CSE 250A HW1

1.1 (a) $P(X,Y|E) = \frac{P(X,Y,E)}{P(E)}$ Definition of conditional probability

$= \frac{P(X|Y,E)\cdot P(Y|E)\cdot P(E)}{P(E)}$ Product rule expansion

$= P(X|Y,E)\cdot P(Y|E)$

(b) $P(X|Y,E) = \frac{P(X,Y,E)}{P(Y,E)}$ Definition of conditional probability

$= \frac{P(E)\cdot P(X|E)\cdot P(Y|X,E)}{P(E)\cdot P(Y|E)}$ Product rule expansion

$= \frac{P(Y|X,E)\cdot P(X|E)}{P(Y|E)}$

(c) From definition of conditional probability: $P(X|E) = \frac{P(X,E)}{P(E)}$

$P(X,Y=y|E) = \frac{P(X,Y=y,E)}{P(E)}$

$\because P(X,E) = \sum_y P(X,Y=y,E)$

$\therefore \frac{P(X,E)}{P(E)} = \frac{\sum_y P(X,Y=y,E)}{P(E)}$

$\therefore P(X|E) = \sum_y (X,Y=y,E)$

1.2 (1) (2) (3) can be expanded based on the definition of conditional probability.

(1) $\frac{P(X,Y,E)}{P(E)} = \frac{P(X,E)}{P(E)}\cdot\frac{P(Y,E)}{P(E)} \Rightarrow P(X,Y,E) = \frac{P(X,E)\cdot P(Y,E)}{P(E)}$

(2) $\frac{P(X,Y,E)}{P(Y,E)} = \frac{P(X,E)}{P(E)} \Rightarrow P(X,Y,E) = \frac{P(X,E)\cdot P(Y,E)}{P(E)}$

(3) $\frac{P(X,Y,E)}{P(X,E)} = \frac{P(Y,E)}{P(E)} \Rightarrow P(X,Y,E) = \frac{P(X,E)\cdot P(Y,E)}{P(E)}$

$\therefore$ (1) (2) (3) are equivalent

1.3 (a) X: There is a fire in the forest.

Y: The weather is hot and dry.

Z: Someone was smoking in the forest just now.

(b) X: There is a fire in the forest.

Y: Someone was smoking in the forest just now.

Z: It is raining now.

(c) X: There is a fire in the forest.

Y: Someone was smoking in the forest just now.

Z: Someone is burning the forest now.

1.4 Given: $P(D=1)=1\%$ $\quad P(T=1|D=0)=5\%$ $\quad P(T=0|D=1)=10\%$

(a) $$P(D=0|T=0) = \frac{P(T=0|D=0)\cdot P(D=0)}{P(T=0)} \quad \text{Bayes rule}$$

$$= \frac{P(T=0|D=0)\cdot P(D=0)}{P(D=0,T=0)+P(D=1,T=0)} \quad \text{Marginalization}$$

$$= \frac{P(T=0|D=0)\cdot P(D=0)}{P(D=0)\cdot P(T=0|D=0)+P(D=1)\cdot P(T=0|D=1)} \quad \text{Product rule}$$

$$= \frac{(1-0.05)\times(1-0.01)}{(1-0.01)\times(1-0.05)+0.01\times 0.10}$$

$$= 99.89\%$$

(b) $$P(D=1|T=1) = \frac{P(T=1|D=1)\cdot P(D=1)}{P(T=1)} \quad \text{Bayes rule}$$

$$= \frac{P(T=1|D=1)\cdot P(D=1)}{1-P(T=0)}$$

$$= \frac{(1-0.10)\times 0.01}{1-[(1-0.01)\times(1-0.05)+0.01\times 0.10]}$$

$$= 15.38\%$$

1.5 (a) $f(x) = \log(x) - (x-1)$

$f'(x) = \frac{1}{x} - 1$

$$f'(x) \begin{cases} >0 & 0<x<1 \\ =0 & x=1 \\ <0 & x>1 \end{cases}$$

$\therefore f(x)$ is maximum at $x=1$

$f(x=\cancel{0}^{\,1}) = \log(1) - 1 + 1 = 0$

$\therefore f(x) = \log(x) - (x-1) \leq 0$

$\therefore \log(x) \leq x-1$

See the plot in later page.

