SE 167/267A
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Signal Processing and Spectral Analysis

HOMEWORK ASSINGMENT 2

TOTAL: 100

DUE DATE: Feb 06, 2024

Consider the same analog signal that we had considered in the first assignment:

$$f(t) = \cos 2000\pi t + 5 \sin 6000\pi t + 10 \cos 12000\pi t + 20 \cos 15000\, \pi t + 10 \sin 20000\, \pi t$$

Fourier Series

1. We know that this function is periodic with fundamental period T_p and fundamental frequency $\not{f}_p$. Start by stating the value of T_p and $\not{f}_p$. Obtain:

$$\int_{\frac{T_p}{2}}^{\frac{T_p}{2}} |f(t)|\, dt$$

   Comment if this function is absolutely integrable and what does this tell you about the function?

2. It won't take you long to realize that $f(t)$ is a well-behaved function and therefore, can be represented as sum of sines and cosines (which is obvious since $f(t)$ is a trigonometric function consisting of sines and cosines by its very definition). We know that the **analog periodic function in time domain** is represented by **discrete frequencies in the frequency domain** $\not{f}_m = n\not{f}_m$. Consider the Fourier representation of the form:

$$f(t) = F_0 + 2\sum_{m=1}^{\infty} |F_m| \cos(\Omega_m t + \theta_m)$$

   - What is the maximum value of the index m (denoted by m_{max}) in the sum above that can exactly represent the signal $f(t)$? That is, this *specific* function $f(t)$ can be represented by finite number of harmonics, such that:

$$f(t) = F_0 + 2\sum_{m=1}^{m_{max}} |F_m| \cos(\Omega_m t + \theta_m)$$

   - Just by looking at the signal $f(t)$, what is Fourier coefficient F_0 for the signal $f(t)$?

3. By visual inspection of the signal $f(t)$, for $1 \le m \le m_{max}$:
   - Obtain and tabulate frequency ν_m, magnitude of Fourier coefficient $|F_m|$, phase of Fourier coefficient θ_m, power density $|F_m|^2$, and the Fourier coefficient F_m.
   - Highlight all the rows that have non-zero Power density. Express your solutions in the tabulated form:

Harmonic m	Frequency $\nu_m = m\nu_p$	Angular Frequency $\Omega_m = 2\pi\nu_m$	Magnitude of Fourier coefficient $\lvert F_m\rvert$	Phase of Fourier coefficient θ_m	Power Density Spectrum $\lvert F_m\rvert^2$	Fourier Coefficient F_m
1						
2						
⋮						
⋮						
m_{max}						

Hint 1: Express the function $f(t)$ as sum of cosines as you did in the last assingment and then use the form of Fourier series given above that is expressed as sum of cosines. **Hint 2:** You can obtain the Fourier Coefficient F_m as $F_m = |F_m|e^{i\theta_m} = |F_m|(\cos\theta_m + i\sin\theta_m)$. **Hint 3:** There are 5 harmonics constituting the function $f(t)$. Therefore, only 5 rows of the table above will have non-zero $|F_m|^2$.

4. Obtain coefficients F_m for $-30 \le m \le 30$ by using the formula for Fourier Transform:
$$F_m = \frac{1}{T_p}\int_{-\frac{T_p}{2}}^{\frac{T_p}{2}} f(t)e^{-i\Omega_m t}dt$$
Also obtain the respective $|F_m|$ and θ_m. You will see that $F_{-m} = \bar{F}_m$.
Hint: I recommend using Mathematica for this problem. Use "Abs" function to obtain the magnitude of complex number and "Arg" to obtain the phase in Mathematica.

5. Plot the **power spectral density** $|F_m|^2$ as a function of the discrete frequency ν_m for $-30 \le m \le 30$. You'll notice that the power spectral density is symmetric about the y axis, and it is non-zero for only the 5 frequencies (that you highlighted in part 3). Also **obtain total average power** by using $f(t)$ and then by using F_m. Do you satisfy Parseval's theorem? Explain.

