ii) $\log\sqrt{x^2+y^2}$

$u = \log\sqrt{x^2+y^2} \Rightarrow \log(x^2+y^2)^{1/2}$

$\Rightarrow \frac{1}{2}\log(x^2+y^2)$

$u = \frac{1}{2}\log(x^2+y^2)$

$\frac{\partial u}{\partial x} = \frac{1}{2}\left[\frac{1}{x^2+y^2}(2x)\right] \Rightarrow \frac{x}{x^2+y^2}$ — ①

$\frac{\partial u}{\partial y} = \frac{1}{2}\left[\frac{1}{x^2+y^2}(2y)\right] \Rightarrow \frac{y}{x^2+y^2}$ — ②

put $x = z$ and $y = 0$. in ① & ②.

① $\Rightarrow \frac{\partial u}{\partial x} = \frac{z}{z^2} = \frac{1}{z}$

② $\Rightarrow \frac{\partial u}{\partial y} = \frac{0}{z^2+0} = 0$

WKT $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f'(z) = \frac{\partial u}{\partial x} \cancel{+} - i\frac{\partial u}{\partial y}$

$f'(z) = \frac{1}{z} - i(0)$

To obtain $f(z)$ integrate wr to z

$f(z) = \int \frac{1}{z}\, dz$

$\boxed{f(z) = \log z + c}$

iii) $u(x,y) = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)((\cos 2x)(2)) - \sin 2x(\sin 2x)(2)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cosh 2y \cos 2x - 2\cos^2 2x - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cosh 2y \cos 2x - 2(\cos^2 x + \sin^2 x)}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial u}{\partial x} = \frac{2\cosh 2y \cos 2x - 2}{(\cosh 2y - \cos 2x)^2}$$

put $x = z, y = 0$.

$$= \frac{2\cosh 2(0)\cos 2z - 2}{(\cosh 2(0) - \cos 2z)^2} \Rightarrow \frac{\cos 2z - 2}{(1 - \cos 2z)^2} = \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{-2}{1 - \cos 2z}$$

$$\frac{\partial u}{\partial y} = \frac{-\sin 2x}{(\cosh 2y - \cos 2x)^2} \times \sinh 2y (2)$$

put $x = z$ $y = 0$.

$$= 0$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$f'(z) = \frac{-2}{1 - \cos 2z} - i(0).$$

$$f'(z) = \frac{-2}{1 - \cos 2z}$$

$$f'(z) = \frac{-\cancel{2}}{\cancel{2}\sin^2 z} \Rightarrow -\operatorname{cosec}^2 z$$

$$f(z) = -\int \operatorname{cosec}^2 z \, dz$$

$$\boxed{f(z) = \cot z + c}$$

7) Imaginary part

i) $\frac{x-y}{x^2+y^2}$ ii) $\cos x \cosh y$ (iii) $e^x \sin y$

i) $v = \frac{x-y}{x^2+y^2}$

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2)(1)-(x-y)(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2+2xy}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2+2xy}{(x^2+y^2)^2}$$ Put $z=x$, $y=0$.

$$\frac{\partial v}{\partial x} = \frac{-z^2+2(0)}{(z^2)^2} \Rightarrow \frac{-z^2}{z^4} \Rightarrow -\frac{1}{z^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)(-1)-(x-y)(2y)}{(x^2+y^2)^2} = \frac{-x^2-y^2-2xy+2y^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2-2xy}{(x^2+y^2)^2}$$ Put $z=x$, $y=0$

$$= \frac{0-z^2-(0)}{z^4} \Rightarrow -\frac{1}{z^2}$$

WKT $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \Rightarrow \cancel{f'(z) = \frac{\partial v}{\partial x} - i\frac{\partial v}{\partial y}}$