(b) $\because \log(x) \leq x-1 \qquad \therefore -\log(x) \geq 1-x$

$$\begin{aligned} \therefore KL(p,q) &= \sum_i p_i \log\left(\frac{p_i}{q_i}\right) \\ &= -\sum_i p_i \log\left(\frac{q_i}{p_i}\right) \\ &\geq \sum_i p_i \left(1 - \frac{q_i}{p_i}\right) = \sum_i (p_i - q_i) = \sum_i p_i - \sum_i q_i = 0 \end{aligned}$$

$\therefore KL(p,q) \geq 0$

(c)
$$\begin{aligned} KL(p,q) &= \sum_i p_i \log\left(\frac{p_i}{q_i}\right) \\ &= -\sum_i p_i \log\left(\frac{q_i}{p_i}\right) \\ &= -2\sum_i p_i \log\left(\sqrt{\frac{q_i}{p_i}}\right) \\ &\geq -2\sum_i p_i \left(\sqrt{\frac{q_i}{p_i}} - 1\right) = -2\sum_i \left(\sqrt{p_i q_i} - p_i\right) \\ &= 2\sum_i p_i - 2\sum_i \sqrt{p_i q_i} \\ &= \sum_i p_i - 2\sum_i \sqrt{p_i q_i} + \sum_i q_i \quad \Leftarrow \sum_i p_i = \sum_i q_i = 1 \\ &= \sum_i \left(p_i - 2\sqrt{p_i q_i} + q_i\right) \\ &= \sum_i \left(\sqrt{p_i} - \sqrt{q_i}\right)^2 \end{aligned}$$

$\therefore KL(p,q) \geq \sum_i \left(\sqrt{p_i} - \sqrt{q_i}\right)^2$

(d) Assume $p = [0.5 \ 0.5]$ and $q = [0.9 \ 0.1]$ satisfies $\sum_i p_i = \sum_i q_i = 1$

$KL(p,q) = 0.5 \log\left(\frac{0.5}{0.9}\right) + 0.5 \log\left(\frac{0.5}{0.1}\right) = 0.5108$

$KL(q,p) = 0.9 \log\left(\frac{0.9}{0.5}\right) + 0.1 \log\left(\frac{0.1}{0.5}\right) = 0.3681 \qquad \therefore KL(p,q) \neq KL(q,p)$

1.7 (a) $I(X,Y) = \sum_x \sum_y P(x,y) \log\left[\frac{P(x,y)}{P(x)P(y)}\right]$

$= -\sum_x \sum_y P(x,y) \log\left[\frac{P(x)P(y)}{P(x,y)}\right]$

$\geq \sum_x \sum_y P(x,y)\left[1 - \frac{P(x)P(y)}{P(x,y)}\right] = \sum_x \sum_y \left[P(x,y) - P(x)P(y)\right]$

$= \sum_x \sum_y P(x,y) - \sum_x P(x) \cdot \sum_y P(y)$

$= 1 - 1$

$= 0$

$\therefore I(X,Y) \geq 0$

(b) If X and Y are independent $\Rightarrow P(x,y) = P(x) \cdot P(y)$

$\therefore I(X,Y) = \sum_x \sum_y P(x,y) \cdot \log(1) = 0$

If $I(X,Y) = 0 \Rightarrow P(x,y) \neq 0$ and $\log\left[\frac{P(x,y)}{P(x) \cdot P(y)}\right] = 0$

