Project Report

Integer Quantum Hall Effect: An Investigation

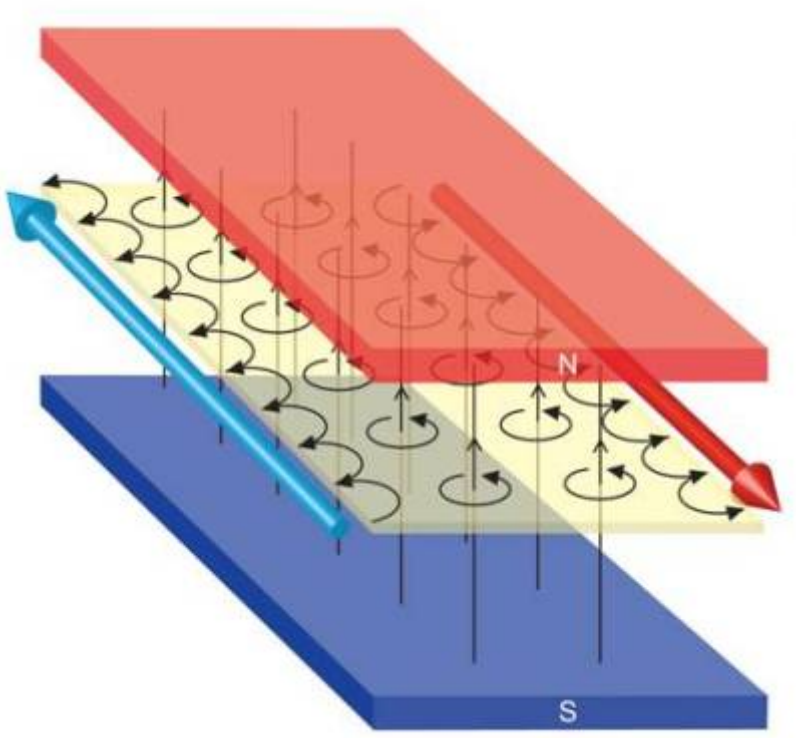

Figure 1: Depiction of quantum Hall effect, Taken from [1]

Archit Bhatnagar (140260010), Rakshit Jain (140260018)
Indian Institute of Technology, Bombay

Contents

Abstract

The report introduces and then investigates why and how does the Integer Quantum Hall effect arise in Two Dimensional Systems at low Temperatures. We first describe the classical picture of Hall Effect using Drude Model, then describe the experimental setup and observations which introduced the Integer QHE. We then go on to the theory of the Integer QHE. We have tried to look at the effect from various theoretical viewpoints, exploring Landau Levels, Edge Modes, Robustness of Integer QHE (the origins of the Hall Plateau), and finally the topological nature of Integer QHE. We also discuss what is Berry Phase and how does it arise in various systems, using it to explain the topological aspect of Integer QHE.

We have mainly followed the discussion from [2] in this report, although we have tried to supplement it using various other references.

List of Symbols Used

- **B**: Magnetic field applied to the sample
- **A**: Magnetic Potential associated with the magnetic field
- Φ: Flux
- $\Phi_0 = \frac{2\pi\hbar}{e}$: Quantum of Flux
- **E**: Electric field applied to the sample
- E: Electric field or Energy (Depending on Context)
- E_F: Fermi Energy for a given system
- V: Electric Potential, Potential Energy (in a Hamiltonian) or Fill Factor (Depending on Context)
- **J**: Current Density in Sample
- $\mathcal{H}$ or H: Hamiltonian
- ΔH: Perturbations in the Hamiltonian
- ρ and σ: Resistivity and Conductivity(Rank-Two Tensors)
- $\omega_B = \frac{eB}{m}$: e and m are the charge (magnitude) and mass of electron
- $\mu_B = \frac{e\hbar}{2m}$: Bohr Magneton
- $l_B = \sqrt{\frac{\hbar}{eB_z}}$: Magnetic Length
- γ: Berry Phase
- $\mathcal{A}$: Berry Connection
- $\mathcal{F}$: Curvature of Berry Connection

Chapter 1

Introduction: The Basics

1.1 Drude Model

Drude model considers all the interactions in a linear relaxation term given by τ. It includes all the friction faced by electrons such as electrons bouncing off from the collisions from ions, impurities and other electrons. The resulting equation of motion in presence of electric and magnetic fields is given by -

$$m\frac{dv}{dt} = -e\mathbf{E} - e\mathbf{v}\times\mathbf{B} - \frac{m\mathbf{v}}{\tau} \tag{1.1}$$

This model is the simplest model considering electrons as billiard balls. We will now derive the Hall coefficients given by the Drude model picture. Since in equilibrium $\frac{dv}{dt} = 0$, we have -

$$\begin{aligned} 0 &= -e\mathbf{E} - e\mathbf{v}\times\mathbf{B} - \frac{m\mathbf{v}}{\tau} \\ -\frac{e\tau}{m}\mathbf{E} &= \frac{e\tau}{m}\mathbf{v}\times\mathbf{B} + \mathbf{v} \end{aligned}$$

Multiplying both the sides by $-ne$ and noting that $\mathbf{J} = -ne\mathbf{v}$, we have for the 2 dimensional geometry as shown in figure 1.1-

$$\begin{bmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & 1 \end{bmatrix}\mathbf{J} = \frac{ne^2\tau}{m}\mathbf{E}$$

From now on we will use notation $\omega_B = \frac{eB}{m}$ and $\sigma_{DC} = \frac{ne^2\tau}{m}$. On inverting the above matrix we calculate the conductivity tensor as -

$$\mathbf{J} = \frac{\sigma_{DC}}{1+\omega_B^2\tau^2}\begin{bmatrix} 1 & -\omega_B\tau \\ \omega_B\tau & 1 \end{bmatrix}\mathbf{E} \tag{1.2}$$

On comparing the above equation to ohm's law $\mathbf{J} = \sigma\mathbf{E}$ we get σ_{Hall} as -

$$\overleftrightarrow{\sigma} = \frac{\sigma_{DC}}{1+\omega_B^2\tau^2}\begin{bmatrix} 1 & -\omega_B\tau \\ \omega_B\tau & 1 \end{bmatrix} \tag{1.3}$$

This is quite similar to conductance tensor of an isotropic material which is given by -

$$\overleftrightarrow{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{bmatrix}$$

The above structure of the conductivity tensor comes from the rotational invariance of the tensor. One can easily show the invariance by -

$$\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{bmatrix} \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{bmatrix}$$

From (1.2) we can calculate resistivity tensor as -

$$\rho = \frac{1}{\sigma_{DC}} \begin{bmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & 1 \end{bmatrix}$$

This leads to very interesting remark, one can calculate ρ_{xy} as -

$$\rho_{xy} = \frac{\omega_B\tau}{\sigma_{DC}} = \frac{B}{ne}$$

It is actually independent of the τ representing some relation of intrinsic properties not depending on how much dirt is present in the system.
The second nice property is actually what we measure in the system. We usually measure Resistance often denoted by **R** which as we will show is related to ρ by geometric factors. We define R_{xy} as -

$$R_{xy} = \frac{V_y}{I_x} = \frac{E_y L}{J_x L w} = \frac{-\rho_{xy}}{w}$$

Where L represents Length along the y axis and w is the width of the sample. So clearly we will be measuring ρ_{xy} when we conduct our experiments. In classical hall effect we also define Hall Resistance(R_H) and Hall Voltage(V_H) as -

$$V_H = \frac{I_x B_z}{nwe}$$
$$R_H = -\frac{E_y}{J_x B_z} = \frac{\rho_{xy}}{B_z}$$

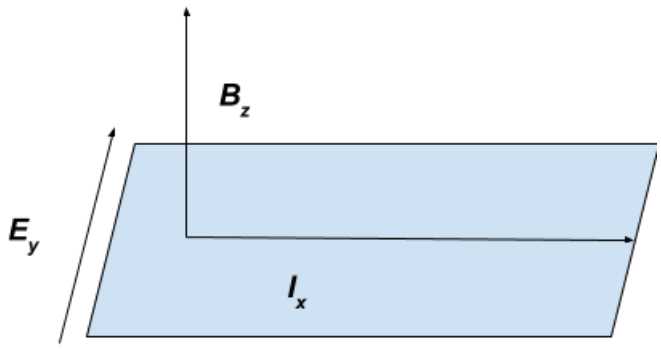

Figure 1.1: Hall Geometry with Magnetic field is applied along the z axis with current flowing along the x axis which leads to a potential difference along y axis

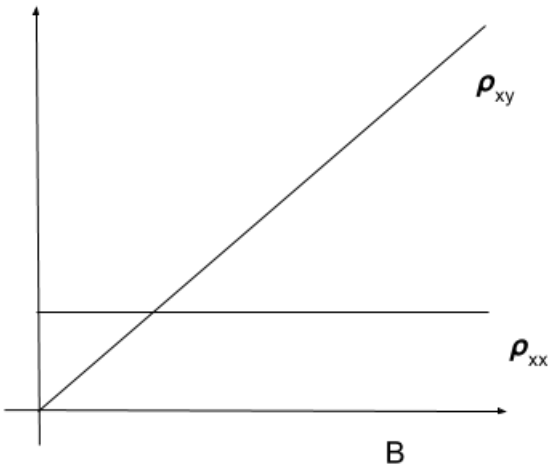

Figure 1.2: Dependence of ρ_{xy} and ρ_{xx} with change in magnetic field's magnitude. Adapted from [2]

Hence in the Drude model, we have hall resistance as -

$$R_H = \frac{\omega_B}{\sigma_{DC} B_z} = \frac{1}{ne} \tag{1.4}$$

Hence Hall resistance depends on charge and density of the conducting particles. Moreover it doesn't depend on the internal relaxation processes represented by τ in drude model.
It is now very clear that in any experiments we will measure Resistivity ρ_{xx} and ρ_{xy} which in drude model are given by -

$$\rho_{xx} = \frac{m}{ne^2\tau}$$
$$\rho_{xy} = \frac{B_z}{ne}$$

We can plot the dependence of ρ_{xx} and ρ_{xy} with Magnetic fields as shown in figure 1.2. These dependence is of much importance as when we will go ahead and see how quantum effects affect these relationships and how resistivity dependence will change with the magnetic field.

We would like to discuss a very important observation we can see from structure of transverse and longitudinal resistivity.

1.2 Quantum Hall Effect

1.2.1 Experimental Setup

Experimentally we measure the resistivity tensor by putting a constant current along the x direction and measuring drop along the current given by V_x and in the transverse direction given by V_H

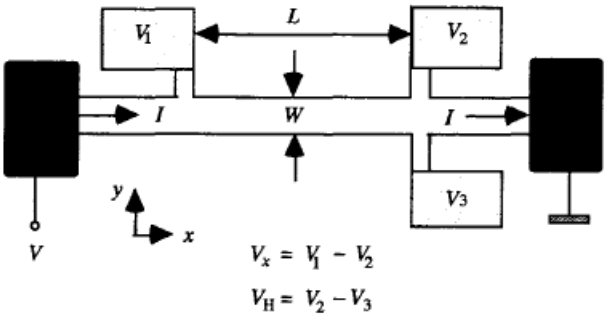

Figure 1.3: A typical hall bar used for measurement of R_H and V_H, Adapted From [3]

1.2.2 Experimental Observations

At low magnetic fields, the measurements of longitudinal voltage almost remains a constant whereas transverse voltage or the Hall voltage increases linearly with the applied magnetic field. These results are clearly in the agreement with the Classical Drude picture as we discussed in the previous sections. However as we increase the magnetic fields we see a oscillatory behavior in longitudinal resistance and plateaus in transverse resistance. These features are quite absent or negligible even in temperatures as low as 77 K, but quite prominent on temperatures near 4K [3].

We will now discuss the ground breaking experiments preformed by von klitzing on the samples made by Pepper and Dorda [4]in 1980. The resistivities looked similar to figure 4. Both the resistivity ρ_{xx} and ρ_{xy} show interesting behavior. Transverse resistivity (ρ_{xy}) appears to be sit on a plateau for some range of a magnetic field and jumping suddenly to the next one as we change the magnetic field. On plateau region the resistivity takes the value of -

$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{v}$$

where v is an integer and hence the name integer quantum hall effect. One of the most phenomenal part of integer quantum hall effect is the values of ρ_{xy} are so robust that they are actually used as a measure of resistivity. One can see this from the fact that v can measured to a accuracy of 1 in 10^9. The constant $\frac{2\pi\hbar}{e^2}$ is also called von klitzing constant as a honor to it's discoverer Klaus von Klitzing. These sudden transitions between the

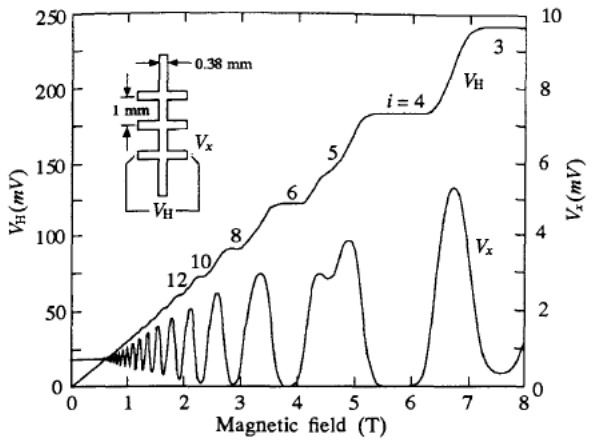

Figure 1.4: Observations of oscillations in longitudinal resistance and plateaus in transverse resistance. Taken from **ME Cage R.F. Dziuba IEEE trans. Instrum. Meas. 1985 IM -34 301**

plateaus occur at specific Magnetic field values -

$$B = \frac{2\pi\hbar n}{ve}$$

where n is the carrier concentration. As we will study the cause of these changes we will talk about Landau levels. This will be a lesser surprising result however it would be difficult to explain the widths of these plateau regions where will consider how dirty your sample actually is. The technical name for this is *disorder*. One has to actually remove this dirt to actually see the physics behind the experiments. However in the case we will see that as we decrease the dirt in the system the plateaus disappear and are expected vanish completely. This is the actual beauty of the integer quantum hall effect that presence of disorder leads to a quantity which is actually very accurate!.
In the coming sections we will talk about how the integer quantum hall effect can be explained by using Quantum Mechanics. We will keep the calculations simple, however the interactions that we will ignore from electron electron interactions will actually lead to fractional quantum hall effect.

