Signal Processing and Spectral Analysis

HOMEWORK ASSINGMENT 1

TOTAL: 150 Points

DUE DATE: January 25, 2024

Consider an analog signal:

$$f(t) = \cos 2000\pi t + 5\sin 6000\pi t + 10\cos 12000\pi t + 20\cos 15000\,\pi t + 10\sin 20000\,\pi t$$

**Basic Characteristics of the Signal**

1. How many distinct frequencies compose this signal? Plot this function and then plot each harmonic separately for time $t \in [-0.002, 0.002]$. Identify by visual inspection, the time period $T_p$ of the function $f(t)$ and mark one full cycle in you plot of $f(t)$.

2. Represent this function in terms of cosines (replace sines as cosines with a phase difference). That is, in the form:

$$f(t) = \sum_i A_i \cos(\Omega_i t + \phi_i).$$

Populate the following table that details the characteristics of harmonics composing this function.

| Harmonics ($Index\ i$) | Angular frequency $\Omega_i$ ($in\ rad/s$) | Frequency $f_i$ ($in\ Hz$) | Time period ($in\ seconds$) | Amplitude $A_i$ | Phase Difference $\phi_i$ |
|---|---|---|---|---|---|
| 1 | | | | | |
| 2 | | | | | |
| ⋮ | | | | | |

3. The fundamental time period of the signal $f(t)$ (denoted by $T_p$) is equal to the least common multiple (LCM) of the time periods of harmonics constituting the signal. Obtain $T_p$ by finding LCM of the time period of various harmonics constituting the signal (fourth column of the table above) and match the obtained time period with the value obtained by visual inspection in part 1.

4. Use the value of $T_p$ obtained in part 3 to get the angular frequency $\Omega$ and frequency $f_p$ of the signal $f(t)$.

5. What is the Nyquist sampling rate $f_N$ for this signal?

**Sampling using $f_s > f_N$**

6. Obtain samples of the function $f(t)$ using sampling frequency $f_s = 2f_N$ between time $t \in [0,0.002]$ and plot the samples over the original analog signal with $x$ axis being time $t$ and the $y$ axis being the value of the function $f(t)$ for $t \in [0,0.002]$. That is, first obtain the set of discrete time axis $\left\{t = nT_s = \frac{n}{f_s}\right\} \in [0,0.002]$ and then obtain the sequence $f[n] = f(t = nT_s)$. In addition, obtain the same plot but between $t \in [0,0.0005]$ (this is a "zoomed" in version and is helpful to visualize better).

7. What is the discrete-time signal obtained after sampling with $f_s = 2f_N$. Obtain analytical expression of the form:

$$f[n] = \sum_i A_i \cos(\omega_i n + \phi_i)$$

8. In the frequency axis $f$ (in Hz) mark the original spectrum. For $f_s = 2f_N$, mark the spectrum of the first two *mirror* and *copy* alias along with the amplitude and phase of the respective frequencies in the alias spectrum (like what we did in class).

9. Although we have sampled at a sampling rate greater than the Nyquist rate in part 6, we can still find the alias or copy signal that yields same samples (as you have marked in part 8). However, sampling greater than Nyquist rate guarantees that we can uniquely reconstruct the original signal using the sampling theorem and that the original frequency spectrum is captured in the sampled signal. Based on part 8, find the: (1) discrete-time signal, and the respective (2) analog signal of the first "*mirror alias*" and the first "*copy alias*" signals.

10. Plot analog form of (a) original signal; (b) first mirror alias signal; (c) first copy alias signal; (d) the samples obtained in Part 6 with $x$ axis being time $t$ and the $y$ axis being the value of these signals for $t \in [0,0.002]$. You should be able to see that the samples are shared by all three analog signals. In addition, obtain the same plot but between $t \in [0,0.0005]$ (this is a "zoomed" in version and is helpful to visualize better).

11. Now, reconstruct the original signal from the samples obtained in part 6 between $t \in [0,0.002]$ using ideal interpolation. You will see that the reconstructed signal matches the original signal and not the alias. This is because we have used sampling rate greater than that of Nyquist rate.

## Sampling using $\mathscr{f}_s < \mathscr{f}_N$

12. **Sampling using $\mathscr{f}_s < \mathscr{f}_N$**: Assume now that we sample the analog signal $f(t)$ using a sampling rate $\mathscr{f}_s = \frac{\mathscr{f}_N}{2}$. What is the discrete-time signal $f[n]$ obtained after sampling?

13. Corresponding to the discrete-time signal $f[n]$ obtained in part 12, what is the analog signal $\hat{f}(t)$. Plot the original function $f(t)$ along with $\hat{f}(t)$ and comment on why $\hat{f}(t) \neq f(t)$. What are the missing frequencies in the reconstructed signal?

## Quantization and Error

14. For the samples $f[n]$ obtained in part 6, quantize the samples $f_q[n]$ using: (a) rounding to the closest integer, and (b) truncating to the closest integer. For each method of quantizing, plot the quantized samples connected by a step function on top of the plot obtained in part 6.

15. Obtain the error sequence $e[n] = f[n] - f_q[n]$ in the quantized signal relative to the original sample for both cases (rounding and truncation). Obtain the quantization error power defined as:

$$P_q = \frac{1}{N}\sum_{n=0}^{N-1} e[n]^2 = \frac{1}{N}\sum_{n=0}^{N-1}\left(f[n] - f_q[n]\right)^2$$

Also obtain the quality of the quantized signal by obtaining the Signal-to-Noise ratio defined by:

$$SNR = 10\log_{10}\frac{P_f}{P_q}$$

Here, $P_f$ is defined as:

$$P_f = \frac{1}{N}\sum_{n=0}^{N-1} f[n]^2$$