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for $p \to q$

Converse :- $q \to p$

Inverse : $\sim p \to \sim q$

contrapositive $\sim q \to \sim p$

$\boxed{p \to q \equiv \sim q \to \sim p}$

* prove $p \to q \equiv \sim q \to \sim p$

p	q	$\sim p$	$\sim q$	$p \to q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

logical equivalance :- Two compound prepositions are said to be logically equivalent if there bi-conditional is a Tautology

$A \equiv B$ (or) $A \Rightarrow B$

$\to$ $A \leftrightarrow B \equiv$ Tautology.

Q) Is $\sim(p \vee q) \equiv \sim p \wedge \sim q$?

p	q	$\sim p$	$\sim q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$	$\sim(p \vee q) \leftrightarrow \sim p \wedge \sim q$
T	T	F	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	F	T
F	F	T	T	T	T	T

$\therefore \sim(p \vee q) \equiv (\sim p \wedge \sim q)$

Note:- There are different terminologies to express the Connective conditional. They are as follows:

(i) if p then q

(ii) p implies q

(iii) if p, q

(iv) p only, if q

(v) p is sufficient for q

(vi) A sufficient condition for q is p

(vii) q if p

(viii) q whenever p

(ix) q when p

(x) q is necessary for p

(xi) A necessary condition for p is q

(xii) q follows from p

(xiii) q unless $\sim p$

* for bi-Conditional statement $(p \leftrightarrow q)$:-

(i) p is necessary & sufficient for q

(ii) if p then q and conversly

(iii) p iff q (p if and only if q)

Q) Construct truth table for $(p \leftrightarrow q) \leftrightarrow (\sim p \vee \sim q)$.

p	q	$\sim p$	$\sim q$	$p \leftrightarrow q$	$\sim p \vee \sim q$	$(p \leftrightarrow q) \leftrightarrow (\sim p \vee \sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	T	T	T

Compound Preposition:-

Because most of the mathematical statements are constructed by combing one or more than one proposition. As simple as that.

Note:- There are different terminologies to express the connective conditional. They are as follows:

(i) if p then q
(ii) p implies q
(iii) if p, q
(iv) p only, if q
(v) p is sufficient for q
(vi) A sufficient condition for q is p
(vii) q if p
(viii) q whenever p
(ix) q when p
(x) q is necessary for p
(xi) A necessary condition for p is q
(xii) q follows from p
(xiii) q unless $\sim p$

* for bi-conditional statement $(p \leftrightarrow q)$:-

(i) p is necessary & sufficient for q
(ii) if p then q and conversly
(iii) p iff q (p if and only if q)

Q) Construct truth table for $(p \leftrightarrow q) \leftrightarrow (\sim p \vee \sim q)$

p	q	$\sim p$	$\sim q$	$p \leftrightarrow q$	$\sim p \vee \sim q$	$(p \leftrightarrow q) \leftrightarrow (\sim p \vee \sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	T	T	T

Compound Prepositions:-

Because most of the mathematical statements are constructed by combing one or more than one proposition. As

Q) Is $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$?

p	q	r	$p \vee q$	$p \vee r$	$q \wedge r$	$p \vee (q \wedge r)$ (A)	$(p \vee q) \wedge (p \vee r)$ (B)	$A \leftrightarrow B$
T	T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T	T
T	F	T	T	T	F	T	T	T
T	F	F	T	T	F	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F	T
F	F	T	F	T	F	F	F	T
F	F	F	F	F	F	F	F	T

$\therefore p \vee (q \wedge r) \equiv (p \vee r) \wedge (p \vee r)$

* **Some equivalent forms:-**

1) $p \to q \equiv \sim p \vee q$
2) $p \to q \equiv \sim q \to \sim p$
3) $p \vee q \equiv \sim p \to q$
4) $p \wedge q \equiv \sim (p \to \sim q)$
5) $\sim (p \to q) \equiv p \wedge \sim q$
6) $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
7) $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
8) $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$

1) $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
2) $p \leftrightarrow q \equiv \sim p \leftrightarrow \sim q \equiv (\sim p \to \sim q) \wedge (\sim q \to$
3) $p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$
4) $\sim (p \leftrightarrow q) \equiv p \leftrightarrow \sim q$

H.W

1) Prove the following

(i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

2) $(p \vee q) \leftrightarrow \sim [(p \vee \sim q) \wedge (\sim p \to q)]$

3) $(p \leftrightarrow \sim q) \leftrightarrow [(p \vee \sim q) \vee (\sim p \wedge q)]$

(i)

p	q	r	p ∨ q	p ∨ r	q ∧ r	p ∨ (q ∧ r)	(p ∨ q) ∧ (p ∨ r)	A ↔ B
T	T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T	T
T	F	T	T	T	F	T	T	T
T	F	F	T	T	F	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F	T
F	F	T	F	T	F	F	F	T
F	F	F	F	F	F	F	F	T

(ii)

p	q	~p	~q	p ∨ q	p ∨ ~q	~p → q	~[(p ∨ ~q) ∧ (~p → q)]
T	T	F	F	T	T	T	F
T	F	F	T	T	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	F	T	T	T

(p ∨ q) ↔ ~[(p ∨ ~q) ∧ (~p → q)]
F
F
T
F

(iii)

p	q	~p	~q	A: p ↔ ~q	p ∨ ~q	~p ∧ q	B: (p ∨ ~q) ∨ (~p ∧ q)	A ↔ B
T	T	F	F	F	T	F	T	F
T	F	F	T	T	T	F	T	T
F	T	T	F	T	T	T	T	T
F	F	T	T	F	T	F	T	F

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* Write converse, inverse & contrapositive for the following statement.

(i) Home Team wins whenever it is raining

q whenever p

p : It is raining

q : The home town win

$p \rightarrow q$:- if it is raining them home team wins

Converse : $q \rightarrow p$

if home team wins then it is raining.

Inverse :- $\sim p \rightarrow \sim q$

~~if home Team~~

if it is not raining then home Team not wins

Contrapositive:- $\sim q \rightarrow \sim p$

if the home team not wins then it is not raining.

* Translating the given english sentences into symbolical form (or) logical expression.

Q) Express the following senstence as a logical expression.

a) You can access the internet from the campus (p) only if your are a student of this college (q) and you are not a 1st year student (r).

Ans:- $p \rightarrow (q \wedge \sim r)$

b) You can not ride a car if you are under 4 feet Tall unless you are older than 18 years

p : you can ride a car

q : you are under 4 feet Tall

r : you are olderthan 18 yrs

$(q \wedge \sim r) \rightarrow p$

~~logical~~

1) $P \wedge T \equiv P$
$P \vee F = P$ } Identity law

2) $P \vee T \equiv T$
$P \wedge F \equiv F$ } Dominant law

3) $\sim(\sim P) \equiv P \rightarrow$ complement law

4) $P \wedge P \equiv P$
$P \vee P \equiv P$ } Idempotent law

5) $P \vee q \equiv q \vee P$
$P \wedge q \equiv q \wedge P$ } commutative law

6) $P \vee (q \vee r) \equiv (P \vee q) \vee r$
$P \wedge (q \wedge r) \equiv (P \wedge q) \wedge r$ } Associative law

7) $P \vee (q \wedge r) \equiv (P \vee q) \wedge (P \vee r) \rightarrow$ Distributive law

8) $\sim (P \vee q) \equiv \sim P \wedge \sim q$
$\sim (P \wedge q) \equiv \sim P \vee \sim q$ } Demorgan's law

9) $P \vee (P \wedge q) \equiv P \rightarrow$ absorption law

10) $P \vee \sim P \equiv T$
$P \wedge \sim P \equiv F$ } negation law

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1) Show that $(P \wedge q) \rightarrow (P \vee q)$ is a Tautology without Truth Table.