1.2.3 Materials

It is quite important to understand the materials that were used for observations of quantum hall effect. Integer quantum hall effect was initially discovered in Si $MOSFET$ or Metal Oxide Semiconductor Field effect transistor which is basically a metal oxide and semiconductor sandwich. In these systems electrons can be trapped in the "inversion" mode and successfully creating a 2D electron gas for hall measurements. Meanwhile the discovery of fractional quantum hall effect was discovered in the samples of GaAs - GaAlAs heterostructure. Electron concentration in both of these systems is roughly of the order 10^{12} cm^{-3}.
Recently quantum hall effect was observed in a 2D material Graphene with relativistic electrons at room temperatures. [5]

1.3 Landau Levels

1.3.1 Introduction

In this section we will deal with the motion of electron in presence of a magnetic field and how it leads to the formation of landau levels. For this discussion we will neglect the role of spin and would also explain its reason in the upcoming section.
The Canonical momentum in the presence of a magnetic field is defined as -

$$\mathbf{p} = m\dot{\mathbf{x}} - e\mathbf{A}$$

This differs from the momentum we generally define as $m\dot{\mathbf{x}}$ and this momentum is now called mechanical momentum We can rearrange the above equation to write -

$$m\dot{\mathbf{x}} = \mathbf{p} + e\mathbf{A}$$

Now the Hamiltonian which is defined in terms of mechanical momentum can be written as -

$$\mathcal{H} = \frac{1}{2}m\dot{\mathbf{x}}^2 = \frac{1}{2m}(\mathbf{p} + e\mathbf{A})^2$$

This should not come as a surprise as magnetic fields don't change the energy of the system as they do no work. However in presence of magnetic fields we lose our canonical properties which were present before . Now the relations of the Poisson brackets are no longer satisfied.

$$\begin{aligned}\{m\dot{\mathbf{x}}_i, m\dot{\mathbf{x}}_j\} &= \{\mathbf{p}_i + e\mathbf{A}_i, \mathbf{p}_j + e\mathbf{A}_j\} \\ &= e\Big(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\Big) \\ &= -e\epsilon_{ijk}B_k \end{aligned} \tag{3.1}$$

1.3.2 Quantization

We will now try to simplify the Hamiltonian in the presence of a magnetic field along the z direction. We will now define new operators -

$$\pi = m\dot{\mathbf{x}} = \mathbf{p} + e\mathbf{A}$$

Now commutation relations follow from (3.1) as -

$$[\pi_x, \pi_y] = -ie\hbar B_z$$

Now we have new operators for raising and lowering operators which are quite similar to harmonic oscillator case, they can be written as -

$$\begin{aligned} a &= \sqrt{\frac{2}{e\hbar B_z}}(\pi_x - i\pi_y) \\ a^\dagger &= \sqrt{\frac{2}{e\hbar B_z}}(\pi_x + i\pi_y) \end{aligned} \tag{3.2}$$

It can followed from the commtator relation in π to get a similar relation in a, which turns exactly like a harmonic oscillator case -

$$[a, a^\dagger] = 1$$

Also the hamiltonian can also be written as -

$$\begin{aligned}\mathcal{H} = \frac{1}{2m}\pi.\pi &= \frac{1}{4m}(e\hbar B_z((a+a^\dagger)^2 - (a-a^\dagger)^2) \\ &= \frac{\hbar\omega_B}{2}(aa^\dagger + a^\dagger a) \\ &= \hbar\omega_B(a^\dagger a + \frac{1}{2}) \end{aligned} \tag{3.3}$$

This is actually harmonic oscillator Hamiltonian in one dimension. This feels weird as our two dimensional system is now converted into a 1 dimensional system. As we move to subsequent section we will deal with degeneracy which this hamiltonian is failing to capture. In this case each energy will have very different eigenfunctions associated leading to degeneracy. A wild degeneracy which will actually lead to all the physics associated with quantum hall effect.

Role of Spin

After calculating the eigenenergies we can now talk about the role of spin as we promised before. Since due to zeeman splitting $\Delta = g\mu_B B$ and the difference between two landau levels is $\hbar\omega_B$. We can see that for free electron both are same as $\mu_B = \frac{e\hbar}{2m}$, if this were true then the energy of electron of Spin Down in $n+1$ landau level will be same as that of Spin Up n landau level. However this is not true as $\omega_B = \frac{eB}{m_{eff}}$ and this leads to zeeman energy playing no effect in the samples we perform the hall experiments on. But there are materials (ZnO with $MnZnO$) where the zeeman field is dominant.

1.3.3 Landau Gauge

For calculation of vector potential we have a gauge degree of freedom which actually leads to different $\mathbf{A}$ such that they satisfy -

$$\nabla \times \mathbf{A} = \mathbf{B}$$

The different $\mathbf{A}$ will lead to different Hamiltonians, but the quantities we observe remain the same. This is very frequent with the Hamiltonians dealing with the magnetic fields. For this case we will choose a specific gauge also called Landau gauge which will help in explaining the physics behind the quantum hall effect. Later we also define symmetric gauge and would use it wherever beneficial. In the landau gauge $\mathbf{A}$ is given by -

$$\mathbf{A} = xB_z\hat{y} \tag{3.4}$$

Now writing the Hamiltonian as we did before -

$$\begin{aligned}\mathcal{H} &= \frac{1}{2m}(\mathbf{p} + e\mathbf{A})^2 \\ &= \frac{1}{2m}(p_x^2 + (p_y + xB_z)^2)\end{aligned}$$

We can see that Hamiltonian does not depend on the y coordinate, thus having a translational symmetry in y direction. Thus for eigenfunctions, it would behave behave like a free particle in y direction. Thus writing the eigenfunction and Hamiltonian as -

$$\begin{aligned}\psi_k(x, y) &= e^{iky} f_k(x) \\ \mathcal{H} &= \frac{1}{2m}(p_x^2 + (\hbar k + xB_z)^2)\end{aligned}$$

Notice that we wrote k dependence on $f(x)$.
We will now define magnetic length as $l_B = \sqrt{\frac{\hbar}{eB_z}}$ and would rewrite the above equation.

$$\mathcal{H} = \frac{1}{2m}p_x^2 + \frac{m\omega_B^2}{2}(x + kl_B^2)^2$$

This is a very familiar equation of a harmonic oscillator displaced by $-kl_B^2$. Hence the eigenfunctions in the functions are actually displaced by momentum vector along y direction. This is actually specific to Landau gauge as in other gauges wave-functions will be different. Hence the energy eigenvalues and eigenfunctions are given by -

$$\begin{aligned}E &= \hbar\omega_B\Big(n + \frac{1}{2}\Big) \\ \psi_k(x, y) &= e^{iky} H_n(x + kl_B) e^{-\frac{(x+kl_B)^2}{2l_B^2}}\end{aligned}$$

where H_n is a Hermite polynomial.
These wavefunctions are localized around $x = -kl_b^2$. However due to large degeneracy, we can actually construct any wavefunction we want to.

Degeneracy in Landau Gauge

The benefit of Landau gauge is that we can see the degeneracy directly from the structure of the wavefunction. Note that for calculating we will assume a 2 dimensional surface of lengths L_x, L_y. Hence in general density of states in 2 dimension is given by -

$$\mathcal{N} = \frac{L_y}{2\pi}\int dk \tag{3.5}$$

Here the limits of the integration can be calculated by -

$$0 \leq x \leq L_x$$

and x is localised around $-kl_B^2$ hence limits on k are given by -

$$-\frac{L_x}{kl_B} \leq k \leq 0$$

Now substituting this limit in (3.5), we have -

$$\mathcal{N} = \frac{L_y L_x}{2\pi k l_B^2} = \frac{AeB}{2\pi\hbar} = \frac{AB}{\Phi_0}$$

Where Φ_0 is given by $\frac{2\pi\hbar}{e}$, also known as quantum of flux. It can be thought of as flux contained in an area of $2\pi l_b^2$. This is a wild degeneracy for each landau level. This will result in much of the physics related to quantum hall effect.

Levels in presence of Electric field

We will now add electric field along the x direction in the same setup to form the new Hamiltonian which looks like -

$$\mathcal{H} = \frac{1}{2m}(p_x^2 + (p_y + xB_z)^2) - eEx$$

Now we will shift our origin to $x = x + mE/eB^2$ and replacing $p_y = \hbar k$ as done earlier will reduce the above equation to -

$$\mathcal{H} = \frac{1}{2m}(p_x^2 + (p_y + xB_z)^2) + eE(kl_B^2 - \frac{eE}{m\omega_B^2}) + \frac{mE^2}{2B^2}$$

Hence the energies are now -

$$E_{n,k} = \hbar\omega_B(n + \frac{1}{2}) + eE(kl_B^2 - \frac{eE}{m\omega_B^2}) + \frac{mE^2}{2B^2}$$

Now the energies depend on both n and k instead of just n in presence of just the magnetic field leading to sloped Landau levels as shown in figure 5

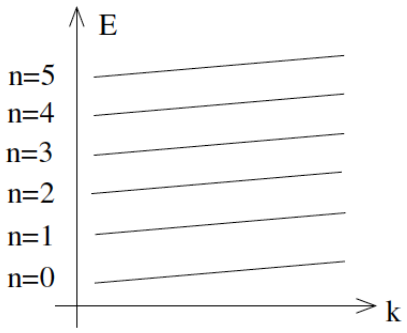

Figure 1.5: Landau levels in presence of an Electric field. Adapted from [2]

Now the wave functions are localized around $x = -kl_B^2 - mE/eB^2$, the last term can be thought of as momentum along the y axis.

1.3.4 Symmetric Gauge

Now having understood Landau gauge properly, we will now deal with symmetric gauge which is quite useful in describing fractional quantum hall effect. In this gauge vector potential is given by -

$$\mathbf{A} = \frac{1}{2}\mathbf{r} \times \mathbf{B} = \frac{-yB_z}{2}\hat{x} + \frac{xB_z}{2}\hat{y} \tag{3.6}$$

This doesn't preserve the translational symmetry in either x or y direction unlike the Landau gauge. However the angular momentum is conserved, making angular momentum a quantum number for this gauge.
We will now approach systematically to find the eigenvalues and energies in presence of Magnetic field.Again we will define $\pi = \mathbf{p} + e\mathbf{A}$ to calculate the Hamiltonian as -

$$\mathcal{H} = \frac{1}{2m}\pi.\pi = \hbar\omega_B\Big(n + \frac{1}{2}\Big)$$

Now we will introduce a conjugate canonical momentum, defined as -

$$\widetilde{\pi} = \mathbf{p} - e\mathbf{A}$$

Now one should note that this momentum we defined above differs from canonical momentum by a - sign, hence it won't be a gauge invariant. We will now focus on the commutator relations of $\widetilde{\pi}$. One can easily show that -

$$[\widetilde{\pi}_x, \widetilde{\pi}_y] = ie\hbar B_z$$

Also cross commutator relations follow -

$$\begin{aligned}[\pi_i, \widetilde{\pi}_i] &= 2ie\hbar\frac{\partial A_i}{\partial X_i} \\ [\pi_x, \widetilde{\pi}_y], [\pi_y, \widetilde{\pi}_x] &= ie\hbar\Big(\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x}\Big)\end{aligned} \tag{3.7}$$

It can clearly seen from the structure of $\mathbf{A}$ (3.6) that in the case of symmetric gauge -

$$[\pi_i, \widetilde{\pi}_j] = 0$$

Thus in the case of symmetric gauge, $\widetilde{\pi}$ will commute with the hamiltonain, hence we can define a second pair of raising and lowering operators as we did before -

$$\begin{aligned}b &= \sqrt{\frac{2}{e\hbar B_z}}(\widetilde{\pi}_x - i\widetilde{\pi}_y) \\ b^\dagger &= \sqrt{\frac{2}{e\hbar B_z}}(\widetilde{\pi}_x + i\widetilde{\pi}_y)\end{aligned}$$

