SE 167/267A: Signal Processing and Spectral Analysis, WI 2024

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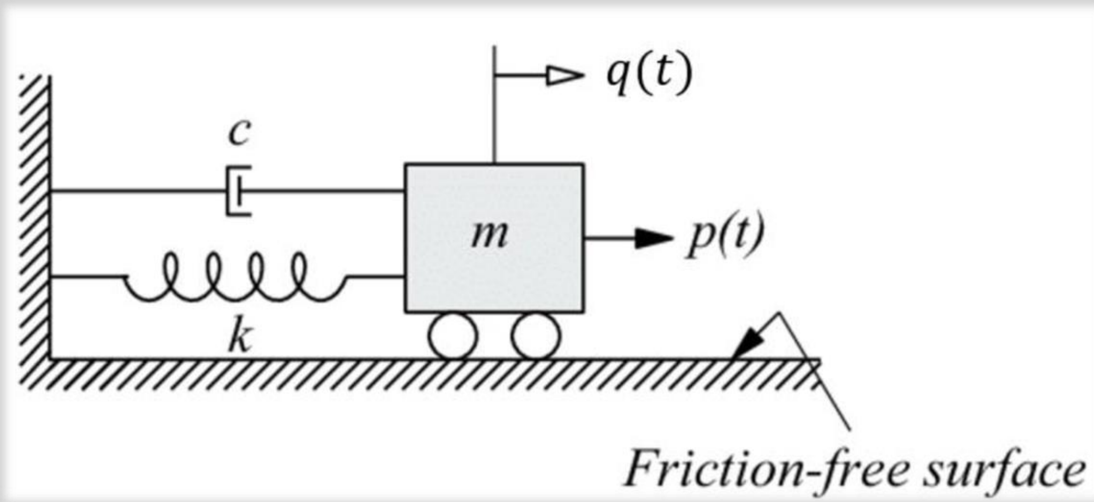
Assignment 4

Due Date: March 19, 2024

SDOF System

Consider a SDOF mass-damper-spring system:



This system is governed by the following differential equation:

$$m\ddot{q}(t) + c\dot{q}(t) + kq(t) = p(t)$$

Subjected to the initial conditions:

$$q(0) \text{ and } \dot{q}(0)$$

Assume:

$m = 1kg$

$c = 0.5\ Nsm^{-1}$

$k = 2\ Nm^{-1}$

$p(t) = e^{-0.12t}(\sin 3t + 5\cos 9\ t)$

Problem 1 (10 points): Obtain the **closed form analytical solution** of the Fourier Transform of load (right now an analog continuous-time function). Note that this is the causal LTI system and therefore time value starts at $0\ s$. Plot:

a. Load $p(t)$ vs. t from $t \in [0,60]\ s$.
b. Absolute value of Fourier Transform of load $|P(\Omega)|$ vs. Ω from $\Omega \in [0,15] rad/s$
c. Phase of Fourier Transform of input $\angle P(\Omega)$ vs. Ω from $\Omega \in [0,15] rad/s$

Comment on the peaks that you observe in Fourier Transform magnitude plot. Where do these peaks occur and what do these peaks represent in your dynamic input load? You can use the following syntax for plotting:

```
Plot[p, {t, 0, 60}, PlotStyle → {Directive[Red, AbsoluteThickness[2]]},
 FrameLabel → {Style["t", Italic], Style["p(t)", Italic]}, PlotLabel → "Load p(t)",
 LabelStyle → {RGBColor[0, 0, 0], FontSize → 12, FontFamily → "Times"}, GridLines → Automatic, Frame → True,
 PlotRange → All]
```

Problem 2 (20 points): Preferably using Mathematica, solve for the impulse response function by solving the differential equation for zero initial conditions subjected to unit impulse loading:

$$m\ddot{h}(t) + c\dot{h}(t) + kh(t) = \delta(t)$$

Subjected to the initial conditions:

$$h(0) = 0 \text{ and } \dot{h}(0) = 0$$

Report the solution that you get by solving this differential equation and plot $h(t)$ vs. t from $t \in [0,60]\ s$. Will this solution change if the external loading changed? Why?

Hint: Use "DSolve" command in Mathematica to solve for the differential equation. You'll get HeavsideTheta[0] and HeavsideTheta[t] as part of your solution. Replace them with "HeavisideTheta[0]−> 0, HeavisideTheta[t]−> 1". Your solution should match the closed form solution of $h(t)$ that we had obtained in class.

Problem 3 (15 points): Using $h(t)$ obtained in problem 2, obtain the **closed form analytical solution** of the Fourier Transform of impulse response (FRF). Note that this is the causal LTI system and therefore time value starts at $0\ s$. Plot:

a. Absolute value of Fourier Transform of FRF $|H(\Omega)|$ vs. Ω from $\Omega \in [0,15] rad/s$
b. Phase of Fourier Transform of FRF $\angle H(\Omega)$ vs. Ω from $\Omega \in [0,15]\ rad/s$

Comment on the peaks that you observe in FRF magnitude plot. You'll see that there is only 1 peak. Where does this peak occur and what does this peak represent in your impulse response? Why is there only 1 peak?