$LHS \equiv (P \wedge q) \rightarrow (P \vee q)$

$= \sim (P \wedge q) \vee (P \vee q)$ $\quad (\because P \rightarrow q \equiv \sim P \vee q)$

$= (\sim P \vee \sim q) \vee (P \vee q)$ $\quad$ By demorgan's law

$= \sim P \vee (\sim q \vee (P \vee q))$

$= \sim P \vee (\sim q \vee (q \vee P))$ $\quad$ By commutative law

$= \sim P \vee ((\sim q \vee q) \vee P)$ $\quad$ By associative law

$= \sim P \vee (T \vee P)$ $\quad$ By Negation law

$= \sim P \vee T = T$ $\quad$ By dominant law

$\therefore$ Hence proved

2) Prove that negation $\sim(p \vee (\sim p \wedge q))$ and $\sim p \wedge \sim q$ are logically equivalent without constructing a truth table.

<u>LHS</u> $\equiv \sim(p \vee (\sim p \wedge q))$

By demorgan's law

$\sim p \wedge (\sim(\sim p \wedge q))$ — Complement law

$\sim p \wedge (p \vee \sim q)$ — Distributive law

$(\sim p \wedge p) \vee (\sim p \wedge \sim q)$ — negation law

$F \vee (\sim p \wedge \sim q)$ — Identity ~~Dominant~~ law

$\sim p \wedge \sim q \equiv$ <u>RHS</u>

<u>Predicate logic</u> :-

→ ~~Predicates~~ If the statement contains a variable as a first part which is subject of the statement and the second part of the statement is printed as a predicate.

<u>ex</u>:- x is greater than 5)

if which x is variable and "is greater than 5" is a predicate.

→ we usually denote the statement by $P(x)$.

→ This statement $P(x)$ is also known as prepositional function of P at x.

→ If we assign certain value to the variable 'x' then the statement $P(x)$ becomes a preposition and then it may have certain truth value.

$P(x)$: x is greater than 5

$P(7) = T$

$P(3) = F$

Universe of dicourse:- (or) Domain of discourse:-

Many mathematical statements . assert that a property is true for all values of a variable in a particular domain is called domain of discourse.

$P(x) \to T$, $P(x) \to F$

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Quantifiers :- Used to Quantify a statement

Two Types

(i) Universal ⟶ given by $\forall x$, $P(x)$ → we read it as for all x, $P(x)$

(ii) Existential

for every x, $P(x)$

for each x, $P(x)$

any x, $P(x)$

for some x, $P(x)$

for atleast one x, $P(x)$

$\forall x, P(x)$ → for every x in domain	→	if there exists atleast one x which is violating the condition
↓ Truthvalue	↓ T	↓ F

Q) What is the truth value of $P(x): x+1>x$ for every $x \in R$

$\forall x, P(x)$ is T

Q) What is the truth value of (i) $P(x): x^2>0$; $x \in Z^+$

(ii) $P(x): x^2>0$; $x \in Z$

Sol:- (i) $\forall x, P(x)$ is T

(ii) $\forall x, P(x)$ is F

$\exists x;\ x=0$

Note:

if $P_1(x), P_2(x) \ldots P_n(x)$ are propositional functions then $P_1(x) \wedge P_2(x) \wedge \ldots \wedge P_n(x)$ is True if every $P(i)$ of x is true.

$P(x): x^2<10$ x is a set of positive integers not exceeding 4.

$P(1)=T$, $P(2)=T$, $P(3)=T$, $P(4)=F$

$\forall x, P(x) = F$

$\forall x, P(x)$ can be read as for some x
atleast one x
There is a x
for arbitary x

Q) $P(x) : x^2 \geq x \quad x \in R$

$P(0) = T \quad P(1) = T \quad P(2) = T \quad - - -$

$\therefore \quad \forall x, P(x) = T$

* $\forall x (x, y) \rightarrow x$ is bounded
y is free

* statements involving predicates & Quantifiers are logically equivalent if they have the same Truth value

Q) Is $\underbrace{\forall x (P(x) \wedge Q(x))}_{A}$ and $\underbrace{\forall x P(x) \wedge \forall x Q(x)}_{B}$ are logically equivalent?

$A \equiv \forall x (P(x) \wedge Q(x))$ is T

$\equiv P(a) \wedge Q(a) \quad [a \in D]$ is T

$\equiv P(a)$ is T and $Q(a)$ is T

$\equiv \forall x P(x)$ is T & $\forall x Q(x)$ is T

$\equiv \underbrace{\forall x P(x) \wedge \forall x Q(x)}_{B}$ is T

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Negation of Quantifiers:-

~ [for every x, $P(x)$ is True] $\equiv$ for atleast one x, $P(x)$ is not True

* $\sim [\forall x, P(x)] \equiv \exists x, \sim P(x)$

* $\sim [\exists x, P(x)] \equiv \forall x, \sim P(x)$

~ [for atleast one x, $P(x)$ is true] $\equiv$ for every x, $P(x)$ is not true

ex:- one Topper in class — ex:- In whole class no one is Topper

Q) write the negation of "every politican, is honest"

P(x): Politicion is honest $\quad [\forall x, P(x)]$

$\sim[\forall x, P(x) \text{ is } T]$

atleast one politican, is not honest $\equiv [\exists x, \sim P(x)]$

Q) Write the negation of "All Indian, are rice Eaters"

P(x): Indians are rice Eaters

$$\sim(\forall x, P(x) \text{ is } T] \equiv \exists x, \sim P(x)$$

Atleast one Indian, is not rice eater.

Q) negation of $[\exists x, x^2=2]$

$$\sim[\exists x, x^2=2] \equiv [\forall x, \sim(x^2=2)]$$

$$\equiv [\forall x, x^2 \neq 2]$$

Q) Express the given statement using predicates and quantifiers.

(i) for every person x, if person x is a student in the class then x has studied mathematics.

P(x): person x is a student in the class

q(x): x has studied mathematics

~~$[\forall x, P(x) \to q(x)]$~~

$$[\forall x, P(x) \to q(x)]$$

(ii) some student in this class has visited Japan.

$[\exists x, P(x)]$

P(x): student visited Japan.

Conclusion & Premice:-

* If the given statement poses 4 (or) 5 sentences such that the last sentense follows all the remaining sentences then we call the last sentence as a conclusion and the remaining sentences are called premices.

Q) "All lions are dangerous, some lions do not drink water, Some dangerous creatures do not drink water." express this statement symbolically using predicates & Quantifiers.

sol:- In the above paragraph first 2 sentences are premices & last sentence is conclusion.

$L(x)$:- lions
$D(x)$:- Dangerous
$W(x)$:- drinks water

$$\forall x\, [L(x) \to D(x)]$$
$$\exists x\, [L(x) \wedge \sim W(x)]$$
$$\therefore \exists x\, [D(x) \wedge \sim W(x)]$$

Q) Consider the statement & express symbolically using predicates & Quantifers.

"All humming birds are richly coloured"
"No, Large birds live on honey"
"Birds that do not live on honey are dull in colour"
"Humming birds are small"

sol:- P: humming birds
Q: Richly coloured
R: Large birds
S: Live on honey

1) $\forall x\, (P(x) \to Q(x))$

2) $\sim [\exists x\, (R(S) \wedge S(x))]$ (or) $\forall x\, (R(x) \to \sim S(x))$

3) $\forall x\, (\sim S(x) \to \sim Q(x))$

4) $\forall x\, (P(x) \to \sim R(x))$

Rules of inference:

→ To validate an argument, we apply rules of ~~interf~~ inference and an argument is ~~set~~ said to be valid if all primises implies the conclusion as True.