Now our hilbert size is increased to account for degeneracy. Now we will define $|0,0\rangle$ as our ground state and subsequent states will be given by -

$$|n,m\rangle = \frac{a^{\dagger n} b^{\dagger m}}{\sqrt{n!m!}} |0,0\rangle$$

We will now try to calculate the wave functions and hence density of states as calculated before. First rewriting the ladder operators as -

$$\begin{aligned}
a &= \sqrt{\frac{2}{e\hbar B_z}}(\pi_x - i\pi_y) \\
&= \sqrt{\frac{2}{e\hbar B_z}}(p_x + eA_x - ip_y - ieA_y) \\
&= \sqrt{\frac{2}{e\hbar B_z}}\Big(- i\hbar\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) + e(A_x - iA_y)\Big) \\
&= \sqrt{\frac{2}{e\hbar B_z}}\Big(- i\hbar\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) + \frac{eB}{2}(-y - ix)\Big)
\end{aligned}$$

Now we will define new complex operators z and ∂ as -

$$\begin{aligned}
z &= x + iy \\
\bar{z} &= x - iy \\
\partial &= \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big) \\
\bar{\partial} &= \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)
\end{aligned}$$

Now writing the ladder operators in terms of new complex variables we see -

$$\begin{aligned}
a &= -i2\sqrt{2}\Big(l_B\bar{\partial} + \frac{z}{l_B}\Big) \\
a^\dagger &= -i2\sqrt{2}\Big(l_B\partial - \frac{\bar{z}}{l_B}\Big) \\
b &= -i2\sqrt{2}\Big(l_B\partial + \frac{\bar{z}}{l_B}\Big) \\
b^\dagger &= -i2\sqrt{2}\Big(l_B\bar{\partial} - \frac{z}{l_B}\Big)
\end{aligned}$$

The ground state is annihilated by both a and b and it turns out it is unique and can be written up to normalization as -

$$\psi_{0,0} \sim e^{-\frac{|z|^2}{4l_B^2}}$$

We will now try to calculate the states corresponding to the first Landau level and represent them as $\psi_{0,m}$. We will obtain these states by multiplying $b^\dagger$ operator on the $\psi_{0,0}$ state -

$$\psi_{0,m} \sim \Big(\frac{z}{l_B}\Big)^m e^{-\frac{|z|^2}{4l_B^2}} \tag{3.8}$$

As predicted intuitively before that under the symmetric gauge angular momentum must be a good quantum number can be checked by -

$$J_z = i\hbar\Big(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\Big)$$

It can be rearranged and written as -

$$J_z = \hbar(z\partial - \bar{z}\bar{\partial})$$

Also it can very easily verified as -

$$J_z\psi_{0,m} = m\hbar\psi_{0,m}$$

The steps stated above can easily be extended to higher Landau levels by applying with $a^\dagger$ and results can be followed as done for the lowest landau levels. Now we should focus on one of the most important property, degeneracy of the landau levels.

Degeneracy in Symmetric gauge

The wavefunction of states in the first landau levels is given by (3.8), We can see that the maximum of the function lies on a circle of $\sqrt{2m}l_B$. Assume a circle of area πR^2 as shown in the figure 6. The total density of states are given by -

$$\mathcal{N} = \frac{R^2}{R_2^2 - R_1^2} = \frac{R^2}{2l_B^2} = \frac{\pi R^2}{2\pi l_B^2} = \frac{AeB}{2\pi\hbar}$$

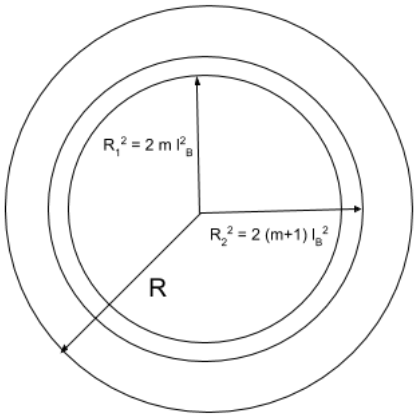

Figure 1.6: Density of states for symmetric gauge

We find the density of states expressions to be same for both the gauges which is expected as the density of the states contribute to conductivity as will be shown in the coming section which is gauge invariant.

Chapter 2

Integer Quantum Hall Effect (QHE)

2.1 Introduction to Integer Quantum Hall Effect

In the previous chapter we saw formation of Landau levels in the presence of Magnetic fields and saw how do they depend on the momenta in y direction in presence of the electric fields. In this chapter we will apply these concepts in order to explain the plateau obtained in the hall measurements and their relation with the density of states obtained in the previous sections.
One should note the uncanny resemblance between the density of states per unit area and hall resistivity -

$$\begin{aligned}\rho_{xy} &= \frac{2\pi\hbar}{e^2}\frac{1}{v} \\ &= \frac{B}{e}\frac{2\pi\hbar}{eBv} \\ &= \frac{B}{e\mathcal{N}v}\end{aligned}$$

This expressions when compared to the expression in the Drude Model, we get the electron density equal to $\mathcal{N}v$. This expressions leads us to believe that the conduction only takes place because of v filled Landau levels.It is quite an amazement that the concepts in the experiments done in 1930s of the SDH oscillations applied directly to the quantum hall effect.
However one should note that the Landau levels are separated by energy $\hbar\omega_B$. If this energy is less than that of thermal fluctuations we will stop seeing the plateau regions, thus these plateaux are only visible in the low temperatures like 4K as shown in the experiments.
To analyse systematically we will define a new term called fill factor denoted by V given by -

$$V = \frac{N}{\mathcal{N}_B}$$

where N denotes the total concentration of the electrons in the system and $\mathcal{N}_B$ denoted the degeneracy of the Landau levels for a given magnetic field. it should be understood

that if we take spin degeneracy into account we will have total number of filled Landau levels as -

$$v = \frac{N}{\mathcal{N}_B} = \left[\frac{V}{2}\right]$$

where [.] denotes the greatest integer function. Also define B_t such that at magnetic field B_t all the electrons are in the first Landau level, Leading to the expressions -

$$N = 2\mathcal{N}_B$$
$$N = \frac{2eB_t}{\hbar}$$
$$B_t = \frac{N\hbar}{2e}$$

Fill factor now can be rewritten as -

$$v = \frac{2B_t}{B}$$

Hence the total number of filled landau levels is $\left[\frac{B_t}{B}\right]$ We can now plot the energies of the highest filled landau levels as a function of the magnetic fields as shown in the figure 7.

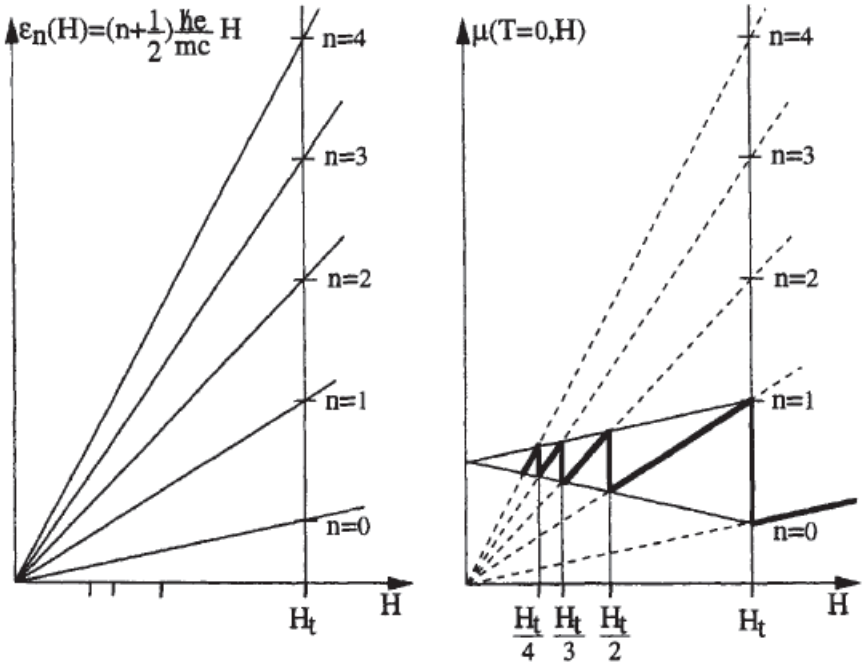

Figure 2.1: Energy of the highest filled Landau level as a function of Magnetic induction H. Adapted from [3]

Now as argued earlier in this section the conductivity is due to filled Landau levels and remains constant when we are between two Landau levels as we are not changing the number of electrons in already filled electron levels. But, have a clear look at the figure 7, one should note that as we change Magnetic field (induction), the shift from one Landau level to another happens almost immediately (Perpendicular straight lines in the diagram)

and would lead to no formation of plateaus at all!.
This is a inherent problem in quantum hall effect where disorder in the system comes into role and effects come form the scenarios we ignore in most of the cases. In the next section we will talk about Edge modes and their significance to the ρ_{xx} and robustness of the quantum hall effect.

2.2 Edge Modes

We have currently described the physics inside the bulk of the material, but what happens at the edge? This is the question this section intends to answer.

Classically, when the electrons are placed in the magnetic field, they start to go in circular orbits. For a fixed magnetic field direction, this revolution happens only in one direction, determined by the magnetic field direction. For now, lets say that the direction is such that the rotation is in anti-clockwise direction. Now, at the edge, the electron orbit now collides with the edge. Since the electron cannot leave the material, the electron then is forced to follow the edge instead of the orbit between the points of intersection between the orbit and the edge. What this eventually results is a skipping motion of the electrons on the edge in which they move only in a single direction (down for the left edge and up for the right, given anticlockwise rotation).

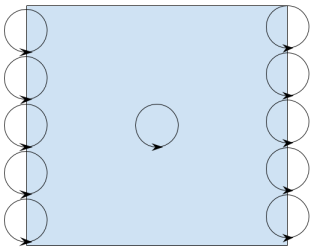

Figure 2.2: The Classical Picture

Now, this is something interesting. We have a chiral behaviour near the edges with the opposite edges carrying currents of opposite chirality. This also ensures that the net current is zero in the absence of a electric field. This is what the Classical picture gives; does it also result from QM? Or does QM refute this?

Turns out QM also preserves this chirality of the edge modes. Assuming the current to be along the y-axis and the magnetic field to be along the z-axis, we can model the edges by a potential $V(x)$ which rises steeply at the edges and remains constant in the bulk region. Choosing the Landau Gauge, we now obtain the Hamiltonian for the electron as:

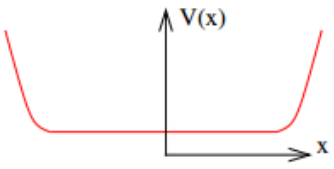

Figure 2.3: Potential Shape (Taken from [2])

$$\mathcal{H} = \frac{1}{2m}\left(p_x^2 + (p_y + eBx)^2\right) + V(x)$$

As we have already seen, in the absence of $V(x)$, the above Hamiltonian leads to a Gaussian wavefunction of width l_B. If the potential is smooth over this distance scale, it is possible for us to taylor expand it about the electron mean position X as $V(x) =$

$V(X) + \frac{\mathrm{d}V}{\mathrm{d}x}\big|_X (x - X) + ...$ Dropping terms of quadratic or higher order and noting that since this is a potential we can always drop the constant terms, we see that only the $\frac{\mathrm{d}V}{\mathrm{d}x}\big|_X x$ term contributes. This suggests a recasting of the Hamiltonian as:

$$\mathcal{H} = \frac{1}{2m}\left(p_x^2 + (p_y + eBx)^2\right) + \left.\frac{\mathrm{d}V}{\mathrm{d}x}\right|_X x$$

But then we can just identify $\frac{\mathrm{d}V}{\mathrm{d}x}\big|_X$ as the electric field divided by electronic charge, by comparing with the Hamiltonian in the Landau Gauge in the presence of Electric field. From the same Hamiltonian's Solution, we see that the electrons now drift along the y-axis with a velocity:

$$v_y(X) = -\frac{1}{eB}\left.\frac{\mathrm{d}V}{\mathrm{d}x}\right|_X$$

In the simple description of the edges we created above, this means that the electrons can move in the y direction only at the edges, for in the bulk $V(x)$ is constant. Also, at the left edge, the potential is decreasing, implying the electrons travel "up" on the left edge. Similarly at the right edge, the potential is increasing, implying the electrons travel "down" on the right edge. We have recovered the chiral edge currents from QM!

One thing to note here is that the above derivation only requires the potential to be steep at the edges to get the chiral edge modes; the specification of the potential in the bulk is not required. This means an arbitrary (well-defined) potential in the bulk can also be employed, without disrupting the edge modes!

This is something interesting. By the above, edge modes are resistant, in fact almost immune, to the perturbations of the potential in the bulk. This is an indication of the robustness of the chiral edge modes. This robustness arises primarily from the chirality of the edge modes themselves. Since at the edge electrons are moving in the same direction, any disorder in the system that tries to scatter these electrons does not succeed, for if these electrons have to travel the opposite way, they have to travel to the other edge, crossing the whole sample. Clearly such a scattering is highly suppressed, making the chiral modes highly robust to any disorder in the system.

At this point, we would like to show some experimental proof for the above said edge modes. This experiment deserves mention, as it is quite unusual and also was helpful in investigating the low-temperature physics of the Quantum Hall effect further. What was done was that electrons were injected on both edges of the material under the above setup, but the electrons injected on one edge (say the left edge) were "warmer" (had a higher temperature) than the other (the right edge). The observation was that the warmer electrons on the left edge were swept upwards on the material and were not dissipated in the bulk, as is to be expected from the above edge mode characterization.[6]

Now, having characterized the robustness of the edge modes, let us now start adding electrons to the above system, by introducing a chemical potential E_F. As we have already shown before, the states *for a given Landau level* are labeled by the momentum along y-axis, but this can well be thought of as the position of the state in the x direction.

