4M

1) State and prove the necessary conditions for analyticity of complex variable function $w=f(z)$.

Statement: The necessary conditions for analyticity of complex variable function $w=f(z)=u(x,y)+iv(x,y)$

be i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ be continuous function in the $xy$ of the region R.

ii) $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ {cr equations hold}.

Proof Since $f(z)$ is a analytic then there exist a unique derivate such that

$$f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

$\because z=x+iy$

$\Delta z=\Delta x+i\Delta y$

$z+\Delta z=(x+\Delta x)+i(y+\Delta y)$

$f(z)=u(x,y)+iv(x,y)$

$f(z+\Delta z)=u(x+\Delta x, y+\Delta y)+iv(x+\Delta x, y+\Delta y)$

$$\Rightarrow f'(z)=\lim_{\Delta x+i\Delta y\to 0+i0}\frac{u(x+\Delta x,y+\Delta y)+iv(x+\Delta x,y+\Delta y)-u(x,y)+iv(x,y)}{\Delta x+i\Delta y}$$

$$f'(z)=\lim_{\Delta x,\Delta y\to 0}\frac{u(x+\Delta x,y+\Delta y)-u(x,y)}{\Delta x+i\Delta y}-\frac{iv(x+\Delta x,y+\Delta y)-v(x,y)}{\Delta x+i\Delta y}$$

Since the limit exist then there exist a same value independent of the path along $\Delta z\to 0$. —(1)

We have to consider to following paths.

i) $\Delta y \to 0$ and then $\Delta x \to 0$.

from ① $f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$

$$\Rightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad -②$$

ii) Let $\Delta x \to 0$ and then $\Delta y \to 0$.

from ① $f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + i \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y}$

$$\Rightarrow \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad -③$$

$\therefore f(z)$ is differentiable then limits are equal.

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ CR equations are necessary condition for $w = f(z)$ be analytic.

② Determine whether the following functions are analytic.

i) $\frac{1}{z}$ ii) $\cosh z$ iii) $\frac{x+iy}{x^2+y^2}$.

i) $\frac{1}{z}$

WKT $z = x+iy$

$$\frac{1}{z}=\frac{1}{x+iy} \Rightarrow \frac{1}{x+iy}\times\frac{x-iy}{x-iy}$$

$$=\frac{x-iy}{x^2+y^2} \qquad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

$$u=\frac{x}{x^2+y^2},\ v=\frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x}=\frac{(x^2+y^2)(1)-x(2x)}{x^2+y^2} \Rightarrow \frac{x^2+y^2-2x^2}{x^2+y^2}=\frac{y^2-x^2}{x^2+y^2} \quad -(1)$$

$$\frac{\partial u}{\partial y}=-\frac{1}{(x^2+y^2)^2}2xy \Rightarrow \frac{-2xy}{(x^2+y^2)^2} \quad -(2)$$

$$\frac{\partial v}{\partial x}=-\left(-\frac{1}{(x^2+y^2)^2}2xy\right) \Rightarrow +\frac{2xy}{(x^2+y^2)^2} \quad -(3)$$

$$\frac{\partial v}{\partial y}=\frac{-(x^2+y^2)(1)-y(2y)}{(x^2+y^2)^2} \Rightarrow \frac{y^2-x^2}{(x^2+y^2)^2} \quad -(4)$$

from (1) & (4) $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$

from (2) & (3) $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$

$\therefore \frac{1}{z}$ is analytic at every where except at origin (0,0).

ii) $\cosh z$.

w.k.t $z=x+iy$ $\qquad (\because \cos(A-B)=\cos A\cos B+\sin A\sin B)$

$$\cosh z=\cosh(x+iy)$$

$$=\cos i(x+iy)$$

$$=\cos(ix-y)$$

$$\cosh z=\cos ix\cos y+\sin ix\sin y$$

$$=\cosh x\cos y+i\sinh x\sin y$$

$u = \cosh x \cos y$ $\quad$ $v = \sinh x \sin y$

$u_x = \sinh x \cos y$ $\quad$ $v_x = \cosh x \sin y$

$u_y = -\cosh x \sin y$ $\quad$ $v_y = \sinh x \cos y$

$v_y = u_x$ & $v_x = -u_y$

∴ CR equations hold and $f(z)$ is analytic.

