1. Find by newton's method the roots of the following equations correct to 2 decimal places.

Sol: 1. given,

$f(x) = \cos x - xe^x$

$f(0) = 1 > 0$; $f(1) = -2.178 < 0$

$\therefore f(0) \cdot f(1) < 0$

$\therefore$ A root lies between 0 and 1.

Take initial root $x_0 = 0.5$

The Newton-iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (n = 0, 1, 2, ----) \rightarrow ①$$

$f'(x) = -\sin x - e^x - xe^x$

1st iteration: $n = 0$ in ①

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{-0.0534}{-2.9525}$$

$x_1 = 0.518$

2nd iteration: $n = 1$ in ①

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.518 - \frac{-0.008}{-3.0435}$$

$x_2 = 0.5178$

3rd iteration:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.5178 - \frac{0}{-3.0421}$$

$x_3 = 0.5178$

$\therefore x_3 \cong x_2$

$\therefore$ A root of $f(x) = 0$ is $x = 0.517$

2. Given,

$f(x) = x^2 + 4\sin x = 0$

$f(-1) < 0,\ f(\cancel{0.5}\ {-2}) > 0,$

$\therefore f(-1) \cdot f(\cancel{0.5}\ {-2}) < 0$

$\therefore$ A root lies in between -1 & $\cancel{0.5}\ {-2}$

Take initial root $x_0 = \cancel{0.5}\ {-2}$

Newton - iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (n = 0, 1, 2, \ldots) \rightarrow ①$$

<u>1st iteration</u>: $n = 0$ in ①

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -2 - \frac{2.1677}{4.5103} = \cancel{0.0194 = x_1}\ {-1.9359}$$

<u>2nd iteration</u>: $n = 1$ in ①

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \cancel{0.0194}\ {-1.9359} - \frac{\cancel{0.0779}\ 0.113}{\cancel{4.038}\ {-5.2999}} = \cancel{0.0001}\ {-1.9337} = x_2$$

<u>3rd iteration</u>: $n = 2$ in ①

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \cancel{0.0001}\ {-1.9337} - \frac{\cancel{0.0004}\ 0.000284}{\cancel{4.0002}\ {-5.287}}\ \cancel{= 0 = x_3} = -1.9337 = x_3$$

$\therefore x_3 \cong x_2$

$\therefore$ A root of $f(x) = 0$ for $x = -1.9337$

3. Given,

$f(x) = \cos x - 3x + 1,\ f(x) = 3x - \cos x - 1$

$f'(x) = 3 + \sin x$

$f(0) = -2 < 0$ and $f(1) = 1.4597$

$\therefore f(0) \cdot f(1) < 0$

$\therefore$ A root lies between 0 and 1

Now take $x_0 = 0.5$

The N-R iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (n = 0, 1, 2, \ldots) \rightarrow ①$$

1st iteration: $n=0$ in ①

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{0.3776}{3.4794} = 0.6085 = x_1$$

2nd iteration: $n=1$ in ①

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.6085 - \frac{0.0051}{3.5717} = 0.6071 = x_2$$

3rd iteration: $n=2$ in ①

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.6071 - \frac{0}{3.5705} = 0.6071 = x_3$$

$\therefore x_2 \cong x_3$

$\therefore$ A root of $f(x)=0$ for $x = 0.6071$

4. Given,

$f(x) = x\log_{10}x - 1.2 = 0$

$f(2) = -1.2 < 0$; $f(3) = 0.2313 > 0$

$f(2) \cdot f(3) < 0$

A root lies between 2 & 3.

Take $x_0 = 3$.

The N-R iterative formula is

$$x_{n+1} = x_n - \frac{f(x_0)}{f'(x_0)} \rightarrow ①$$

$f'(x) = \log_{10}e + \log_{10}x = 0.4343 + \log_{10}x$.

1st iteration: $n=0$ in ①

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{1.431}{0.911} = 2.7461$$

2nd iteration: $n=1$ in ①

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.7406$$

3rd iteration: $n=2$ in ①

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.7406 \qquad \therefore x_3 \cong x_2.$$

$\therefore$ A root lies of $f(x)=0$ for $x = 2.7406$

5. Given,

$f(x) = \log x - \cos x$

$f'(x) = \frac{1}{x} + \sin x$

$f(1) = -0.5403 < 0$; $f(2) = 1.1092 > 0$

$\therefore f(1) \cdot f(2) < 0$

$\therefore$ A root lies in between 1 and 2.