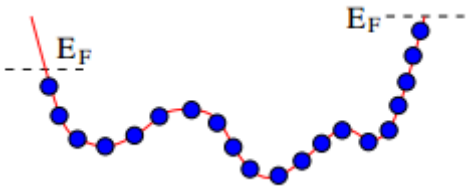

Figure 2.4: States of a Landau Level arranged on a potential $V(x)$ with chemical potential difference at the two edges (Taken from [2])

We are still using the same potential $V(x)$. We are working for now only with a Landau level.

Notice that the bulk of the material now acts as an insulator, while the edges act as metal, since the band gap between the conduction and the valence band is wide in the bulk, but the edge crosses this band gap. This is a general phenomenon in such systems, called the bulk-boundary correspondence. More about this will be covered later; for now, the above info is sufficient.

The Hall voltage will introduce a voltage difference between the two edges. This will in turn introduce a potential energy difference $\Delta\mu$ in between the two edges. This will affect the currents flowing through the edge modes which is what we calculate below.

The current can just be calculated by a summation over the velocities along the y-axis as:

$$I_y = -e \int \frac{\mathrm{d}k}{2\pi} v_y(k)$$

But under the level of approximation we are operating, we can just replace k by $k = \frac{x}{l_B^2}$. This gives:

$$I_y = \frac{e}{2\pi l_B^2} \int \mathrm{d}x \frac{1}{eB} \frac{\mathrm{d}V}{\mathrm{d}x} = \frac{\Delta\mu}{2B\pi l_B^2}$$

Using the definition of $l_B = \sqrt{\frac{\hbar}{eB}}$, we can write $2B\pi l_B^2 = 2B\pi \frac{\hbar}{eB} = 2\pi\frac{\hbar}{e}$, giving

$$I_y = \frac{e}{2\pi\hbar}\Delta\mu$$

But the Hall Voltage relates to $\Delta\mu$ as $eV_H = \Delta\mu$, giving ($V_H = V_x$),

$$I_y = \frac{e^2}{2\pi\hbar} V_x \implies \sigma_{xy} = \frac{e^2}{2\pi\hbar}$$

This is indeed the expected conductivity due to a single Landau level. Note that we never needed any assumptions regarding the shape of the potential in the bulk; only the

difference between the potential $V(x)$ at the edges was required. This leaves us free to put any kind of potential in the bulk. This incidentally quells a possible contradiction of the above, for when the above current is calculated directly from the Hamiltonian, due to the presence of an electric field (due to the Hall Voltage), all the degenerate states of a single Landau level contribute to the current in the Landau level. If we had the restriction that our $V(x)$ can only be constant in the bulk, this would have been trouble, for then in the above calculation, only the edge modes would have contributed to the current. Fortunately, this restriction is not there; we are free to choose the form of the $V(x)$ inside the bulk region. For this case, we might choose the $V(x)$ given by the electric field, but the above calculation is more general. *We are not bound by a choice of $V(x)$ in bulk*. The upshot is that the total current through a Landau level is constant, despite the various forms $V(x)$ might take in the bulk!

For multiple Landau levels, we can play the same game. Since the potential difference for each level is the same, we might as well write the conductivity as:

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar} n$$

where n is the number of filled Landau Levels (i.e. Levels up to the chemical potential E_F). We have recovered the Integer QHE relation from the above approx. using Edge modes. Note that there is a egde mode on each edge for each Landau Level; thus there are n chiral edge modes on each edge.

Note that the fact that the above only depends on the difference in the potential at the edges is highly dependent on two things:

- The edges always act as metals.
- There is no state inside the bulk which can act as a metal, for a given Landau level.

To show that these two hold good, we need some prior mathematical development. Thus, these will be deferred till that development is done.

2.3 Hall Plateaus - Robustness of the Hall State

The above sections established the Landau Level Formalism, and then showed that *if an integer number of Landau levels are filled*, then the observed values of the Hall resistivity in the Integer QHE are obtained. But we have been (as mentioned in the end of a previous section) neglecting one piece of the puzzle: The Hall Plateaux.

In the Integer QHE, the hall resistance is constant for a range of values of the magnetic field. While we have demonstrated how this constant changes, we are yet to show how for range of values of the Magnetic field does the Hall resistivity remain a constant. In fact, our derivation rebels against this. It asks for the Landau levels to be either completely

filled or completely empty, but this is only possible if $B = \frac{n\Phi_0}{\nu}$ for some integer ν [where the number density is n and the quantum of flux is Φ_0].

What happens when $B \neq \frac{n\Phi_0}{\nu}$? Well, we then have on our hands some non-equilibrium distribution of electrons in the Landau level. Clearly we cannot now expect to have the integer constant applying here as well. Yet, that is the whole point of Integer QHE. Where did we go wrong?

The hint lies in the derivation. While discussing the Edge Modes, we made a startling discovery: the form of the potential in the bulk is immaterial to the conductivity relation. This gives us a hint: Integral QHE is pretty robust to *disorder*.

Disorder is the key word, for then we saw disorder only in x direction. Now, we ask how would the states be affected if we now introduce disorder in the y direction?

So, now we have a potential $V(x, y)$ varying (possibly) in both axes. What will this Potential, representing the disorder in the system, do to the Landau State? Since V is arbitrary, it will most likely break the degeneracy between states in a Landau Level. But wee don't want the Landau level themselves to be disrupted. Hence we ask for the disorder to be restricted by:

$$V << \hbar\omega_B$$

Thus, we ask for disorder, but only slight, not in truckloads. As mentioned before, this will then break the degeneracy between the states in a Landau Level, thus implying a "Broadening" of the Landau Level.

What more can the disorder do? Well, it could affect the states themselves. What can this effect this? One is the Broadening of the Landau States that we mentioned; the other relates to the shape of the random potential $V(x, y)$. To investigate this, we take the following approximation:

$$|\nabla V| << \frac{\hbar\omega_B}{l_B}$$

The disorder is of a special kind: It varies much only on the length scales much greater than l_B. Using the symmetric Gauge, we then have the operators for the center of the cyclotron motion of a electron, which is essentially now in a constant potential, as

$$X = x - \frac{\pi_y}{m\omega_B} \quad Y = y - \frac{\pi_x}{m\omega_B}$$

where $\boldsymbol{\pi} = \mathbf{p} + e\mathbf{A}$ is the gauge invariant mechanical momentum. Then by the Heisenberg picture, the time evolution of the above operators is (given the Hamiltonian without the disorder as H):

$$i\hbar\dot{X} = [X, H + V] = [X, V] = [X, Y]\frac{\partial V}{\partial Y} = il_B^2\frac{\partial V}{\partial Y}$$

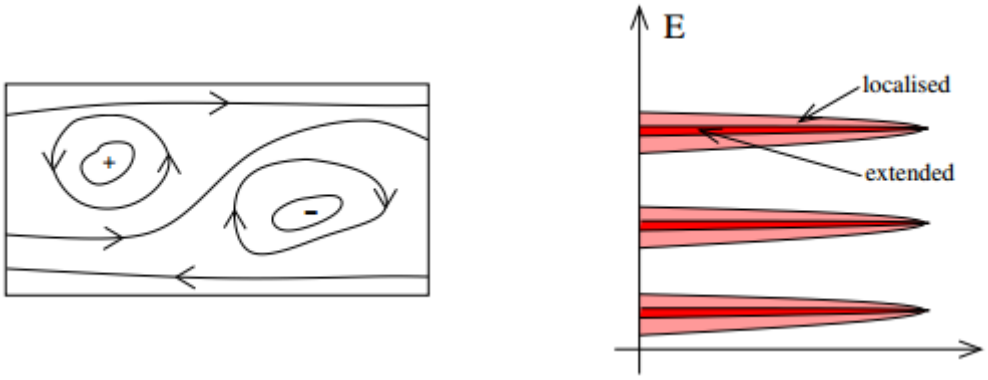

Figure 2.5: Left: Possible type of Potential $V(x)$. Note the Localized states near the peak/trough. Right: Broadening of Landau Levels, with the Localized and Extended states marked. (Taken from [2])

$$i\hbar\dot{Y} = [Y, H+V] = [Y, V] = [Y, X]\frac{\partial V}{\partial X} = -il_B^2\frac{\partial V}{\partial X}$$

Thus the electron drifts in the direction perpendicular to ∇V, that is, it travels on equipotentials. But this means that there are now two kinds of states for each Landau Levels: "Extended" and "Localized". Extended states, as the name suggests, extend from one end of the sample to the other; while the localized states are confined to a certain area. This results from the shape of the potential on the sample: Near peaks and troughs we get localized states, and near saddle points we get extended states.

Conductivity cares only about extended states; currents in localized states do not contribute to conductivity, simply because they do not transport any charge through the sample. The fact that Localized states only occur near the peaks/troughs of the potential $V(x)$ implies that the localized states exist at the ends of the broadened Landau levels and the extended states in the middle.

Above configuration implies that extended states are enough to account for the conductivity seen in the Integer QHE. Now, suppose that we are exactly at the Landau Level energy (i.e. $B = \frac{n\Phi_0}{\nu}$) and now we start to decrease B for fixed n. This means that each Landau level accommodates lesser electrons; thus Fermi Energy increases. But, now we start filling up the Localized states, which means that the conductivity does not change, as the extended states are untouched. This explains plateaux seen in Integer QHE.

But we hit another roadblock. We mentioned that each state in a Landau level contributes current in Integer QHE, under the pure sample condition. Current is same in all the states, which is evident from the potential's form $V(x) = -e\frac{V_H}{L_x}x$ (due to the Hall Voltage). Since $v_y = \frac{1}{eB}\frac{\partial V}{\partial x} = -\frac{V_H}{BL_x}$, each state contributes ev_y current and thus same current.

For disordered sample, this means that since localized states do not carry their share of current, we need extended states to carry extra current to compensate for that, so that conductivity values remain same. Can we show this? Yes but for sack of brevity (or whatever of that remains for this report), we refer the reader to [7] for the above.

Chapter 3

Topology and Integer QHE

3.1 Kubo's Formula

We have now explained the Integer QHE and how it emerges. But now we go furthur; we derive a formula for the Hall conductivity, useful in many quantum systems with similar properties. This formula is the famous Kubo's Formula, and here we are actually just using it for our purpose; the actual Kubo's formula is much more general. Anyway, we get on with its derivation.

We have a multi-particle Hamiltonian H_0, which describes the system perfectly in the absence of an electric field. This hamiltonian can be anything; the non-interacting single particle Hamiltonian we saw earlier, or some other Hamiltonian including the effects of the interactions between the particles. We take its eigenstates as $|\psi_m\rangle$ with $H_0 |\psi_m\rangle = E_m |\psi_m\rangle$, with the generalized index m.

This Hamiltonian is then perturbed by an electric field, giving rise to the perturbative term ΔH. Choosing the electric potential to be zero, the electric field is given by $\mathbf{E} = -\partial_t \mathbf{A}$. The new Hamiltonian is $H = H_0 + \Delta H$, where

$$\Delta H = -\mathbf{J} \cdot \mathbf{A}$$

Here, $\mathbf{J}$ is the quantum operator for electric current density.

We now perform certain "magic" tricks. First, despite the knowledge that we will apply a constant DC electric field, we assume for now an AC electric field and then take its frequency $\omega \to 0$, to make it a DC electric field. Second, we use complex forms; since we intend only to go to first order, the real part is easily derivable. Then $\mathbf{E}(t) = \mathbf{E}\mathrm{e}^{-i\omega t}$ and $\mathbf{A}(t) = \frac{\mathbf{E}}{i\omega}\mathrm{e}^{-i\omega t}$. Thus,

$$\Delta H = i\frac{\mathbf{J} \cdot \mathbf{E}}{\omega}\mathrm{e}^{-i\omega t}$$

Assuming that this perturbation is small, we intend to calculate $\langle \mathbf{J} \rangle$. We work in the

interaction picture; thus the operators and states evolve as:

$$\mathcal{O}(t) = \mathrm{e}^{\frac{i\Delta Ht}{\hbar}}\mathcal{O}\mathrm{e}^{-\frac{i\Delta Ht}{\hbar}} \qquad |\psi(t)\rangle_I = \exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}\Delta H(t')\mathrm{d}t'\right)|\psi(t_0)\rangle_I$$

Now, we prepare the system in a specific state $|\psi_0\rangle$ at $t \to -\infty$. Then at time t, the expectation value of the current density is given by (taking $U(t) = U(t, t_0 \to -\infty)$:

$$\begin{aligned}
\langle \mathbf{J}(t)\rangle &= \langle\psi_0(t)|\,\mathbf{J}(t)\,|\psi_0(t)\rangle \\
&= \langle\psi_0|\,U^{-1}(t)\mathbf{J}(t)U(t)\,|\psi_0\rangle \\
&\approx \langle\psi_0|\left(\mathbf{J}(t) + \frac{i}{\hbar}\int_{-\infty}^{t}\mathrm{d}t'\,[\Delta H(t'), \mathbf{J}(t)]\right)|\psi_0\rangle \\
&= \langle\psi_0|\left(\mathbf{J}(t) + \frac{1}{\hbar\omega}\int_{-\infty}^{t}\mathrm{d}t'\,[\mathbf{J}(t), \mathbf{J}(t')\cdot\mathbf{E}]\mathrm{e}^{-i\omega t'}\right)|\psi_0\rangle
\end{aligned}$$

where we have expanded the unitary operator U and kept only leading terms. The above is called the Kubo's Formula. We now just redecorate and rearrange it here so that it is more useful to us in the Hall effect context.

