VSAQ

A.Y: 2020-21

Course code: 17MA1301A

Branch: IT

Unit - I

1. Write the real and imaginary parts of $\sin z$

A. $w = f(z) = \sin z = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + \cos x \, i \sinh y$

$= \sin x \cosh y + i \cos x \sinh y$

$\therefore \; u = \sin x \cosh y, \; v = \cos x \sinh y$

2. Define limit, continuity, Derivative of a complex function

A. **Limit** A function $w = f(z)$ is said to tend to a limit 'l' as z approaches a point z_0, if for every real ε, we can find a +ve real δ such that $|f(z) - l| < \varepsilon$ for $|z - z_0| < \delta$ i.e. $\underset{z \to z_0}{Lt} f(z) = l$

Continuity: A function $w = f(z)$ is said to be continuous at $z = z_0$ if $\underset{z \to z_0}{Lt} f(z) = f(z_0)$

Derivative: A function $w = f(z)$ is said to be differentiable if $\underset{\delta z \to 0}{Lt} \dfrac{f(z+\delta z) - f(z)}{\delta z} = f'(z)$ exists and has the same value for all different ways in which $\delta z \to 0$

3. What are C-R eqns (i) Cartesian form (ii) Polar form

(A) (i) Cartesian form: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(ii) polar form: $u_r = \frac{1}{r} v_\theta$, $v_r = -\frac{1}{r} u_\theta$

4. Define (a) analytic function (b) entire function with example

A. Analytic function: A function $f(z)$ which possesses a unique derivative at all points of a region R is called an Analytic function

Entire function: A function which is analytic every where in the complex plane is known as an entire function

eg: z^3, e^z, $\sin z$, $\cos z$ are examples of entire function

5. Define Singularpoint.

A. A point at which an analytic function ceases to possesses a derivative is called a Singular point of $f(z)$.

6. Write Laplace equation in (i) cartesian form (ii) polar form

A (i) Cartesian form: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(ii) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

7. Write the complex form of $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

A. $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

8: Define Harmonic function and S.T $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic

Sy: Harmonic function: Any function $\phi(x,y)$ in 2 variables satisfying Laplace equation is called Harmonic function (ie) $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

$u = \frac{1}{2}\log(x^2+y^2)$ $\qquad u_y = \frac{1}{2}\frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2}$

$u_x = \frac{1}{2}\frac{1}{x^2+y^2} \times 2x = \frac{x}{x^2+y^2}$

$u_{xx} = \frac{x^2+y^2-2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$ $\qquad u_{yy} = \frac{x^2+y^2(1) - y(2y)}{(x^2+y^2)^2}$

$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$\therefore u_{xx} + u_{yy} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0$

$\therefore$ u is Harmonic.

9. Is $|z|^2$ analytic function?

A given $f(z) = |z|^2 = |x+iy|^2 = (\sqrt{x^2+y^2})^2 = x^2+y^2$

$\therefore u = x^2+y^2,\ v = 0$ $\qquad u_x = 2x,\ u_y = 0$

$v_x = 0,\ v_y = 0$

Clearly $u_x = v_y,\ u_y = -v_x$ only at $(0,0)$

Hence $|z|^2$ is not analytic

10. Determine 'p' such that the function $f(z) = \frac{1}{2}\log_e(x^2+y^2) + i\tan^{-1}\left(\frac{px}{y}\right)$ be an analytic function

Sol: $u = \frac{1}{2}\log_e(x^2+y^2)$ $\qquad v = \tan^{-1}\left(\frac{px}{y}\right)$

$u_x = \frac{1}{2}\frac{1}{x^2+y^2}(2x) = \frac{x}{x^2+y^2}$, $u_y = \frac{1}{2}\frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2}$

$v_x = \frac{1}{1+\left(\frac{px}{y}\right)^2}\left(\frac{p}{y}\right)$, $\quad v_y = \frac{1}{1+\left(\frac{px}{y}\right)^2} px\left(-\frac{1}{y^2}\right)$

$\therefore f(z)$ is Analytic $\therefore u_x = v_y$ and $u_y = -v_x$

$\frac{x}{x^2+y^2} = \frac{-x}{\frac{y^2+p^2x^2}{y^2}} \times \frac{p}{y^2} \Rightarrow \frac{x}{x^2+y^2} = \frac{-pxy^2}{y^2+p^2x^2} \times \frac{1}{y^2}$