Problem 4 (20 points): Preferably using Mathematica, solve for the response/output $q(t)$ by solving the differential equation for zero initial conditions subjected to unit impulse loading:

$$m\ddot{q}(t) + c\dot{q}(t) + kq(t) = p(t)$$

Subjected to the initial conditions:

$$q(0) = 0 \text{ and } \dot{q}(0) = 0$$

Report the solution that you get by solving this differential equation and plot $q(t)$ vs. t from $t \in [0,60]\ s$. Clearly mention the terms corresponding to the transient response and the steady state response.

Hint: Use "DSolve" command in Mathematica to solve for the differential equation. Notice that Mathematica may give you complex solution. However, we know that we have a real response function. We obtain that using the following command (after you have solved the differential equation):

```
q = Chop[FullSimplify[ComplexExpand[Re[q]]]]
```

Problem 5 (15 points): Using $q(t)$ obtained in problem 4, obtain the **closed form analytical solution** of the Fourier Transform of output/response, denoted by $Q(\Omega)$. Note that this is the causal LTI system and therefore time value starts at $0\ s$. Plot:

a. Absolute value of Fourier Transform of output $|Q(\Omega)|$ vs. Ω from $\Omega \in [0,15] rad/s$
b. Phase of Fourier Transform of output $\angle Q(\Omega)$ vs. Ω from $\Omega \in [0,15] rad/s$

Comment on the peaks that you observe in magnitude of the Fourier Transform plot. You'll see that there are 3 peaks. Where does each of these peaks occur and what does each of these peaks represent in your output? Why are there 3 peaks peak?

Problem 6 (20 points): Obtain the cross-power spectral density (PSD) of input and output $S_{yx}(\Omega)$, auto PSD of input $S_{xx}(\Omega)$ and output $S_{yy}(\Omega)$. Plot each of their magnitude (absolute) plot and phase plot for $\Omega \in [0,15] rad/second$. Comment on peaks observed in each of these absolute plots and what these peaks represent.

Problem 7 (50 points): Sample the input and output at frequency of $f_s = 100\ Hz$ between $t \in [0,100]\ s$ and obtain discrete input $p[n]$ and output $q[n]$. When we subject a system to input in the real world (such as using shake table), the control system usually almost perfectly applies discrete input. However, unlike input, acquiring output data is usually subjected to measurement noise. We simulate the noisy output sequence $\hat{q}[n]$ that by adding zero-mean Gaussian noise:

$$\hat{q}[n] = q[n] + N\left(\mu = 0, \sigma = \frac{\max q[n]}{30}\right)$$

Using the input sequence $p[n]$ and the noisy output sequence $\hat{q}[n]$, obtain:

1. Plot discrete version of load $p[n]$ and noisy output $\hat{q}[n]$ vs time vector for $t_n \in [0,100]\ s$.
2. Obtain DFT (using FFT algorithm) for the input sequence and output sequence. That is, obtain, $P[\Omega_k]$ and $\hat{Q}[\Omega_k]$ where $\Omega_k = \frac{2\pi}{N} f_s$ for $k = 0, \ldots, N$. Assume N equal to the number of points in the sequence $p[n]$ (or $\hat{q}[n]$ since they are same length sequences). Limiting your

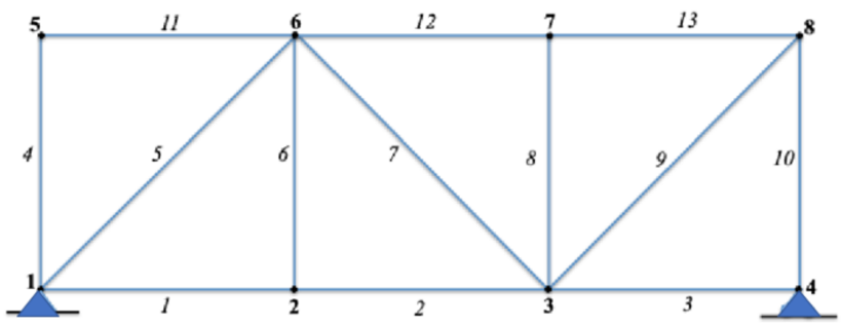
frequencies to $\Omega_k \in [0,15]\ Hz$, plot the magnitude and phase of $P[\Omega_k]$ and $\hat{Q}[\Omega_k]$. Here is a small snippet to help you:

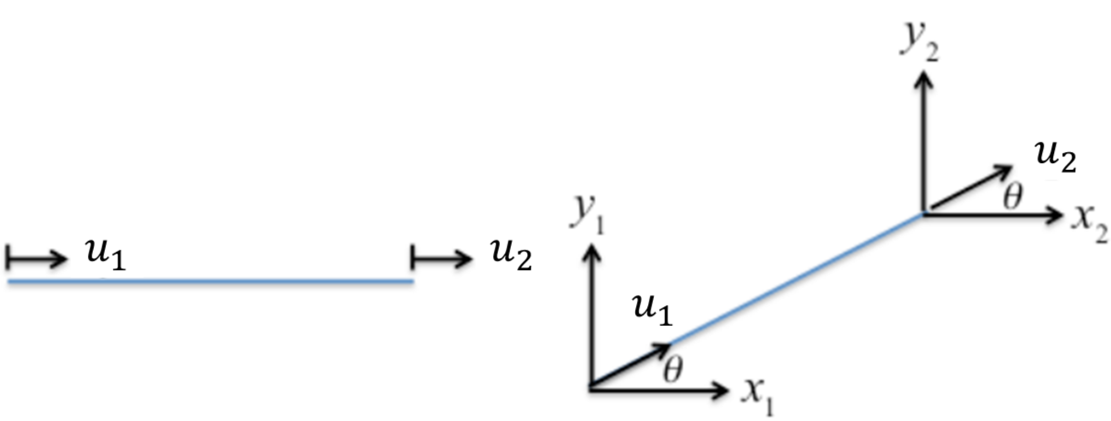
```
FreqVector = Range[0, 2 Pi (1 - 1 / Length[Qvector]), 2 Pi / Length[Qvector]] * (1 / Ts);
FreqVector = DeleteCases[FreqVector, x_ /; x > 15];
Qvector = Take[Qvector, Length[FreqVector]];
```

3. Obtain the estimated FRF sequence $H[\Omega_k]$. Limiting your frequencies to $\Omega_k \in [0,15]\ Hz$ plot the absolute value $|H[\Omega_k]|$ and phase $\angle H[\Omega_k]$.
4. Obtain the estimated cross power spectral density (PSD) sequence $S_{yx}[\Omega_k]$. Limiting your frequencies to $\Omega_k \in [0,15]\ Hz$ plot the absolute value $|S_{yx}[\Omega_k]|$ and phase $\angle S_{yx}[\Omega_k]$.
5. Comment on the impact of measurement noise in estimating FRF and cross-PSD.

MDOF System (A structural System)

Consider a **two-dimensional** truss that has in-plane deformation only. This truss has hinge support at nodes 1 and 4 (allows for free rotation—like a pin—but restricts horizontal and vertical motion). Each element of the structure, numbered by an italic number, is a homogeneous linear elastic truss member with Young's modulus E, density ρ, and cross-sectional area A. Each node is numbered in boldface. The horizontal and vertical members have length L and the diagonal members have length $\sqrt{2}L$. Truss members only develop axial member forces (tension or compression). Therefore, a single member has two local displacement degrees of freedom, shown on the left-hand side of the figure below as $u_1(t)$ and $u_2(t)$:





As you have studied in SE130B, SE201A, SE276A, or SE201B, the single element on the left side of this figure, characterized in local coordinates, may be transformed to global inertial coordinates in the plane (clearly you can see that many of the structural members are not horizontally oriented). The matrix $\boldsymbol{W}_e$ transforms the single element e's <u>*local coordinates*</u> $\boldsymbol{u}_e(t) = [u_1(t), u_2(t)]$ into <u>*global element coordinates*</u> $\boldsymbol{X}_e(t) = [x_1(t), y_1(t), x_2(t), y_2(t)]^T$. That is, the <u>*local to global*</u> transformation matrix is given by:

$$\boldsymbol{X}_e(t) = \boldsymbol{W}_e.\boldsymbol{u}_e(t)$$

Notice that $\boldsymbol{W}_e$ is a 4×2 matrix. The mass matrix $\boldsymbol{M}_{e,local}$ and stiffness matrix $\boldsymbol{K}_{e,local}$ for each element e in <u>*local coordinate*</u> system is given as:

$$\boldsymbol{M}_{e,local} = \rho A_e L_e \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}; \ \boldsymbol{K}_{e,local} = \frac{EA_e}{L_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Here, A_e, L_e denote area and length of element e. The Young's modulus E and the density ρ is assumed to be same for all the elements. We can then transform the local mass and stiffness matrices to their global version.

$$\boldsymbol{M}_{e,global} = \boldsymbol{W}_e.\boldsymbol{M}_{e,local}.\boldsymbol{W}_e^T$$
$$\boldsymbol{K}_{e,global} = \boldsymbol{W}_e.\boldsymbol{K}_{e,local}.\boldsymbol{W}_e^T$$