The first term is the current in the absence of the electric field, which we take here to be zero. Then the above can be written as:

$$\langle J_i(t)\rangle = \sum_j \frac{1}{\hbar\omega}\int_{-\infty}^{t}\mathrm{d}t'\ \langle\psi_0|\,[J_i(t), J_j(t')]\,|\psi_0\rangle\, E_j\mathrm{e}^{-i\omega t'}$$

Since the system is invariant under time-translation (meaning that the choice of zero point of time is completely arbitrary), the above quantity must depend only on the time difference $t'' = t - t'$. Using this, we can then change to the variable t'', obtaining:

$$\langle J_i(t)\rangle = \sum_j \frac{1}{\hbar\omega}\left(\int_0^{\infty}\mathrm{d}t''\,\mathrm{e}^{i\omega t''}\ \langle\psi_0|\,[J_i(t''), J_j(0)]\,|\psi_0\rangle\right)E_j\mathrm{e}^{-i\omega t}$$

The t dependence lies outside the integral, implying a electric field of frequency ω induces a current of frequency ω, essentially indicating a linear response. This is what the Kubo's formula really tries to get at.

Note that we have related two vectors in the above equation, which means that we can define a matrix relating $\langle J_i(t)\rangle$ and E_j from the above. Clearly the above matrix is the frequency dependent conductivity matrix:

$$\sigma_{ij} = \frac{1}{\hbar\omega}\int_0^{\infty}\mathrm{d}t\,\mathrm{e}^{i\omega t}\ \langle\psi_0|\,[J_i(t), J_j(0)]\,|\psi_0\rangle$$

Then in the previous coordinate systems, the Hall conductivity is given by:

$$\sigma_{xy} = \frac{1}{\hbar\omega}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{i\omega t}\ \langle\psi_0|\,[J_x(t), J_y(0)]\,|\psi_0\rangle$$

Using the time evolution of the operators in the interaction picture, we can rewrite the above as (we denote values at $t = 0$ by suppressing the (0) bit):

$$\begin{aligned}
\sigma_{xy} &= \frac{1}{\hbar\omega}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{i\omega t}\ \langle\psi_0|\,[J_x(t), J_y]\,|\psi_0\rangle \\
&= \frac{1}{\hbar\omega}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{i\omega t}\ \left[\,\langle\psi_0|\,J_x(t)J_y\,|\psi_0\rangle - \langle\psi_0|\,J_yJ_x(t)\,|\psi_0\rangle\,\right] \\
&= \frac{1}{\hbar\omega}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{i\omega t}\ \left[\langle\psi_0|\,\mathrm{e}^{i\frac{H_0t}{\hbar}}J_x\mathrm{e}^{-i\frac{H_0t}{\hbar}}J_y\,|\psi_0\rangle - \langle\psi_0|\,J_y\mathrm{e}^{i\frac{H_0t}{\hbar}}J_x\mathrm{e}^{-i\frac{H_0t}{\hbar}}\,|\psi_0\rangle\right] \\
&= \frac{1}{\hbar\omega}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{i\omega t}\ \sum_n\left[\langle\psi_0|\,\mathrm{e}^{i\frac{H_0t}{\hbar}}J_x\,|\psi_n\rangle\,\langle\psi_n|\,\mathrm{e}^{-i\frac{H_0t}{\hbar}}J_y\,|\psi_0\rangle - \langle\psi_0|\,J_y\mathrm{e}^{i\frac{H_0t}{\hbar}}\,|\psi_n\rangle\,\langle\psi_n|\,J_x\mathrm{e}^{-i\frac{H_0t}{\hbar}}\,|\psi_0\rangle\right] \\
&= \frac{1}{\hbar\omega}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{i\omega t}\ \sum_n\left[\langle\psi_0|\,J_x\,|\psi_n\rangle\,\langle\psi_n|\,J_y\,|\psi_0\rangle\,\mathrm{e}^{i\frac{E_0-E_n}{\hbar}t} - \langle\psi_0|\,J_y\,|\psi_n\rangle\,\langle\psi_n|\,J_x\,|\psi_0\rangle\,\mathrm{e}^{i\frac{E_n-E_0}{\hbar}t}\right] \\
&= -\frac{1}{\hbar\omega}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{i\omega t}\ \sum_n\left[\langle\psi_0|\,J_y\,|\psi_n\rangle\,\langle\psi_n|\,J_x\,|\psi_0\rangle\,\mathrm{e}^{i\frac{E_n-E_0}{\hbar}t} - \langle\psi_0|\,J_x\,|\psi_n\rangle\,\langle\psi_n|\,J_y\,|\psi_0\rangle\,\mathrm{e}^{i\frac{E_0-E_n}{\hbar}t}\right]
\end{aligned}$$

The above is clearly a troublesome integral, due to the term $\mathrm{e}^{i\omega t}$ in the integral. But we can side-step this by using another standard "magic" trick: replace ω with $\omega + i\epsilon$, and then let ϵ go to zero. Doing this, we get the integral to be:

$$\begin{aligned}
\int_0^\infty \mathrm{d}t\, \mathrm{e}^{i\omega t}\mathrm{e}^{i\frac{E_n-E_0}{\hbar}t} \xrightarrow[\omega\to\omega+i\epsilon]{} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{i\omega t-\epsilon t}\mathrm{e}^{i\frac{E_n-E_0}{\hbar}t} &= \frac{i\hbar}{\omega\hbar + i\epsilon\hbar + E_n - E_0} \\
\overset{\epsilon\to0}{\Longrightarrow} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{i\omega t}\mathrm{e}^{i\frac{E_n-E_0}{\hbar}t} &= \frac{i\hbar}{\omega\hbar + E_n - E_0}
\end{aligned}$$

$$\begin{aligned}
\text{Similarly, } \int_0^\infty \mathrm{d}t\, \mathrm{e}^{i\omega t}\mathrm{e}^{i\frac{E_0-E_n}{\hbar}t} \xrightarrow[\omega\to\omega+i\epsilon]{} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{i\omega t-\epsilon t}\mathrm{e}^{i\frac{E_0-E_n}{\hbar}t} &= \frac{i\hbar}{\omega\hbar + i\epsilon\hbar + E_0 - E_n} \\
\overset{\epsilon\to0}{\Longrightarrow} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{i\omega t}\mathrm{e}^{i\frac{E_0-E_n}{\hbar}t} &= \frac{i\hbar}{\omega\hbar + E_0 - E_n}
\end{aligned}$$

Also, note that in the original integral, we see that putting $n = 0$ gives $\langle\psi_0|\,J_y\,|\psi_0\rangle\,\langle\psi_0|\,J_x\,|\psi_0\rangle - \langle\psi_0|\,J_x\,|\psi_0\rangle\,\langle\psi_0|\,J_y\,|\psi_0\rangle = 0$. Thus $n = 0$ does not contribute to the integral. Then

$$\sigma_{xy} = -\frac{i}{\omega}\sum_{n\neq0}\left[\frac{\langle\psi_0|\,J_y\,|\psi_n\rangle\,\langle\psi_n|\,J_x\,|\psi_0\rangle}{\hbar\omega + E_n - E_0} - \frac{\langle\psi_0|\,J_x\,|\psi_n\rangle\,\langle\psi_n|\,J_y\,|\psi_0\rangle}{\hbar\omega + E_0 - E_n}\right]$$

Finally, we apply the promised $\omega \to 0$ limit to the above. Doing so requires expanding the following:

$$\frac{1}{\hbar\omega + E_n - E_0} \approx \frac{1}{E_n - E_0} - \frac{\hbar\omega}{(E_n - E_0)^2} + \mathcal{O}(\omega^2)$$

Then we get for σ_{xy},

$$\begin{aligned}\sigma_{xy} = -2\sum_{n\neq 0} \frac{i}{\omega} \frac{\langle\psi_0| J_y |\psi_n\rangle \langle\psi_n| J_x |\psi_0\rangle}{E_n - E_0} \\ + i\hbar \sum_{n\neq 0} \frac{\langle\psi_0| J_y |\psi_n\rangle \langle\psi_n| J_x |\psi_0\rangle - \langle\psi_0| J_x |\psi_n\rangle \langle\psi_n| J_y |\psi_0\rangle}{(E_n - E_0)^2} + \mathcal{O}(\omega)\end{aligned}$$

Note that the first term can be eliminated by gauge invariance. For this, we need to just add a term $-\frac{\mathbf{E}}{\hbar\omega}$ to the magnetic potential. This gives a term opposite to the above in the conductivity formula by a similar calculation as done above, which then eliminates the first term. To do all this, we just need to have $-\frac{\mathbf{E}}{\hbar\omega}$ as a derivative of a scalar function; that is, we must have $\nabla \times -\frac{\mathbf{E}}{\hbar\omega} = 0$. But this means $\nabla \times \mathbf{E} = 0 \implies \partial_t \nabla \times \mathbf{A} = 0 \implies \partial_t \mathbf{B} = 0$, which is true, since the magnetic field is constant in time. This validates our argument. Then we can finally write the Hall conductivity as:

$$\sigma_{xy} = i\hbar \sum_{n\neq 0} \frac{\langle\psi_0| J_y |\psi_n\rangle \langle\psi_n| J_x |\psi_0\rangle - \langle\psi_0| J_x |\psi_n\rangle \langle\psi_n| J_y |\psi_0\rangle}{(E_n - E_0)^2}$$

A Little Note on the units

On verifying the units of the above calculation, we get absurd results. The problem lies in the interpretation of the current density $\mathbf{J}$. While for a 3-D system, this is current per unit area and thus has dimensions of $\frac{\mathrm{A}}{\mathrm{m}^2}$, in the Hall effect case, we have a 2-D system on our hands. This means that the current density referred to here is the current surface density, with units $\frac{\mathrm{A}}{\mathrm{m}}$. This means that the perturbation term ΔH is actually then

$$\Delta H = -a\mathbf{J} \cdot \mathbf{A}$$

where a is the area of the sample. This yields the following formula for the Hall Conductivity:

$$\sigma_{xy} = ia\hbar \sum_{n\neq 0} \frac{\langle\psi_0| J_y |\psi_n\rangle \langle\psi_n| J_x |\psi_0\rangle - \langle\psi_0| J_x |\psi_n\rangle \langle\psi_n| J_y |\psi_0\rangle}{(E_n - E_0)^2}$$

3.2 The topological origin of Integer QHE

Lets review what we have done till now. We started with the classical Hall effect, discovered its anomalous behaviour in the low temperature case, leading to the Integer Quantum Hall Effect. We then discussed the Quantum formulation of the problem, talking about the Landau Levels, the emergence of edge modes, and the role of disorder in the emergence of the Hall plateaux. We then went ahead and derived a relation for the Hall conductivity for a generalized quantum system using the Kubo's Formula.

Here we explore a different theme. Instead of computing stuff, we show how topology is related to the Hall conductivity, and then show how the seemingly distinct properties we discussed are in fact very much related. We considered a rectangular setup before; we now consider a torus instead. Topologically, a torus is homeomorphic to a rectangle, for a torus is just a rectangle with opposite sides identified. The lengths of these sides will be taken to be L_x and L_y.

We thread a uniform magnetic field B through the torus. Now, the geometry of a torus gives a quantization condition on the magnetic field. This can be found easily by considering the behaviour of the wavefunction over a torus.

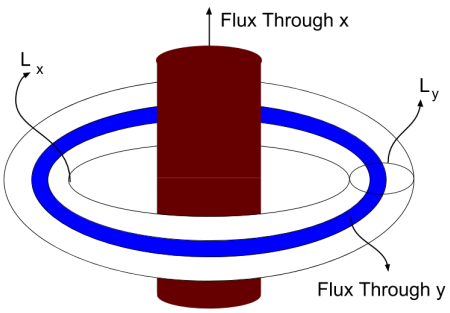

Figure 3.1: Geometry of the system

Given a wavefunction $\psi(x, y)$, what are the periodic boundary conditions that we must put on the wavefunction? We can find a minimal one by asking how does the wavefunction behave as one goes around the two independent circular paths possible on a torus. Let us introduce the translation operators $T(\mathbf{d})$ to quantify this ($T(\mathbf{d})$ translates ψ by $\mathbf{d}$):

$$T(\mathbf{d}) = \mathrm{e}^{-\mathbf{d}\cdot\mathbf{p}/\hbar} = \mathrm{e}^{-\mathbf{d}\cdot(i\nabla + e\mathbf{A}/\hbar)}$$

The appropriate Boundary conditions are then simply $T_x\psi(x, y) = \psi(x, y)$ and $T_y\psi(x, y) = \psi(x, y)$, where $T_x = T(\mathbf{d} = (L_x, 0))$ & $T_y = T(\mathbf{d} = (0, L_y))$. Note that the operators are not gauge invariant. We choose the Landau gauge to work with. Then $T_x = \mathrm{e}^{-iL_x\frac{\partial}{\partial x}}$ and $T_y = \mathrm{e}^{-iL_y\frac{\partial}{\partial y} - ieBL_y x/\hbar}$.

