UNIT-II

The Predicate Calculus

Predicate

A part of a declarative sentence describing the properties of an object is called a predicate. The logic based upon the analysis of predicate in any statement is called predicate logic.

Consider two statements:
John is a bachelor
Smith is a bachelor.

In each statement ‖is a bachelor‖ is a predicate. Both John and Smith have the same property of being a bachelor.
In the statement logic, we require two different symbols to express them and these symbols do not reveal the common property of these statements.
In predicate calculus these statements can be replaced by a single statement "x is a bachelor".

A predicate is symbolized by a capital letters which is followed by the list of variables. The list of variables is enclosed in parenthesis.
If P stands for the predicate "is a bachelor", then $P(x)$ stands for "x is a bachelor", where x is a predicate variable.

The domain for $P(x)$: x is a bachelor, can be taken as the set of all human names. Note that $P(x)$ is not a statement, but just an expression. Once a value is assigned to x, $P(x)$ becomes a statement and has the truth value. If x is Ram, then $P(x)$ is a statement and its truth value is true.

Quantifiers

Quantifiers: Quantifiers are words that are refer to quantities such as 'some' or 'all'.

Universal Quantifier: The phrase 'forall' (denoted by $\forall$) is called the universal quantifier.

For example, consider the sentence ‖All human beings are mortal‖.
Let $P(x)$ denote 'x is a mortal'.
Then, the above sentence can be written as

$(\forall x \in S)P(x)$ or $\forall x P(x)$

where S denote the set of all human beings.

$\forall x$ represents each of the following phrases, since they have essentially the same for all x

For every x
For each x.

Existential Quantifier: The phrase 'there exists' (denoted by $\exists$) is called the existential quantifier.

For example, consider the sentence

For example, consider the sentence

"There exists x such that $x^2 = 5$.

This sentence can be written as

$$(\exists x \in R)P(x) \text{ or } (\exists x)P(x),$$

where $P(x) : x^2 = 5$.

$\exists x$ represents each of the following phrases

There exists an x
There is an x
For some x
There is at least one x.

Example: Write the following statements in symbolic form:

(i). Something is good
(ii). Everything is good
(iii). Nothing is good
(iv). Something is not good.

Solution: Statement (i) means "There is atleast one x such that, x is good".

Statement (ii) means "Forall x, x is good".

Statement (iii) means, "Forall x, x is not good".

Statement (iv) means, "There is atleast one x such that, x is not good.

Thus, if $G(x) : x$ is good, then

statement (i) can be denoted by $(\exists x)G(x)$

statement (ii) can be denoted by $(\forall x)G(x)$

statement (iii) can be denoted by $(\forall x)\neg G(x)$

statement (iv) can be denoted by $(\exists x)\neg G(x)$.

Example: Let $K(x) : x$ is a man

$L(x) : x$ is mortal
$M(x) : x$ is an integer
$N(x) : x$ either positive or negative

Express the following using quantifiers:

- All men are mortal
- Any integer is either positive or negative.

Solution: (a) The given statement can be written as

for all x, if x is a man, then x is mortal and this can be expressed as

$(x)(K(x) \rightarrow L(x))$.

(b) The given statement can be written as

for all x, if x is an integer, then x is either positive or negative and this can be expressed as $(x)(M(x) \rightarrow N(x))$.

Free and Bound Variables

Given a formula containing a part of the form $(x)P(x)$ or $(\exists x)P(x)$, such a part is called an x-bound part of the formula. Any occurrence of x in an x-bound part of the formula is called a bound occurrence of x, while any occurrence of x or of any variable that is not a bound occurrence is called a free occurrence. The smallest formula immediately following $(\forall x)$ or $(\exists x)$ is called the scope of the quantifier.