The global mass and stiffness matrices for element e, denoted by $\boldsymbol{M}_{e,global}$, $\boldsymbol{K}_{e,global}$ are 4×4 in size. Now we assemble the global mass and stiffness of elements to obtain total system mass matrix $\boldsymbol{M}_{system}$ and $\boldsymbol{K}_{system}$ (they are obviously in global coordinate system). The system mass matrix and system stiffness matrix should have row/column dimensions that are equal to the number of nodes times the number of global degrees of freedom per node. That is, we have 8 nodes and each node (ignoring supports for now) has 2 degrees of freedom (x and y)—this makes $\boldsymbol{M}_{system}$ and $\boldsymbol{K}_{system}$ to have size of 16×16. However, realize that nodes 1 and 4 are fixed. Therefore, the $x(t)$ and $y(t)$ for nodes 1 and 4 are 0. Therefore, we get rid of these 4 degrees of freedom since these are known to be 0 because of constraint/support. We now delete the respective rows and columns of the matrices $\boldsymbol{M}_{system}$ and $\boldsymbol{K}_{system}$ and global degree of freedom vector $\boldsymbol{X}(t)$. The mass and stiffness matrices $\boldsymbol{M}_{system}$ and $\boldsymbol{K}_{system}$ have a size of 12×12 and the $\boldsymbol{X}(t)$ has length of 12 (corresponding to the 12 unknown degrees of freedom).

In the following page, I illustrate how I have defined the global degrees of freedom (corresponding to each joint) these rows (corresponding to the constraint degrees of freedom at the support) deleted. I am also providing you with the system's mass and stiffness matrix $\boldsymbol{M}_{system}$ and $\boldsymbol{K}_{system}$. We assume proportional damping $\boldsymbol{C}_{system} = 0.7\ \boldsymbol{M}_{system} + 0.0001\ \boldsymbol{K}_{system}$.

System's degrees of freedom before considering support degrees of freedoms

$$
\boldsymbol{X} = \begin{bmatrix} \left.\begin{matrix} x_1 \\ y_1 \end{matrix}\right\} \text{DOF corresponding to the node 1 (hinge support)} \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ \left.\begin{matrix} x_4 \\ y_4 \end{matrix}\right\} \text{DOF corresponding to the node 4 (hinge support)} \\ x_5 \\ y_5 \\ x_6 \\ y_6 \\ x_7 \\ y_7 \\ x_8 \\ y_8 \end{bmatrix}
$$

Deleting rows 1, 2, 7 and 8 (these rows corresponds to DOF for hinge support at nodes 1 and 4)

$$
\boldsymbol{X} = \begin{bmatrix} \left.\begin{matrix} x_2 \\ y_2 \end{matrix}\right\} \text{DOF corresponding to the } \textbf{node 2} \\ \left.\begin{matrix} x_3 \\ y_3 \end{matrix}\right\} \text{DOF corresponding to the } \textbf{node 3} \\ \left.\begin{matrix} x_5 \\ y_5 \end{matrix}\right\} \text{DOF corresponding to the } \textbf{node 5} \\ \left.\begin{matrix} x_6 \\ y_6 \end{matrix}\right\} \text{DOF corresponding to the } \textbf{node 6} \\ \left.\begin{matrix} x_7 \\ y_7 \end{matrix}\right\} \text{DOF corresponding to the } \textbf{node 7} \\ \left.\begin{matrix} x_8 \\ y_8 \end{matrix}\right\} \text{DOF corresponding to the } \textbf{node 8} \end{bmatrix}
$$

Similarly, the 1,2,7, and 8 rows and columns of the mass and stiffness matrix can be obtained. This yields the final form of the system's mass and stiffness matrix $\boldsymbol{M}_{system}$ and $\boldsymbol{K}_{system}$

$$
\boldsymbol{K}_{system} = \begin{pmatrix}
\frac{2Ae}{L} & 0 & -\frac{Ae}{L} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{Ae}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{Ae}{L} & 0 & 0 & 0 & 0 \\
-\frac{Ae}{L} & 0 & \frac{Ae}{\sqrt{2}L} + \frac{2Ae}{L} & 0 & 0 & 0 & -\frac{Ae}{2\sqrt{2}L} & \frac{Ae}{2\sqrt{2}L} & 0 & 0 & -\frac{Ae}{2\sqrt{2}L} & -\frac{Ae}{2\sqrt{2}L} \\
0 & 0 & 0 & \frac{Ae}{\sqrt{2}L} + \frac{Ae}{L} & 0 & 0 & \frac{Ae}{2\sqrt{2}L} & -\frac{Ae}{2\sqrt{2}L} & 0 & -\frac{Ae}{L} & -\frac{Ae}{2\sqrt{2}L} & -\frac{Ae}{2\sqrt{2}L} \\
0 & 0 & 0 & 0 & \frac{Ae}{L} & 0 & -\frac{Ae}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{Ae}{L} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{Ae}{2\sqrt{2}L} & \frac{Ae}{2\sqrt{2}L} & -\frac{Ae}{L} & 0 & \frac{Ae}{\sqrt{2}L} + \frac{2Ae}{L} & 0 & -\frac{Ae}{L} & 0 & 0 & 0 \\
0 & -\frac{Ae}{L} & \frac{Ae}{2\sqrt{2}L} & -\frac{Ae}{2\sqrt{2}L} & 0 & 0 & 0 & \frac{Ae}{\sqrt{2}L} + \frac{Ae}{L} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{Ae}{L} & 0 & \frac{2Ae}{L} & 0 & -\frac{Ae}{L} & 0 \\
0 & 0 & 0 & -\frac{Ae}{L} & 0 & 0 & 0 & 0 & 0 & \frac{Ae}{L} & 0 & 0 \\
0 & 0 & -\frac{Ae}{2\sqrt{2}L} & -\frac{Ae}{2\sqrt{2}L} & 0 & 0 & 0 & 0 & -\frac{Ae}{L} & 0 & \frac{Ae}{2\sqrt{2}L} + \frac{Ae}{L} & \frac{Ae}{2\sqrt{2}L} \\
0 & 0 & -\frac{Ae}{2\sqrt{2}L} & -\frac{Ae}{2\sqrt{2}L} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{Ae}{2\sqrt{2}L} & \frac{Ae}{2\sqrt{2}L} + \frac{Ae}{L}
\end{pmatrix}
$$