Now take $x_0 = 1.5$

N-R iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \rightarrow ①$$

1st iteration: $n=0$ in ①

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.298$$

2nd iteration: $n=1$ in ①

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.302$$

3rd iteration: $n=2$ in ①

$$x_3 = 1.302 \quad \because \boxed{x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}}$$

$\therefore x_3 \cong x_2$

$\therefore$ A root of $f(x) = 0$ is $x = 1.302$

2. Given, data in tabular form,

	x_0	x_1	x_2	x_3
x:	5	6	9	11
y:	12	13	14	16
	y_0	y_1	y_2	y_3

The Lagrange's interpolation formula is

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

At $x = 10$.

$$y = \frac{4\times1\times-1}{-1\times-4\times-6} \times 12 + \frac{5\times1\times-1}{1\times-3\times-5} \times 13 + \frac{5\times4\times-1}{4\times3\times-2} \times 14 + \frac{5\times4\times1}{6\times5\times2} \times 16$$

$\therefore y = 14.66$ (or) $\boxed{y(10) = 14.66}$

3. Given, data in tabular form::

	x_0	x_1	x_2	x_3	x_4
x:	0	1	3	5	9
y:	-18	0	0	248	13104
	y_0	y_1	y_2	y_3	y_4

$\therefore y = f(x)$

Lagrange's interpolation formula.

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} y_3 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} y_4.$$

At $x = x$;

$$y = \frac{(x-1)(x-2)(x-5)(x-9)}{(0-1)(0-3)(0-5)(0-9)} \times (-18) + 0 + 0 + \frac{(x-0)(x-1)(x-3)(x-9)}{(5-0)(5-1)(5-3)(5-9)} \times 248 +$$

$$\frac{(x-0)(x-1)(x-3)(x-5)}{(9-0)(9-1)(9-3)(9-5)} \times 13104$$

$$y = \frac{x^4 - 18x^3 + 104x^2 - 222x + 135}{135} \times -18 + 0 + 0 +$$

$$\frac{x^4 - 13x^3 + 39x^2 - 27x}{-160} \times (248) + \frac{x^4 - 9x^3 + 63x^2 - 15x}{576} \times 13104$$

$y =$

4. Given data in divided difference table

$x = x,\ f(x) = y$

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	0	648			
			$\to 28$		
				$\to -1$	
x_1	2	704			$\to 0$
			$\to 25$		
				$\to -1$	
x_2	3	729	$\to 21$		
x_3	6	792			

Newton's divided difference interpolation formula.

$$y = y_0 + (x - x_0) \cdot f(x_0, x_1) + (x - x_0)(x - x_1)\, f(x_0, x_1, x_2) +$$
$$(x - x_0)(x - x_1)(x - x_2) \cdot f(x_0, x_1, x_2, x_3)$$

At $x = x$;

$$y = 648 + (x - 0)28 + (x-0)(x-2)(-1) + (x-0)(x-2)(x-3) \cdot 0$$

$$y = 648 + 28x - x^2 + 2x.$$

$$\therefore y = -x^2 + 30x + 648.$$

5. Given data in ~~divided~~ difference table

year x	pop in thou's y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1961	19.96				
		19.69			
			-0.53		
1971	39.65			-0.23	
		19.16			-0.01
			-0.76		
1981	58.81			-0.24	
		18.4			
			-1		
1991	77.21				
		17.4			
2001	94.61				

To estimate f(1991). Since 1991 is near the end of the table we use newton backward formula.

$$y = f(x) = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n +$$

$$\frac{P(P+1)(P+2)(P+3)}{4!}\nabla^4 y_n, \quad P = \frac{x - x_n}{h} = \frac{\overset{1991}{\cancel{2001}} - \overset{2001}{\cancel{1991}}}{10} = -1$$

$$y = 94.61 + (-1)(17.4) + \frac{(-1)(-1+1)}{2!}(-1) + \frac{(-1)(-1+1)(-1+2)}{3!}(-0.24) +$$

$$\frac{(-1)(-1+1)(-1+2)(-1+3)}{3!}(-0.01).$$

$$y = 94.61 - 17.4 = 77.21$$

The population in the year 1991 is 77.21.