Consider the following formulas:

- $(x)P(x, y)$
- $(x)(P(x) \rightarrow Q(x))$
- $(x)(P(x) \rightarrow (\exists y)R(x, y))$
- $(x)(P(x) \rightarrow R(x)) \vee (x)(R(x) \rightarrow Q(x))$
- $(\exists x)(P(x) \wedge Q(x))$
- $(\exists x)P(x) \wedge Q(x)$.

In (1), $P(x, y)$ is the scope of the quantifier, and occurrence of x is bound occurrence, while the occurrence of y is free occurrence.

In (2), the scope of the universal quantifier is $P(x) \rightarrow Q(x)$, and all concrescences of x are bound.

In (3), the scope of (x) is $P(x) \rightarrow (\exists y)R(x, y)$, while the scope of $(\exists y)$ is $R(x, y)$. All occurrences of both x and y are bound occurrences.

In (4), the scope of the first quantifier is $P(x) \rightarrow R(x)$ and the scope of the second is $R(x) \rightarrow Q(x)$. All occurrences of x are bound occurrences.

In (5), the scope $(\exists x)$ is $P(x) \wedge Q(x)$.

In (6), the scope of $(\exists x)$ is $P(x)$ and the last of occurrence of x in $Q(x)$ is free.

Negations of Quantified Statements

(i). $\neg(x)P(x) \Leftrightarrow (\exists x)\neg P(x)$

(ii). $\neg(\exists x)P(x) \Leftrightarrow (x)(\neg P(x))$.

Example: Let $P(x)$ denote the statement "x is a professional athlete" and let $Q(x)$ denote the statement "x plays soccer". The domain is the set of all people.

(a). Write each of the following proposition in English.

- $(x)(P(x) \rightarrow Q(x)$
- $(\exists x)(P(x) \wedge Q(x))$
- $(x)(P(x) \vee Q(x))$

(b). Write the negation of each of the above propositions, both in symbols and in words.

Solution:

(a). (i). For all x, if x is an professional athlete then x plays soccer.

"All professional athletes plays soccer" or "Every professional athlete plays soccer".

(ii). There exists an x such that x is a professional athlete and x plays soccer.

”Some professional athletes paly soccer”.

(iii). For all x, x is a professional athlete or x plays soccer.

”Every person is either professional athlete or plays soccer”.

(b). (i). In symbol: We know that

$$\neg(x)(P(x) \rightarrow Q(x)) \Leftrightarrow (\exists x)\neg(P(x) \rightarrow Q(x)) \Leftrightarrow (\exists x)\neg(\neg(P(x)) \vee Q(x))$$

$$\Leftrightarrow (\exists x)(P(x) \wedge \neg Q(x))$$

There exists an x such that, x is a professional athlete and x does not paly soccer.
In words: ”Some professional athlete do not play soccer”.

(ii). $\neg(\exists x)(P(x) \wedge Q(x)) \Leftrightarrow (x)(\neg P(x) \vee \neg Q(x))$

In words: ”Every people is neither a professional athlete nor plays soccer” or All people either not a professional athlete or do not play soccer”.

(iii). $\neg(x)(P(x) \vee Q(x)) \Leftrightarrow (\exists x)(\neg P(x) \wedge \neg Q(x))$.

In words: ”Some people are not professional athlete or do not paly soccer”.

Inference Theory of the Predicate Calculus

To understand the inference theory of predicate calculus, it is important to be famil-iar with the following rules:

Rule US: Universal specification or instaniation

$$(x)A(x) \Rightarrow A(y)$$

From $(x)A(x)$, one can conclude $A(y)$.

Rule ES: Existential specification

$$(\exists x)A(x) \Rightarrow A(y)$$

From $(\exists x)A(x)$, one can conclude $A(y)$.

Rule EG: Existential generalization

$$A(x) \Rightarrow (\exists y)A(y)$$

From $A(x)$, one can conclude $(\exists y)A(y)$.

Rule UG: Universal generalization

$$A(x) \Rightarrow (y)A(y)$$

From $A(x)$, one can conclude $(y)A(y)$.

