CSE250A HW1

1.1 (a) $P(X,Y|E) = \dfrac{P(X,Y,E)}{P(E)}$ Definition of conditional probability

$= \dfrac{P(X|Y,E) \cdot P(Y|E) \cdot P(E)}{P(E)}$ Product rule expansion

$= P(X|Y,E) \cdot P(Y|E)$

(b) $P(X|Y,E) = \dfrac{P(X,Y,E)}{P(Y,E)}$ Definition of conditional probability

$= \dfrac{P(E) \cdot P(X|E) \cdot P(Y|X,E)}{P(E) \cdot P(Y|E)}$ Product rule expansion

$= \dfrac{P(Y|X,E) \cdot P(X|E)}{P(Y|E)}$

(c) From definition of conditional probability: $P(X|E) = \dfrac{P(X,E)}{P(E)}$

$P(X, Y=y|E) = \dfrac{P(X,Y=y,E)}{P(E)}$

$\because P(X,E) = \sum_y P(X,Y=y,E)$

$\therefore \dfrac{P(X,E)}{P(E)} = \dfrac{\sum_y P(X,Y=y,E)}{P(E)}$

$\therefore P(X|E) = \sum_y (X, Y=y, E)$

1.2 (1) (2) (3) can be expanded based on the definition of conditional probability.

(1) $\dfrac{P(X,Y,E)}{P(E)} = \dfrac{P(X,E)}{P(E)} \cdot \dfrac{P(Y,E)}{P(E)} \Rightarrow P(X,Y,E) = \dfrac{P(X,E) \cdot P(Y,E)}{P(E)}$

(2) $\dfrac{P(X,Y,E)}{P(Y,E)} = \dfrac{P(X,E)}{P(E)} \Rightarrow P(X,Y,E) = \dfrac{P(X,E) \cdot P(Y,E)}{P(E)}$

(3) $\dfrac{P(X,Y,E)}{P(X,E)} = \dfrac{P(Y,E)}{P(E)} \Rightarrow P(X,Y,E) = \dfrac{P(X,E) \cdot P(Y,E)}{P(E)}$

$\therefore$ (1) (2) (3) are equivalent

1.3 (a) X: There is a fire in the forest.

Y: The weather is hot and dry.

Z: Someone was smoking in the forest just now.

(b) X: There is a fire in the forest.

Y: Someone was smoking in the forest just now.

Z: It is raining now.

(c) X: There is a fire in the forest.

Y: Someone was smoking in the forest just now.

Z: Someone is burning the forest now.

1.4 Given: $P(D=1)=1\%$ $\quad P(T=1 \mid D=0)=5\%$ $\quad P(T=0 \mid D=1)=10\%$

(a) $$P(D=0 \mid T=0) = \frac{P(T=0 \mid D=0)\cdot P(D=0)}{P(T=0)} \quad \text{Bayes rule}$$

$$= \frac{P(T=0 \mid D=0)\cdot P(D=0)}{P(D=0, T=0) + P(D=1, T=0)} \quad \text{Marginalization}$$

$$= \frac{P(T=0 \mid D=0)\cdot P(D=0)}{P(D=0)\cdot P(T=0 \mid D=0) + P(D=1)\cdot P(T=0 \mid D=1)} \quad \text{Product rule}$$

$$= \frac{(1-0.05)\times(1-0.01)}{(1-0.01)\times(1-0.05) + 0.01\times 0.10}$$

$$= 99.89\%$$

(b) $$P(D=1 \mid T=1) = \frac{P(T=1 \mid D=1)\cdot P(D=1)}{P(T=1)} \quad \text{Bayes rule}$$

$$= \frac{P(T=1 \mid D=1)\cdot P(D=1)}{1-P(T=0)}$$

$$= \frac{(1-0.10)\times 0.01}{1-[(1-0.01)\times(1-0.05)+0.01\times 0.10]}$$

$$= 15.38\%$$

1.5 (a) $f(x) = \log(x) - (x-1)$

$f'(x) = \frac{1}{x} - 1$

$$f'(x) \begin{cases} >0 & 0<x<1 \\ =0 & x=1 \\ <0 & x>1 \end{cases}$$

$\therefore f(x)$ is maximum at $x=1$

$f(x=\overset{1}{\cancel{0}}) = \log(1) - 1 + 1 = 0$

$\therefore f(x) = \log(x) - (x-1) \leq 0$

$\therefore \log(x) \leq x-1$

See the plot in later page.

