactly (n- 1) edges and a Hamiltonian cycle must contain exactly n edges. (ii) If a vertex v in G has degree k, then a Hamiltonian path must contain at least one edge incident on v and atmost two edgs incident on v. A Hamiltonian cycle will of course contain exactly two edges incident on v. In particular both edges incident on a vertex of degree two will be contained in every Hamiltonian cycle i.e., there cannot be three or more edges incident with one vertex in a Hamiltonian cycle.

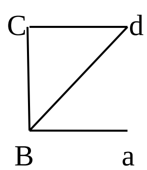
(iii) No cycle that does not contain all the vertices of G can be formed when building a Hamiltonian path or cycle.

(iv) once the Hamiltonian cycle we are building has passed through a vertex v, then all other unused edges incident on v can be included in a Hamiltonian cycle.



the path through the vertices of G form Hamiltonian path i.e. a-b-c-d-e-f-g.

however G has no Hamiltonian cycle.



In this graph A-B-C-D is a Hamiltonian path as every edge taken exactly once but we can not form Hamiltonian cycle.

Grinberg’s Theorem: Let G be a simple plane graph with n vertices. Suppose that C is a Hamiltonian cycle in G then with respective to C, $\sum_{i=3}^{n}(i-2)(r_i - r_i') = 0$.

Problem: Use Grinberg’s theorem to show that there are no planar Hamiltonian graphs with regions of degree 5,8 and 9 with exactly one region of degree 9.

Solution: Let us suppose there is a Hamiltonian graph with regions of degree 5, 8 and 9 where the region of degree 9 is only one. In Hamiltonian graph there will be a Hamiltonian cycle and given there are regions of degree 5,8 and 9. Then using Grinberg’s theorem we have $\sum_{i=3}^{n}(i-2)(r_i - r_i') = 0$ for i = 5,8 and 9 such that $(r_9 - r_9') = \pm 1$. Therefore (5-2) $(r_5 - r_5') + (8-2)(r_8 - r_8') + (9-2)(r_9 - r_9') = 0$

$3\,(r_5 - r_5') + 6\,(r_8 - r_8') + 7(r_9 - r_9') = 0$; where $(r_9 - r_9') = \pm 1$

Therefore $3\,(r_5 - r_5') + 6\,(r_8 - r_8') = \pm 7$; which implies 3 divides ± 7, which is contradiction. Hence there is no Planar Hamiltonian graph of region of degree 5, 8 and 9 with exactly one region of degree 9 .

Graph coloring and Chromatic number:

Definition: A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Definition: The Chromatic number of a graph is the least number of colors needed for a coloring of this graph. The Chromatic number of a graph G is denoted by $\chi(G)$.

Rules of Chromatic Number:

1) $\chi(G) \leq |V|$, where $|V|$ is the number of vertices of G.
2) K_n is the complete graph on n vertices and chromatic number is $\chi(K_n) = n$.
3) If some subgraph of G requires k colors then $\chi(G) \geq k$.
4) If degree(v) = d, then atmost d colors are required to color the vertices adjacent to v.
5) $\chi(G) = \text{maximum}\{ \chi(c) /\ c \text{ is connected component of } G\}$
6) every k-chromatic graph has atleast k vertices v such that degree (v) $\geq k - 1$.
7) For any graph G, $\chi(G) \leq 1 + \Delta(G)$; where $\Delta(G)$ is the largest degree of any vertex of G.

what is the chromatic number of cycle graph?

The chromatic number of a cycle graph is either 2 or 3 depending on whether its length is even or odd.

What is chromatic number of $K_{3,3}$?

As per the rule we have $\chi(G) \leq 1 + \Delta(G)$ where $\Delta(G)$ is highest degree of any vertex of graph G, then for $\chi(G) \leq 1 + 3 = 4$.

What is chromatic number of complete bipartite graph $K_{m,n}$?

$\chi(G) = 2$. The number of colors needed may seem to depend on m and n. however only two colors are needed because $K_{m,n}$ is a complete bipartite.