Equivalence formulas:

$E_{31} : (\exists x)[A(x) \vee B(x)] \Leftrightarrow (\exists x)A(x) \vee (\exists x)B(x)$

$E_{32} : (x)[A(x) \wedge B(x)] \Leftrightarrow (x)A(x) \wedge (x)B(x)$

$E_{33} : \neg(\exists x)A(x) \Leftrightarrow (x)\neg A(x)$

$E_{34} : \neg(x)A(x) \Leftrightarrow (\exists x)\neg A(x)$

$E_{35} : (x)(A \vee B(x)) \Leftrightarrow A \vee (x)B(x)$

$E_{36} : (\exists x)(A \wedge B(x)) \Leftrightarrow A \wedge (\exists x)B(x)$

$E_{37} : (x)A(x) \rightarrow B \Leftrightarrow (x)(A(x) \rightarrow B)$

$E_{38} : (\exists x)A(x) \rightarrow B \Leftrightarrow (x)(A(x) \rightarrow B)$

$E_{39} : A \rightarrow (x)B(x) \Leftrightarrow (x)(A \rightarrow B(x))$

$E_{40} : A \to (\exists x)B(x) \Leftrightarrow (\exists x)(A \to B(x))$

$E_{41} : (\exists x)(A(x) \to B(x)) \Leftrightarrow (x)A(x) \to (\exists x)B(x)$

$E_{42} : (\exists x)A(x) \to (x)B(X) \Leftrightarrow (x)(A(x) \to B(X))$.

Example: Verify the validity of the following arguments:

"All men are mortal. Socrates is a man. Therefore, Socrates is mortal".

or

Show that $(x)[H(x) \to M(x)] \wedge H(s) \Rightarrow M(s)$.

Solution: Let us represent the statements as follows:

$H(x)$: x is a man
$M(x)$: x is a mortal
s : Socrates

Thus, we have to show that $(x)[H(x) \to M(x)] \wedge H(s) \Rightarrow M(s)$.

{1}	(1)	$(x)[H(x) \to M(x)]$	Rule P
{1}	(2)	$H(s) \to M(s)$	Rule US, (1)
{3}	(3)	$H(s)$	Rule P
{1, 3}	(4)	$M(s)$	Rule T, (2), (3), and I_{11}

Example: Establish the validity of the following argument:"All integers are ratio-nal numbers. Some integers are powers of 2. Therefore, some rational numbers are powers of 2".

Solution: Let $P(x)$: x is an integer

$R(x)$: x is rational number

$S(x)$: x is a power of 2

Hence, the given statements becomes

$$(x)(P(x) \to R(x)), (\exists x)(P(x) \wedge S(x)) \Rightarrow (\exists x)(R(x) \wedge S(x))$$

Solution:

{1}	(1)	$(\exists x)(P(x) \wedge S(x))$	Rule P
{1}	(2)	$P(y) \wedge S(y)$	Rule ES, (1)
{1}	(3)	$P(y)$	Rule T, (2) and $P \wedge Q \Rightarrow P$
{1}	(4)	$S(y)$	Rule T, (2) and $P \wedge Q \Rightarrow Q$
{5}	(5)	$(x)(P(x) \to R(x))$	Rule P
{5}	(6)	$P(y) \to R(y)$	Rule US, (5)
{1, 5}	(7)	$R(y)$	Rule T, (3), (6) and $P, P \to Q \Rightarrow Q$
{1, 5}	(8)	$R(y) \wedge S(y)$	Rule T, (4), (7) and $P, Q \Rightarrow P \wedge Q$
{1, 5}	(9)	$(\exists x)(R(x) \wedge S(x))$	Rule EG, (8)

Hence, the given statement is valid.

Example: Show that $(x)(P(x) \to Q(x)) \wedge (x)(Q(x) \to R(x)) \Rightarrow (x)(P(x) \to R(x))$.