6. Using Mathematica, reconstruct $f(t)$ by using the values of F_m for $-30 \le m \le 30$ obtained in part 4, such that:
$$f(t) = \sum_{m=-30}^{30} F_m e^{i\Omega_m t}$$
Plot the reconstructed function along with the original function $f(t)$ for $t \in [0, 0.0005]$.

Fourier Transform

7. Consider an aperiodic function:

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ \cos 2000\pi t + 5 \sin 6000\pi t + 10 \cos 12000\pi t + 20 \cos 15000\,\pi t + 10 \sin 20000\,\pi t & \text{for } 0 \le t \le 0.002 \\ 0 & \text{for } t > 0.002 \end{cases}$$

Using Mathematica:

- Obtain Fourier Coefficient $F(\not{f})$ defined as:

$$F(\not{f}) = \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi \not{f} t} dt$$

- Plot the **energy spectral density** and the **phase spectral density** (that is the phase $\theta(\not{f})$ of $F(\not{f})$ as a function of $\not{f}$) as a function of frequency $\not{f} \in [-20000, 20000]$.

You will notice that unlike the discrete power spectral density plot obtained in part 5, the energy spectral density plot is a continuous function of frequency $\not{f}$. The "peaks" in the energy spectral density corresponds to the "major frequency content" of the signal.
Hint: Use "Abs" function to obtain the magnitude of complex number and "Arg" to obtain the phase.

8. Reconstruct the original signal by considering the following three cases:
   - Case 1: by limiting the frequency content (bandlimited signal) $\not{f} \in [-7500, 7500]$.
   - Case 2: by limiting the frequency content (bandlimited signal) $\not{f} \in [-10000, 10000]$.
   - Case 3: by limiting the frequency content (bandlimited signal) $\not{f} \in [-20000, 20000]$.

➢ Plot the reconstructed signal for all three cases along with the original function defined in part 7.
➢ Comment on what you observe about the reconstruction when the bandwidth increases.

Hint: Define the reconstructed function in Mathematica as a "dynamic function" of time and use "NIntegrate" to define the integral:

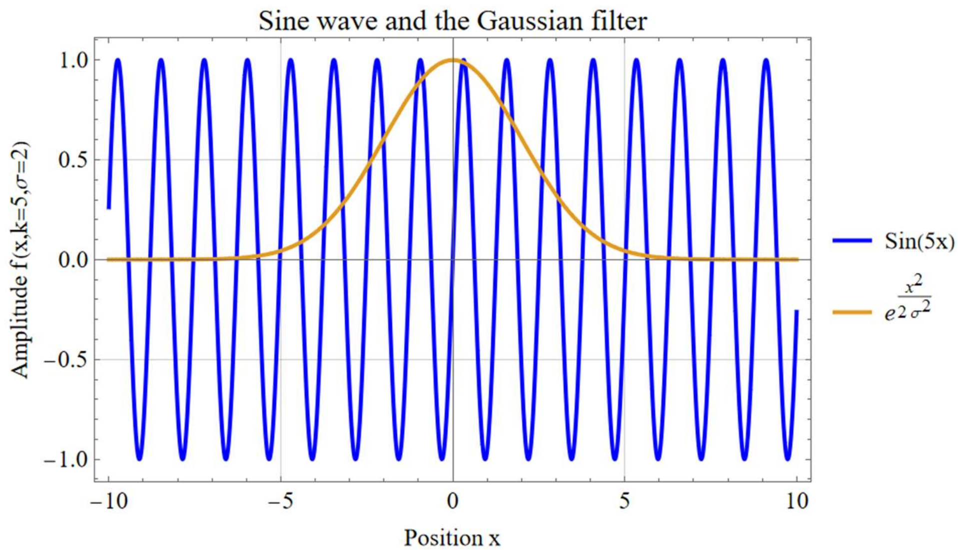
$$F(\not{f}) = \int_{\not{f}_{min}}^{\not{f}_{max}} f(t)\, e^{-i2\pi \not{f} t} dt$$

Fourier Transform of matter wave.

