1. Prove that $\Delta = E-1$, $\nabla = 1-E^{-1}$, $\delta = E^{1/2} - E^{-1/2}$, $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

$\Delta = E\nabla = \nabla E = \delta E^{1/2}$.

$E = e^{hD}$, $\mu^2 = 1 + \frac{\delta^2}{4}$, $\Delta\nabla = \nabla\Delta = \delta^2$, $(1+\Delta)(1-\nabla) = 1$

Sol: i, $\Delta = E-1$

Proof:

we know that $\Delta f(x) = f(x+h) - f(x)$

$$\Delta f(x) = E[f(x)] - f(x)$$

$$\Delta f(x) = f(x)[E-1]$$

$$\boxed{\therefore \Delta = E-1}$$

ii, $\nabla = 1 - E^{-1}$

Proof:

we know that $\nabla f(x) = f(x) - f(x-h)$

$$= f(x) - E^{-1}[f(x)]$$

$$\nabla f(x) = f(x)\cdot[1-E^{-1}]$$

$$\boxed{\therefore \nabla = 1-E^{-1}}$$

iii, $\delta = E^{1/2} - E^{-1/2}$

Proof:

we know that $\delta f(x) = f(x + h/2) - f(x - h/2)$

$$= E^{1/2}(f(x)) - E^{-1/2} f(x)$$

$$\delta f(x) = f(x)\cdot(E^{1/2} - E^{-1/2})$$

$$\boxed{\therefore \delta = E^{1/2} - E^{-1/2}}$$

iv) $\mu = \frac{1}{2}\left[\epsilon^{1/2} + \epsilon^{-1/2}\right]$

Proof: We know that,

$$\mu(f(x)) = \frac{1}{2}\left[f(x+h/2) + f(x-h/2)\right]$$

$$= \frac{1}{2}\left[\epsilon^{1/2} f(x) + \epsilon^{-1/2} f(x)\right]$$

$$\mu f(x) = f(x)\left[\frac{\epsilon^{1/2} + \epsilon^{-1/2}}{2}\right]$$

$$\boxed{\therefore \mu = \frac{1}{2}\left[\epsilon^{1/2} + \epsilon^{-1/2}\right]}$$

v) $\Delta = \epsilon\nabla$

Proof:

We know that, $\Delta f(x) = f(x+h) - f(x)$

$$\epsilon\nabla f(x) = \epsilon(\nabla f(x)) = \epsilon\left[f(x) - f(x-h)\right]$$

$$= f(x+h) - f(x+h-h)$$

$$= f(x+h) - f(x)$$

$$\epsilon\nabla f(x) = \Delta f(x)$$

$$\boxed{\therefore \Delta = \epsilon\nabla}$$

vi) $\Delta = \nabla\epsilon$

Proof:

$$\nabla\epsilon = \nabla\left[\epsilon f(x)\right] = \nabla f(x+h)$$

$$= f(x+h) - f(x+h-h)$$

$$= f(x+h) - f(x)$$

$$\nabla\epsilon f(x) = \nabla f(x)$$

$$\boxed{\therefore \nabla\epsilon = \nabla}$$

vii) $\Delta = \delta\epsilon^{1/2}$

$$\delta\left[\epsilon^{1/2} f(x)\right] = \delta\left(f(x+h/2)\right) = f\left(x+\frac{h}{2}+\frac{h}{2}\right) - f\left(x+\frac{h}{2}-\frac{h}{2}\right)$$