Solution:

$\{1\}$	(1) $(x)(P(x) \to Q(x))$	Rule P
$\{1\}$	(2) $P(y) \to Q(y)$	Rule US, (1)
$\{3\}$	(3) $(x)(Q(x) \to R(x))$	Rule P
$\{3\}$	(4) $Q(y) \to R(y)$	Rule US, (3)
$\{1, 3\}$	(5) $P(y) \to R(y)$	Rule T, (2), (4), and I_{13}
$\{1, 3\}$	(6) $(x)(P(x) \to R(x))$	Rule UG, (5)

Example: Show that $(\exists x)M(x)$ follows logically from the premises

$(x)(H(x) \to M(x))$ and $(\exists x)H(x)$.

Solution:

$\{1\}$	(1) $(\exists x)H(x)$	Rule P
$\{1\}$	(2) $H(y)$	Rule ES, (1)
$\{3\}$	(3) $(x)(H(x) \to M(x))$	Rule P
$\{3\}$	(4) $H(y) \to M(y)$	Rule US, (3)
$\{1, 3\}$	(5) $M(y)$	Rule T, (2), (4), and I_{11}
$\{1, 3\}$	(6) $(\exists x)M(x)$	Rule EG, (5)

Hence, the result.

Example: Show that $(\exists x)[P(x) \wedge Q(x)] \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$.

Solution:

$\{1\}$	(1) $(\exists x)(P(x) \wedge Q(x))$	Rule P
$\{1\}$	(2) $P(y) \wedge Q(y)$	Rule ES, (1)
$\{1\}$	(3) $P(y)$	Rule T, (2), and I_1
$\{1\}$	(4) $(\exists x)P(x)$	Rule EG, (3)
$\{1\}$	(5) $Q(y)$	Rule T, (2), and I_2
$\{1\}$	(6) $(\exists x)Q(x)$	Rule EG, (5)
$\{1\}$	(7) $(\exists x)P(x) \wedge (\exists x)Q(x)$	Rule T, (4), (5) and I_9

Hence, the result.

Note: Is the converse true?

$\{1\}$	(1) $(\exists x)P(x) \wedge (\exists x)Q(x)$	Rule P
$\{1\}$	(2) $(\exists x)P(x)$	Rule T, (1) and I_1
$\{1\}$	(3) $(\exists x)Q(x)$	Rule T, (1), and I_1
$\{1\}$	(4) $P(y)$	Rule ES, (2)
$\{1\}$	(5) $Q(s)$	Rule ES, (3)

Here in step (4), y is fixed, and it is not possible to use that variable again in step (5). Hence, the *converse is not true.*

Example: Show that from $(\exists x)[F(x) \wedge S(x)] \to (y)[M(y) \to W(y)]$ and $(\exists y)[M(y) \wedge \neg W(y)]$ the conclusion $(x)[F(x) \to \neg S(x)]$ follows.

{1}	(1)	$(\exists y)[M(y) \wedge \neg W(y)]$	Rule P
{1}	(2)	$[M(z) \wedge \neg W(z)]$	Rule ES, (1)
{1}	(3)	$\neg[M(z) \to W(z)]$	Rule T, (2), and $\neg(P \to Q) \Leftrightarrow P \wedge \neg Q$
{1}	(4)	$(\exists y)\neg[M(y) \to W(y)]$	Rule EG, (3)
{1}	(5)	$\neg(y)[M(y) \to W(y)]$	Rule T, (4), and $\neg(x)A(x) \Leftrightarrow (\exists x)\neg A(x)$
{1}	(6)	$(\exists x)[F(x) \wedge S(x)] \to (y)[M(y) \to W(y)]$	Rule P
{1, 6}	(7)	$\neg(\exists x)[F(x) \wedge S(x)]$	Rule T, (5), (6) and I_{12}
{1, 6}	(8)	$(x)\neg[F(x) \wedge S(x)]$	Rule T, (7), and $\neg(x)A(x) \Leftrightarrow (\exists x)\neg A(x)$
{1, 6}	(9)	$\neg[F(z) \wedge S(z)]$	Rule US, (8)
{1, 6}	(10)	$\neg F(z) \vee \neg S(z)$	Rule T, (9), and De Morgan's laws
{1, 6}	(11)	$F(z) \to \neg S(z)$	Rule T, (10), and $P \to Q \Leftrightarrow \neg P \vee Q$
{1, 6}	(12)	$(x)(F(x) \to \neg S(x))$	Rule UG, (11)