(iii) $\dfrac{x+iy}{x^2+y^2}$

$u = \dfrac{x}{x^2+y^2}$ $\quad$ $v = \dfrac{y}{x^2+y^2}$

$u_x = \dfrac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} \Rightarrow \dfrac{y^2-x^2}{(x^2+y^2)^2}$

$u_y = -\dfrac{xy(2)}{(x^2+y^2)^2} \Rightarrow \dfrac{-2xy}{(x^2+y^2)^2}$

$v_x = \dfrac{-2xy}{(x^2+y^2)^2}$

$v_y = \dfrac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} \Rightarrow \dfrac{x^2+y^2-2y^2}{(x^2+y^2)^2} \Rightarrow \dfrac{x^2-y^2}{(x^2+y^2)^2}$

$\dfrac{-(y^2-x^2)}{(x^2+y^2)^2}$

Not analytic.

3) Determine the value of p such that

$f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(px/y)$

Given $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(px/y)$

$u = \frac{1}{2}\log(x^2+y^2)$ , $v = \tan^{-1}(px/y)$

$u_x = \frac{1}{2}\left[\frac{1}{x^2+y^2}(2x)\right] = \frac{x}{x^2+y^2}$

$u_y = \frac{1}{2}\left[\frac{1}{x^2+y^2}(2y)\right] = \frac{y}{x^2+y^2}$

$v_x = \frac{1}{1+(px/y)^2}(p/y) = \frac{1}{\frac{y^2+(px)^2}{y^2}}(p/y) \Rightarrow \frac{py}{y^2+(px)^2}$.

$v_y = \frac{-px}{y^2+(px)^2}$

$u_x = v_y$ & $u_y = -v_x$

$f(z)$ is analytic when $\boxed{p=-1}$

4) prove that function $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $z \neq 0$

$f(0) = 0$ is continuous and the C-R equations are satisfied at the origin, yet $f'(0)$ does not exist.

Sol

We have to prove that $f(z)$ is continuous at $z = 0$.

CR equations hold at origin

$f'(0)$ does not exist

i) $\lim_{z\to 0} f(z) = f(0)$.

$\lim_{z\to 0} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$

$= \lim_{z\to 0}\left[\dfrac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2}\right] \Rightarrow \lim_{\substack{x\to 0\\ y\to 0}}\left[\dfrac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2}\right]$

Take $y = mx$ such that $x\to 0$.

$\lim_{x\to 0}\left[\dfrac{x^3-(mx)^3+i(x^3+(mx)^3)}{x^2+(mx)^2}\right] = \lim_{x\to 0}\left[\dfrac{x^3-m^3x^3+i(x^3-m^3x^3)}{x^2+m^2x^2}\right]$

$\lim_{x\to 0}\left[\dfrac{x^{\cancel{3}}(1-m^3)}{\cancel{x^2}(1+m^2)} + i\dfrac{x^{\cancel{3}}(1-m^3)}{\cancel{x^2}(1+m^2)}\right] \Rightarrow \lim_{x\to 0}\left[\dfrac{x(1-m^3)}{1+m^2} + \dfrac{x(1-m^3)}{1+m^2}\right]$

$= 0 \Rightarrow f(0)$.

ii) To show CR equations hold $(0,0)$.