6. Given, data in difference table

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	
1.0	0	0.128			
			0.288	0.048	
1.2	0.128	0.416			0
			0.336	0.048	
1.4	0.544	0.752			0
			0.384		
1.6	1.296	1.136		0.048	
1.8	2.432		0.432		
		1.568			
2.0	4				

To find dy/dx and d^2y/dx^2 at $x=1.1$
use newton - forward formula, since it is near the beginning of the table.

$$y = y_0 + p\Delta y_0 + \frac{p^2-p}{2}\Delta^2 y_0 + \frac{p^3-3p^2+2p}{6}\Delta^3 y_0 + \cdots$$

where $p = \frac{x-x_0}{h}$, $\frac{dp}{dx} = \frac{1}{h}$, $h = 0.2$

$$\frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \cdots\right]$$

$$p = \frac{1.1-1.0}{0.2} = 0.1$$

$$\frac{dy}{dx} = \frac{1}{0.2}\left[0.128 + \frac{2(0.1)-1}{2}(0.288) + \frac{3(0.1)^2-6(0.1)+2}{6}(0.048)\right]$$

$$\frac{dy}{dx} = \frac{1}{0.2}\overset{0.02424}{\left[\cancel{0.04896}\right]} \cancel{= 0.2448} = 0.1212$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6p-6}{6}\Delta^3 y_0 + \cdots\right]$$

$$= \frac{1}{(0.2)^2}\left[0.288 + \frac{6(0.1)-6}{6}(0.048)\right]$$

$$= 6.12$$

7. given, $f(x) = \int_0^5 \frac{dx}{4x+5}$, $n = 10$

$a=0$, $b=5$, $h = \frac{b-a}{n} = \frac{5-0}{10} = 0.5$

x	y	
$x_0 = 0$	$1/4(0)+5 = 0.2$	y_0
$x_1 = x_0 + h = 0.5$	$1/4(0.5)+5 = 0.1428$	y_1
$x_2 = x_0 + 2h = 0 + 2(0.5) = 1$	$1/4(1)+5 = 0.1111$	y_2
$x_3 = x_0 + 3h = 0 + 3(0.5) = 1.5$	$1/4(1.5)+5 = 0.0909$	y_3
$x_4 = x_0 + 4h = 0 + 4(0.5) = 2$	$1/4(2)+5 = 0.0769$	y_4
$x_5 = x_0 + 5h = 0 + 5(0.5) = 2.5$	$1/4(2.5)+5 = 0.0666$	y_5
$x_6 = x_0 + 6h = 0 + 6(0.5) = 3$	$1/4(3)+5 = 0.0588$	y_6
$x_7 = x_0 + 7h = 0 + 7(0.5) = 3.5$	$1/4(3.5)+5 = 0.0526$	y_7
$x_8 = x_0 + 8h = 0 + 8(0.5) = 4$	$1/4(4)+5 = 0.0476$	y_8
$x_9 = x_0 + 9h = 0 + 9(0.5) = 4.5$	$1/4(4.5)+5 = 0.0434$	y_9
$x_{10} = x_0 + 10h = 0 + 10(0.5) = 5$	$1/4(5)+5 = 0.04$	y_{10}

(a) By using trapezoidal rule,

$$I = \frac{h}{2}\left[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)\right]$$

$$I = \frac{0.5}{2}\left[(0.2 + 0.04) + 2(0.1428 + 0.1111 + 0.0909 + 0.0769 + 0.0666 + 0.0588 + 0.0526 + 0.0476 + 0.0434)\right]$$

$$I = \frac{0.5}{2}(\cancel{2.8202}\ 1.6214) = \cancel{0.70505}\ 0.40535$$

(b) By using Simpson's 1/3rd rule;

$$I = \frac{h}{3}\left[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)\right]$$

$$I = \frac{0.5}{3}\left[(0.2 + 0.04) + 4(0.1428 + 0.0909 + 0.0666 + 0.0526 + 0.0434) + 2(0.1111 + 0.0769 + 0.0588 + 0.0476)\right]$$