Here, e denote Young's Modulus (in Mathematica, E is reserved for exponential). A denote area of cross-section (assumed constant for all the elements). L denotes length of horizontal and vertical elements ($\sqrt{L}$ is the length of the diagonal elements).

$$
\boldsymbol{M}_{system} = \begin{pmatrix}
\frac{2AL\rho}{3} & 0 & \frac{AL\rho}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{AL\rho}{3} & 0 & 0 & 0 & 0 & 0 & \frac{AL\rho}{6} & 0 & 0 & 0 & 0 \\
\frac{AL\rho}{6} & 0 & \frac{1}{3}\sqrt{2}\,AL\rho + \frac{2AL\rho}{3} & 0 & 0 & 0 & \frac{AL\rho}{6\sqrt{2}} & -\frac{AL\rho}{6\sqrt{2}} & 0 & 0 & \frac{AL\rho}{6\sqrt{2}} & \frac{AL\rho}{6\sqrt{2}} \\
0 & 0 & 0 & \frac{1}{3}\sqrt{2}\,AL\rho + \frac{AL\rho}{3} & 0 & 0 & -\frac{AL\rho}{6\sqrt{2}} & \frac{AL\rho}{6\sqrt{2}} & 0 & \frac{AL\rho}{6} & \frac{AL\rho}{6\sqrt{2}} & \frac{AL\rho}{6\sqrt{2}} \\
0 & 0 & 0 & 0 & \frac{AL\rho}{3} & 0 & \frac{AL\rho}{6} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{AL\rho}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{AL\rho}{6\sqrt{2}} & -\frac{AL\rho}{6\sqrt{2}} & \frac{AL\rho}{6} & 0 & \frac{1}{3}\sqrt{2}\,AL\rho + \frac{2AL\rho}{3} & 0 & \frac{AL\rho}{6} & 0 & 0 & 0 \\
0 & \frac{AL\rho}{6} & -\frac{AL\rho}{6\sqrt{2}} & \frac{AL\rho}{6\sqrt{2}} & 0 & 0 & 0 & \frac{1}{3}\sqrt{2}\,AL\rho + \frac{AL\rho}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{AL\rho}{6} & 0 & \frac{2AL\rho}{3} & 0 & \frac{AL\rho}{6} & 0 \\
0 & 0 & 0 & \frac{AL\rho}{6} & 0 & 0 & 0 & 0 & 0 & \frac{AL\rho}{3} & 0 & 0 \\
0 & 0 & \frac{AL\rho}{6\sqrt{2}} & \frac{AL\rho}{6\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{AL\rho}{6} & 0 & \frac{AL\rho}{3\sqrt{2}} + \frac{AL\rho}{3} & \frac{AL\rho}{3\sqrt{2}} \\
0 & 0 & \frac{AL\rho}{6\sqrt{2}} & \frac{AL\rho}{6\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{AL\rho}{3\sqrt{2}} & \frac{AL\rho}{3\sqrt{2}} + \frac{AL\rho}{3}
\end{pmatrix}
$$

Here, A denote area of cross-section (assumed constant for all the elements); L denotes length of horizontal and vertical elements ($\sqrt{L}$ is the length of the diagonal elements), and ρ denotes the density (assumed constant for all the elements).

Obtaining the system's mass and stiffness in this final form is a task that is taught in a finite element or structural analysis courses. Here, I give you the $\boldsymbol{M}_{system}, \boldsymbol{K}_{system}$, and $\boldsymbol{C}_{system} = 0.7\,\boldsymbol{M}_{system} + 0.0001\,\boldsymbol{K}_{system}$. Assume $E = 70\,GPa$, $A = 0.0016\,m^2$, $L = 20\,m$, and $\rho = 2700\,kg\,m^{-3}$. Given these system's property in spatial domain, we start by obtaining the properties in the modal domain:

Problem 8 (20 points): Obtain the following:

1. Since there are 12 DOF, we'll have 12 natural frequencies ω_m and modal damping ratio ζ_m, where $m = 1,2,\ldots,12$. Obtain the natural frequencies (square root of the eigenvalues), and modal damping ratios.
2. Obtain the mass-normalized mode shapes $\boldsymbol{\psi}_m$ and construct the model matrix $\boldsymbol{\Psi}_{\text{system}}$ consisting of $\boldsymbol{\psi}_m$ as its columns.