$\Rightarrow \cancel{y^2+p^2x^2 = px^2+y^2}$

$\Rightarrow \frac{1}{x^2+y^2} = \frac{-p}{y(y^2+p^2x^2)}$

~~only~~ when $p = -1$

11. Determine the value of 'k' so that $x^3 + 3kxy^2$ may be Harmonic

A. Let $u = x^3 + 3kxy^2$ $\quad u_y = 6kxy$

$u_x = 3x^2 + 3ky^2$ $\quad u_{yy} = 6kx$

$u_{xx} = 6x$

$\because u$ is Harmonic $\therefore u_{xx} + u_{yy} = 0$

$\Rightarrow 6x + 6xk = 0 \quad 6x(k+1) = 0 \Rightarrow k+1 = 0$

$\Rightarrow k = -1$

12. P.T $\int_C \frac{dz}{z-a} = 2\pi i$ where c is the circle $|z-a| = r$

Sol: $\because$ c is $|z-a| = r \Rightarrow z-a = re^{i\theta} \Rightarrow z = a + re^{i\theta}$ and 'θ': 0 to 2π

$\therefore dz = ire^{i\theta}\, d\theta$

$$\therefore \int_C \frac{dz}{z-a} = \int_{\theta=0}^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}}\, d\theta = i\int_0^{2\pi} d\theta = 2\pi i.$$

13. State Cauchy's theorem or Cauchy's integral theorem

A. If $f(z)$ is an Analytic function and $f(z)$ is continuous at each point with in and on a closed curve C then $\int_C f(z)\, dz = 0$

14. State Cauchy's integral form

A If $f(z)$ is Analytic with in and on a closed curve C and if 'a' is any point with in C, then

$$\oint_C \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a)$$

15. P.T $\int_C (z-a)^n\, dz = 0 \; (n \neq -1)$ where c is the circle $|z-a| = r$

A. $|z-a| = r \Rightarrow z-a = re^{i\theta}$ $\quad dz = ire^{i\theta}\, d\theta, \quad \theta$: 0 to 2π

$$\int_C (z-a)^n\, dz = \int_0^{2\pi} (re^{i\theta})^n\, ire^{i\theta}\, d\theta$$

$$= i r^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} \, d\theta \qquad (n \neq -1)$$

$$= i r^{n+1} \left[\frac{e^{i(n+1)\theta}}{i(n+1)} \right]_{\theta=0}^{2\pi} = \frac{r^{n+1}}{n+1} \left(e^{i(n+1)\theta} \right)_{\theta=0}^{2\pi}$$

$$= \frac{r^{n+1}}{n+1} \left[e^{i(n+1)2\pi} - 1 \right] = 0$$

$$\because e^{i(2n+2)\pi} = \cos(2n+2)\pi + i\sin(2n+2)\pi$$
$$= 1 + i(0) = 1$$

Unit-II

(1) What is Taylor series expansion of $f(z)$

(A) If $f(z)$ is analytic inside a circle C with Centre at 'a' then for z inside C,

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots$$

or

$$f(z) = a_0 + a_1 (z-a) + a_2 (z-a)^2 + \cdots + a_n (z-a)^n + \cdots$$

where $a_n = \frac{f^n(a)}{n!}$

(2) Write Laurent series expansion of $f(z)$

(A) If $f(z)$ is analytic in the ring shaped region R bounded by two concentric circles C and C_1 of radii r and r_1 $(r > r_1)$ and with centre at 'a' then for all z in R

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \cdots$$

where $a_n = \frac{1}{2\pi i} \oint_R \frac{f(t)}{(t-a)^{n+1}} dt$ and $a_{-n} = \frac{1}{2\pi i} \oint_C \frac{f(t)\, dt}{(t-a)^{-n+1}}$

(3) Expand $\cos z$ in Taylor series about $z = \pi/2$

Sol Given $f(z) = \cos z \qquad a = \pi/2 \qquad f(\frac{\pi}{2}) = 0$

$f'(z) = -\sin z \qquad f'(\pi/2) = -1$

$f''(z) = -\cos z \qquad f''(\pi/2) = 0$

$f'''(z) = \sin z \qquad f'''(\pi/2) = 1$ etc

$\therefore \cancel{\cos z}\, f(z) = f(\pi/2) + (z-\pi/2) f'(\pi/2) + (z-\pi/2)^2 f''(\pi/2) + \cdots$

$\therefore \cos z = 0 + (z-\pi/2) \times -1 + \frac{(z-\pi/2)^2}{2!} \times 0 + \frac{(z-\pi/2)^3}{3!} \times 1 + \cdots$

(4) Expand $\frac{1-\cos z}{z^3}$ in Laurent series about $z=0$

(A) $f(z) = \frac{1-\cos z}{z^3} = \frac{1}{z^3}\left[1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)\right]$

$\left(\because \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$

$\therefore f(z) = \frac{1}{z^3}\left[\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \frac{z^8}{8!} + \cdots\right]$

$= \frac{1}{2!\,z} - \frac{z}{4!} + \frac{z^3}{6!} - \frac{z^5}{8!} + \cdots$

contains both positive and negative powers of z

(5) Define zero of an Analytic function $f(z)$ and find zero of $\left(\frac{z+1}{z^2+1}\right)^2$