$$I = \frac{0.5}{3}(2.414) = 0.40233$$

8. To compute the value of $\log_e 5$ on direct integration

$$I = \int_0^5 \frac{dx}{4x+5}$$

Let $u = 4x+5$, $du = 4dx$

$\Rightarrow 1/4\, du = dx$

$$\Rightarrow I = \int_5^{25} \frac{1}{u} \cdot \frac{1}{4}\, du$$

$$\Rightarrow I = \int_5^{25} \frac{1}{4u} \cdot du.$$

$$\Rightarrow I = \frac{1}{4} \int_5^{25} \frac{1}{u} \cdot du.$$

$$I = \frac{1}{4} (\log u)_5^{25} + C$$

$$\cancel{I = \frac{1}{4} \log(4x+5)}.$$

$$I = \frac{1}{4} (\log 25 - \log 5)$$

$$= \frac{1}{4} [5 \log 5 - \log 5]$$

$$I = \frac{4}{4} \log 5$$

$$I = \log 5$$

From trapezoidal rule, $\log 5 = 0.40535$

From sympson's 1/3 rd rule, $\log 5 = 0.40233$

8. here, $f(x) = \int_0^{\pi/2} \sin x . dx$, $n = 10$, ($\because$ 11 ordinates to be taken)

$a = 0, b = \pi/2$, $h = \frac{b-a}{n} = \frac{\pi/2 - 0}{10} = \frac{180^\circ}{20^\circ} = 9^\circ$

x	y	
$x_0 = 0$	$\sin 0^\circ = 0$	y_0
$x_1 = x_0 + h = 9^\circ$	$\sin 9^\circ = 0.1564$	y_1
$x_2 = x_0 + 2h = 18^\circ$	$\sin 18^\circ = 0.3090$	y_2
$x_3 = x_0 + 3h = 27^\circ$	$\sin 27^\circ = 0.4539$	y_3

$x_4 = x_0 + 4h = 36°$ $\quad \sin 36° = 0.5877$ $\quad y_4$

$x_5 = x_0 + 5h = 45°$ $\quad \sin 45° = 0.7071$ $\quad y_5$

$x_6 = x_0 + 6h = 54°$ $\quad \sin 54° = 0.8090$ $\quad y_6$

$x_7 = x_0 + 7h = 63°$ $\quad \sin 63° = 0.8910$ $\quad y_7$

$x_8 = x_0 + 8h = 72°$ $\quad \sin 72° = 0.9510$ $\quad y_8$

$x_9 = x_0 + 9h = 81°$ $\quad \sin 81° = 0.9876$ $\quad y_9$

$x_{10} = x_0 + 10h = 90°$ $\quad \sin 90° = 1$ $\quad y_{10}$

By using trapezoidal rule,

$$I = \frac{h}{2}\left[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)\right]$$

$$= \frac{(\pi/20)}{2}\left[(0+1) + 2(0.1564 + 0.3090 + 0.4539 + 0.5877 + 0.7071 + 0.8090 + 0.8910 + 0.9510 + 0.9876)\right]$$

$$= \cancel{1.0205}\ 0.9978$$

By using 1/3 rd simpson's rule,

$$I = \frac{h}{3}\left[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)\right]$$

$$I = \frac{(\pi/20)}{3}\left[(0+1) + 4(0.1564 + 0.4539 + 0.7071 + 0.8910 + 0.9876) + 2(0.3090 + 0.5877 + 0.8090 + 0.9510)\right]$$

$$I = 0.0523\,[1 + 18.0974] = 0.9998$$

9. Find an app value of y

(a) when $x = 0.2$ from $dy/dx = 1 - 2xy$, $y(0) = 0$

(b) when $x = 1.1$, 1.2 from $dy/dx = \log(xy)$

Sol: Given that, $dy/dx = 1-2xy$, $y(0)=0$

$y'(x) = 1-2xy$, $x_0=0$, $y_0=0$.

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \cdots \rightarrow (1)$$

$y(x_0)=0$.

$\cancel{y'(x_0) = x_0^2+y_0^2 = 0+0=0}$

$\cancel{y''(x_0) = 2x+2yy' = 2x_0+2y_0y_0' = 2(0)+2(0)(0)=0}$

$\cancel{y'''(x_0) = 2+2yy''+2y'y' = 2+2y_0y_0''+2y_0'y_0'}$

$\cancel{= 2+2(0)(0)+2(0)(0) = 2}$

$y'(x_0) = 1-2xy = 1-2x_0y_0 = 1-2(0)(0) = 1 = y_0'$

$y''(x_0) = 1-2y-2xy' = 1-2y_0-2x_0y_0'$.