$$= f(x+h) - f(x)$$

$$\boxed{\therefore \delta\epsilon^{1/2} = \Delta}$$

viii, $E = e^{hD}$

<u>Proof</u>: we know that, $E[f(x)] = f(x+h)$

$$E[f(x)] = f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \cdots$$

$$f(x+h) = f(x) + hD \cdot f(x) + \frac{h^2}{2!} D^2 f(x) + \cdots$$

$$f(x+h) = f(x)\left[1 + hD + \frac{h^2D^2}{2!} + \cdots\right]$$

$$E[f(x)] = e^{hD} f(x)$$

$$\boxed{\therefore E = e^{hD}}$$

ix, $\mu^2 = 1 + \frac{\delta^2}{4}$

$$1 + \frac{\delta^2}{4} = 1 + \frac{(E^{1/2} - E^{-1/2})^2}{4}$$

$$= \frac{4 + E + E^{-1} - 2}{4}$$

$$= \frac{E + E^{-1} + 2}{4}$$

$$= \left(\frac{E^{1/2} + E^{-1/2}}{2}\right)^2$$

$$\frac{1+\delta^2}{4} = \mu^2$$

$$\boxed{\therefore \mu^2 = 1 + \frac{\delta^2}{4}}$$

x, $\Delta\nabla = \nabla\Delta = \delta^2$

$$\Delta\nabla = (E-1)(1-E^{-1})$$

$$= E - 1 - 1 + E^{-1}$$

$$= E + \frac{1}{E} - 2$$

$$= \frac{E^2 + 1 - 2E}{E}$$

$$= \frac{(E-1)^2}{E}$$

$$= \left(\frac{E-1}{E^{1/2}}\right)^2$$

$$= \left[E^{1/2} - E^{-1/2}\right]^2$$

$$\boxed{\therefore \delta N = \delta^2}$$

$$\nabla\Delta = (1-E^{-1})(E-1)$$

$$= E-1-1+E^{-1}$$

$$= E-2+\frac{1}{E}$$

$$= \frac{(E-1)^2}{E}$$

$$= \left(\frac{E-1}{E^{1/2}}\right)^2$$

$$= (E^{1/2} - E^{-1/2})^2$$

$$\boxed{\therefore \nabla\Delta = \delta^2}$$

Si, $(1+\Delta)(1-\nabla) = 1$

$\Rightarrow (E)(E^{-1}) = 1$

2. What is the value of $\Delta^n(e^x)$, $\Delta(1/x^2+5x+6)$ where $h=1$

Sol: $\Delta^n(e^x) = \Delta^{n-1}\Delta(e^x)$

$$= \Delta^{n-1}(e^{(x+h)} - e^x)$$

$$= \Delta^{n-1}(e^{x+1} - e^x)$$

$$= \Delta^{n-1}(ee^x - e^x)$$

$$= e^x \Delta^{n-1}(e-1)$$

$$= (e-1)\Delta^{n-1} e^x$$

$$\Delta^{n-2}(\Delta^2 e^x) = \Delta^{n-2}(\Delta(e-1)e^x)$$

$$= \Delta^{n-2}(e-1(e^{x+h} - e^x))$$

$$= \Delta^{n-2}(e-1(e^{x+1} - e))$$

$$\Delta^n(e^x) = \Delta^{n-2}(e-1)^2(e^x)$$

$\Delta^n(e^x) = (e-1)^n e^x$

iii) $\Delta(1/x^2+5x+6)$

$$\Rightarrow \Delta\left(\frac{1}{x^2+5x+6}\right) = \frac{1}{(x+h)^2+5(x+h)+6} - \frac{1}{x^2+5x+6}$$

$h=1$

$$\Delta\left(\frac{1}{x^2+5x+6}\right) = \frac{1}{(x+1)^2+5(x+1)+6} - \frac{1}{x^2+5x+6}$$

$$= \frac{1}{x^2+1+2x+5x+5+6} - \frac{1}{x^2+5x+6}$$

$$= \frac{1}{x^2+7x+12} - \frac{1}{x^2+5x+6}$$

$$= \frac{1}{(x+3)(x+4)} - \frac{1}{(x+3)(x+2)}$$

$$= \frac{(x+2)-(x+4)}{(x+2)(x+3)(x+4)}$$

$$\Delta\left(\frac{1}{x^2+5x+6}\right) = \frac{-2}{(x+2)(x+3)(x+4)}$$

3. Define interpolation

The process of finding the values of y corresponding to for any value of $x = x_i$ between x_0 & x_n is called interpolation.

(or)

It is the technique of estimating the value of a function for any intermediate value of the independent variable.

4. Newton's formula formula:

Let the function $y=f(x)$ takes the values $y_0, y_1, \ldots, y_n$ corresponding to $x = x_0, x_1, x_2, \ldots, x_n$. Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$ where p is any real number, then $y = f(x) = f(x_0+ph)$

$$y = f(x) = f(x_0+ph) = E^p f(x_0) = (1+\Delta)^p y_0$$

$[\because \Delta = E-1,\ E = 1+\Delta)$
$f(x_0) = y_0$

$$(1+\Delta)^p y_0 = \left[1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \cdots\right] y_0$$

$$(1+\Delta)^p y_0 = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

where $p = \frac{x-x_0}{h}$

Backward formula:

If it given by $y = f(x) = f(x_n+ph) = E^p f(x_n) = E^p y_n$
$= (E-1)^{-p} y_n$.