$\therefore P(x,y) = P(x) \cdot P(y)$ X and Y are independent

CSE 250A HW-1

Problem 1.5: Kullback-Leibler Distance

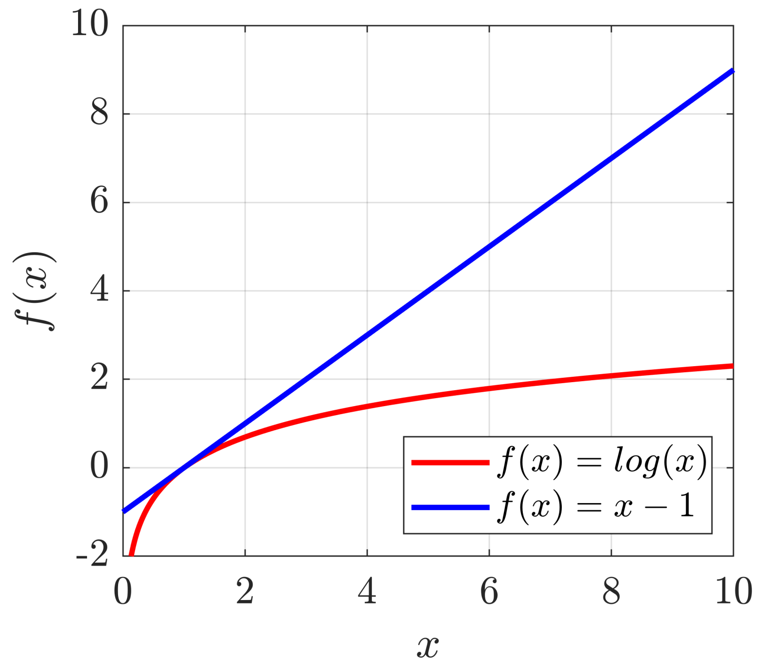
(a) Plot the function.

```
% Plot settings.
set(0, 'DefaultAxesFontSize', 15);
set(0, 'DefaultTextFontSize', 15);

set(0, 'DefaultTextInterpreter', 'latex');
set(0, 'DefaultLegendInterpreter', 'latex');
set(0, 'DefaultAxesTickLabelInterpreter', 'latex');

% Plot the function.
x = 0:0.01:10;

figure('Position', [0, 0, 400, 350]);
hold on;
plot(x, log(x), 'Color', 'r', 'LineWidth', 2);
plot(x, x-1, 'Color', 'b', 'LineWidth', 2);
grid on;
box on;
xlim([0 10]);
xticks(0:2:10);
ylim([-2 10]);
yticks(-2:2:10);
xlabel('$x$');
ylabel('$f(x)$');
legend('$f(x)=log(x)$', '$f(x)=x-1$', 'Location', 'southeast');
```



Problem 1.7: Hangman

(a) Read the word list and compute the prior probability, then print out the 15 most frequent and the 14 least frequent 5-letter words.

Initialization and read word list file:

```
% Obtain the name and path of the word list file.
% "uigetfile" command: Open a dialog box that lists files with particular file type
in the current folder.
[FileName, FolderName, FilterIndex] = uigetfile('*.txt');
FullFileName = fullfile(FolderName, FileName); % fullfile(folder, subfolder, file).

% Open the file based on the given file name and returns to an integer file
identifier equal to or greater than 3.
% "fopen" command: FileID = fopen(filename, permission). 'r' represents read only.
WordListFile = fopen(FullFileName, 'r');

% Read the word list and convert it into a table.
WordList = textscan(WordListFile, '%s %d', 'Delimiter', ' ');
WordList = table(cell2mat(WordList{1}), WordList{2}, 'VariableNames', {'Word',
'Count'});

% Close the word list file.
fclose(WordListFile);
```

Compute the prior probability:

```
% Note: At least one operand should be double type in division.
% Therefore, floating-point division is performed instead of integer division.
WordList.Probability = double(WordList{:,2})/sum(WordList{:,2});
```

Find and print the 15 most frequent words:

```
% Find the 15 largest frequencies from the 2nd column in the word list and their
corresponding row indices.
[MaxFreqs, MaxIndices] = maxk(WordList{:,2}, 15); % [MaxValues, Indices] =
maxk(Vector, NumberOfValues).

% Save the rows corresponding to the 15 largest frequencies to a table.
WordsMaxFreq = WordList(MaxIndices,:);
disp(WordsMaxFreq);
```

Word	Count	Probability
THREE	273077	0.035627
SEVEN	178842	0.023333
EIGHT	165764	0.021626
WOULD	159875	0.020858
ABOUT	157448	0.020542
THEIR	145434	0.018974

```
WHICH    142146      0.018545
AFTER    110102      0.014365
FIRST    109957      0.014346
FIFTY    106869      0.013943
OTHER    106052      0.013836
FORTY     94951      0.012388
YEARS     88900      0.011598
THERE     86502      0.011286
SIXTY     73086     0.0095352
```

Find and print the 14 least frequent words:

```
% Find the 14 smallest frequencies from the 2nd column in the word list and their corresponding row indices.
[MinFreqs, MinIndices] = mink(WordList{:,2}, 14); % [MinValues, Indices] = mink(Vector, NumberOfValues).

% Save the rows corresponding to the 14 smallest frequencies to a table.
WordsMinFreq = WordList(MinIndices,:);
disp(WordsMinFreq);
```