Rule	Tatalogical form	Name of rule
1) $P \rightarrow Q$ P $\therefore Q$	$[(P \rightarrow Q) \wedge P] \rightarrow Q$	Modus Poremis
2) $P \rightarrow Q$ $\sim Q$ $\therefore \sim P$	$[(P \rightarrow Q) \wedge \sim Q] \rightarrow \sim P$	Modus Tollens
3) $P \rightarrow Q$ $Q \rightarrow R$ $\therefore P \rightarrow R$	$[(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R)$	hypothetical Syllogism
4) $P \vee Q$ $\sim P$ $\therefore Q$	$[(P \vee Q) \wedge \sim P] \rightarrow Q$	Disjunctive syllogism
5) P $\therefore P \vee Q$	$P \rightarrow (P \vee Q)$	Addition
6) $P \wedge Q$ $\therefore P$	$(P \wedge Q) \rightarrow P$	Simplification
7) P Q $\therefore P \wedge Q$	$((P) \wedge (Q)) \rightarrow (P \wedge Q)$	Conjunction
8) $P \vee Q$ $\sim P \vee R$ $\therefore Q \vee R$	$[(P \vee Q) \wedge (\sim P \vee R)] \rightarrow Q \vee R$	Resolution

1) state which rule of inference is used in given argument

"If it rains today, then we will not have college, if we don't have college Today then we will have college Tommorow.

P = it rains today, Q = have college, R = college Tommorow

∴ If it rains today we will have college Tommorow.

$$\begin{array}{l} P \rightarrow Q \\ Q \rightarrow R \\ \hline \therefore P \rightarrow R \end{array}$$

$$((P \rightarrow Q) \land (Q \rightarrow R)) \rightarrow (P \rightarrow R)$$

∴ The Given argument follows hypothetical syllogism.

∴ It is a Tautology.

2) Validate the given Argument

"It is not sunny this afternoon & it is colder than yesterday. we will go swimming only if it is sunny, if we don't go swimming only if it is Sunny, If we don't go swimming Then we will go to movie and if we go to movie then we will be at home by sunset. This leads to the conclusion that we will be in home by Sunset".

P = It is sunny this afternoon
Q = Colder than yesterday
R = Go Swimming
S = Go to movie
T = At home by sunset.

formulation

$\longrightarrow$

$$\begin{array}{l} \sim P \wedge Q \\ R \rightarrow P \\ \sim R \rightarrow S \\ S \rightarrow T \\ \hline \therefore T \end{array}$$

Validation :-

Assertion		Reason
1)	$\sim P \wedge Q$	Premise
2)	$\sim P$	By simplification
3)	$R \rightarrow P$	Premise
4)	$\sim R$	Modus Tollen (by 2 & 3)
5)	$\sim R \rightarrow S$	Premise
6)	S	Modus pollen (By 4 & 5)
7)	$S \rightarrow t$	Premise
8)	t	Modus pollen (by 6 & 7)

$\therefore$ The Given argument the conclusion 'T' is followed from all the primises by applying rules of inference. Hence, The Given argument is Valid.

3) Validate the Given pattern.

$$\begin{array}{l} \sim R \rightarrow (S \rightarrow \sim T) \\ \sim R \vee W \\ P \rightarrow S \\ \sim W \\ \hline \therefore t \rightarrow P \end{array}$$

Assertions	Reason
1) $\sim R \vee W$	premise
2) $\sim W$	premise
3) $\sim R$	Disjunctive syllogism (by 1 & 2)
4) $\sim R \rightarrow (S \rightarrow \sim T)$	premise
5) $S \rightarrow \sim T$	Modus pollen (by 3 & 4)
6) $\sim P \rightarrow S$	Premise
7) $\sim P \rightarrow \sim t$	(By 5 & 6) hypothetical syllogism
8) $(t \rightarrow P)$	Contrapositive

∴ Given pattern the conclusion "$t \rightarrow P$" is followed from all primises. By applying rules of inference. Hence, The Given pattern is Valid.

H.W

Q) Prove (or) Disprove Argument

"If you send me email message then I will finish writing a program, If you don't send me an email then I will go to sleep early. If I go to sleep early then will feel refreshed. which leads to the conclusion If I don't finish writing the program then I will feel Refreshed.

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Rules of Inference of Quantified statements:-

Rule	Name
1) $\dfrac{\forall x, P(x)}{\therefore P(c)}$	universal Instantiation
2) $\dfrac{\text{for any } c, P(c)}{\therefore \forall x, P(x)}$	universal Generalization
3) $\dfrac{\exists x, P(x)}{\therefore P(c)}$	Existential Instatiation
4) $\dfrac{\text{for any } c, P(c)}{\therefore \exists x, P(x)}$	Existential Generalization.

ex:- Prove or disprove the validity of following statement.

Every living thing is a plant or an animal. David's dog is alive and is not a plant. All animals have hearts. Hence, David's dog has a heart.

$P(x)$: x is plant

$q(x)$: x is an animal

a: David's Dog

$r(x)$: x having heart

$\forall x\,[P(x) \vee q(x)]$

$\sim P(a)$

$\forall x\,[q(x) \rightarrow r(x)]$

$\therefore r(a)$

Assertion	Reasons
$\forall x [P(x) \vee q(x)]$	premise
$P(a) \vee q(a)$	for some a in domain by universal Instantiation
$\sim P(a)$	premise
$q(a)$	By disjunctive syllogism
$\forall x [q(x) \rightarrow r(x)]$	premise
$q(a) \rightarrow r(a)$	for some a, by universal instantiation
$r(a)$	by ~~modden~~ modus Tollen's syllogism.

$\therefore$ The statement is Valid.

Q) All integers are rational numbers, some integers are powers of two, therefore some rational numbers are powers of two.

Sol :-

x : integer

$P(x)$ = Rational number

$Q(x)$ = Power's of 2

1) $\forall x [P(x)]$

2) $\exists x [Q(x)]$

---

(3) $\therefore \exists P(x) \rightarrow Q(x)$

Methods of proofs:-

* A proof is a valid argument that establish the truth of a mathematical statement.

* The following are the different methods of proofs.

→ **Trivial proof:-**

Trivial proof of $p \to q$

→ The proof of $p \to q$ is T if the Truth value of Q is T

→ **Vacous proof** It P is shown to be flase, then If $p \to q$ is T, for any Truth value of q

→ **direct proof:-**

A direct proof of statement is constructed by assuming p and q and showing that q must be True.

Q) Give direct proof for the statement "If n is odd numbers then n^2 is odd.

If n is odd integer, then n^2 is odd

p: n is odd

q: n^2 is odd

By direct proof, assume P as True

then n is odd integer

$n = 2k+1$

where $n^2 = (2k+1)^2$

$= 4k^2+4k+1$

$= 2(2k^2+2k+1)$

$= 2n+1$ where $n = 2k^2+2k$

p is True
q is True

Hence, $p \to q$ is True

∴ Given statement is valid.

→ Proof by Contraposition (or) indirect proof:-

In this method, we prove $p \to q$ is T, by considering its contrapositive.

ex:- Prove that if 'n' is an integer and $3n+2$ is odd then 'n' is odd.

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Prove that if $n = ab$, then $a \le \sqrt{n}$ or $b \le \sqrt{n}$ where a & b are positive integers.