(A) A zero of an analytic function $f(z)$ is that value of z for which $f(z) = 0$

Here $f(z) = \frac{(z+1)^2}{(z^2+1)^2} = 0 \Rightarrow (z+1)^2 = 0 \Rightarrow z = -1$

$\therefore$ $z=-1$ is a zero of order 2

(6) Define isolated singularity

(A) A point $z=a$ is called isolated singularity of $f(z)$ if it is analytic at each point in its neighbourhood

(7) write the poles of $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$

(A) $f(z)$ has simple pole at $z=-2$
and pole of order 2 at $z=1$

(8) Find the nature and location of singularities of
(a) $\dfrac{z-\sin z}{z^2}$ (b) $\dfrac{1}{\sin z-\cos z}$ (c) $\dfrac{e^{1/z}}{z^2}$

(A) (a) $f(z) = \dfrac{z-\sin z}{z^2} = \dfrac{1}{z^2}\left[z-\left[z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-\dfrac{z^7}{7!}+\cdots\right]\right]$ The singular point of $f(z)$ $=0$

$\left(\because \sin x = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\cdots\right)$

$\therefore f(z) = \dfrac{1}{z^2}\left[\dfrac{z^3}{3!}-\dfrac{z^5}{5!}+\dfrac{z^7}{7!}-\cdots\right]$

$= \dfrac{z}{3!}-\dfrac{z^3}{5!}+\dfrac{z^5}{7!}-\cdots$

$\therefore$ There are no negative powers of z

$\therefore$ $z=0$ is removable singularity

(b) Poles of $f(z) = \dfrac{1}{\sin z-\cos z}$ are given by

writing $\sin z-\cos z=0 \Rightarrow \sin z = \cos z$

$\Rightarrow \dfrac{\sin z}{\cos z}=1 \Rightarrow \tan z = 1 \Rightarrow z=\pi/4$

clearly $z=\pi/4$ is simple pole

(c) $f(z) = \frac{e^{1/z}}{z^2}$

The singular point of $f(z)$ is $z^2 = 0 \Rightarrow z = 0$

~~Hence is~~ a pole of ~~order~~ order 2

$$f(z) = \frac{e^{1/z}}{z^2} = \frac{1}{z^2}\left[1 + \frac{1}{1!}\frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots\right]$$

$$\left(\because e^x = 1 + x + \frac{x^2}{2!} + \cdots\right)$$

$$\therefore f(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{2!}\frac{1}{z^4} + \cdots$$

$$= z^{-2} + z^{-3} + \frac{1}{2!}z^{-4} + \cdots$$

$\because$ There are infinite number of negative powers of z

$\therefore$ $z = 0$ is called essential singularity

9. Define Residue of $f(z)$

A The coefficient of $(z-a)^{-1}$ in the expansion of $f(z)$ is called residue of $f(z)$ around an isolated singularity

10. State Residue theorem

A. If $f(z)$ is analytic in a closed curve C except at a finite number of singular points with in C, then

$$\oint_C f(z)\,dz = 2\pi i \times [\text{sum of the residues at the singularities}]$$

11. Write the $\mathrm{Res}\, f(z)$ at the (i) simple pole $z = a$
(ii) at a pole of order n

A. (i) $\mathrm{Res}\, f(a) = \lim_{z \to a} (z-a) f(z)$

(ii) $\mathrm{Res}\, f(a) = \lim \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}(z-a)^n f(z)\right]_{z=a}$

12. What is a bilinear transform

A $w = \frac{az+b}{cz+d}$ where a, b, c, d are complex constants

and $ad - bc \neq 0$

13. When do you say that $w = f(z)$ is conformal

A. If $f(z)$ is analytic and $f'(z) \neq 0$ in a region R of the z-plane then the mapping $w = f(z)$ is conformal

14. What is the critical point of a Transformation $w = f(z)$

A. A point at which $f'(z) = 0$ is called critical point

15. Find the fixed points or invariant points of $f(z) = \frac{z-1}{z+1}$

A. Fixed points of $f(z)$ are given by writing $f(z) = z$

$$\therefore \quad z = \frac{z-1}{z+1} \Rightarrow z^2 + z = z - 1$$
$$\Rightarrow z^2 = 1 \Rightarrow z = \pm 1$$

are the invariant points or fixed points

16. Find the critical points of $w = z + \frac{1}{z}$

A The critical points are given by writing $\frac{dw}{dz} = 0$ or $f'(z) = 0$

$$\therefore f'(z) = \frac{dw}{dz} = 1 - \frac{1}{z^2} = 0$$
$$\Rightarrow z^2 - 1 = 0 \Rightarrow z = \pm 1$$