Note the form of the above operators. $\mathrm{e}^{-iL_x\frac{\partial}{\partial x}} = \mathrm{e}^{-ip_xL_x/\hbar}$ is just the translation operator in the usual rectangular space. Thus, the boundary conditions give:

$$T_x\psi(x,y) = \psi(x+L_x,y) = \psi(x,y)$$
$$T_y\psi(x,y) = \mathrm{e}^{-ieBL_yx/\hbar}\psi(x,y+L_y) = \psi(x,y)$$

Now, we apply the two operators in succession. This gives us:

$$T_yT_x = \mathrm{e}^{-ieBL_xL_y/\hbar}T_xT_y$$

To have the two sides match,

$$\frac{eBL_xL_y}{\hbar}\,\epsilon\,2\pi\mathbb{Z}$$

This is called the Dirac Quantization condition.

Now, we are ready to work with the system. Given a torus, we can now thread two fluxes in it, Φ_x and Φ_y. Introducing such small fluxs amounts to the following changes in the gauge potential:

$$A_x = \frac{\Phi_x}{L_x} \quad A_y = \frac{\Phi_y}{L_y} + Bx$$

We have discussed a similar geometry in the Aharonov - Bohm Effect [See Appendix for details]. There, we saw that the spectrum of the system is only sensitive to the fractional part of $\frac{\Phi_x}{\Phi_0}$, where $\Phi_0 = \frac{2\pi\hbar}{e}$, and that the system can undergo spectral flow.

Now, the above change in the potential causes a perturbation in the Hamiltonian:

$$\Delta H = -\sum_{i=x,y}\frac{J_i\Phi_i}{L_i}$$

Let us see what this affects the ground state $|\psi_0\rangle$. Assuming the ground state to be non-degenerate, to first order, we get the new state as:

$$|\psi_0'\rangle = |\psi_0\rangle + \sum_{n\neq 0}|\psi_n\rangle\frac{\langle\psi_n|\,\Delta H\,|\psi_0\rangle}{E_n-E_0} = |\psi_0\rangle - \sum_{i=x,y}\frac{\Phi_i}{L_i}\sum_{n\neq 0}|\psi_n\rangle\frac{\langle\psi_n|\,J_i\,|\psi_0\rangle}{E_n-E_0}$$

We can write this as (considering only infinitesimal changes in Φ_i):

$$\left|\frac{\partial\psi_0}{\partial\Phi_i}\right\rangle = -\frac{1}{L_i}\sum_{n\neq 0}|\psi_n\rangle\frac{\langle\psi_n|\,J_i\,|\psi_0\rangle}{E_n-E_0}$$

Note that this seems very similar to the terms in the Kubo's Formula. Using the above to write the Kubo's Formula:

$$\left\langle \frac{\partial\psi_0}{\partial\Phi_y}\middle|\frac{\partial\psi_0}{\partial\Phi_x}\right\rangle = \frac{1}{L_xL_y}\sum_{n\neq0}\sum_{m\neq0}\frac{\langle\psi_0|\,J_i\,|\psi_n\rangle}{E_n-E_0}\langle\psi_n|\psi_m\rangle\frac{\langle\psi_m|\,J_i\,|\psi_0\rangle}{E_m-E_0} = \frac{1}{L_xL_y}\sum_{n\neq0}\frac{\langle\psi_0|\,J_i\,|\psi_n\rangle\langle\psi_n|\,J_i\,|\psi_0\rangle}{(E_n-E_0)^2}$$

The area of the sample is L_xL_y, giving:

$$\begin{aligned}\sigma_{xy} &= i\hbar L_xL_y\sum_{n\neq0}\frac{\langle\psi_0|\,J_y\,|\psi_n\rangle\langle\psi_n|\,J_x\,|\psi_0\rangle-\langle\psi_0|\,J_x\,|\psi_n\rangle\langle\psi_n|\,J_y\,|\psi_0\rangle}{(E_n-E_0)^2}\\ &= i\hbar\left[\left\langle\frac{\partial\psi_0}{\partial\Phi_y}\middle|\frac{\partial\psi_0}{\partial\Phi_x}\right\rangle-\left\langle\frac{\partial\psi_0}{\partial\Phi_x}\middle|\frac{\partial\psi_0}{\partial\Phi_y}\right\rangle\right]\\ &= i\hbar\left[\frac{\partial}{\partial\Phi_y}\left\langle\psi_0\middle|\frac{\partial\psi_0}{\partial\Phi_x}\right\rangle-\frac{\partial}{\partial\Phi_x}\left\langle\psi_0\middle|\frac{\partial\psi_0}{\partial\Phi_y}\right\rangle\right]\end{aligned}$$

This way of writing the conductivity gives us an insight; that the conductivity depends on how the states behave when the system is taken on a tour in the parameter space of (Φ_x,Φ_y). This is very reminiscent of some concept we discussed before: The Berry Phase. With this, we finally lay bare the topological aspect of the Integer QHE.

Quantization as a result of Topology

As we discussed before, the spectrum of the quantum system is only sensitive to the fractional part of $\frac{\Phi_i}{\Phi_0}$, that is, on Φ_i mod Φ_0. This means we can think of the parameter space of the pertubed Hamiltonian as being periodic: (Φ_x,Φ_y) define a torus by the above restriction. We introduce the variables θ_i to parameterise this torus:

$$\theta_i = \frac{2\pi\Phi_i}{\Phi_0}$$

Now, we find the Berry connection $\mathcal{A}(\mathbf{\Phi})$. We get:

$$\mathcal{A}_i(\mathbf{\Phi}) = -i\,\langle\phi_0|\,\frac{\partial}{\partial\theta_i}\,|\phi_0\rangle$$

The curvature of the Berry Connection is then:

$$\begin{aligned}\mathcal{F}_{xy} &= \frac{\partial \mathcal{A}_x}{\partial \theta_y} - \frac{\partial \mathcal{A}_y}{\partial \theta_x} \\ &= -i\left[\frac{\partial}{\partial \theta_y}\left\langle \phi_0 \middle| \frac{\partial \phi_0}{\partial \theta_x}\right\rangle - \frac{\partial}{\partial \theta_x}\left\langle \phi_0 \middle| \frac{\partial \phi_0}{\partial \theta_y}\right\rangle\right] \\ &= -\frac{i\Phi_0^2}{4\pi^2}\left[\frac{\partial}{\partial \Phi_y}\left\langle \phi_0 \middle| \frac{\partial \phi_0}{\partial \Phi_x}\right\rangle - \frac{\partial}{\partial \Phi_x}\left\langle \phi_0 \middle| \frac{\partial \phi_0}{\partial \Phi_y}\right\rangle\right] \\ &= \frac{\hbar^2}{ie^2}\left[\frac{\partial}{\partial \Phi_y}\left\langle \phi_0 \middle| \frac{\partial \phi_0}{\partial \Phi_x}\right\rangle - \frac{\partial}{\partial \Phi_x}\left\langle \phi_0 \middle| \frac{\partial \phi_0}{\partial \Phi_y}\right\rangle\right]\end{aligned}$$

But this expression was encountered as the formula for Hall conductivity. Using that expression:

$$\sigma_{xy} = -\frac{e^2}{\hbar}\mathcal{F}_{xy}$$

A Nice formula relating the Berry Phase with Hall Conductivity! But, so far, we haven't explained (using this) as to how the Hall conductivity is quantized.

Suppose we now average over all the fluxes. This yields the following:

$$\sigma_{xy} = -\frac{e^2}{\hbar}\int \frac{\mathrm{d}^2\theta}{(2\pi)^2}\,\mathcal{F}_{xy}$$

But we have met such integrals in the Berry Phase formalism. There, we have discussed that for the general case, on taking the surface integral of the curvature throughout the space, we obtain an integer called the Chern Number:

$$C = \frac{1}{2\pi}\int_S \mathcal{F}_{ij}\mathrm{d}S^{ij}, \quad C\,\epsilon\,\mathbb{Z}$$

This means that the above integral converts to a single integer, giving:

$$\sigma_{xy} = -\frac{e^2}{2\pi\hbar}C$$

Exactly the Integer QHE relation we were looking for! This finally relates the integral quantization of Hall Effect to the topological considerations of the parameter space, showing that the phenomenon of Integer QHE is essentially topological and fairly independent of the material properties of the sample.

General Remarks: Topology and Integer QHE

We now make some general remarks regarding the stability of the Hall Effect. The above Chern Number cannot vary if the Hamiltonian varies smoothly (as we did by introducing

a small potential $V(x)$ in the previous sections). It is because it is a topological invariant, and thus is immune to smooth perturbations in the parameter space of the Hamiltonian. This explains the robustness of the Integer QHE, leading to the Hall Plateaux.

We also mentioned two conditions regarding our treatment of the edge modes earlier. These conditions were:

- The edges always act as metals.
- There is no state inside the bulk which can act as a metal, for a given Landau level.

To show the above, we again take the help of the topological invariance of the Chern Number. Note that the Chern number characterizes the Hall conductivity and therefore can be taken as characterizing a quantum Hall state. If the chern number is zero, there is essentially no QHE and the material is then a trivial insulator, such as vacuum.

Now, consider what happens at the edge. The sample is moving from a non-zero chern number state to a zero chern number state when it crosses the edge (from inside the sample to outside in the vacuum). Since any continuous perturbative effects cannot cause level crossing for a quantum system (a result from Perturbation Theory) and since the Chern number is invariant under smooth perturbations of the Hamiltonian, we see that on crossing the edge, the change in the Chern number must necessarily cause a level crossing at the edge. This means that at the edge, the material's Conduction and Valence Band are connected; essentially then the edge modes are metallic. Incidentally, this also shows that any other internal state cannot be metallic, for then the Chern number must change inside the sample. [8]

This means that the edge modes are essentially like express highways; they always run in one direction (chirality) and are immune to any changes in the hinterlands (inside the sample), while at the same time they allow uninterrupted conduction of electrons due to their metallic character.

Finally, we have been able to unify what had been a unconnected set of observations and explanations. This unification has been brought about by the deep topological origins of the Integer Quantum Hall Effect, and help us see what the topological properties of the parameter space can mean in the real space.

Appendix A

Berry Phase

The Discussion here follows both [2] and [9].

A.1 Origin of Berry Phase

All Quantum States are Rays in Hilbert space, that is,

$$|\psi_1\rangle \sim |\psi_2\rangle \iff |\psi_1\rangle = \mathrm{e}^{i\alpha} |\psi_2\rangle , \quad \alpha \epsilon \mathfrak{R} \tag{A.1}$$

Thus, there is always an ambiguity of phase (Which cannot be observed) in each state. This ambiguity is the origin of Berry Phase.

Consider the Hamiltonian H depending on parameters $\mathbf{q} = (q_1, q_2, q_3, ...)$. Let us consider a continuous circuit in the parameter space parameterized by t such that $\mathbf{q}(0) = \mathbf{q}(T)$. The state evolves as per the Schrödinger equation :

$$H(\mathbf{q}(t)) |\psi(t)\rangle = i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle$$

Let the natural basis of the system be denoted by $n(\mathbf{q}(t))$, each with energy $E_n(\mathbf{q}(t))$. Then

$$H(\mathbf{q}(t)) |n(\mathbf{q}(t))\rangle = E_n(\mathbf{q}(t)) |n(\mathbf{q}(t))\rangle$$

Note that above equation does not imply any relation between the phases of the eigenstates at different parameter values $\mathbf{q}(t)$. We can make any choice, provided the choice leads to a single valued phase at each point in C.

The system was prepared in the eigenstate $n(\mathbf{q}(0))$ at $t = 0$. Then, by the adiabatic approximation, it will go to $n(\mathbf{q}(t))$ at t. Then we can write the state of the system as $|\psi\rangle$ as:

$$|\psi(t)\rangle = \exp\left[-\frac{i}{\hbar}\int_0^t \mathrm{d}t' E_n(\mathbf{q}(t))\right] \exp(i\gamma_n(t))\,|n(\mathbf{q}(t))\rangle$$

The first exponential is the familiar dynamical phase. It is the second exponential that is of interest. The phase $\gamma_n(t)$ cannot be written as a function of $\mathbf{q}(t)$ and is also not single-valued, i.e. $\gamma_n(0) \neq \gamma_n(T)$. This difference in the phase on traversing a circuit in the parameter space is the berry phase.

To substantiate the above claims, we require that the above form of $|\psi\rangle$ solve the Schrödinger equation. Doing so leads to the following for $\gamma_n(t)$:

$$\begin{aligned}
H(\mathbf{q}(t))\,|\psi(t)\rangle &= i\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle \\
\implies E_n\,|\psi(t)\rangle &= E_n\,|\psi(t)\rangle + \\
&\qquad \exp\left[-\frac{i}{\hbar}\int_0^t \mathrm{d}t' E_n(\mathbf{q}(t))\right] \exp(i\gamma_n(t)) \left(i\frac{\mathrm{d}}{\mathrm{d}t}\gamma_n\,|n(\mathbf{q}(t))\rangle + |\nabla_{\mathbf{q}} n(\mathbf{q}(t))\rangle \cdot \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{q}(t)\right) \\
\implies \frac{\mathrm{d}}{\mathrm{d}t}\gamma_n &= i\,\langle n(\mathbf{q}(t))|\nabla_{\mathbf{q}} n(\mathbf{q}(t))\rangle \cdot \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{q}(t)
\end{aligned}$$

Then the state after one round trip on C is:

$$|\psi(t)\rangle = |\psi(0)\rangle \exp\left\{-\frac{i}{\hbar}\int_0^T \mathrm{d}t' E_n(\mathbf{q}(t))\right\} \exp(i\gamma_n(T))$$

where,

$$\gamma_n(T) = i\oint_C \langle n(\mathbf{q}(t))|\nabla_{\mathbf{q}} n(\mathbf{q}(t))\rangle \cdot \mathrm{d}\mathbf{q}(t)$$

This is the Berry Phase; the *non dynamical* phase acquired by the state on traversing a circuit in the parameter space. This phase does not depend on how the circuit was traversed (provided that it was slow enough to guarantee adiabatic approximation), and neither does it depend on any dynamical quantity - it is purely due to geometric reasons. Normalization of n ensures that the berry phase is real.

