Math 131: Linear Algebra

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2.A. Span and linearly independence

- linear combination and span
- linearly independent
- Properties of linearly independence

In this section, V is a vector space over $\mathbb{F}$. You may take $\mathbb{F}$ as $\mathbb{R}$ or $\mathbb{C}$.

4.1 Linear combination

Definition 1 Let V be a vector space over $\mathbb{F}$. A *linear combination* of a set $\{v_1, \ldots, v_m\}$ of vectors in V is a vector of the form

$$a_1 v_1 + \ldots + a_m v_m,$$

where $a_1, \ldots, a_m \in \mathbb{F}$.

Example 1 *In $\mathbb{C}^3$, is $(1, 2, 0)$ a linear combination of $(1, 0, 0)$ and $(0, 1, 0)$?*
Is $(1, 2, 3)$ a linear combination of $(1, 0, 0)$ and $(0, 1, 0)$?

Solution: $(1, 2, 0)$ *a linear combination of* $(1, 0, 0)$ *and* $(0, 1, 0)$: *If we write*

$$(1, 2, 0) = a(1, 0, 0) + b(0, 1, 0)$$

for some $a, b \in \mathbb{C}$, then

$$(1, 2, 0) = (a, b, 0).$$

Hence $a = 1, b = 2$ and $(1, 2, 0) = 1 \cdot (1, 0, 0) + 2(0, 1, 0)$ is indeed a linear combination.

$(1, 2, 3)$ *is NOT a linear combination of* $(1, 0, 0)$ *and* $(0, 1, 0)$. *Because the equation*

$$(1, 2, 3) = a(1, 0, 0) + b(0, 1, 0) = (a, b, 0)$$

has no solution of a, b.

Example 2 *In $\mathbb{C}^3$, is $(17, -4, 2)$ a linear combination of $(2, 1, -3)$ and $(1, -2, 4)$? Is $(17, -4, 5)$ a linear combination of $(2, 1, -3)$ and $(1, -2, 4)$?*

Solution: $(17, -4, 2)$ *is a linear combination of* $(2, 1, -3)$ *and* $(1, -2, 4)$ *and* $(17, -4, 5)$ *is NOT a linear combination of* $(2, 1, -3)$ *and* $(1, -2, 4)$. *This is because the equation*

$$(17, -4, 2) = a(2, 1, -3) + b(1, -2, 4)$$

has a solution. We solve it here. The equation

$$(17, -4, 2) = a(2, 1, -3) + b(1, -2, 4) = (a + b, a - 2b, -3a + 4b)$$

implies 3 new equations from each component:

$$17 = 2a + b \tag{1}$$
$$-4 = a - 2b \tag{2}$$
$$2 = -3a + 4b \tag{3}$$

The equation (1) implies that $b = 17 - 2a$. Plug it into (2), we get

$$-4 = a + (-2)(17 - 2a) = -34 + 5a$$

$$30 = 5a$$

$$a = 6$$

Hence $b = 17 - 2 \cdot 6 = 5$. We plug in $a = 6, b = 5$ into equation (3).

By a similar argument, we can check that $(17, -4, 5)$ is NOT a liner combination of $(2, 1, -3)$ and $(1, -2, 4)$.

Definition 2 Let V be a vector space over $\mathbb{F}$. The set of all linear combinations of a set of vectors $\{v_1, \ldots, v_m\}$ in V is called the *span* of $\{v_1, \ldots, v_m\}$, denoted by $\mathtt{Span}(v_1, \ldots, v_m)$. In other words,

$$\mathtt{Span}(v_1, \ldots, v_m) = \{a_1 v_1 + \ldots + a_m v_m \mid a_i \in \mathbb{F}\}.$$

The span of the empty set is defined to be $\{0\}$. If $\mathtt{Span}(v_1, \ldots, v_m)$ equals V, we say that $\{v_1, \ldots, v_m\}$ *spans/generates* V.

Definition 3 (Optional) Generalized definition of span.

Let S be a subset of V.

$$\mathtt{Span}(S) = \left\{ \sum_{i=1}^{k} \lambda_i v_i \,\middle|\, k \in \mathbb{N}, v_i \in S, \lambda_i \in \mathbb{F} \right\}.$$

Given a vector space V, if $V = \mathtt{Span}(S)$ for some subsets S, then S is called a generating/spanning set of V.

Example 3 In $\mathbb{R}^3$, let $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, and $\vec{k} = (0, 0, 1)$.