$y''(x_0) = 1-2(0)-2(0)(1) = 1 = y_0''$

$y'''(x_0) = 1-2y'-2xy''-2y' = 1-2y_0'-2x_0y_0''-2y_0'$

$y'''(x_0) = 1-2(1)-2(0)(1)-2(1) = 1-2-2 = -3 = y_0'''$

$y^{IV}(x_0) = 1-4y''-2xy'''-2y'' = 1-6y_0''-2x_0y_0'''$

$= 1-6(1)-2(0)(-3)$

$y^{IV}(x_0) = 1-6 = -5 = y_0^{IV}$

Substituting values in (1)

$$y(x) = 0 + (x-0)1 + \frac{(x-0)^2}{2!}(1) + \frac{(x-0)^3}{3!}(-3) + \frac{(x-0)^4}{4!}(-5) + \cdots$$

$$= x - \frac{x^2}{2!} + \frac{x^3}{6}(-3) + \frac{x^4}{24}(-5)$$

$y(0.2) = 0.1756$

(b) Given that $dy/dx = \log(xy)$, $y(1) = 2$.

$y'(x) = \log(xy)$, $x_0 = 1$, $y_0 = 2$

$$y(x) = y(x_0) + (x - x_0) y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \cdots \rightarrow (1)$$

$y(x_0) = 2$

$y'(x_0) = \log xy = \log(x_0 y_0) = \log(2) = 0.3010$

$$y''(x_0) = \frac{1}{xy}(xy' + y) = \frac{x_0 y_0' + y_0}{x_0 y_0} = \frac{(1)(0.3010) + 2}{(1)(2)}$$

$$= 1.1505$$

$$y'''(x_0) = \frac{-1}{x^2} + \frac{y(y'') - (y')^2}{y^2} = -0.4474$$

~~$y^{IV}(x_0) =$~~

$$y(x) = 2 + (x-1)(0.3010) + \frac{(x-1)^2}{2}(1.1505) + \frac{(x-1)^3}{3!}(-0.4474) + \cdots$$

(-0.4474)

$$= 2 + 0.3010x - 0.3010 + \frac{(x^2 + 1 - 2x)}{2}(1.1505) + \frac{(x^3 - 1 - 3x^2 + 3x)}{6} + \cdots$$

$y(1.1) = 2.0357$

$y(1.2) = 2.0826$

10. Solve by modified euler's method $\frac{dy}{dx} = y + e^x$, $y(0) = 0$ at $x = 0.2$ in steps of 0.1.

Sol: Given that, $f(x,y) = y + e^x$, $x_0 = 0$, $y_0 = 0$.

here given step-size, $h = 0.1$

$$n = \frac{0.2 - 0}{0.1} = 2$$

To find $y(0.1)$

The euler's formula is

$$y_{n+1} = y_n + hf(x_0, y_0) \rightarrow (1)$$

put $n=0$ in ①

$y_1 = y_0 + h f(x_0, y_0)$

$= 0 + 0.1(0+e^0) = 0.1(1)$

$y_1 = 0.1$ for $x_1 = 0+0.1 = 0.1$

We now improve y_1 using modified euler's formula by.

$$y_{n+1}^{(P+1)} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(1)}) \rightarrow ②$$

put $n=0$ in ②

$$y_1^{(P+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)^{i}] \rightarrow Ⓐ$$

put $p=0$ in Ⓐ, $i=0$.

$$y_1^{(1)} = 0 + \frac{0.1}{2}[1 + 1.2051]$$

$$y_1^{(1)} = 0.110255.$$

put $p=1$ and $i=1$ in Ⓐ

$$y_1^{(2)} = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1)^{(1)}]$$

$$= 0 + \frac{0.1}{2}[1 + 1.2154]$$

$$y_1^{(2)} = 0.11077$$

put $p=2$ and $i=2$ in Ⓐ

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0), f(x_1, y_1)^{(2)}]$$

$$= 0 + \frac{0.1}{2}[1 + 1.2159]$$

$$y_1^{(3)} = 0.11079$$

$\therefore y_1^{(3)} = y_1^{(2)}$ hence $y_1 = 0.1107$ hence $y_1 = 0.1107$

for $x_1 = 0.1$ i.e $y(0.1) = 0.1107$.