$u = \dfrac{x^3-y^3}{x^2+y^2}$ $\quad v = \dfrac{x^3+y^3}{x^2+y^2}$

$\left(\dfrac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{x\to 0}\dfrac{u(x,0)-u(0,0)}{x-0} \Rightarrow \dfrac{x-0}{x-0} = 1$

$\left(\dfrac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y\to 0}\dfrac{u(0,y)-u(0,0)}{y-0} \Rightarrow \dfrac{-y-0}{y-0} = -1$

$\left(\dfrac{\partial v}{\partial x}\right)_{(0,0)} = \lim_{x\to 0}\dfrac{v(x,0)-v(0,0)}{y-0} = \dfrac{x-0}{x-0} = 1$

$\left(\dfrac{\partial v}{\partial y}\right)_{(0,0)} = \lim_{y\to 0}\dfrac{v(0,y)-v(0,0)}{y-0} = \dfrac{y-0}{y-0} = 1$

$u_x = v_y$, $u_y = -v_x$.

To show that $f'(z)$ does not exist

$$\underset{z\to 0}{lt}\ \frac{f(z)-f(0)}{z-0} \Rightarrow \underset{z\to 0}{lt}\left[\frac{\frac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2}}{(x+iy)}\right] - 0$$

$$\underset{z\to 0}{lt}\left[\frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2)(x+iy)}\right]$$

taking $y=mx \Rightarrow x\to 0$

$$= \underset{x\to 0}{lt}\left[\frac{(x^3-(mx)^3)+i(x^3+(mx)^3)}{(x^2+(mx)^2)(x+i(mx))}\right]$$

$$= \underset{x\to 0}{lt}\left[\frac{x^3-m^3x^3}{(x^2+m^2x^2)(x+imx)} + \frac{i(x^3+x^3m^3)}{(x^2+m^2x^2)(x+imx)}\right]$$

$$= \underset{x\to 0}{lt}\left[\frac{\cancel{x^3}(1-m^3)}{\cancel{x^3}(1+m^2)(1+im)} + i\frac{\cancel{x^3}(1+m^3)}{\cancel{x^3}(1+m^2)(1+im)}\right]$$

$$= \frac{1-m^3}{(1+m^2)(1+im)} + i\frac{1+m^3}{(1+m^2)(1+im)}$$

$$= \frac{1+m^3+i(1+m^3)}{(1+m^2)(1+im)}$$

$f'(z)$ does not exist as it is not unique depends on $m$.

5) prove that

i) $u = x^3-3xy^2+3y^2+1$ is harmonic and find its harmonic conjugate

ii) $u(r,\theta) = r^2\cos 2\theta$.

iii) $x^3-3xy^2 = c$ find orthogonal trajectories.

$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$u_x = 3x^2 - 3y^2 + 6x \qquad u_y = -6xy - 6y$

$u_{xx} = 6x + 6 \qquad u_{yy} = -6x - 6$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow 6x + 6 - 6x - 6 = 0.$

To find Harmonic conjugate

Assume that $f(z)$ is analytic

$u_x = v_y \qquad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$u_y = -v_x$

$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ — (1)

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$ (Put $z = x$, $y = 0$.

$\frac{\partial u}{\partial x} = 3z^2 + 6z$

$\frac{\partial u}{\partial y} = -6xy - 6y \qquad (1) \Rightarrow f'(z) = 3z^2 + 6z$

$= 0 \qquad \int f'(z) = \int 3z^2 + 6z \, dz$

$f(z) = z^3 + 3z^2 + c$

$z = x + iy$

$\Rightarrow (x+iy)^3 + 3(x+iy)^2 + c$

$= x^3 + (iy)^3 + 3x^2(iy) + 3x(iy)^2 + 3x^2 + 3y^2 + i6xy$

$= x^3 + i^3y^3 + 3x^2iy - 3xy^2 + 3x^2 - 3y^2 + i6xy + c$

$= x^3 - 3xy^2 + 3x^2 - 3y^2 + c + i(3x^2y + 6xy - y^3)$

$u(r\theta) = r^2\cos 2\theta$

$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

$\frac{\partial u}{\partial r} = 2r\cos 2\theta \qquad \frac{\partial u}{\partial \theta} = -r^2\sin 2\theta(2)$

$\frac{\partial^2 u}{\partial r^2} = 2\cos 2\theta \qquad \frac{\partial^2 u}{\partial \theta^2} = -4r^2\cos 2\theta$

$2\cos 2\theta + \frac{1}{r}(2r\cos 2\theta) + \frac{1}{r^2}(-4r^2\cos 2\theta)$

$= 0.$

To find harmonic conjugate.