Note that the above form requires a derivative of the eigenstate, thus requiring a locally single-valued basis for the system. We can get around this by using the Stokes' theorem (in 3D) or its higher dimensional analog. For now, we deal with a three dimensional parameter space. Then by application of Stokes' theorem, we get:

$$\begin{aligned}
|\gamma_n(T)\rangle &= -\mathrm{Im}\int\int_C \mathrm{d}\mathbf{S}\cdot\nabla\times\langle n|\nabla n\rangle = -\mathrm{Im}\int\int_C \mathrm{d}\mathbf{S}\cdot\langle\nabla n|\times|\nabla n\rangle \\
\implies |\gamma(T)\rangle &= -\mathrm{Im}\int\int_C \mathrm{d}\mathbf{S}\cdot\sum_{m\neq n}\langle\nabla n|m\rangle\times\langle m|\nabla n\rangle
\end{aligned}$$

The off-diagonal terms are $\langle m|\nabla n\rangle = \frac{\langle m|\nabla H|n\rangle}{E_n - E_m}, \quad m \neq n$. Then we can write the above as:

$$|\gamma_n(T)\rangle = -\int\int_C \mathrm{d}\mathbf{S}\cdot\mathbf{V}_n(\mathbf{q})$$

$$\mathbf{V}_n(\mathbf{q}) = \mathrm{Im}\sum_{m\neq n}\frac{\langle n(\mathbf{q})|\,\nabla_{(\mathbf{q})}H\,|m(\mathbf{q})\rangle \times \langle m(\mathbf{q})|\,\nabla_{(\mathbf{q})}H\,|n(\mathbf{q})\rangle}{(E_m(\mathbf{q}) - E_n(\mathbf{q}))^2}$$

This form indicates that the phase relations between the eigenstates are now immaterial, and that any basis can be used, irrespective of its local properties.

The discussion from here can be extended to the case of higher dimensions using the theory of differential forms, or could explore the case of degeneracies. We however do not go further in either direction; rather we discuss some interesting properties of a quantity called Berry Connection, which will come useful later, and then move on to work on a toy model for berry phase related calculations. We end this discussion with the Aharonov - Bohm effect.

A.2 Properties of the Berry connection

The Berry Connection can be defined as the Matrix element of the del operator in $\mathbf{q}$ space:

$$\mathcal{A} = -i\,\langle n(\mathbf{q}(t))|\nabla_{\mathbf{q}} n(\mathbf{q}(t))\rangle \implies \gamma_n(T) = -\oint_C \mathcal{A}\cdot\mathrm{d}\mathbf{q}(t)$$

Now, we discussed that the choice of the phases of the eigenstates is upto us. What does this imply for the Berry Connection? To explore this, we choose another basis $|n'(\mathbf{q}(t))\rangle$, which is related by the phase (any scalar function) $\omega(\mathbf{q}(t))$ as:

$$|n'(\mathbf{q}(t))\rangle = \mathrm{e}^{i\omega(\mathbf{q}(t))}\,|n(\mathbf{q}(t))\rangle$$

Then by the definition of the Berry Connection,

$$\mathcal{A}' = -i\,\langle n'(\mathbf{q}(t))|\nabla_{\mathbf{q}} n'(\mathbf{q}(t))\rangle = \mathcal{A} + \nabla_{\mathbf{q}}\omega(\mathbf{q})$$

This seems more like the Gauge Transformation in Electromagnetic Potential. Let us explore this analogy a bit furthur. The electromagnetic potential A_μ gave rise to a Field Strength Tensor $F_{\mu\nu}$, given as

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$$

We intend to introduce an analogous tensor for Berry connection. The fact that both these tensors will remain invariant under Gauge Transformation means that these tensors tell us something about the physics of the system. In particular the Berry Phase tensor will tell us about how the system behaves under changes in its parameters.

So, on with it. Using the analogy, we define this tensor for the Berry phase, which we call the curvature of the connection, as:

$$\mathcal{F}_{ij}(\mathbf{q}) = \frac{\partial \mathcal{A}_i}{\partial \mathrm{q}_j} - \frac{\partial \mathcal{A}_j}{\partial \mathrm{q}_i}$$

As we noted, the above is Gauge invariant. But it is not the only Gauge invariant quantity in town; the Berry phase itself is Gauge invariant (it has to be, for we can observe it). This is due to the definition of the berry phase in terms of the Berry Connection and the fact that for any scalar function $\oint_C \nabla_{\mathbf{q}} \cdot \mathrm{d}\mathbf{q}(t) = 0$. Morever, it can be connected to the curvature of the connection by the higher dimensional version of the Stokes Law (mentioned before) as:

$$\gamma = -\int_S \mathcal{F}_{ij} \mathrm{dS}^{ij}$$

Note that this means the two formalisms we have here match up: the Curvature of the connection $\mathcal{F}$ is the related to the $\mathbf{V}_n(\mathbf{q})$ the same way the magnetic field is connected to the Field Strength Tensor:

$$\mathcal{F}_{ij} = -\epsilon_{ijk}(V_n(\mathbf{q}))_k$$

Of course, this only works in three dimensions, i.e. when there are only three parameters, for $\mathbf{V}_n(\mathbf{q})$ is defined only in three dimensions. But then we have a good way to generalize it from above.

Enough of this blabber! Lets see the berry phase in action.

A.3 The Toy Model: Electron in a Magnetic field

We will consider here only the spin's interaction with the magnetic field. The spin of the Electron is $^1/_2$, so the spin operator in this case will be $\frac{\hbar}{2}$ times the Pauli Matrices. Then the hamiltonian is given by:

$$H = -\frac{\hbar}{2}\mathbf{B} \cdot \boldsymbol{\sigma}$$

The eigenstates of this system will be either along the magnetic field $|\uparrow\rangle$, with energy $E_\uparrow = -\frac{\hbar}{2}B$ or opposite to it $|\downarrow\rangle$, with energy $E_\downarrow = \frac{\hbar}{2}B$.

Using this, we can find $\mathbf{V}_\uparrow$ and $\mathbf{V}_\downarrow$. First, we need $\nabla_{\mathbf{B}} H = -\frac{\hbar}{2}\boldsymbol{\sigma}$. Then we can write:

$$\mathbf{V}_\uparrow = \mathrm{Im}\left[\frac{\langle\uparrow|\nabla_{\mathbf{B}}H|\downarrow\rangle \times \langle\downarrow|\nabla_{\mathbf{B}}H|\uparrow\rangle}{(E_\uparrow - E_\downarrow)^2}\right] = \mathrm{Im}\left[\frac{\langle\uparrow|\boldsymbol{\sigma}|\downarrow\rangle \times \langle\downarrow|\boldsymbol{\sigma}|\uparrow\rangle}{4B^2}\right]$$
$$\mathbf{V}_\downarrow = \mathrm{Im}\left[\frac{\langle\downarrow|\nabla_{\mathbf{B}}H|\uparrow\rangle \times \langle\uparrow|\nabla_{\mathbf{B}}H|\downarrow\rangle}{(E_\uparrow - E_\downarrow)^2}\right] = \mathrm{Im}\left[\frac{\langle\downarrow|\boldsymbol{\sigma}|\uparrow\rangle \times \langle\uparrow|\boldsymbol{\sigma}|\downarrow\rangle}{4B^2}\right]$$

Note that $\mathbf{V}_\uparrow = -\mathbf{V}_\downarrow$. Now, rotating the system such that $\mathbf{B}$ is along z-axis, we see that $|\uparrow\rangle \equiv |+\rangle$ & $|\downarrow\rangle \equiv |-\rangle$. Then the above is just:

$$\mathbf{V}_\uparrow = \mathrm{Im}\left[\frac{\langle+|\boldsymbol{\sigma}|-\rangle \times \langle-|\boldsymbol{\sigma}|+\rangle}{4B^2}\right] = \mathrm{Im}\left[\frac{\langle+|\boldsymbol{\sigma}|-\rangle \times \langle+|\boldsymbol{\sigma}|-\rangle^*}{4B^2}\right]$$

Using $\sigma_x|\pm\rangle = |\mp\rangle$, $\sigma_y|\pm\rangle = \pm i|\mp\rangle$ & $\sigma_z|\pm\rangle = \pm|\pm\rangle$, we then have

$$\begin{aligned}
&\boldsymbol{\sigma}|-\rangle = |+\rangle\hat{\mathrm{x}} - i|+\rangle\hat{\mathrm{y}} - |-\rangle\hat{\mathrm{z}} \implies \langle-|\boldsymbol{\sigma}|+\rangle = \hat{\mathrm{x}} - i\hat{\mathrm{y}} \\
&\implies \langle+|\boldsymbol{\sigma}|-\rangle \times \langle+|\boldsymbol{\sigma}|-\rangle^* = (\hat{\mathrm{x}} - i\hat{\mathrm{y}}) \times (\hat{\mathrm{x}} + i\hat{\mathrm{y}}) = 2i\hat{\mathrm{z}} \\
&\implies \mathbf{V}_\uparrow = \mathrm{Im}\left[\frac{\langle+|\boldsymbol{\sigma}|-\rangle \times \langle+|\boldsymbol{\sigma}|-\rangle^*}{4B^2}\right] = \frac{B\hat{\mathrm{z}}}{2B^3} \implies \mathbf{V}_\downarrow = -\frac{B\hat{\mathrm{z}}}{2B^3}
\end{aligned}$$

Reverting back to the original coordinates, we then have:

$$\mathbf{V}_\uparrow = \frac{\mathbf{B}}{2B^3} \implies (\mathcal{F}_\uparrow)_{ij} = -\epsilon_{ijk}\frac{B_k}{2B^3} \quad \mathbf{V}_\downarrow = -\frac{\mathbf{B}}{2B^3} \implies (\mathcal{F}_\downarrow)_{ij} = \epsilon_{ijk}\frac{B_k}{2B^3}$$

But this seems interesting! We took the parameter space as the space of all possible magnetic fields, computed the quantity $\mathbf{V}$, and then find it to be like a magnetic monopole of magnetic charge $\pm\frac{1}{2}$, but in the space of magnetic fields! Note that it sits at the point $B = 0$, where we expect this all to fail,since at $B = 0$ the states are degenerate. In some sense then, the presence of this singularity is expected and it is what is dominating the physics of the Berry phase.

Finally, we have everything to calculate the Berry phase for a given curve in the space of magnetic fields (i.e. the parameter space). Using the analogy of the magnetic monopole, we immediately see that for any given curve C, the Berry Phase is given by the solid angle (Ω) C subtends on the origin of the magnetic field space (i.e. on the monopole) by:

$$\gamma_\uparrow = -\frac{\Omega}{2}, \quad \gamma_\downarrow = \frac{\Omega}{2}$$

As a side note, we also see that we can choose either Ω or $4\pi - \Omega$ as the solid angle. But the wavefunction depends on γ only through $\exp(i\gamma) = \exp\left(\pm i\frac{\Omega}{2}\right)$. Since $\exp\left(\pm i\frac{4\pi-\Omega}{2}\right) = \exp\left(\pm i\frac{4\pi-\Omega}{2}\right) = \exp\left(\mp i\frac{-\Omega}{2}\right)$, and since we reversed the orientation of the curve by taking the above choice, we have the $-$ sign for the $\uparrow$ state and vice-versa.

Note that this only works when the "magnetic monopole" has half-integer charges, thus creating a quantization in the Berry Phase. This also means that over any *closed surface* S

$$\int\int_S \mathcal{F}_{ij} \mathrm{d}S^{ij} = 2\pi C \text{ for integer } C$$

C is also called the Chern number. This is valid in all systems, not just this one.

A.4 Aharonov - Bohm Effect

We will show here how Aharonov - Bohm effect is related to the Berry Phase. But first, let us elucidate what Aharonov - Bohm effect is.

Classically, the electric and magnetic fields *in the locality of the particle* determines its dynamical behaviour. This is so because in the Newton's Second Law, only forces enter and since only fields can generate a force, we have the above result. Thus in the classical theory, Electric and magnetic potentials just act as a mathematical construct; having no physical relation to the particle's dynamics, except via the fields. This is manifest in the fact that the potentials are not uniquely determined.

Things differ once we reach the quantum world. In QM, the Hamiltonian depends on the potentials, hence we should expect that we would see the dynamics of the particle depending on the potentials. This is the philosophy behind the Aharonov - Bohm effect.

Note that the fact that the Hamiltonian depends on the potentials does not infact negate the Gauge Transformations of the potential, for the Hamiltonian is itself gauge invariant.