Note: Mathematica may occasionally give Eigen values in descending order. If that's the case, use "Reverse" command to obtain Eigen Values and the corresponding Eigen Vectors. Secondly, Mathematica yields Eigen vectors in "rows". Remember that the modal matrix or the mass normalized model matrix $\boldsymbol{\Psi}_{\text{system}}$ contains mass-normalized mode shapes $\boldsymbol{\psi}_m$ as its "columns". Once you obtain these model properties, you can check for the correctness of the solution by checking the orthogonality condition.

$$\boldsymbol{\Psi}^T_{\text{system}}.\boldsymbol{M}_{system}.\boldsymbol{\Psi}_{\text{system}} = \boldsymbol{I}$$

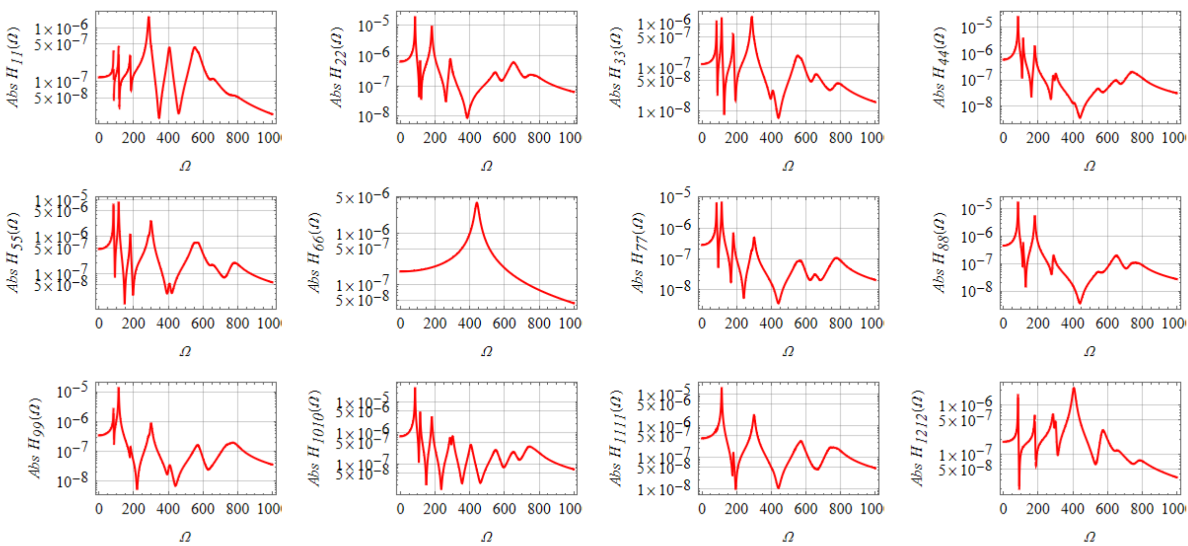
$$\boldsymbol{\Psi}^T_{\text{system}}.\boldsymbol{K}_{system}.\boldsymbol{\Psi}_{\text{system}} = \omega_m^2\boldsymbol{I}$$

Here, $\boldsymbol{I}$ is the identity matrix; and $\omega_m^2\boldsymbol{I}$ is a diagonal matrix with ω_m^2 as the m^{th} diagonal element.