- $\mathtt{Span}(\vec{i})$ is the x-axis. $\mathtt{Span}(\vec{j})$ is the y-axis.
- $\mathtt{Span}(\vec{i}, \vec{j})$ is the xy plane.
- $\mathtt{Span}(\vec{i}, \vec{j}, \vec{k})$ is the entire vector space $\mathbb{R}^3$.

Theorem 1 *Let V be a vector space over $\mathbb{F}$. The span of $\{v_1, \ldots, v_m\}$ in V is the smallest subspace of V containing the vectors v_i for all i.*

Sketch of the proof: We need to prove two parts. First of all, $\mathtt{Span}(v_1, \ldots, v_m)$ *is a subspace* *of V. Secondly, if W is a subspace of V which contains all v_i, then every linear combination of $\{v_1, \ldots, v_m\}$ is also in W. In other words, $\mathtt{Span}(v_1, \ldots, v_m) \subseteq W$.*

One of the useful corollary of the above theorem is the following lemma, which can be proved based on definition of span directly.

Lemma 1 *If $w \in \mathtt{Span}(v_1, \ldots, v_m)$, then*

$$\mathtt{Span}(w, v_1, \ldots, v_m) = \mathtt{Span}(v_1, \ldots, v_m).$$

Proof is similar to the one of worksheet 3, problem 4. The idea is to prove the inclusion of two direction. $\mathtt{Span}(v_1, \ldots, v_m) \subseteq \mathtt{Span}(w, v_1, \ldots, v_m)$ is trivial. The other direction is based on the definition of span.

Example 4 For the positive integer $1 \leq i \leq n$, let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{F}^n$ be the vector with i-th coordinator equal to 1. We can check that

$$\mathtt{Span}(e_1, \ldots, e_n) = V.$$

Hence $\{e_1, \ldots, e_n\}$ spans $\mathbb{F}^n$.

Example 5 Recall that

$$P_n(\mathbb{F}) = \{p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_k t^k \mid k \in \mathbb{N}, k \leq n, a_0, a_1, \ldots, a_k \in \mathbb{F}\}.$$

We can check that $P_n = \mathtt{Span}(1, t, \ldots, t^n)$.

Definition 4 A vector space V is called *finite-dimensional* if there is a set of finitely many elements in V which spans the space.

A vector space is called *infinite-dimensional* if it is not finite-dimensional.

Example 6

- $\mathbb{F}^n$, $\mathtt{Mat}_{n\times m}(\mathbb{F})$ and $P_n(\mathbb{F})$ are finite-dimensional. We can find finite sets which span these spaces. For example, $\mathbb{F}^n$ is spanned by the standard basis $\{e_i\}$ in Example 4. Could you find finite sets to span $\mathtt{Mat}_{n\times m}(\mathbb{F})$ and $P_n(\mathbb{F})$?
- $P(\mathbb{F})$, $\mathbb{R}^{\mathbb{R}}$ are infinite-dimensional. We need further results to prove this.

4.2 Linearly independent

Definition 5 A set $\{v_1, \ldots, v_m\}$ of vectors in V is called *linearly independent* if the only choice of $a_1, \ldots, a_m \in \mathbb{F}$ that makes

$$a_1 v_1 + \ldots + a_m v_m = 0$$

is $a_1 = a_2 = \cdots = a_m = 0$. The empty set $\emptyset$ is also declared to be linearly independent.

A set of vectors $\{v_1, \ldots, v_m\}$ in V is called *linearly dependent* if it is NOT linearly independent.

Equivalently, there exist $a_1, \ldots, a_m \in \mathbb{F}$ and at least one $a_i \neq 0$, such that

$$a_1 v_1 + \ldots + a_m v_m = 0.$$

We say this is a linear or linearly independent relation among $\{v_1, \ldots, v_m\}$.

Definition 6 (Optional) Generalized definition of linearly independent for a subset $S \subset V$.

Let $S \subseteq V$ be a subset. Then S is *linearly independent* if for any collection of finitely many elements $\{v_1, \ldots, v_m\} \subseteq S$ for some $m \in \mathbb{N}$, $\{v_1, \ldots, v_m\}$ is linearly independent.

Lemma 2 *Some basic facts about linearly independence. We omit the proofs here.*

1. *Let $v \in V$ be an element. Then $\{v\}$ is linearly independent if and only if $v \neq 0$. This follows from Proposition 1 in section 1.B.*
2. *A subset of two elements, $\{v_1, v_2\} \subset V$, is linearly independent if and only if they are not proportional.*
3. *A subset $\{v_1, \ldots, v_m\}$ is linearly dependent, then for any $w \in V$, the larger set $\{v_1, \ldots, v_m, w\}$ is also linearly dependent. This is because there are numbers $a_i \in \mathbb{F}$, not all are zeros, such that*

$$a_1 v_1 + \cdots a_m v_m = 0.$$

Then we take $b = 0$, we have

$$a_1 v_1 + \cdots a_m v_m + bw = 0,$$

which gives a linear relation between elements $\{v_1, \ldots, v_m, w\}$.