$$(1-\nabla)^{-p} y_n = \left[1 + p\nabla + \frac{p(p+1)}{2}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \cdots\right] y_n$$

$$= y_n + p\nabla y_n + \frac{p(p+1)}{2}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

where $p = \frac{x-x_n}{h}$

Gauss forward formula:

If it given by $y = f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_{-1} +$

$$\frac{(p+1)(p)(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-1} +$$

$$\frac{(p+2)(p+1)(p)(p-1)(p-2)}{5!}\Delta^5 y_{-2} + \cdots$$

where $p = \frac{x - x_0}{h}$

Gauss backward formula:

It is given by $y = f(x) = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} +$

$\frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p)(p-1)(p+1)}{4!}\Delta^4 y_{-2} + \cdots$

where $p = \frac{x - x_0}{h}$

Stirling's formula:

It is the average of gauss forward and backward interpolation.

$y = y_0 + \frac{p}{2}[\Delta y_0 + \Delta y_{-1}] + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}[\Delta^3 y_{-1} + \Delta^3 y_{-2}] +$

$\frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$, where $p = \frac{x - x_0}{h}$

Bessel formula:

$y = \frac{y_0 + y_1}{2} + (p - \frac{1}{2})\Delta y_0 + \frac{p(p-1)}{2}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) +$

$\frac{(p-1/2)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\left[\frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right] + \cdots$

Lagrange's interpolation formula:

If $y = f(x)$ takes the values $y_0, y_1, \cdots y_n$ corresponding to $x = x_0, x_1, \cdots x_n$ not equally spaced then

$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} \times y_0 +$

$\frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} \times y_1 + \frac{(x_2-x_0)(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)\cdots(x_2-x_n)} y_2 +$

$\cdots$

Newton's divided difference formula:

It is given by $y = f(x) = y_x = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3) + \ldots$

5. What is Numerical differentiation

It is the process of calculating the derivative of a function at some particular value of independent variable by means of set of given values of that function.

6. Newton Cotes Quadrature formula:

It is given by $I = \int_a^b f(x) \cdot dx$

$$I = h\left[n y_0 + \frac{n^2}{2} \Delta y_0 + \left(\frac{n^3}{6} - \frac{n^2}{4}\right) \Delta^2 y_0 + \ldots\right] \rightarrow (1)$$

where $h = \frac{b-a}{n}$.

7. Numerical integration:

$$I = \int_{x_0}^{x_0+nh} \left[y_0 + p\Delta y_0 + \frac{p^2-p}{2} \Delta^2 y_0 + \frac{p^3-3p^2+2p}{6} \Delta^3 y_0 + \ldots\right] dx$$

$$I = \int_0^n \left[y_0 + p\Delta y_0 + \frac{p^2-p}{2} \Delta^2 y_0 + \ldots\right] h\,dp$$

8. Trapezoidal rule: by taking five ordinates, n=4

$$I = \frac{h}{2}\left[(y_0 + y_4) + 2(y_1 + y_2 + y_3)\right]$$

9. Simpson's 1/3 rule for n=6

$$I = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

10. Initial value problem:

Problems in which all the conditions are given only at initial point.

11. Write euler's recursive formula:

$y_{n+1} = y_n + h f(x_n, y_n)$ $(n = 0, 1, 2, ---)$

$dy/dx = f(x, y)$, $y(x_0) = y_0$

12. Taylor's series method:

$$y(x) = y_0 + (x - x_0) y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \frac{(x - x_0)^3}{3!} y_0''' + ---$$

13. The order of convergence of N-R method.

Since the error at each step is proportional to the square of previous error and hence this method has quadratic convergence (2nd order)

14. The condition for convergence of N-R method:

It is given by $|f(x) \cdot f''(x)| < |f'(x)|^2$

15. Diagonal dominance condition of Gauss-Siedel I.M:

If a_1, b_2, c_3 are large as compare to other co-efficients then solving these for (x, y, z) then system becomes

$x = \frac{1}{a_1}[d_1 - b_1 y - c_1 z]$ $(\because |a_1| > |b_1| + |c_1|)$

$y = \frac{1}{b_2}[d_2 - a_2 y - c_2 z]$ $(\because |b_2| > |a_2| + |c_2|)$

$z = \frac{1}{c_3}[d_3 - a_3 x - b_3 y]$ $(\because |c_3| > |a_3| + |b_3|)$

It is condition for diagonal dominance.