sessional.

) State and prove Cauchy's integral theorem.

Stmt: If $f(z)$ is a analytic and $f'(z)$ is continuous within or on the curve C then $\oint_C f(z)dz = 0$.

Proof $f(z)$ is analytic. CR equations hold

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$f(z) = u(x,y) + iv(x,y)$.

$z = x + iy$

$d(z) = dx + idy$

$$\oint_C f(z)dz = \oint_C (u+iv)(dx+idy) = \oint_C (udx - vdy) + i\oint_C (vdx + udy).$$

from Green's theorem $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

$$\Rightarrow \oint_C f(z)dz = \iint_R \left(-\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\right) dxdy + i\iint_R \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dxdy$$

$$= \iint_R \left(-\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\right) dxdy + i\iint_R \left(-\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\right) dxdy$$

$= 0$

Hence proved.

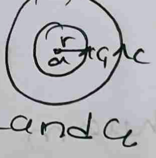
2) state and prove cauchy's integral formula.

Proof

If $f(z)$ is a analytic within and on the closed curve c and a be point with in c then $\oint_c \frac{f(z)}{z-a} dz = 2\pi i f(a)$.

$\Rightarrow$ consider the function $\frac{f(z)}{z-a}$ which is analytic insid closed curve c

Now draw a small circle c_1 with a as center and r as radius which lies entirely inside c.



$\therefore \oint_c \frac{f(z)}{z-a}$ is analytic b/w c and c_1

The eqn of circle is $|z-a|=r$

$$z-a = re^{i\theta}$$

$$z = a+re^{i\theta}$$

$$dz = ire^{i\theta} d\theta.$$

$$\oint_c \frac{f(z)}{z-a} dz = \oint_c \frac{f(a+re^{i\theta})}{re^{i\theta}} (ire^{i\theta} d\theta)$$

If θ varies from 0 to 2π

$$= \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta})}{\cancel{re^{i\theta}}} (i\cancel{re^{i\theta}} d\theta)$$

$$= i\int_{\theta=0}^{2\pi} f(a+re^{i\theta}) d\theta$$

$\therefore$ the unit $r \to 0$

$= i\int_0^{2\pi} f(a)\, d\theta.$

$= if(a)\,(\theta)_0^{2\pi}$

$= if(a)\,(2\pi)$

$= \underline{2\pi i f(a)}.$

Hence proved.

2) i) $\oint_C \frac{z^2+1}{z(2z+1)} dz$ where c is $|z|=1$

Sol $z(2z+1)=0.$

$z = 0, -1/2$ $\quad |0|<1$, $|1/2|<1$

Both the singular points lie inside c.

$\therefore \oint_C \frac{z^2+1}{z(2z+1)} dz = \oint_C \frac{\frac{z^2+1}{z}}{2z+1} dz + \oint_C \frac{\frac{z^2+1}{2z+1}}{z-0} dz.$

$= \oint_C \frac{\frac{z^2+1}{z}}{2(z-(-1/2))} dz + \oint_C \frac{\frac{z^2+1}{2z+1}}{z-0} dz$

$= 2\pi i f(-1/2) + 2\pi i f(0)$

$= 2\pi i \left(\frac{(-1/2)^2+1}{-1/2}\right) + 2\pi i \left(\frac{0+1}{0+1}\right)$

$= \cancel{2}\pi i \left(\frac{1+4}{\cancel{4}_2}\right)\cancel{4}$

$= \frac{-5\pi i}{2} + 2\pi i$

$= -\pi i/2$

ii) $\oint_C \frac{\sin \pi z + \cos \pi z}{(z-1)(z-2)} dz$.

$|z|=4$

singular point $(z-1)(z-2)=0$

$z = 1, 2$

$\Rightarrow |1| < 4, \ |2| < 4$.

lies inside circle

$$\oint_C \frac{\sin\pi z+\cos\pi z}{(z-1)(z-2)}dz = \oint_C \frac{\frac{\sin\pi z+\cos\pi z}{z-1}}{z-2}dz + \oint_C \frac{\frac{\sin\pi z+\cos\pi z}{z-2}}{z-1}dz$$

$= 2\pi i f(2) + 2\pi i f(1)$.

$= 2\pi i \left[\frac{\sin\pi(2)+\cos\pi(2)}{2-1}\right] + 2\pi i \left[\frac{\sin\pi(1)+\cos\pi(1)}{\cancel{z}-2}\right]$

$= 2\pi i \left[0 + \frac{1}{1}\right] + 2\pi i \left[\frac{0-1}{-1}\right]$

$= 2\pi i + 2\pi i$

$= 4\pi i$.

iii) $\oint_C \frac{\log z}{(z-1)^3} dz$ $\quad |z-1| = 1/2$.

singular point $(z-1)^3 = 0$.