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

$\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$

$\frac{\partial v}{\partial \theta} = r(2r\cos 2\theta) \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}(-r^2 2\sin 2\theta)$

$= 2r^2\cos 2\theta \quad -(1) \qquad = 2r\sin 2\theta \quad -(2)$

integrate (1) w.r.to $\theta$ ($r = c$)

$v(r\theta) = \int 2r^2\cos 2\theta \, d\theta$

$= 2r^2 \frac{\sin 2\theta}{2} + f(r)$

$v(r\theta) = r^2\sin 2\theta + f(r) \quad -(3)$

diff 3 partialy with r.to r

$\frac{\partial v}{\partial r} = 2r\sin 2\theta + f'(r) \quad -(4)$

from (2) & (4) $\quad \cancel{2r\sin 2\theta} = \cancel{2r\sin 2\theta} + f'(r)$

$f'(r) = 0 \qquad f(r) = c.$

$v(r\theta) = r^2\sin 2\theta.$

iii) $x^3 - 3xy^2 = c$

let $u = x^3 - 3xy^2$

Assume that $f(z)$ is analytic then it satisfy CR equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = -6xy \quad \text{--- (2)}$$

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2.$$

integrate $\frac{\partial v}{\partial y}$ w.r.to y

$$v(x,y) = \int (3x^2 - 3y^2)\, dy$$

$$= 3x^2y - \frac{3y^3}{3} + \phi(x)$$

$$v(x,y) = 3x^2y - y^3 + \phi(x) \quad \text{--- (3)}$$

diff 3 w.r.to x partially

$$\frac{\partial v}{\partial x} = 6xy + \phi'(x).$$

$$\Rightarrow \cancel{6xy} + \phi'(x) = \cancel{6xy} \quad (\because \text{CR equation})$$

$$\phi'(x) = k \quad \text{--- (4)}$$

(4) in (3)

$$v(x,y) = 3x^2y - y^3 + k$$

$\therefore$ The orthogonal trajectory of given family of curve is $3x^2y - y^3 + k$.

7) Determine the analytic function whose real part is i) $e^x(x\cos y - y\sin y)$ ii) $\log\sqrt{x^2+y^2}$

iii) $\sin 2x/(\cosh 2y - \cos 2x)$

$VSU - U \int\int U$

Given $U = e^x(x\cos y - y\sin y)$

Let $f(z)$ be the analytic such that

$\cancel{\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \frac{\partial V}{\partial x} = -\frac{\partial V}{\partial y}}$ $\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$ , $\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$

$\frac{\partial U}{\partial x} = e^x x\cos y - e^x y\sin y$

$= e^x\cos y + e^x x\cos y - (e^x y\sin y)$ — ①

$\frac{\partial U}{\partial y} = -e^x x\sin y - e^x(\sin y + y\cos y)$ — ②

put $z = x$ and $y = 0$ in ① & ②

$\frac{\partial U}{\partial x} = e^z + ze^z - (0). \Rightarrow e^z(1+z)$

$\frac{\partial U}{\partial y} = -(0) - e^x(0+0) = 0.$

WKT $f'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} \Rightarrow \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$

$f'(z) = e^z(1+z) - i(0).$

$f'(z) = e^z(1+z)$

$f(z) = \int e^z(1+z)\,dz$

c is complex constant

$= (1+z)e^z - (1)(e^z) + c$

$= e^z(1+z-1) \Rightarrow ze^z + c$