Word	Count	Probability
BOSAK	6	7.8279e-07
CAIXA	6	7.8279e-07
MAPCO	6	7.8279e-07
OTTIS	6	7.8279e-07
TROUP	6	7.8279e-07
CCAIR	7	9.1326e-07
CLEFT	7	9.1326e-07
FABRI	7	9.1326e-07
FOAMY	7	9.1326e-07
NIAID	7	9.1326e-07
PAXON	7	9.1326e-07
SERNA	7	9.1326e-07
TOCOR	7	9.1326e-07
YALOM	7	9.1326e-07

The results make sense since the most frequent words represent numbers or some commonly used words, and I have never seen any one of the 14 least frequent words ever before.

(b) Find the best next guess and report the corresponding probability.

Case 1:

```
% Input the evidence.
TrueEvidence = table([ ], [ ], 'VariableNames', {'Letter', 'Index'});
FalseEvidence = ' ';

% Calculate and report the best next guess and report the corresponding probability.
[BestGuess, MaxProbability] = Hangman(TrueEvidence, FalseEvidence, WordList);
```

```
Best next guess: E
Probability: 0.53942
```

Case 2:

```
% Input the evidence.
TrueEvidence = table([ ], [ ], 'VariableNames', {'Letter', 'Index'});
FalseEvidence = 'EA';

% Calculate and report the best next guess and report the corresponding probability.
[BestGuess, MaxProbability] = Hangman(TrueEvidence, FalseEvidence, WordList);
```

```
Best next guess: O
Probability: 0.53403
```

Case 3:

```
% Input the evidence.
TrueEvidence = table(['A'; 'S'], [1; 5], 'VariableNames', {'Letter', 'Index'});
FalseEvidence = ' ';

% Calculate and report the best next guess and report the corresponding probability.
[BestGuess, MaxProbability] = Hangman(TrueEvidence, FalseEvidence, WordList);
```

```
Best next guess: E
Probability: 0.77154
```

Case 4:

```
% Input the evidence.
TrueEvidence = table(['A'; 'S'], [1; 5], 'VariableNames', {'Letter', 'Index'});
FalseEvidence = 'I';

% Calculate and report the best next guess and report the corresponding probability.
[BestGuess, MaxProbability] = Hangman(TrueEvidence, FalseEvidence, WordList);
```

```
Best next guess: E
Probability: 0.7127
```

Case 5:

```
% Input the evidence.
TrueEvidence = table(['O'; 'O'], [3; 3], 'VariableNames', {'Letter', 'Index'});
FalseEvidence = 'AEMNT';

% Calculate and report the best next guess and report the corresponding probability.
[BestGuess, MaxProbability] = Hangman(TrueEvidence, FalseEvidence, WordList);
```

```
Best next guess: R
Probability: 0.74539
```

Case 6:

```
% Input the evidence.
TrueEvidence = table([], [], 'VariableNames', {'Letter', 'Index'});
FalseEvidence = 'EO';

% Calculate and report the best next guess and report the corresponding probability.
[BestGuess, MaxProbability] = Hangman(TrueEvidence, FalseEvidence, WordList);
```

```
Best next guess: I
Probability: 0.63656
```

Case 7:

```
% Input the evidence.
TrueEvidence = table(['D'; 'I'], [1; 4], 'VariableNames', {'Letter', 'Index'});
FalseEvidence = ' ';

% Calculate and report the best next guess and report the corresponding probability.
[BestGuess, MaxProbability] = Hangman(TrueEvidence, FalseEvidence, WordList);
```

```
Best next guess: A
Probability: 0.82068
```

Case 8:

```
% Input the evidence.
TrueEvidence = table(['D'; 'I'], [1; 4], 'VariableNames', {'Letter', 'Index'});
FalseEvidence = 'A';

% Calculate and report the best next guess and report the corresponding probability.
[BestGuess, MaxProbability] = Hangman(TrueEvidence, FalseEvidence, WordList);
```

```
Best next guess: E
Probability: 0.75207
```