$p: n = ab$

$q: a \le \sqrt{n}$ or $b \le \sqrt{n}$

$p \to q \equiv \sim q \to \sim p$

$\sim q \equiv \sim(a \le \sqrt{n} \text{ or } b \le \sqrt{n})$

$\equiv a > \sqrt{n}$ and $b > \sqrt{n}$

$\equiv ab > n \to F$

$\sim p \equiv \sim(n = ab)$

$n \neq ab \to F$

$\sim q \to \sim p$ is T

→ Proof by Contradiction:-

In this method, we doesn't prove a result directly and it is similar to indirect proof

Q) show that atleast four of any 22 days must fall on the same day of the week.

Given statement is, $x \ge 4$

by proof by contradiction $\sim(x \ge 4)$ is T

$x < 4$

$\Rightarrow x \le 3$ (This means MMM, TTT, WWW, ThThTh, FFF, SSS, SuSuSu)

So, our contradiction fails

in total 21 and 1 day left

$21 < 22$

So $x \ge 4$ is T

→ Proof of equivalance:-

If the given statement is in the form $p \leftrightarrow q$ then its equivalent form $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$ is true then the given statement is True.

Q) 'n' is an integer and 'n' is odd iff 'n^2' is odd

$p \leftrightarrow q$

$\equiv (p \to q) \wedge (q \to p)$

$p \to q$

if 'n' is odd then 'n^2' is add

let $n = 2k+1$

$n^2 = 2(2k^2+2k)+1$

$= 2n+1$

so $p \to q$ is T

$q \to p$

if n^2 is odd then n is odd

$q \to p \equiv \sim p \to \sim q$

∴ $\sim p$ is 'n' is even $n = 2k$ → False F

∴ $\sim q$ is 'n^2' is even $n^2 = 4k^2$

$= 2(2k^2) = 2n = $ even → F

$\sim p$ is F & $\sim q$ is F

$\sim p \to \sim q$ is T

so, $p \leftrightarrow q$ is T.

* Exhaustive Proof:-

Some theorems can be proved by examining a relatively small number of examples such proofs are called exhauste proofs.

Q) Prove that $(n+1)^3 \geq 3^n$, for $n \leq 4$, n is a +ve Integer.

for $n=1$ | $n=2$ | $n=3$ | $n=4$

$(1+1)^3 \geq 3^1$ T | $(1+2)^3 \geq 3^2$ T | $(1+3)^3 \geq 3^3$ T | $(4+1)^3 \geq 4^3$ T

$\therefore (n+1)^3 \geq 3^n$ is T

→ **Proof by Cases:-**

It must cover all possible cases that arise in a theorem.

Q) If n is integer, prove that $n^2 \geq n$.

n is +ve integer
n is -ve integer
n = 0

1) n = 0 then $0^2 \geq 0$ is T
2) n = +ve integer then $n \geq 1 \Rightarrow n^2 \geq n$ T
3) n = -ve integer then $n \leq -1$ but $n^2 \geq 0$, it follows $n^2 \geq n$

→ **Existence Proof:-**

It is a proof of preposition of the form $\exists x, P(x)$.

→ An existence proof can be given by finding an element 'a' such that P(a) is True.

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Fallacies:- ~~An~~ incorrect arguments are called fallacies.

→ Usually fallacies resemble the rules of inference but they are based of contigency rather than tautology.

→ There are 3 types of fallacies.

They are

(i) Affiriming the Consequent → $\dfrac{\begin{matrix} p \to q \\ q \end{matrix}}{\therefore p}$

(ii) Denying the Antecedent → $\dfrac{\begin{matrix} p \to q \\ \sim p \end{matrix}}{\therefore \sim q}$

(iii) Non-sequiter fallacy → $\dfrac{p}{\therefore q}$

Q) The earth is spherical ⟶ moon is spherical

The earth is not spherical, therefore moon is not spherical

→ Denying the antecedent fallasy

→ So, Not Valid.

Q) A triangle has 3 sides, Therefore triangle is a square.

→ It follows Non-sequiter fallasy

→ So, Invalid.

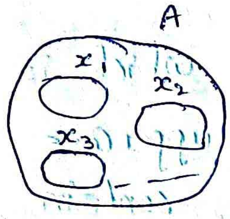
Basic Counting Techniques

→ 2 types

(i) sum rule

(ii) Product rule.

a) Sum rule:-



→(Cardinality of A)

$|A|$ = number of elements in A

x_1, x_2, x_3 are dijoint subsets of A.

$$|A| = |x_1| + |x_2| + \dots |x_n|$$

$$= \sum_{i=1}^{n} |x_i|$$

$$|A| = |x_1 \cup x_2 \cup x_3 \dots \cup x_n| = |x_1| + |x_2| + \dots + |x_n|$$

if any task can be done in one of the n_1 ways (or) in one of the n_2 ways in which none of the set of n_1 ways is same as the set of n_2 ways.

→ Then the task can be completed in $n_1 + n_2$ ways.

Q) Suppose that a member of faculty ~~are~~ (or) a student is choosen for university committee as a representative and if there are 50 faculty and 150 students then how many different choice are there for the representative such that he/her not both faculty & student.

Sol:- 50 (or) 150 ⟶ 50+150 = 200.

b) Product rule

suppose that a procedure is broken down into sequence of 2 tasks such that first task can be done in n_1 ways and 2nd task can be done in n_2 ways. Then there are $n_1 \times n_2$ ways to complete the procedure.

Q) Suppose a company with 2 employee's A & B rents a building with 12 offices then how many ways are there to assign different offices to A & B.

Sol:- A and B ~~2x12 = 24~~

12 11 $12 \times 11 = 144$

Q) Each user on Computer system has a password which is 6-8 characters where each character is an upper case letter or a digit and each password must contain atleast 1 digit then how many possible passwords are there.

P_8 36 36 36 36 36 36 36 36 (or) P_7 36 36 36 36 36 36 36 (or) P_6 36 36 36 36 36 36

$= \cancel{((26\times7)+10) + ((36\times6)+10) + ((36\times5)+10)}$

$= (36^8 - 26^8) + (36^7 - 26^7) + (36^6 - 26^6)$

Cases with both alphabet & digits

case with only alphabet

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Q) How many diff liscence plates are there that involves 1,2 (or) 3 letters followed by 1,2,3 (or) 4 digits.

$[\underline{26} \text{ (or) } \underline{26}\,\underline{26} \text{ (or) } \underline{26}\,\underline{26}\,\underline{26}]\ \underline{10}$

$= [26+26^2+26^3]10 + [26+26^2+26^3]10^2 + [26+26^2+26^3]10^3 + [26+26^2+26^3]10^4$

$= [26+26^2+26^3][10+10^2+10^3+10^4]$

Q) How many numbers can be formed using the digits 1,3,4,5,6,8,9 if no repetation is allowed.

sol:- 7

or 7 6

or 7 6 5

or 7 6 5 4

or 7 6 5 4 3

or 7 6 5 4 3 2

or 7 6 5 4 3 2 1

$= 7 + 7\times6 + 7\times6\times5 + 7\times6\times5\times4 + 7\times6\times5\times4\times3$

$+ 7\times6\times5\times4\times3\times2 + 7\times6\times5\times4\times3\times2\times1$

=

Q) How many three digit numbers are there which are even and have no repeated digits.

all numbers

9 9 8

(it can be 0)

$9\times9\times8$ –

~~$9\times8(5)$~~

~~360~~

odd number

8 8 5

$8\times8\times5$

$8(81-40)$

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[or]

only even

Case I:- with zero at last

9 8 $\boxed{0}$

$8(9+32)$

Case II:- without zero at last

8 8 4

$9\times8 + 8\times8\times4$

$8(9+32)$

$8(41)$

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The principle of inclusive & Exclusive:-

(or)

The inclusion & Exclusion Principle:-

Suppose that a task can be done in n_1 (or) n_2 ways but some of the set of n_1 ways to do the task are same as some of the set of n_2 ways to do the task, in this ~~called~~ case we can't apply sum rule to count no. of ways.