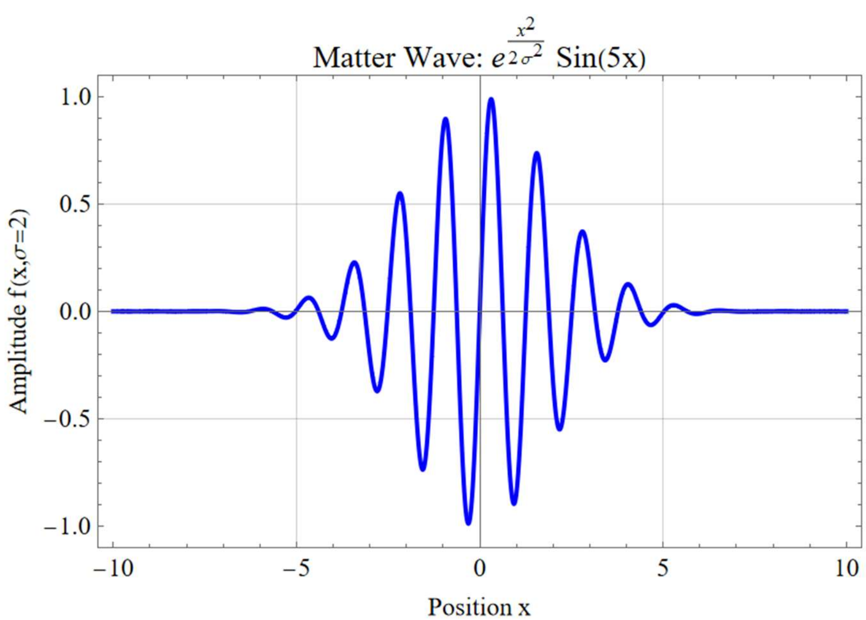
Consider a signal that is function of 1D space variable x:

$$f(x, k, \sigma) = e^{-\frac{(x-\mu)^2}{2\sigma^2}} \sin kx$$

Here, $e^{-\frac{x^2}{2\sigma^2}}$ superimposes "Bell Shape" over the standard sine wave. The variable k defines the wave number and has units of $radians/m$; μ defines the mean and σ defines the standard deviation of the Gaussian filter. Figure below illustrates the sine wave with $k = 5$ (that is, $\sin 5x$) and the Gaussian filter with $\sigma = 2$ and $\mu = 0$. We restrict to $\mu = 0$ in this case. The standard deviation σ is related to the width/variability of the Gaussian filter.



In the following figure, you can see the function $f(x, k = 5, \sigma = 2)$. This is also a form of a snapshot of matter wave in Quantum mechanics.



I am providing you with a Mathematica code named MatterWave_SE167_HW2.nb. Run the code and comment on the following:

9. In the Mathematica code, when you run the "Manipulate" command, you can visually see how the signal $f(x, k, \sigma)$ is changing as you vary the wave number k and the standard deviation σ of the Gaussian filter.
    - Keep the standard deviation slider to maximum and then gradually change the variable k. Report your observation (you'll see why k is called as wave number).
    - Keep the wave number slider k to maximum value and gradually change the standard deviation σ slider. Report your observation and provide a logical explanation of your observation.
10. I have used another manipulate command to visualize the effect of change of wave number k and standard deviation σ of the Gaussian filter on the Fourier Transform of the signal. Clearly, this is an aperiodic function as a result of imposing Gaussian filter on the periodic sine wave. Note that the program will be slow when you change the slider in this case since for every new k and σ, Mathematica will dynamically evaluate the Fourier Transform.
    - Keep the standard deviation slider to maximum and then gradually change the variable k. What happens to the Energy density spectrum when k increases. Provide a logical explanation of your observation.
    - Keep the wave number slider k to maximum value and gradually change the standard deviation σ slider. What happens to the Energy density spectrum when k increases. Provide a logical explanation of your observation.