Hence, the result.

Example: Show that $(x)(P(x) \vee Q(x)) \Rightarrow (x)P(x) \vee (\exists x)Q(x)$. (May. 2012)

Solution: We shall use the indirect method of proof by assuming $\neg((x)P(x) \vee (\exists x)Q(x))$ as an additional premise.

{1}	(1)	$\neg((x)P(x) \vee (\exists x)Q(x))$	Rule P (assumed)
{1}	(2)	$\neg(x)P(x) \wedge \neg(\exists x)Q(x)$	Rule T, (1) $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$
{1}	(3)	$\neg(x)P(x)$	Rule T, (2), and I_1
{1}	(4)	$(\exists x)\neg P(x)$	Rule T, (3), and $\neg(x)A(x) \Leftrightarrow (\exists x)\neg A(x)$
{1}	(5)	$\neg(\exists x)Q(x)$	Rule T, (2), and I_2
{1}	(6)	$(x)\neg Q(x)$	Rule T, (5), and $\neg(\exists x)A(x) \Leftrightarrow (x)\neg A(x)$
{1}	(7)	$\neg P(y)$	Rule ES, (5), (6) and I_{12}
{1}	(8)	$\neg Q(y)$	Rule US, (6)
{1}	(9)	$\neg P(y) \wedge \neg Q(y)$	Rule T, (7), (8)and I_9
{1}	(10)	$\neg(P(y) \vee Q(y))$	Rule T, (9), and $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$
{11}	(11)	$(x)(P(x) \vee Q(x))$	Rule P
{11}	(12)	$(P(y) \vee Q(y))$	Rule US
{1, 11}	(13)	$\neg(P(y) \vee Q(y)) \wedge (P(y) \vee Q(y))$	Rule T, (10), (11), and I_9
{1, 11}	(14)	F	Rule T, and (13)

which is a contradiction.Hence, the statement is valid.

Example: Using predicate logic, prove the validity of the following argument: ”Every husband argues with his wife. x is a husband. Therefore, x argues with his wife”.

Solution: Let $P(x)$: x is a husband.

$Q(x)$: x argues with his wife.

Thus, we have to show that $(x)[P(x) \to Q(x)] \wedge P(x) \Rightarrow Q(y)$.

$\{1\}$	(1)	$(x)(P(x) \to Q(x))$	Rule P
$\{1\}$	(2)	$P(y) \to Q(y)$	Rule US, (1)
$\{1\}$	(3)	$P(y)$	Rule P
$\{1\}$	(4)	$Q(y)$	Rule T, (2), (3), and I_{11}

Example: Prove using rules of inference

Duke is a Labrador retriever.
All Labrador retriever like to swim.
Therefore Duke likes to swim.

Solution: We denote

$L(x)$: x is a Labrador retriever.

$S(x)$: x likes to swim.

d: Duke.

We need to show that $L(d) \wedge (x)(L(x) \to S(x)) \Rightarrow S(d)$.

$\{1\}$	(1)	$(x)(L(x) \to S(x))$	Rule P
$\{1\}$	(2)	$L(d) \to S(d)$	Rule US, (1)
$\{2\}$	(3)	$L(d)$	Rule P
$\{1, 2\}$	(4)	$S(d)$	Rule T, (2), (3), and I_{11}.