$$f'(z) = \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x}\right)$$

$$f'(z) = \left(\frac{-1}{z^2}\right) + i\left(\frac{-1}{z^2}\right)$$

$$f(z) = \int \left(-\frac{1}{z^2}\right) + i\left(-\frac{1}{z^2}\right) dz$$

$$f(z) = \frac{1}{z} + i\left(\frac{1}{z}\right) + k.$$

$\cosh(0) = 1$

$\sinh(0) = 0.$

ii) $\cos x \cosh y$

$v = \cos x \cosh y$

$\frac{\partial v}{\partial x} = -\sin x \cosh y$

put $x = z$ $y = 0$

$\frac{\partial v}{\partial y} = \cos x \sinh y$

put $z = x, y = 0$

$= \cos z \sinh(0)$

$\frac{\partial v}{\partial x} = -\sin z \cosh(0)$

$\frac{\partial v}{\partial y} = 0$

$\frac{\partial v}{\partial x} = -\sin z$

w.k.t $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$

$= (0) + i(-\sin z)$

$f'(z) = -i\sin z$

$f(z) = -\int i \sin z \, dz$

$= -i(-\cos z) + k$

$f(z) = i\cos z + k.$

iii) $v = e^x \sin y$

$\frac{\partial v}{\partial x} = e^x \sin y$ $\qquad$ $\frac{\partial v}{\partial y} = e^x \cos y$

$x = z, y = 0.$ $\qquad$ $x = z, y = 0.$

$\frac{\partial v}{\partial x} = 0$ $\qquad$ $= e^z \cos(0)$

$\frac{\partial v}{\partial y} = e^z.$

WKT $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$

$f'(z) = e^z + i(0)$

$\boxed{f(z) = e^z + k}$

Q) 5) prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

Given that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

WKT $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

$LHS = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\frac{\partial^2}{\partial z \partial \bar{z}} f(z) f(\bar{z})$

$= 4\frac{\partial}{\partial z} f(z) f'(\bar{z})$

$= 4 f'(z) f'(\bar{z})$

$= 4|f(z)|^2$

LHS = RHS//.

~~sessional~~

(9) Evaluate $\int_0^{1+i} (x^2 - iy)\, dz$ along the path $y = x$ & $y = x^2$.

Given $\int_0^{1+i} (x^2 - iy)\, dz$

$z = x + iy$ Give 0 to $1+i$

$dz = dx + i dy$ $(0,0)$ to $(1,1)$

along the path $y = x$

$$\int_0^{1+i} (x^2 - iy)\, dz = \int_0^1 (x^2 - ix)(dx + i dy)$$

$$= \int_0^1 (x^2 - ix)(dx + i dy)$$

$$= \int_0^1 (x^2 - ix)\, dx\, (1+i)$$

$$= (1+i) \int_0^1 (x^2 - ix)\, dx$$

$$\Rightarrow (1+i) \left[\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]\right]_0^1 \Rightarrow (1+i)\left[\frac{1}{3} - i/2\right] + c_{ll}$$

along the path $y = x^2$

$dy = 2x dx$

$$\int_0^{1+i} (x^2 - iy)\, dz = \int_0^1 (y - iy)(dx + i dy)$$

$$= \int_0^1 (x^2 - ix^2)(dx + i\, 2x dx)$$

$$= \int_0^1 (x^2 - ix^2)\, dx\, (1 + 2ix)$$

$$= \int_0^1 (x^2 - i x^2)(1 + 2ix)\, dx$$

$$\int_0^1 (x^2 + 2x^3 i - i x^2 + 2x^3)\,dx$$

$$\int_0^1 x^2 + 2x^3 + i(2x^3 - x^2)\,dx$$

$$= \left[\frac{x^3}{3} + \frac{2x^4}{4} + i\left(\frac{2x^4}{4} - \frac{x^3}{3} \right) \right]_0^1$$

$$= \frac{1}{3} + \frac{1}{2} + i\left(\frac{1}{2} - \frac{1}{3} \right)$$

$$= \frac{2+3}{6} + i\left(\frac{3-2}{6} \right) + c \Rightarrow 5/6 + i(1/6) + c$$

(8).