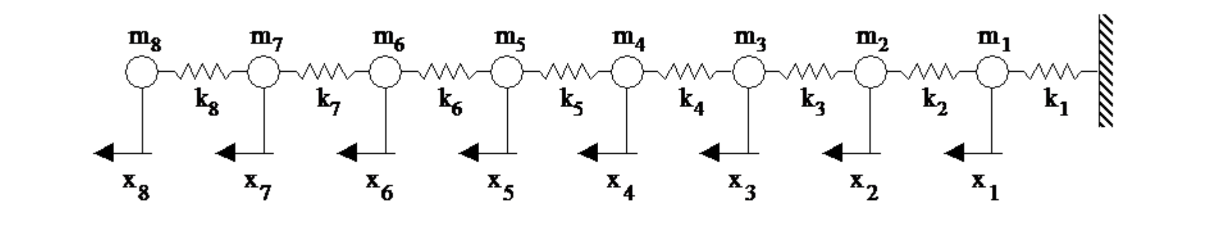
SE 165/265 Homework 3
Assigned April 17th, 2024; Due April 24th, 2024 (11:59 pm Pacific Time)

Assignment Goal:
1. Extract features base on modal parameters.
2. Reinforce material presented in Lecture 6 and Lecture 7.

Task 1: We will calculate the mode shapes and resonant frequencies for an 8 DOF lumped mass system

A. Form the mass matrix, m, for 8 degree of freedom (DOF) system shown below. Display your matrix on the command window.



$m_1=m_2\ldots=m_8=0.4194$kg
$k_1=k_2\ldots=k_8=56700$N/m

Note that we will specify the mass matrix such that there is no inertial coupling (off-diagonal terms). Therefore, the mass matrix will only have diagonal elements and these element are identical to the respective lumped mass (i.e. m(1,1)=m(2,2)….=m(8,8)=0.4194 Kg, all off-diagonal terms = 0).

Remember, for an 8 DOF systems the mass matrix will be of dimension 8 rows x 8 columns.

B. Form the stiffness matrix, k, for the 8 DOF system. Display your matrix on the command window.

The definition of an element of the stiffness matrix, k(i,j), is the force applied at node (or DOF) i when a unit displacement is applied at node j such that all other node displacement are held fixed. The degree of freedoms for this system are shown with the arrows labeled x_1, x_2,….x_8. With this definition k(1,1)=k_1+k_2,, k(2,2)=k_2+k_3, and k(8,8)=k_8

Now k(1,2) is the force that must be applied at node 1 when a unit displacement is applied at node 2 such that node 1 is held fixed. A unit displacement of node 2 will apply a force of 1*k_2 to node 1. We must then apply a force of (-1*k_2) to this node to hold it fixed. Therefore, k(1,2)= -k_2. Knowing that the stiffness matrix will be symmetric, we can define k(2,1)=k(1,2).

Lets examine k(1,3), which is the force that must be applied at node 1 when a unit displacement is applied at node 3 and all other nodes are held fixed. Because node 2 is held fixed, no force is transmitted to node 1 as a result of the unit displacement at node 3. Therefore, k(1,3)=0 and because of symmetry, k(3,1)=0.

It turns out that this stiffness matrix will be banded with only diagonal terms and terms in the first off-diagonal positions.

C. Solve for the system eigenvalues (resonant frequencies) and eigenvectors (mode shapes)

Begin with the equation of motion for our system, noting that we have neglected damping.

$$[\boldsymbol{m}]\{\ddot{\boldsymbol{x}}\} + [\boldsymbol{k}]\{\boldsymbol{x}\} = \{\boldsymbol{f}(\boldsymbol{t})\}$$

Assuming the displacement vector {x} is described by the harmonic function $\{x\} = \{x\}e^{i\omega t}$ and $\{\ddot{x}\} = -\omega^2\{x\}$, the equation of motion becomes:

$$-\omega^2[m]\{x\} + [k]\{x\} = \{0\}$$

Note an eigenvalue problem is defined as:

$$[[A] - \lambda[I]\{x\} = \{0\}$$

Where λ are the eigenvalues, {x} are the eigenvectors, and [*I*] is the identity matrix (ones on diagonal, zeros on off-diagonal terms). In our case:

$$[A] = [m]^{-1}[k]$$

and

$$\lambda = -\omega^2$$

Now that we have reviewed the theory behind the problem, we can go back to the Matlab assignment.

- Solve for eigenvalue (λ) and eigenvectors (ϕ) using MATLAB command **eig** such that:

  $$[\phi, \lambda] = eig(k, m)$$

  $[\phi]$ will be an 8x8 matrix with each column corresponding to the eigenvectors or modes shapes.
  $[\lambda]$ will be an 8x8 matrix whose <u>diagonal</u> values are the eigenvalues.

- Note the eigenvectors and eigenvalues may have small imaginary components due to finite precision computing. We are only interested in the real portions of the components so use the "*real*" command to eliminate the imaginary parts of $[\phi]$ and $[\lambda]$.

- Knowing that the diagonal values of $[\lambda]$ correspond to the eigenvalues, save them into a new vector $\{\lambda_d\}$.

- Your eigenvalues $\{\lambda_d\}$ should already be sorted in an ascending order. Check if they are. If they are not, sorting them in an ascending order and sort the eigenvectors accordingly (i.e., sort the columns of $[\phi]$ using the indexes obtained after sorting $\{\lambda_d\}$.