(b) $\because \log(x) \leq x-1 \qquad \therefore -\log(x) \geq 1-x$

$$\begin{aligned} \therefore KL(p,q) &= \sum_i p_i \log\left(\frac{p_i}{q_i}\right) \\ &= -\sum_i p_i \log\left(\frac{q_i}{p_i}\right) \\ &\geq \sum_i p_i \left(1-\frac{q_i}{p_i}\right) = \sum_i (p_i - q_i) = \sum_i p_i - \sum_i q_i = 0 \end{aligned}$$

$\therefore KL(p,q) \geq 0$

(c)
$$\begin{aligned} KL(p,q) &= \sum_i p_i \log\left(\frac{p_i}{q_i}\right) \\ &= -\sum_i p_i \log\left(\frac{q_i}{p_i}\right) \\ &= -2\sum_i p_i \log\left(\sqrt{\frac{q_i}{p_i}}\right) \\ &\geq -2\sum_i p_i \left(\sqrt{\frac{q_i}{p_i}} - 1\right) = -2\sum_i(\sqrt{p_i q_i} - p_i) \\ &= 2\sum_i p_i - 2\sum_i \sqrt{p_i q_i} \\ &= \sum_i p_i - 2\sum_i \sqrt{p_i q_i} + \sum_i q_i \quad \Leftarrow \sum_i p_i = \sum_i q_i = 1 \\ &= \sum_i (p_i - 2\sqrt{p_i q_i} + q_i) \\ &= \sum_i (\sqrt{p_i} - \sqrt{q_i})^2 \end{aligned}$$

$\therefore KL(p,q) \geq \sum_i (\sqrt{p_i} - \sqrt{q_i})^2$

(d) Assume $p = [0.5\ 0.5]$ and $q = [0.9\ 0.1]$ satisfies $\sum_i p_i = \sum_i q_i = 1$

$KL(p,q) = 0.5 \log\left(\frac{0.5}{0.9}\right) + 0.5\log\left(\frac{0.5}{0.1}\right) = 0.5108$

$KL(q,p) = 0.9\log\left(\frac{0.9}{0.5}\right) + 0.1\log\left(\frac{0.1}{0.5}\right) = 0.3681 \qquad \therefore KL(p,q) \neq KL(q,p)$

1.7 (a) $I(X,Y) = \sum_x \sum_y P(x,y) \log\left[\frac{P(x,y)}{P(x)P(y)}\right]$

$= -\sum_x \sum_y P(x,y) \log\left[\frac{P(x)P(y)}{P(x,y)}\right]$

$\geq \sum_x \sum_y P(x,y)\left[1 - \frac{P(x)P(y)}{P(x,y)}\right] = \sum_x \sum_y \left[P(x,y) - P(x)P(y)\right]$

$= \sum_x \sum_y P(x,y) - \sum_x P(x) \cdot \sum_y P(y)$

$= 1 - 1$

$= 0$

$\therefore I(X,Y) \geq 0$

(b) If X and Y are independent $\Rightarrow P(x,y) = P(x) \cdot P(y)$

$\therefore I(X,Y) = \sum_x \sum_y P(x,y) \cdot \log(1) = 0$

If $I(X,Y) = 0 \Rightarrow P(x,y) \neq 0$ and $\log\left[\frac{P(x,y)}{P(x) \cdot P(y)}\right] = 0$

$\therefore P(x,y) = P(x) \cdot P(y)$ X and Y are independent