In this situation we use a technique called principle of inclusion & Exclusion principle.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Q) How many bit strings of length eight start with '1' bit or end with 2 bits '0'.

$\underset{}{1}\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}$ (or) $\underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ 0\ 0$

2^7 $\qquad$ 2^6 $\qquad$ $1\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ 0\ 0$

2^5

Total $= 2^7 + 2^6 - 2^5$

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Pegion-Hole Principle:-

It states that if there are more pegions than pegion hole then there must be atleast one pegion hole with atleast 2 (or) more pegions in it.

Note:- if there are k+1 (or) more objects are to be placed in 'k' no.of boxes, then there is atleast one box containing 2 (or) more objects.

→ If there are 13 people then atleast 2 must celebrate their birthday in same month.

Q) How many students are there in a class to garuntee that atleast 2 students gets the same score in an exam & if the exam is grabed on the scale from 0-100 marks.

Ans:- 102

Permutations & Combinations:-

The permutations of a set of distinct objects is an ordered arrangement of these objects.

An ordered arrangement of 'r' elements of a set of 'n' objects is called 'r' permutation and is denoted by $P(n,r)$

$$P(n,r) = \frac{n!}{(n-r)!}$$

ex:- {a b c}

2 permutation ab ba
ac ca
bc cb

3 permutation abc acb bca cab cba bac

Q) How many ways are there to select a first prize winner, second prize winner & 3rd prize winner from 100 different people who contested in a game.

$^{100}P_3$

* ~~the~~ An unordered ~~arrange~~ selection of 'r' elements from a set of 'n' distinct objects is called an "r-combination" and is denoted by $C(n,r)$

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

ex:- {a b c}

2 combination ab
bc
ca

3 combination abc

Q) How many hands of 5 cards can be delt from standard deck of 52 cards.

$^{52}C_5$

* $C(n,r) = C(n, n-r)$ / $^nC_r = {}^nC_{n-r}$

Q) suppose that there are nine faculty and 11 students in a dept. How many ways are there to select a committee with 3 faculty & 4 students.

$^{9}C_3 \times {}^{11}C_4$

Enumerating P & C with repetations:-

The no. of r permutations of set of n objects with repeation is n^r

Q) How many strings of length r can be formed from english alphabets

26^r

→ The no. of r combination of n objects with repeation is given by $C(n+r-1, r) = C(n+r-1, n-r)$

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Q) ~~r combinations from a set of n elements when repeatation of elements is allowed, is given by C (~~

Q) Ice cream shops has 4 diff types of icecreams then how many diff ways are there to choose six icecreams.

$n=4$ $r=6$

$C(9, 6) = {}^9C_6$

Q) enumerate the no. of ways of putting 20 indistingustable balls into 5 boxes such that each box is non empty.

$n = 20$ $r=5$

$\cancel{{}^{20}C_5 + {}^{20}C_4 + {}^{20}C_3 + {}^{20}C_2 + {}^{20}C_1 = 9309}$

$\boxed{1}\ \boxed{1}\ \boxed{1}\ \boxed{1}\ \boxed{1}$ ${}^{15}C_5$

$x_1 + x_2 + x_3 + x_4 + x_5 = 20$

$x_1 + x_2 + x_3 + x_4 + x_5 = 15$

$n = 5$; $r = 15$

$C(n+r-1, r) = C(19, 15)$

Q) what are the no. of solutions does the equation $x_1+x_2+x_3=11$, where x_1, x_2, x_3 are non -ve integers.

$x_1+x_2+x_3=11$

$n=3,\ r=11$

$c(3+11-1,\ 11)$

Q) $x_1+x_2+x_3+x_4+x_5=20$, ∋ $x_1\geq 2,\ x_2\geq 1,\ x_3\geq 3,\ x_4\geq 1,\ x_5\geq 2$

$2+1+3+1+2=9$

$y_1+y_2+y_3+y_4+y_5=11$

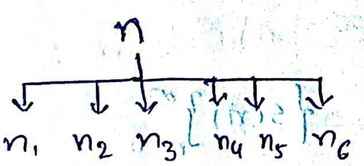
$n=5,\ r=11$

$c(15,11)$

* $\dfrac{n!}{n_1!\,n_2!\,n_3!\cdots n_k!}$ repetations n_1, n_2, n_3 ——

14/12/21

Q) In how many ways can 14 men be partioned into 6 teams such that first team has 3 members, 2nd team has 2 members & 3rd team has 3 members and 4, 5, 6th teams each have 2 members.



$= \dfrac{14!}{3!\,2!\,3!\,2!\,2!\,2!} = \dfrac{14!}{(3!)^2(2!)^4}$

Multinomial:-

More generally for the powers of multinomial $(x_1+x_2+x_3+\cdots+x_k)^n$ in which the ~~co~~ number of multinomial co-efficients can be determined as $P(n,q_1,q_2,\ldots q_k)$

$$P(n,q_1,q_2,q_3,\ldots,q_k) = \frac{n!}{q_1!\,q_2!\cdots q_k!}$$

where $q_1, q_2 \ldots q_k$ are non-negative integers such that $q_1+q_2+\cdots+q_k=n$.

Q) find the co-efficient of $x_1^2 x_3 x_4^3 x_5^4$ in $(x_1+x_2+x_3+x_4+x_5)^{10}$

$P(2,0,1,3,4)$ $\dfrac{10!}{2!\,0!\,1!\,3!\,4!}$

* if x_1 is raised to power q_1, x_2 to pow q_2 - - x_k to pow q_k then the numbers of terms in selection of 'n' objects with repetation from 'k' distinct types is given by

$C(n+k-1, n)$

for above an

$n = 10$; $k = 5$

$C(14, 10)$

* Generating functions

Sequence of real numbers such as $\{a_r\}_{r=0}^{\infty}$ is a function

$S: Z^+ \longrightarrow R$.

Sequence is represented as $<S_n>$ (or) $\{s(n)\}_{n=0}^{\infty}$

ex:- $\{2^r\}_{r=0}^{\infty} = \{1, 2, 4 - - - \}$

* To $\{a_r\}_{r=0}^{\infty}$ we define a series as $A(x) = a_0 + a_1 x + a_2 x^2 + -$

$A(x) = \sum_{r=0}^{\infty} a_r x^r$

and is called as formal power series (or) Generating function.

→ Here a_r is corresponding co-efficient of x^r

Consider a generating fn $B(x) = \sum_{r=0}^{\infty} b_r x^r$

prop of Generating fn

sum of Two generating function (say $A(x), B(x)$) is a Generat function similarly for difference, multiplication, divison.

To G.F are equal if $a_r = b_r$

Q) find the sequence of G.F of $f(x) = 2x^2 (1-x)^{-1}$

$f(x) = 2x^2 (1+x+x^2+x^3 \text{———})$

$= 2x^2 + 2x^3 + 2x^4 \text{———}$

$\therefore$ required sequence is $(0, 0, 2, 2, 2, \text{———})$

Q) find the G.F for the seq $(1, 2, 3, 4 \text{———})$

$f(x) = 1 + 2x + 3x^2 + 4x^3 - \text{———}$

$= (1-x)^{-2}$

Q) find the G.F for the seq $(0, 1, 2, 3, 4 \text{———})$

$f(x) = 0 + x + 2x^2 + 3x^3 + 4x^4 \text{———}$

$= x(1 + 2x + 3x^2 + 4x^3 \text{———})$

$= x(1-x)^{-2}$

Q) find G.F for a_r, where a_r is non-ve integral solution for an equation $e_1 + e_2 + e_3 + e_4 + e_5 = r$; $0 \le e_1 \le 3$; $0 \le e_2 \le 2$, $0 \le e_3 \le 2$; $1 \le e_4 \le 5$; e_5 is odd such that $1 \le e_5 \le 10$.