- Next we will normalize the sorted eigenvectors [ϕ] with respect to the mass matrix using the following MATLAB command:

$$\phi_{norm} = \phi / sqrt(\phi' * m * \phi)$$

Hint: To check that you have done everything correctly up to this point, check the orthogonality conditions with the following MATLAB commands:

$$\phi_{norm}{}' * m * \phi_{norm} = I\ ,?$$
$$\phi'_{norm} * k * \phi_{norm} = \lambda_d\ ,?$$

- Finally, convert the eigenvalues λ_d to **frequencies in Hertz** by first taking the square root of λ_d and then dividing by 2π (corresponding to conversion of 2π radians per cycle). Display the frequencies in the command window.

- Make a plot of the resonant frequency in Hertz on the vertical axis as a function of the mode number on the horizontal axis.

- Make one plot for each mode shape (columns of ϕ_n), where mode shape amplitude is plot on vertical axis and node number (DOF) is plotted on horizontal axis (there are 8 mode shapes in total). Add labels and a title to your plots.

D. Analyze damaged system and compare with the modal properties from the undamaged system to those from the damaged system.

Introduce a 10% reduction in the spring stiffness at each location **(one spring at a time)**. Modify your stiffness matrix accordingly and repeat the modal analysis.

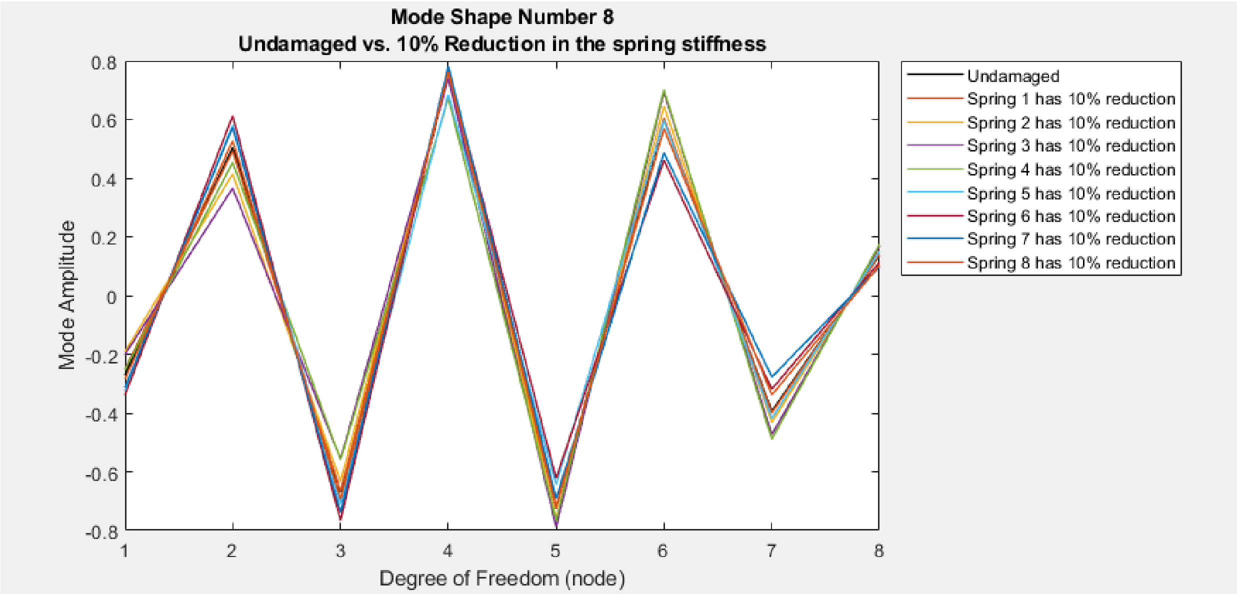
You will repeat this for all 8 springs, where only one spring has its stiffness reduced, and all other 7 have the original stiffness.

For the undamaged case (from part C) and all 8 damage conditions that were just created on part D, overlay plots of:

- Resonant frequencies vs mode number (Fig. 7.32 in reference),
- Mode shape for 1st mode,
- And mode shape for 8th mode on top of corresponding plots for undamaged condition (Figure 7.35 in reference book).

These 3 new figures should have the 8 different damage conditions on top of the undamaged condition. As usual, add axes labels, titles, and a legend to all plots.

Hint: Plot of Mode shape for 8th mode.



E. Calculate the Modal Assurance Criteria (MAC) Metric for comparing undamaged and damaged mode shapes.

For the case where a 10% reduction in stiffness has occurred in **spring 1**, calculate the matrix of modal assurance criteria values between undamaged modes ϕ_x and damaged modes ϕ_y that are given by:

$$\mathrm{MAC}\left(\{\phi_x\},\{\phi_y\}\right)=\frac{\left|\{\phi_x\}^T\{\phi_y^*\}\right|^2}{\left(\{\phi_x\}^T\{\phi_x^*\}\right)\left(\{\phi_y\}^T\{\phi_y^*\}\right)}$$

Where * designated complex conjugate (our modes will have zero imaginary part), ϕ_x is mode from undamaged structure and ϕ_y is mode from damaged structure. (Results are shown in Table 7.6 of the reference book).

Display your result on the command window.