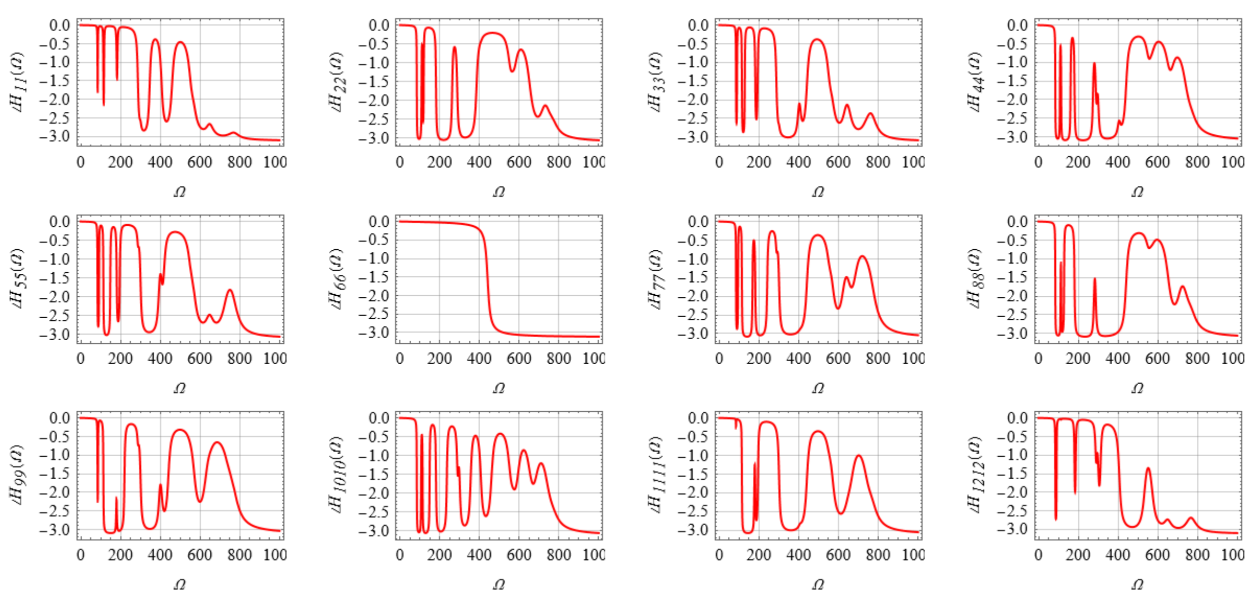
Problem 9 (20 points): Either using the spatial quantities $\boldsymbol{M}_{system}$, $\boldsymbol{K}_{system}$, and $\boldsymbol{C}_{system}$ or using the model properties obtained earlier, obtain the following:

1. Obtain the FRF matrix $\boldsymbol{H}(\Omega)$. Plot the absolute values $|H_{mm}(\Omega)|$ and the phase $\angle H_{mm}(\Omega)$ of the **diagonal** elements $H_{mm}(\Omega)$ where $m = 1,2,\dots,12$. You'll realize that all your natural frequencies are under 1000 rad/s. Therefore, limit $\Omega \in [0,1000]rad/s$.
2. Obtain the accelarance matrix $\boldsymbol{A}(\Omega)$ and plot the absolute values $|A_{mm}(\Omega)|$ and the phase $\angle A_{mm}(\Omega)$ of the diagonal elements $A_{mm}(\Omega)$ where $m = 1,2,\dots,12$. Limit $\Omega \in [0,1000]rad/s$.

Hint: Here is the absolute and phase of FRF for diagonal elements of $\boldsymbol{H}(\Omega)$ that I get. I used "LogPlot" to plot the absolute values and "Plot" function to obtain the phase plot.

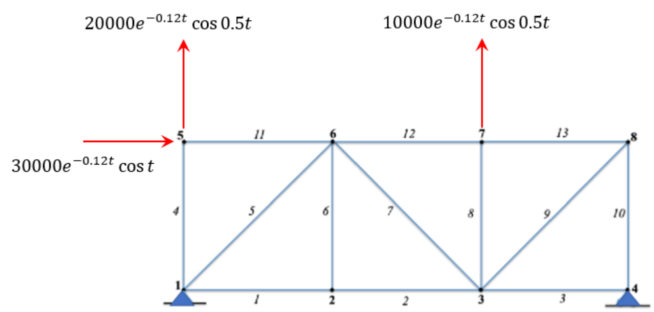


This is the phase plot:



Problem 10 (20 points): Consider the spatial force vector $\boldsymbol{F}(t)$

$$\boldsymbol{F}(t) = \begin{bmatrix} 0\\ 0\\ 0\\ 0\\ 30000e^{-0.12t}\cos t\\ 20000e^{-0.12t}\cos 0.5t\\ 0\\ 0\\ 0\\ 10000\,e^{-0.12t}\cos 0.5t\\ 0\\ 0 \end{bmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{Forces on node 2}\\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{Forces on node 3}\\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{Forces on node 5}\\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{Forces on node 6}\\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{Forces on node 7}\\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{Forces on node 8} \end{array}$$



Assume a zero initial condition on all the nodes. That is, $\boldsymbol{X}(0) = \boldsymbol{0}$ and $\dot{\boldsymbol{X}}(0) = \boldsymbol{0}$. The linear but coupled differential equations governing this system is given by:

$$\boldsymbol{M}_{system}\ddot{\boldsymbol{X}}(t) + \boldsymbol{C}_{system}\dot{\boldsymbol{X}}(t) + \boldsymbol{K}_{system}\boldsymbol{X}(t) = \boldsymbol{F}(t)$$

Obtain the uncoupled equations in terms of the modal degrees of freedom q_m defined by:

$$\boldsymbol{X}(t) = \boldsymbol{\Psi}_{system}.\boldsymbol{q}(t)$$

You should get 12 second order equations that are uncoupled with dependent variable $q_m(t)$. Make sure to appropriately transform the force vector from spatial domain to model domain. **Print/display** these 12 differential equations.