$A(x) = A_1(x) A_2(x) A_3(x) A_4(x) A_5(x)$

$= (1+x+x^2+x^3)(1+x+x^2)(1+x+x^2)(x+x^2+x^3+x^4+x^5)$
$(x+x^3+x^5+x^7+x^9)$

1) Generating function a_r = no. of non-negative integral solutions.

A) $a_r = e_1 + e_2 + \text{———} + e_n = r$ wher $0 \le e_1 \le 1$

$$A(x) = A_1(x) \cdot A_2(x) \text{ --- } A_n(x)$$
$$= (1+x)(1+x) \text{ --- } (1+x)$$
$$= (1+x)^n$$
$$= (1+x+x^2+ \text{ --- })^n = \frac{1}{(1-x)^n}$$

2) find Generating function for a_r which is equal to no. of ways the sum obtained when 2 distinguishable (both are not same) dies are rolled.

sol:- Generating functions for 2 dies $\Rightarrow e_1, e_2$

$$\therefore e_1 + e_2 = r$$
$$1 < e_i \leq 6$$
$$\therefore (x + x^2 + x^3 + x^4 + x^5 + x^6)^2$$

Note:-

1) $(1-x)^{-n} = \Sigma\, C(n+r-1, r)\, x^r$

2) $(1+x)^n = \Sigma\, C(n, r)\, x^r$

3) $\dfrac{1}{1-ax} = \displaystyle\sum_{r=0}^{\infty} a^r x^r$

4) $\dfrac{1}{(1-x)^n} = \Sigma\, C(n+r-1, r)\, (-a)^r x^r$

5) $\dfrac{1}{1-x^k} = \Sigma\, x^{kr}$

6) $\dfrac{1}{a-x} = \dfrac{1}{a} \Sigma \left(\dfrac{x}{a}\right)^r$

7) $1 + x + x^2 + \text{ --- } + x^n = \dfrac{1-x^{n+1}}{1-x}$

1) find the Co-efficient of x^{12} in $x^3(1-2x)^{10}$

$\Rightarrow x^3 \sum ({}^{10}C_r)(-2x)^r$

$\sum ({}^{10}C_r)(-2)^r x^{r+3}$

$r = 9$

x^{12} coefficient : $({}^{10}C_9)(-2)^9$

2) find the Co-efficient of x^9 in $(1+x^3+x^8)^{10}$
$(0 \quad 3 \quad 8)$

$x^9 \Rightarrow \frac{10!}{3!\,7!}$ (or) ${}^{10}C_3$

3) x^{16} in $(1+x^4+x^8)^{10}$
$(0, 4, 8)$

$x^{16} = {}^{10}C_4 + {}^{10}C_2 + \frac{10!}{2!\,1!\,7!}$

4) x^4 Co-efficient in $(x^2+x^3+x^4+x^5)(x+x^2+x^3+x^4+x^5+x^6+x^7)$
$(1+x+x^2+ - - - x^{15})^3$

1st expression → minimum = x^2

$\therefore x^2(1+x+x^2+x^3)$

2nd expression → minimum = x

$\therefore x(1+x+x^2+x^3 - - - x^6)$

3rd expression → minimum = 1

$\therefore (1+x+x^2 - - - x^{15})$

$\therefore x^3 \left(\frac{1-x^4}{1-x}\right)\left(\frac{1-x^7}{1-x}\right)\left(\frac{1-x^{16}}{1-x}\right)$

$= \frac{x^3}{(1-x)^3}(1-x^4)(1-x^7)(1-x^{16})$

$= x^3(1-x^4)(1-x^7)(1-x^{16}) \sum C(3-1+r, r)x^r$

$= (1-x^4)(1-x^7)(1-x^{16}) \sum C(2+r, r) x^{r+3}$

$= (1-x^4)(1-x^7) \sum C(2+r, r) x^{r+3}$. for $r = 12$

$= (1-x^7-x^4+x^{11}) \sum C(2+r, r) x^{r+3}$ for $r=12$

20/2/21

Recurence relation:-

~~Recurense~~ relation is a formula that relates for any individual $n \geq 1$ and it is relation among the terms of the sequence $\{a_n\}$. ~~which~~

* A recurence relation which is in the form of

$$c_0(n)a_n + c_1(n)a_{n-1} + \cdots + c_k(n)a_{n-k} = f(n) \quad \text{—①} \quad \forall n \geq k$$

is called linear recurence relation

where $c_0, c_1, \ldots c_k$ are all functions of n.

→ if the recurence relation ① the RHS $f(n) = 0$ then the recurence relation is called homogenous recurence relation otherwise it is called inhomogenous recurence relation.

Q) verify $a_n = c_1 2^n + c_2 5^n$ is a solution of recurence relation,

$$a_n - 7a_{n-1} + 10a_{n-2} = 0$$

Sol:- $a_n - 7a_{n-1} + 10_{n-2}$

$$c_1 2^n + c_2 5^n - 7(c_1 2^{n-1} + c_2 5^{n-1}) + 10(c_1 2^{n-2} + c_2 5^{n-2})$$

$$c_2 5^n + c_1 2^n - 7c_1 2^{n-1} - 7c_2 5^{n-1} + 10c_1 2^{n-2} + 10c_2 5^{n-2}$$

$$c_1 2^{n-2}(4 - 14 + 10) + c_2 5^{n-2}(25 - 35 + 10)$$

$$= 0 = \text{RHS}$$

Hence Proved.

21/12/2...

Q) solve the recurence relation $a_n = a_{n-1} + f(n), \forall n \geq 1$ by substitution method.

put $n=1$

$a_1 = a_0 + f(1)$

$n=2$ $\quad a_2 = a_1 + f(2) = a_0 + f(1) + f(2)$

$n=3$ $\quad a_3 = a_2 + f(3) = a_0 + f(1) + f(2) + f(3)$

$\vdots$

$a_n = a_n + f(1) + f(2) + \cdots$

$= a_n + \sum_{k=1}^{n} f(k)$

Q) solve the recurence relation $a_n = a_{n-1} + n \ \forall n \geq 1$; $a_0 = 2$

put $n=1$

$a_1 = a_0 + 1 = 2+1 = 3 = \frac{1(1+1)}{2} + 2$

$n=2$ $\quad a_2 = a_1 + n = 3+2 = 5 = \frac{2(2+1)}{2} + 2$

$\vdots$

$a_n = \frac{n(n+1)}{2} + 2$

Q) solve the r.r $a_n = a_{n-1} + 2n+1$; $a_0 = 1$

$n=1$

$a_1 = a_0 + 2 + 1 = 1+2+1 = 4 = (1+1)^2$

$n=2$ $\quad a_2 = a_1 + 4 + 1 = 4+4+1 = 9 = (2+1)^2$

$n=3$ $\quad a_3 = a_2 + 6 + 1 = 9+6+1 = 16 = (3+1)^2$

$\vdots$

$a_n = (n+1)^2$

* Generating functions:-

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_{k-1} x^{k-1} + \sum_{n=k}^{\infty} a_n x^n$$

$$\left[A(x) - a_0 - a_1 x - a_2 x^2 - \cdots - a_{k-1} x^{k-1}\right] = \sum_{n=k}^{\infty} a_n x^n$$

$$\sum_{n=k}^{\infty} a_{n-1} x^n = x\left[A(x) - a_0 - a_1 x - \cdots - a_{k-2} x^{k-2}\right]$$

$$\sum_{n=k}^{\infty} a_{n-2} x^n = x^2\left[A(x) - a_0 - a_1 x - \cdots - a_{k-3} x^{k-3}\right]$$