The reader might accuse us of cheating in the above argument, for isn't the dynamics of a classical particle dependent on the potentials as well, just that it is through the fields? Then how is the quantum particle different? This is easy to see with an example. Consider a solenoid placed in the middle of the slits in a double-slit experiment. Now, for a solenoid, the magnetic field outside it is zero, but since there is a flux through any surface enclosing the solenoid, the magnetic potential is non-zero. Now, when we perform the double-slit experiment, we will see a shift in the interference pattern on the screen as we change the current in the solenoid (and thus the magnetic potential it exerts outside), but the magnetic field remains zero outside. The change in the magnetic potential is manifesting in the change of the interference pattern through the Hamiltonian.

Wait a second! Isn't the magnetic field inside the solenoid changing? Could it not be that the magnetic field inside is affecting the particle? Infact, that is what is happening. However, this should by now illustrate the difference between the quantum and the classical case: *Classical particles are affected by local fields only, quantum particles are affected by non-local fields, albeit through the local potentials they exert.* This should not surprise us, for we know by the Bell's Theorem that quantum mechanics is essentially a non-local theory. We deem it easier to think in terms of potentials in this case, for they affect only locally (which makes life simpler).

Let us now think of where this effect of potential can reside. We know that the electromagnetic potential changes under gauge transformations, but the physics doesn't. Another

quantity with a similar behaviour is the phase of a wavefunction: Changing the phase does not change the physics described by the wavefunction. Is it then too much a stretch to assume a coupling between the two? Such a coupling would preserve the Gauge invariance of Electromagnetic phenomena, while also provide us information about the effect of the Electromagnetic potentials through interference experiments. This completes the Aharonov - Bohm effect: The *local* electromagnetic potentials are coupled and thus affect the phase of the wavefunction. Shifts in the potentials manifest then as phase shifts in the wavefunction and can be realized through interference experiments (of the kind mentioned before).

Enough of this philosophical discussion! We turn now to first analyze the interesting example we mentioned earlier: the Solenoid. We explore some aspects of it and then turn to show how Aharonov - Bohm effect in this system can be explained through Berry Phase (we have kind of indicated this by showing the coupling of the phase with the potentials).

A.4.1 Particles Moving around a Solenoid/Flux Tube

The system here is only a solenoid, with a quantum particle forced to lie only in a ring of radius r around it. As mentioned earlier, the potential outside the ring is non-zero, and it can be calculated using the Stokes Law:

$$\Phi = \int \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \oint_C \mathbf{A} \cdot \mathrm{dl} \implies \mathbf{A} = \frac{\Phi}{2\pi r}\hat{\phi}$$

where $\Phi = BA$ is the flux through the solenoid of area A with Magnetic field B in it. Now the Hamiltonian for the particle is:

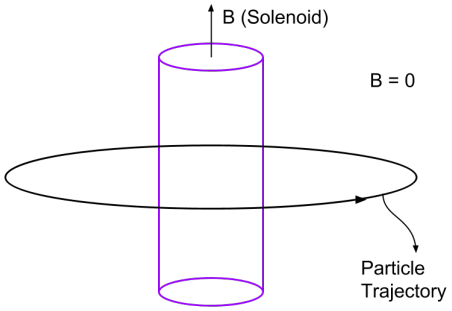

Figure A.1: Particle Moving Around A Solenoid

$$H = \frac{1}{2m}(\mathbf{p} + e\mathbf{A})^2$$

Since the particle is only allowed to move in the $\hat{\phi}$ direction, we might as well write the above as:

$$H = \frac{1}{2m}(p_\phi + eA_\phi)^2 = \frac{1}{2mr^2}\left(-i\hbar\frac{\partial}{\partial\phi} + \frac{e\Phi}{2\pi}\right)^2$$

The energy eigenstates to this can be found easily. They are simply:

$$\psi = \frac{1}{\sqrt{2\pi r}} \mathrm{e}^{in\phi}, \quad n \,\epsilon\, \mathbb{Z} \text{ as the wavefunction must be single-valued}$$

The energy values are:

$$E_n = \frac{\hbar^2}{2mr^2} \left(n + \frac{\Phi}{\Phi_0} \right)^2$$

where we have denoted $\Phi_0 = \frac{2\pi\hbar}{e}$. If Φ is an integer multiple of Φ_0, then the spectrum remains the same, but this condition is no way required to be satisfied. When the flux is not a multiple of Φ_0, we get a shifted spectrum. This is the effect of the Magnetic potential we have been looking for, the nonlocal interaction of the magnetic field and the particle that we discussed before.

There is something more interesting going on here, however. If we change the flux so slowly that the adiabatic approximation is valid, as Φ crosses integer multiples of Φ_0, we see that by the virtue of the approximation the states start morphing into each other; essentially the spectrum "flows" outwards as Φ increases. However, for integer values of $\frac{\Phi}{\Phi_0}$, the spectrum remains the same. We note this point for now, and see what comes out of it later.

Now, onto the last main event of this discussion of Berry Phase - Aharonov-Bohm effect in the solenoid system.

A.4.2 Aharonov-Bohm Effect from the Berry Phase

Here, we take the same geometry, but now confine the particle in a small box at $\mathbf{R}$ from the solenoid, instead of on a ring. Let the Hamiltonian of the particle in the box be given by $H(\mathbf{p}, \mathbf{r} - \mathbf{R})$ *in the absence of any current in the solenoid.* $\mathbf{r}$ is the position coordinate of the particle and $\mathbf{R}$ is the coordinate of the center of the box. Let the energy eigenstates of this Hamiltonian be $|n(\mathbf{R})\rangle$, where

$$H(\mathbf{p}, \mathbf{r} - \mathbf{R}) \, |n(\mathbf{R})\rangle = E_n \, |n(\mathbf{R})\rangle \,, \quad \langle \mathbf{r} | n(\mathbf{R}) \rangle = \psi_n(\mathbf{r} - \mathbf{R})$$

When there is a current in the solenoid, this induces a magnetic potential in the box $\mathbf{A}$, which changes the Hamiltonian to $H(\mathbf{p} - Q\mathbf{A}, \mathbf{r} - \mathbf{R})$, if the particle has charge Q. We assume a solution of the form $\langle \mathbf{r} | n(\mathbf{R}, \mathbf{A}) \rangle = \mathrm{e}^{\alpha(\mathbf{r})} \psi_n(\mathbf{r} - \mathbf{R})$ to this modified Hamiltonian, where its energy eigenstates are given by $|n(\mathbf{R}, \mathbf{A})\rangle \,, |n(\mathbf{R}, \mathbf{A} = 0)\rangle = |n(\mathbf{R})\rangle$. The Energy eigenvalues remain the same, due to the fact that the magnetic field can do no work. Now, to find the function α in the ansatz, we see that we must satisfy:

$$(\mathbf{p} - Q\mathbf{A}) \mathrm{e}^{\alpha(\mathbf{r})} \psi_n(\mathbf{r} - \mathbf{R}) = \mathrm{e}^{\alpha(\mathbf{r})} \mathbf{p} \psi_n(\mathbf{r} - \mathbf{R})$$

This can be enforced by:

$$
\begin{aligned}
(\mathbf{p}-Q\mathbf{A})\mathrm{e}^{\alpha(\mathbf{r})}\psi_n(\mathbf{r}-\mathbf{R}) &= (-i\hbar\nabla - Q\mathbf{A})\mathrm{e}^{\alpha(\mathbf{r})}\psi_n(\mathbf{r}-\mathbf{R}) \\
&= -Q\mathbf{A}\mathrm{e}^{\alpha(\mathbf{r})}\psi_n(\mathbf{r}-\mathbf{R}) - i\hbar\nabla\left(\mathrm{e}^{\alpha(\mathbf{r})}\psi_n(\mathbf{r}-\mathbf{R})\right) \\
&= -Q\mathbf{A}\mathrm{e}^{\alpha(\mathbf{r})}\psi_n(\mathbf{r}-\mathbf{R}) - i\hbar\nabla\left(\mathrm{e}^{\alpha(\mathbf{r})}\right)\psi_n(\mathbf{r}-\mathbf{R}) - \mathrm{e}^{\alpha(\mathbf{r})}i\hbar\nabla\psi_n(\mathbf{r}-\mathbf{R}) \\
&= -\left(Q\mathbf{A} + i\hbar\nabla\alpha(\mathbf{r})\right)\mathrm{e}^{\alpha(\mathbf{r})}\psi_n(\mathbf{r}-\mathbf{R}) + \mathrm{e}^{\alpha(\mathbf{r})}\mathbf{p}\psi_n(\mathbf{r}-\mathbf{R}) \\
\text{Thus, } (\mathbf{p}-Q\mathbf{A})\mathrm{e}^{\alpha(\mathbf{r})}\psi_n(\mathbf{r}-\mathbf{R}) &= \mathrm{e}^{\alpha(\mathbf{r})}\mathbf{p}\psi_n(\mathbf{r}-\mathbf{R}) \\
\implies Q\mathbf{A} + i\hbar\nabla\alpha(\mathbf{r}) &= 0 \\
\implies \nabla\alpha(\mathbf{r}) &= \frac{iQ}{\hbar}\mathbf{A}
\end{aligned}
$$

Now, since the Hamiltonian was a function of $\mathbf{r}-\mathbf{R}$, so the nabla operator was wrt to this vector, not $\mathbf{r}$ (Physically, this means the particle is confined in the box and thus its momentum is a function of the relative coordinate in the box). Then, by integrating the above, we get $\alpha(\mathbf{r}) = \frac{iQ}{\hbar}\int_{\mathbf{R}}^{\mathbf{r}}\mathbf{A}(\mathbf{r}')\cdot\mathrm{d}\mathbf{r}'$. This gives the wavefunction in the box in the presence of a magnetic potential as:

$$
\langle\mathbf{r}|n(\mathbf{R},\mathbf{A})\rangle = \mathrm{e}^{\alpha(\mathbf{r})}\psi_n(\mathbf{r}-\mathbf{R}) = \exp\left[\frac{iQ}{\hbar}\int_{\mathbf{R}}^{\mathbf{r}}\mathbf{A}(\mathbf{r}')\cdot\mathrm{d}\mathbf{r}'\right]\psi_n(\mathbf{r}-\mathbf{R})
$$

Everything is set now. We now first identify the parameters that we wanna vary in this problem as the position of the box $\mathbf{R}$. We take the box on a curve C looping around the solenoid. The Berry Phase for this curve can then be found from the Berry connection as:

$$
\mathcal{A} = -i\left\langle n(\mathbf{R},\mathbf{A})\right|\nabla_{\mathbf{R}}\left|n(\mathbf{R},\mathbf{A})\right\rangle = -i\iiint\mathrm{d}^3\mathbf{r}\,\psi_n^*(\mathbf{r}-\mathbf{R})\left[-\frac{iQ}{\hbar}\mathbf{A}(\mathbf{R})\psi_n(\mathbf{r}-\mathbf{R}) + \nabla_{\mathbf{R}}\psi_n(\mathbf{r}-\mathbf{R})\right]
$$

The wavefunction is normalized, thus $\iiint\mathrm{d}^3\mathbf{r}\,\psi_n^*(\mathbf{r}-\mathbf{R})\psi_n(\mathbf{r}-\mathbf{R}) = 1$ and $\iiint\mathrm{d}^3\mathbf{r}\,\psi_n^*(\mathbf{r}-\mathbf{R})\nabla_{\mathbf{R}}\psi_n(\mathbf{r}-\mathbf{R}) = 0$. This gives

$$
\mathcal{A} = -\frac{Q}{\hbar}\mathbf{A}
$$

Wow! We made an analogy before between the Berry connection and the magnetic potential, and we see the analogy becoming a reality here. This is indeed very satisfactory. Also, note that the Berry connection is *same for all states*, and so will the Berry Phase be. Now, the Berry phase itself is given by:

$$
\gamma = -\oint_C\mathcal{A}\cdot\mathrm{d}\mathbf{R} = \frac{Q\Phi}{\hbar}
$$

This is the promised way to obtain Aharonov - Bohm effect through the Berry Phase. Berry Phase adds up to the phase of the wavefunction and is itself dependent on the magnetic potential (through the magnetic flux), which is what we required for the Aharonov - Bohm Effect.

Bibliography

[1] Joseph G Checkelsky, Jianting Ye, Yoshinori Onose, Yoshihiro Iwasa, and Yoshinori Tokura. Dirac-fermion-mediated ferromagnetism in a topological insulator. *Nature Physics*, 8(10):729–733, 2012.

[2] David Tong. Lectures on the Quantum Hall Effect. 2016.

[3] John H. Davies. *The Physics of Low-dimensional Semiconductors: An Introduction*. Cambridge University Press, 1997.

[4] K. v. Klitzing, G. Dorda, and M. Pepper. New method for high-accuracy determination of the fine-structure constant based on quantized hall resistance. *Phys. Rev. Lett.*, 45:494–497, Aug 1980.

[5] Yuanbo Zhang, Yan-Wen Tan, Horst L Stormer, and Philip Kim. Experimental observation of the quantum hall effect and berry's phase in graphene. *Nature*, 438(7065):201–204, 2005.

[6] G. Granger, J. P. Eisenstein, and J. L. Reno. Observation of chiral heat transport in the quantum hall regime. *Phys. Rev. Lett.*, 102:086803, Feb 2009.

[7] B. I. Halperin. Quantized hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential. *Phys. Rev. B*, 25:2185–2190, Feb 1982.

[8] M. Z. Hasan and C. L. Kane. Colloquium. *Rev. Mod. Phys.*, 82:3045–3067, Nov 2010.

[9] Michael V Berry. Quantal phase factors accompanying adiabatic changes. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 392, pages 45–57. The Royal Society, 1984.