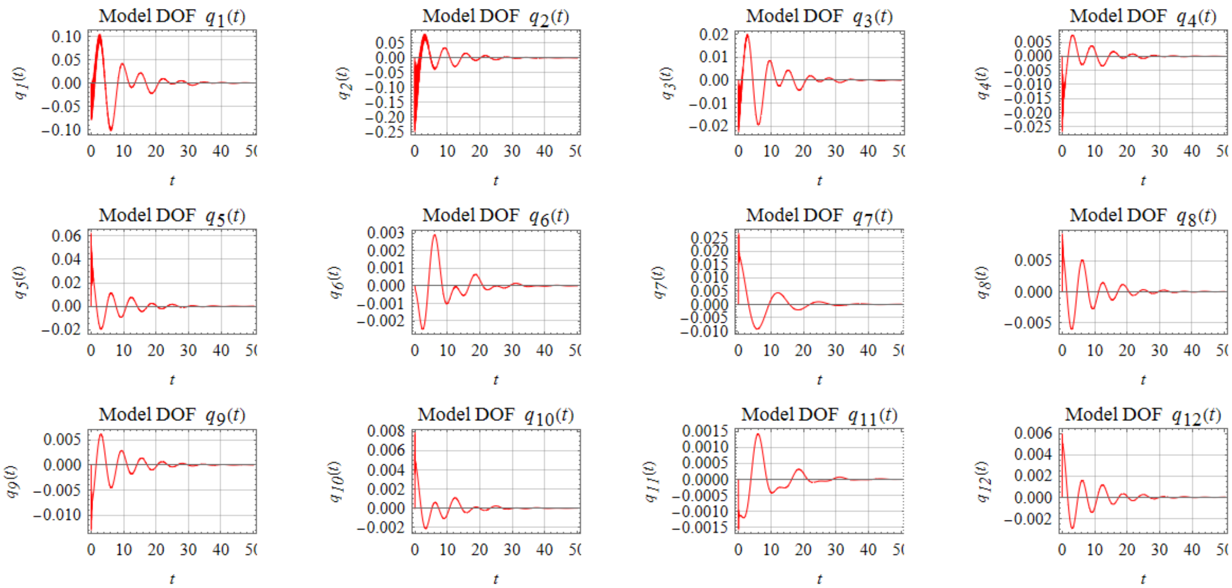
Problem 11 (20 points): Solve for the modal degrees of freedom and:

1. Plot $q_m(t)$ vs. t for $t \in [0,50]\ s$.
2. Using $\boldsymbol{X}(t) = \boldsymbol{\Psi}_{system}.\boldsymbol{q}(t)$ obtain the solution for spatial degrees of freedom and plot each of the 12 spatial degrees of freedom vs. t for $t \in [0,50]\ s$.

Hint: The modal DOF should be real in this case. When you solve the 12-differential equation in Mathematica, it yields a complex output. You can simplify it. I used the following code snippet to simplify my solution for the modal DOF (you can do something similar for spatial DOF as well).

```
modalDOFsolt =
  Table[
   Chop[
    ComplexExpand[Re[Chop[TrigReduce[Chop[TrigExpand[FullSimplify[Chop[ComplexExpand[Re[modalDOFsolt[[i]]]]]]]]]]]],
   {i, 1, numDOF}];
```

Here is what I get for model degrees of freedom plot:



Problem 12 (20 points): Obtain the Fourier Transform of the input force vector $\boldsymbol{F}(t)$ analytically (denoted by $\mathbb{F}(\Omega)$). Since there are 12 DOF, you'll have 12 elements in the vector $\mathbb{F}(\Omega)$. Plot (you'll get 12 plots corresponding to forces on each node in x and y direction):

1. Absolute value of Fourier Transform of load $|\mathbb{F}(\Omega)|$ vs. Ω from $\Omega \in [0,5] rad/s$
2. Phase of Fourier Transform of input $\angle\mathbb{F}(\Omega)$ vs. Ω from $\Omega \in [0,15] rad/s$

We now know the FRF matrix $\boldsymbol{H}(\Omega)$ (which is a 12×12 matrix) and the Fourier transform vector of the input load vector, denoted by $\mathbb{F}(\Omega)$ (which is a 12×1 vector). Using $\boldsymbol{H}(\Omega)$ and $\mathbb{F}(\Omega)$, obtain the Fourier Transform $\mathbb{X}(\Omega)$ of the output (which is a 12×1 vector). Plot:

1. Absolute value of Fourier Transform of output $|\mathbb{X}(\Omega)|$ vs. Ω from $\Omega \in [0,1000] rad/s$
2. Phase of Fourier Transform of output $\angle\mathbb{X}(\Omega)$ vs. Ω from $\Omega \in [0,1000]\ rad/s$

Although many elements in the force vector are zero—leading to the respective Fourier transforms being 0, we still have non-zero response at each degree of freedom (as well as non-zero Fourier Transform of the output). Comment on what that's the case and what are the *general* dominant frequencies in the output $\boldsymbol{X}(t)$.