Case 9:

```
% Input the evidence.
TrueEvidence = table(['U'; 'U'], [2; 2], 'VariableNames', {'Letter', 'Index'});

FalseEvidence = 'AEIOS';

% Calculate and report the best next guess and report the corresponding probability.
[BestGuess, MaxProbability] = Hangman(TrueEvidence, FalseEvidence, WordList);
```

```
Best next guess: Y
Probability: 0.62697
```

Hangman function:

```
function [BestGuess, MaxProbability] = Hangman(TrueEvidence, FalseEvidence,
WordList)
% Input:
% 1. TrueEvidence (nx2) = Correctly guessed letters (column 1) and the correponding
indices (column 2).
% 2. FalseEvidence (1xn) = Incorrectly guessed letters.
% 3. WordList = All the word list including counts and prior probabilities.
% Output:
% BestGuess = Best guess of the letter.
% MaxProbablity = Probability that the best guess is correct.

% Evidence check: Find all the possible words that satisfies the given evidence.
% Initialize evidence mask values for all the words as 1.
EvidenceMask = ones(size(WordList, 1), 1);

for WordIndex = 1:size(WordList, 1) % Loop over all the possible words.
    % Evidence check 1: Check correctly guessed letters appear at correct indices.
    for TrueEvidenceIndex = 1:size(TrueEvidence, 1) % Loop over each correctly
guessed letters.
        % If the word can match the correct guess, keep the previous mask value.
        if WordList{WordIndex, 1}(TrueEvidence{TrueEvidenceIndex, 2}) ==
TrueEvidence{TrueEvidenceIndex,1}
            EvidenceMask(WordIndex) = EvidenceMask(WordIndex);
        % If the word can not match the correct guess, set the mask value to 0.
        else
            EvidenceMask(WordIndex) = 0;
        end
    end

    % Evidence check 2: Check correctly guessed letters or incorrectly guessed
letters do not appear at any other indices.
    if EvidenceMask(WordIndex, 1) == 1
        % Create a sub-word after removing correctly guessed letters (at correct
indices).
        SubWord = WordList{WordIndex, 1};
        SubWord(TrueEvidence{:,2}) = [];
        % If any correctly guessed letters or incorrectly guessed letters appear at
any other indices, set the mask value to 0.
        if any(ismember(SubWord, [TrueEvidence{:,1}; FalseEvidence']))
            EvidenceMask(WordIndex) = 0;
        end
    end
end

% Initialization for the candidate word list as an empty table.
CandidateWordList = table([], [], 'VariableNames', {'Word', 'Count'});

% Generate the candidate word list.
for WordIndex = 1:size(WordList, 1)
    if EvidenceMask(WordIndex) == 1
```

```
        CandidateWordList = [CandidateWordList; {WordList{WordIndex, 1},
WordList{WordIndex, 2}}];
    end
end

% Create a list for all the letters except those have been correctly guessed.
LetterList = 'ABCDEFGHIJKLMNOPQRSTUVWXYZ';
RemovedIndices = zeros(size(TrueEvidence, 1), 1);

for TrueEvidenceIndex = 1:size(TrueEvidence, 1) % Loop over each correctly guessed
letters.
    RemovedIndices(TrueEvidenceIndex) = find(LetterList ==
TrueEvidence{TrueEvidenceIndex, 1});
end

LetterList(RemovedIndices) = [];

% Initialization of a matrix for the count of letters in the candidate letter list.
LetterCount = zeros(size(CandidateWordList, 1) , size(LetterList, 2));

% Caculate the letter count matrix.
for WordIndex = 1:size(CandidateWordList, 1)
    for LetterIndex = 1:size(LetterList, 2)
        if contains(CandidateWordList{WordIndex, 1}, LetterList(LetterIndex))
            LetterCount(WordIndex, LetterIndex) = LetterCount(WordIndex,
LetterIndex) + 1;
        end
    end
end

% Update the letter count considering the number of each word.
LetterCount = LetterCount .* double(CandidateWordList{:,2});

% Calcuate the correct guess probability for each letter.
Probability = sum(LetterCount)/sum(CandidateWordList{:, 2});

% Obtain the maximum probability and the corresponding index for the letter.
[MaxProbability, MaxLetterIndex] = max(Probability);

% Select the letter corresponding to the maximum probability as the best guess.
BestGuess = LetterList(MaxLetterIndex);

disp(['Best next guess: ', BestGuess]);
disp(['Probability: ', num2str(MaxProbability)]);

end
```