To find $y(0.2)$ i.e. y_2 for $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

The initial value of y_2 is obtained by putting $n=1$ in E.F. ①

$y_2 = y_1 + hf(x_1, y_1)$

$y_2 = 0.1107 + 0.1\, f(0.1, 0.1107)$

$y_2 = 0.1107 + 0.1\,(1.2159)$

$y_2 = 0.2322$ for $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

This value of y_2 is now improved using M.E.F.

$y_{n+1}^{p+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(i)})\right] \rightarrow$ ②

put $n=1$ in ②, $i=0$, $p=0 \rightarrow$ ②

$y_2^{(1)} = y_1 + \frac{0.1}{2}\left[f(0.1, 0.1107) + f(0.2, 0.2322)^{(0)}\right]$

$y_2^{(1)} = 0.1107 + \frac{0.1}{2}\left[1.2159 + 1.4536\right]$

$y_2^{(1)} = 0.2441$ for $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

put $n=2$ in ②, $i=1$, $p=1$ in ②

$y_2^{(2)} = y_1 + \frac{0.1}{2}\left[f(0.1, 0.1107) + f(0.2, 0.2441)\right]$

$= 0.1107 + \frac{0.1}{2}\left[1.2159 + 1.4655\right]$

$y_2^{(2)} = 0.2447$

put $n=3$ in ② $i=2$, $p=2$ in ②

$y_2^{(3)} = y_1 + \frac{0.1}{2}\left[f(0.1, 0.1107) + f(0.2, 0.2447)\right]$

$= 0.1107 + \frac{0.1}{2}\left[1.2159 + 1.4661\right]$

$y_2^{(3)} = 0.2448 \quad \therefore y_2^{(3)} = y_2^{(2)}$

Hence $y_2 = 0.2448$ for $x_2 = 0.2$

$\therefore y(0.2) = 0.2448.$

11. Solve by R.K method of 4th order for y at $x = 1.2$ from $dy/dx = \frac{2xy + e^x}{x^2}$:

Sol: Given $\frac{dy}{dx} = \frac{2xy + e^x}{x^2}$, $y(1) = 2$

Comparing the given equation with $dy/dx = f(x,y)$, $y(x_0) = y_0$

Then, $f(x,y) = \frac{2xy + e^x}{x^2}$, $x_0 = 1$, $y_0 = 2$

$y = y_0 + k$ for $x = x_0 + h$

where $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\Rightarrow k_1 = h f(x_0, y_0)$, $h = x - x_0 = 1.2 - 1 = 0.2$

$\Rightarrow k_1 = 0.2\left(\frac{2x_0y_0 + e^{x_0}}{x_0^2}\right) = 0.2\left(\frac{2(1)(2) + 2.7182}{1}\right)$

$k_1 = 1.3436$

$\Rightarrow k_2 = 0.2 f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 f\left(1 + \frac{0.2}{2}, 2 + \frac{1.3436}{2}\right)$

$= 0.2 f(1.1, 2.6718)$

$= 0.2\left(\frac{2(1.1)(2.6718) + e^{1.1}}{(1.1)^2}\right)$

$\Rightarrow k_2 = \cancel{0.2}\ 1.4681$

$\Rightarrow k_3 = 0.2 f\left(x_0 + h/2, y_0 + \frac{k_2}{2}\right) = 0.2 f\left(1 + \frac{0.2}{2}, 2 + \frac{1.4681}{2}\right)$

$= 0.2 f(1.1, 2.734)$

$k_3 = 0.2\left(\frac{2(1.1)(2.734) + e^{1.1}}{(1.1)^2}\right)$

$k_3 = 1.4907$

$K_4 = hf(x_0+h, y_0+k_3) = 0.2f(1+0.2, 2+1.4907)$

$K_4 = 0.2f(1.2, 3.4907)$

$$K_4 = 0.2\left(\frac{2(1.2)(3.4907)+e^{1.2}}{(1.2)^2}\right)$$

$K_4 = 0.2(8.12347)$

$K_4 = 1.6246$

$\Rightarrow K = \frac{1}{6}(K_1+2K_2+2K_3+K_4)$

$K = \frac{1}{6}(1.3436+2(1.4681)+2(1.4907)+1.6246)$

$K = 1.4809$

$y = y_0 + K = 2+1.4809 = 3.4809$

$x = x_0 + h = 1+0.2 = 1.2$

$\therefore y(1.2) = 3.4809$