MATH 131–HOMEWORK 7

Due on Wednesday, Feb 27, before class.

Please read section 3.B. 3.C.

Question 1 (10 points) Suppose U, V and W are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim(\texttt{range}\, ST) \leq \min\{\dim(\texttt{range}\, S), \dim(\texttt{range}\, T)\}.$$

Question 2 (10 points) Suppose that V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in L(W, V)$ such that ST is the identity map on V.

Question 3 (10 points) Suppose $T \in \mathcal{L}(P_2(\mathbb{R}), P_4(\mathbb{R}))$ is the linear map defined by

$$Tp = x^2 p.$$

(1) Find the matrix of T with respect to the standard basis.
(2) Verify the fundamental theorem of linear maps.

Question 4 (10 points) Let $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$. Let $\mathcal{E} = \{e_1, \ldots, e_n\}$ be a basis of V, and $\mathcal{F} = \{f_1, \ldots, f_m\}$ be a basis of W. Show that there are identities of matrices as following:

$$[S + T]_{\mathcal{F} \leftarrow \mathcal{E}} = [S]_{\mathcal{F} \leftarrow \mathcal{E}} + [T]_{\mathcal{F} \leftarrow \mathcal{E}},$$

and

$$[\lambda S]_{\mathcal{F} \leftarrow \mathcal{E}} = \lambda [S]_{\mathcal{F} \leftarrow \mathcal{E}}.$$

Question 5 (10 points) Suppose $D \in \mathcal{L}(P_3(\mathbb{R}), P_2(\mathbb{R}))$ is the differential map defined by

$$Dp = p'.$$

Find a basis $\mathcal{E}$ of $P_3(\mathbb{R})$ and a basis $\mathcal{F}$ of $P_2(\mathbb{R})$ such that the matrix of D with respect to these bases is

$$[D]_{\mathcal{F} \leftarrow \mathcal{E}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Question 6 (10 points) Find linear maps $S, T \in \mathcal{L}(\mathbb{R}^2)$ such that $ST \neq TS$.