$z = 1$

$|z-1| < 1/2$

All 3 point lies inside C

$\oint_C \frac{\log z}{(z-1)^3} = \frac{2\pi i f''(1)}{2!}$

$f(3) = \log 3$

$f'(3) = \frac{1}{3}$

$f''(3) = \frac{-1}{3^2}$

$= 2\pi i\left[\frac{-1/1}{2}\right]$

$= \Rightarrow i\pi$

3) find taylor series expansion at $f(z) = \frac{2z^3+1}{(z-1)^3}dz$

$\Rightarrow \frac{2z^3+1}{z^2+z}$ about the point $z=i$

Given $f(z) = \frac{2z^3+1}{z^2+z}$ ND > DD.

$$
\begin{array}{r|l}
z^2+z & 2z^3+1 \;(2z-2 \\
 & \underline{\cancel{2z^3} \pm 2z^2} \\
 & 1-\cancel{2z^2} \\
 & \underline{\mp \cancel{2z^2} \mp 2z} \\
 & 1+2z
\end{array}
$$

$2z^3+1 = (z^2+z)(2z-2) + 1+2z$

$\frac{2z^3+1}{z^2+z} = 2z-2+\frac{1+2z}{z^2+z}$

Now $\frac{1+2z}{z(z+1)} = \frac{A}{z} + \frac{B}{z+1}$

$1+2z = A(z+1) + Bz$

$z=-1 \Rightarrow 1+2(-1) = A(-1+1) - B.$

$-1 = -B$

$\boxed{B=1}$

$z=0.$

$\boxed{A=1}$

$\frac{2z^3+1}{z^2+z} = 2z-2+\frac{1}{z}+\frac{1}{z+1}$

$\therefore f(z) = 2z - 2 + \frac{1}{z} + \frac{1}{z+1}$

Given $z = i$

$z - i = t$

$z = t + i$

$f(z) = 2(t+i) - 2 + \frac{1}{(t+i)} + \frac{1}{(t+i+1)}$

$= 2(t+i) - 2 + \frac{1}{i\left(\frac{t}{i}+1\right)} + \frac{1}{(1+i)\left(\frac{t}{i}+1\right)}$

$= 2(t+i) - 2 + \frac{1}{i}\left(\frac{t}{i}+1\right)^{-1} + \frac{1}{1+i}\left(\frac{t}{1+i}+1\right)^{-1}$

$= 2(t+i) - 2 + \frac{1}{i}\left(1 - t/i + (t/i)^2 + \cdots\right) + \frac{1}{1+i}\cdot\left(1 - t/_{1+i} + t/_{(1+i)}^2 + \cdots\right)$

$= 2(t+i) - 2 + \frac{1}{i} - t/i^2 + t^2/i^3 + \cdots + \frac{1}{(1+i)}\left\{1 - \frac{t}{(1+i)^2} + \frac{t^2}{(1+i)^3} + \cdots\right.$

4) $f(z) = \frac{z^2 - 6z - 1}{(z-1)(z-3)(z+3)}$ in the region $3 < |z+2| < 5$

Laurent series.

Given $f(z) = \frac{z^2 - 6z - 1}{(z-1)(z-3)(z+2)}$ region $3 < |z+2| < 5$

$f(z) = \frac{z^2 - 6z - 1}{(z-1)(z-3)(z+2)} = \frac{A}{z-1} + \frac{B}{z-3} + \frac{C}{z+2}$.

$z^2 - 6z - 1 = A(z-3)(z+2) + B(z-1)(z+2) + C(z-1)(z-3)$

Put $z = 3$ $\quad 9 - 18 - 1 = 0 + B(2)(5)$

$\Rightarrow -10 = 10B$

$\boxed{B=-1}$

put $z=1$

$1-6-1 = A(-2)(3) + B(0)$

$-6 = A(-6)$

$\boxed{A=1}$

put $z=-2$

$4+12-1 = C(-3)(-5)$

$15 = 15C$

$\boxed{C=1}$

$f(z) = \frac{1}{z-1} - \frac{\cancel{B}}{z-3} + \frac{1}{z+2}$

Given $3<|z+2|<5 \Rightarrow z = t-2$

let $z+2=t$ Then the region r $3<|t|<5$.

$\frac{3}{|t|}<1, \frac{|t|}{5}<1$

$f(z) = \frac{1}{t-3} - \frac{1}{\cancel{t}-5} + \frac{1}{t}$

$= \frac{1}{t(1-3/t)} - \frac{1}{-5(1-t/5)} + \frac{1}{t}$

$= \frac{1}{t}(1-3/t)^{-1} + \frac{1}{5}(1-t/5)^{-1} + \frac{1}{t}$ $\boxed{t = z+2}$

$= \frac{1}{t}(1+(3/t)+(3/t)^2+\cdots) + \frac{1}{5}(1+(t/5)+(t/5)^2+\cdots) + \frac{1}{t}$

$= \left[\frac{1}{t} + \frac{3}{t^2} + \frac{9}{t^3} + \cdots\right] + \left[\frac{1}{5} + t/5^2 + t^2/5^3 + \cdots\right] + \frac{1}{t}$