Q) Express the given recurence relation.

$$a_n + 5a_{n-1} + 3a_{n-2} = 0 \quad \forall\, n \geq 2;\ a_0 = 1,\ a_1 = -1 \quad \text{—①}$$

in the form of $A(x) = \frac{P(x)}{Q(x)}$, such that $A(x)$ generates $\{a_n\}$

Sol: WKT Generating function

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

multiply every term of eq ① with x^n and do the summation for $n=2$ to ∞

$$\sum_{n=2}^{\infty} a_n x^n + 5\sum_{n=2}^{\infty} a_{n-1} x^n + 3\sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$A(x) - a_0 - a_1 x + 5x\left[A(x) - a_0\right] + 3x^2\left[A(x)\right] = 0$$

$$\left[A(x)\right](1 + 5x + 3x^2) - a_0(1+5x) - a_1 x = 0$$

$$A[x] = \frac{a_0(1+5x) + a_1 x}{1 + 5x + 3x^2} \qquad (\because a_0 = 1;\ a_1 = -1)$$

$$= \frac{1(1+5x) + (-1)x}{1+5x+3x^2}$$

$$A[x] = \frac{1+4x}{1+5x+3x^2} = \frac{P(x)}{Q(x)}$$

Q) Solve recurence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0 \quad \forall\, n \geq 2$ using generating functions.

sol:- WKT generating function

$$A[x] = \sum_{n=0}^{\infty} a_n x^n$$

Given r.r

$$a_n - 7a_{n-1} + 10a_{n-2} = 0 \quad \text{--- (1)}$$

multiply every term of eq (1) with x^n and do the summation from $n=2$ to ∞

$$\sum_{n=2}^{\infty} a_n x^n - 7\sum_{n=2}^{\infty} a_{n-1} x^n + 10\sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$[A(x) - a_0 - a_1 x] - 7x[A(x) - a_0] + 10x^2[A(x)] = 0$$

$$A(x)[1 - 7x + 10x^2] = a_1 x + a_0(1 - 7x)$$

$$A(x) = \frac{a_0(1-7x) + a_1 x}{1 - 7x + 10x^2}$$

$$A(x) = \frac{a_0(1-7x) + a_1 x}{(1-5x)(1-2x)} = \frac{C_1}{1-2x} + \frac{C_2}{1-5x}$$

$$\sum_{r=0}^{\infty} a_r x^r = C_1 \cdot \sum 2^r x^r + C_2 \sum 5^r x^r$$

$$\boxed{a_r = C_1 2^r + C_2 5^r}$$

* (not given in qn)

if $a_0 = 1;\ a_1 = 2$

$$A(x) = \cancel{\frac{1-7x+2x}{1-7x+10x^2}} = \cancel{\frac{1-5x}{1-7x+10x^2}}$$

put $r = 0$, $a_0 = C_1 + C_2 \Rightarrow C_1 + C_2 = 1$

$r = 1$, $a_1 = 2C_1 + 5C_2 \Rightarrow 2C_1 + 5C_2 = 2$

$3C_2 + 2(C_1 + C_2) = 2$

$3C_2 = 0$

$\boxed{C_2 = 0} \Rightarrow \boxed{C_1 = 1}$

$$\boxed{\therefore a_r = 2^r}$$

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5) solve the recurence relation $a_n - 9a_{n-1} + 26a_{n-2} - 24a_{n-3} = 0$ $\forall n \geq 3$ — ①

$a_0 = 0;\ a_1 = 1;\ a_2 = -1$

multiply every term of eq ① with x^n and do the Summation for $n=3$ to ∞.

$$\sum_{n=3}^{\infty} a_n x^n - 9\sum_{n=3}^{\infty} a_{n-1} x^n + 26\sum_{n=3}^{\infty} a_{n-2} x^n - 24\sum_{n=3}^{\infty} a_{n-3} x^n = 0$$

$$\left[A(x) - a_0 - a_1 x - a_2 x^2\right] - 9x\left[A(x) - a_0 - a_1 x\right] + 26x^2\left[A(x) - a_0\right] + x^3 \cdot 24\left[A(x)\right] = 0$$

$$A(x)\left[1 - 9x + 26x^2 + 24x^3\right] - a_0\left[1 - 9x + 26x^2\right] - a_1 x(1 - 9x) - a_2 x^2 = 0$$

$$A(x) = \frac{a_0(1 - 9x + 26x^2) + a_1(x - 9x^2) + a_2 x^2}{1 - 9x + 26x^2 + 24x^3}$$

$$= \frac{0 + 1(x - 9x^2) - x^2}{1 - 9x + 26x^2 + 24x^3} = \frac{x - 10x^2}{(1-2x)(1-3x)(1-4x)}$$

$$= \frac{c_1}{1-2x} + \frac{c_2}{1-3x} + \frac{c_3}{1-4x}$$

$$\sum_{n=0}^{\infty} a_n x^n = c_1 \sum_{n=0}^{\infty} 2^n x^n + c_2 \sum_{n=0}^{\infty} 3^n x^n + c_3 \sum_{n=0}^{\infty} 4^n x^n$$

$$\boxed{a_n = 2^n c_1 + 3^n c_2 + 4^n c_3}$$

$a_0 = 0$ put $n=0$

$a_0 = c_1 + c_2 + c_3 = 0$ — ①

$a_1 = 1$ put $n=1$

$a_1 = 2c_1 + 3c_2 + 4c_3 = 1$

$\Rightarrow c_2 + 2c_3 = 1$ $(\because c_1 + c_2 + c_3 = 0)$

$\Rightarrow c_2 = 1 - 2c_3$ — ②

$a_2 = -1$ put $n=2$

$a_2 = 4c_1 + 9c_2 + 16c_3 = -1$ $(\because$ from ①$)$

$5c_2 + 12c_3 = -1$ — ③

sub ② in ①

$5(1-2c_3)+12c_3=-1$ $\Rightarrow$ $c_2 = 1-2c_3$

$5-10c_3+12c_3=-1$ $= 1-2(-3)$

$5+2c_3=-1$ $\boxed{c_2 = 7}$

$2c_3 = -6$ $\Rightarrow$ $c_1+c_2+c_3=0$

$\boxed{c_3=-3}$ $c_1+7-3=0$

$\boxed{c_1=-4}$

$$\boxed{a_n = -4\cdot 2^n + 7\cdot 3^n - 3\cdot 4^n}$$

Solving recurence relations using Characteristic roots

let recurence relation be

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} - - - c_k a_{n-k} = 0$$

corresponding characteristic polynomial of given r.r equation.
is $c(t) = t^k + c_1 t^{k-1} + c_2 t^{k-2} + --- + c_k$ then we find characterstic
~~To write a solution for the given r.r~~
roots for $c(t)$ to write a solution for given r.r

Note:- A characteristic polynomial $c(t)$ can also be defined as

$$c(t) = t^k \cdot q\left(\tfrac{1}{t}\right)$$

Q) solve the r.r using characteristic roots $a_n - 7a_{n-1} + 12a_{n-2} = 0$ $\forall n \geq 2$ and given ①
$a_0 = 2$; $a_1 = 5$

multiply every term of eq ① with x^n and do the summation for $n=2$ to ∞.

$$\sum_{n=2}^{\infty} a_n x^n - 7\sum_{n=2}^{\infty} a_{n-1}x^n + 12\sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

$$[A(x)-a_0-a_1x] - 7x[A(x)-a_0] + 12x^2[A(x)] = 0$$

$$A(x)[1-7x+12x^2] - a_0[1-7x] - a_1 x = 0$$

$$A(x) = \frac{a_0(1-7x)+a_1x}{1-7x+12x^2} = \frac{2-9x}{(1-4x)(1-3x)} = \frac{P(x)}{Q(x)}$$

$\therefore$ The characteristic ~~equation~~ polynomial

$$e(t) = t^2 \cdot Q\left(\frac{1}{t}\right)$$

$$= t^2 \cdot \left(1 - 7\left(\frac{1}{t}\right) + 12\left(\frac{1}{t}\right)^2\right)$$

$$= t^2 \left(\frac{t^2 - 7t + 12}{t^2}\right)$$

$$= t^2 - 7t + 12$$

$$\boxed{= (t-3)(t-4)}$$

$\therefore$ $t = 3$ & 4 are characteristic roots.

Hence

$$a_n = c_1 \cdot 3^n + c_2 \cdot 4^n$$

given $a_0 = 2$ $\quad$ $a_1 = 5$

put $n = 0$ $\quad$ put $n = 1$

$a_0 = c_1 + c_2 = 2$ $\quad$ $a_1 = 3c_1 + 4c_2 = 5$

$c_2 + 3(c_1 + c_2) = 5$

$c_2 + 3(2) = 5$

$\boxed{c_2 = -1} \Rightarrow \boxed{c_1 = 3}$

$$\therefore \boxed{a_n = (-1) \cdot 3^n + 3 \cdot 4^n}$$

Note:- if $t = k$ is a characteristic root of multiplicity 2

i.e $(t-k)^2$ then $a_n = c_1 k^n + c_2 n k^n = (c_1 + nc_2) k^n$

if $t = k$ is a c.r of multiplicity 3 i.e $(t-k)^3$

$$a_n = c_1 k^n + c_2 n k^n + c_3 n^2 k^n = (c_1 + nc_2 + n^2 c_3) k^n$$

Inhomogenous recurence relations :-

* finding particular solution for given inhomogenous recurence relation *

$f(n)$	characteristic polynomial	Particular soln (a_n^p)
1) Da^n	$c(a) \neq 0$	Aa^n
2) Da^n	a is root of $c(t)$ with multiplicity 'm'	$An^m a^n$
3) $Dn^s a^n$	$c(a) \neq 0$	$(A_0 + A_1 n + A_2 n^2 + \dots A_s n^s) a^n$
4) $Dn^s a^n$	a is root with multiplicity 'm'	$n^m (A_0 + A_1 n + \dots A_s n^s) a^n$
5) Dn^s	$c(1) \neq 0$	$(A_0 + A_1 n + A_2 n^2 + \dots A_s n^s)$
6) Dn^s	1 is root of $c(t)$ with multiplicity m	$n^m (A_0 + A_1 n + \dots A_s n^s)$

complete solution of inhomogenous r.r is

$$a_n = a_n^h + a_n^p$$

where a_n^h is a solution of homgenous r.r

a_n^p is a particular solution

Q) solve the r.r $a_n - 6a_{n-1} + 8a_{n-2} = n \cdot 4^n \quad \forall n \geq 2 ; a_0 = 1, a_1 = 2$

let $a_n - 6a_{n-1} + 8a_{n-2} = 0$

$$c(t) = t^2 - 6t + 8$$

$$= (t-2)(t-4)$$

characteristic roots are $t = 2 \; \& \; 4$

Hence $a_n^h = c_1 2^n + c_2 4^n$ — ⓐ

Given function $f(n) = n4^n$, which is in the form of $Dn^s a^n$

where $D = 1$; $s = 1$; $a = 4$

$a = 4$ is a root.

So $a_n^p = \cancel{(A_0 + A_1 n)}$ ~~$n^1 4^n$~~ $n^1(A_0 + A_1 n) 4^n$ — ⓑ

where A_0 & A_1 are arbitary Constants

Since a_n^P is a solution of eq (1) we have

$$a_n^P - 6a_{n-1}^P + 8a_{n-2}^P = n \cdot 4^n$$

$$n(A_0 + A_1 n)4^n - 6\left[(n-1)(A_0 + A_1(n-1))4^{n-1}\right] + 8\left[(n-2)(A_0 + A_1(n-2))4^{n-2}\right] = n4^n$$

Put $n=1$ in (b)

$$(A_0 + A_1)4 + \cancel{8}^{2}(-A_0 + A_1)\frac{1}{\cancel{4}} = 4$$

$$6A_1 - A_0 = 4 \quad -(1)$$

put $n=2$ in (b)

$$2(A_0 + 2A_1)4^2 - 6\left[(1)(A_0 + A_1)4^1\right] + 0 = 2 \cdot 4^2$$

$$32A_0 + 64A_1 - 24A_0 - 24A_1 = 32$$

$$12A_0 + 40A_1 = 32 \quad -(2)$$

$$12(6A_1 - 4) + 40A_1 = 32 \qquad \text{(from (1))}$$

$$72A_1 - 48 + 40A_1 = 32$$

$$112A_1 = 80$$

$$A_1 = \frac{5}{7} \Rightarrow A_0 = \frac{2}{7}$$

put $n=0$ in (a)

$$a_0 = c_1 \cdot 2^0 + c_2 \cdot 4^0$$

$$1 = c_1 + c_2 \quad -(3)$$

put $n=1$ in (a)

$$a_1 = 2c_1 + 4c_2$$

$$2 = 2c_2 + 2(c_1 + c_2) \qquad (\because \text{from (3)})$$

$$2 = 2c_2 + 2$$

$$\boxed{c_2 = 0} \Rightarrow \boxed{c_1 = 1}$$

$$\therefore \quad a_n = a_n^h + a_n^P$$

$$\boxed{a_n = 2^n + n\left(\frac{2}{7} + \frac{5n}{7}\right)4^n}$$

Q) solve $F_n = F_{n-1} + F_{n-2}$ $\forall$ $n \geq 2$

given

$$F_n - F_{n-1} - F_{n-2} = 0 \quad \text{--- (1)}$$

$$c(t) = t^2 - t - 1$$

$$\boxed{a_n = c_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n}$$

(or) $t = \frac{1 \pm \sqrt{5}}{2}$

multiply every term of eq (1) with x^n and do the summation for $n=2$ to ∞

$$\sum_{n=2}^{\infty} F_n x^n - \sum_{n=2}^{\infty} F_{n-1} x^n - \sum_{n=2}^{\infty} F_{n-2} x^n = 0$$

$$[f(x) - F_0 - F_1 x] - x[f(x) - F_0] - x^2 [f(x)] = 0$$

$$f(x)[1 - x - x^2] = F_0(1-x) - F_1 x$$

$$f(x) = \frac{F_0(1-x) - F_1 x}{1 - x - x^2}$$

$$c(t) = t^2 \cdot g\left(\frac{1}{t}\right)$$

$$= t^2 \cdot \left(1 - \frac{1}{t} - \frac{1}{t^2}\right)$$

$$= t^2 - t - 1$$

$t = \frac{1 \pm \sqrt{5}}{2}$ are roots

$$\boxed{F_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n}$$