Example 7 For the positive integers $1 \leq i \leq n$, let $e_i = (0, \ldots, 0, 1, 0, \cdots, 0) \in \mathbb{F}^n$ be the vector with i-th coordinator equal to 1. We can check that $\{e_i\}_{i=1,\ldots,n}$ is linearly independent.

Solution: For any $a_1, \ldots, a_n \in \mathbb{F}$ such that $a_1 e_1 + \ldots + a_n e_n = 0$, we will show that all $a_i = 0$. This is because

$$a_1 e_1 + \ldots + a_n e_n = (a_1, a_2, \ldots, a_n) = (0, 0, \ldots, 0).$$

Hence $a_i = 0$ for every i.

Example 8 The set $\{(2, 3, 1), (1, -1, 2), (7, 3, c)\}$ is linearly dependent in $\mathbb{F}^3$ if and only if $c = 8$.

Solution: $\Rightarrow$: We first assume that the set is linearly dependent and show that $c = 8$. The set is linearly dependent implies that there are $x, y, z \in \mathbb{F}$ such that at least one of x, y, z is not equal to 0 and

$$\begin{aligned}
x(2,3,1) + y(1,-1,2) + z(7,3,c) = 0 &= (0,0,0) && (4)\\
\text{Equivalently, } (2x + y + 7z, 3x - y + 3z, x + 2y + zc) &= (0,0,0) && (5)\\
\Rightarrow \qquad 2x + y + 7z &= 0; && (6)\\
3x - y + 3z &= 0; && (7)\\
x + 2y + zc &= 0. && (8)
\end{aligned}$$

From equation (6) and (7), we get $x = -2z$ and $y = -3z$. Plug into (8), we obtain

$$x + 2y + zc = -2z - 6z + zc = z(c - 8) = 0$$

If $z = 0$, then $x = y = 0$, which contradicts with the fact that at least one of the

x, y, z is not equal to 0. Hence $z \neq 0$. Hence $c = 8$.

$\Leftarrow$: Now we assume that $c = 8$, we reverse the above calculation. Any $z \neq 0$, $x = -2z$, $y = -3z$ will give a linear dependent relation. For instance, we take $z = 1$, $x = -2$ and $y = -3$ and check that

$$(-2)(2, 3, 1) + (-3)(1, -1, 2) + 1(7, 3, 8) = 0.$$

This gives a linearly dependence relation among the given vectors.

Remark 1 In the above example, linearly dependence means there are nontrivial (linear) relations between these vectors. Later we will see that in $\mathbb{F}^3$, a general set of 3 elements should be linearly independent. We will also know that any set in $\mathbb{R}^3$ of length greater than 3 is linearly dependent.

In the above example, when $c = 8$, we have

$$(-2)(2, 3, 1) + (-3)(1, -1, 2) + 1(7, 3, 8) = 0.$$

Equivalently, the last vector $(7, 3, 8) = 2(2, 3, 1) + 3(1, -1, 2)$ is a linear combination of the first two vectors $(2, 3, 1)$ and $(1, -1, 2)$. Geometrically, these three vectors lie on a plane, which is the subspace $\mathtt{Span}((2, 3, 1), (1, -1, 2))$. In Math 10A, we can find the plane equation

$$7x - 3y - 5z = 0.$$

Proposition 1 *(Linearly dependence lemma) Suppose $\{v_1, \ldots, v_m\}$ is a linearly dependent set in V. Then there exists $j \in \{1, 2, \ldots, m\}$ such that the following hold:*

(a) $v_j \in \mathtt{Span}(v_1, \ldots, v_{j-1})$.

(b) If v_j is removed from the set $\{v_1, \ldots, v_m\}$, then

$$\mathtt{Span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m) = \mathtt{Span}(v_1, \ldots, v_m).$$

Sketch of the proof: Since $\{v_1, \ldots, v_m\}$ is a linearly dependent. There exist some $a_i \in \mathbb{F}$, not all equal to 0, such that

$$a_1 v_1 + \ldots + a_m v_m = 0.$$

Let j be the largest index such that $a_j \neq 0$. Then for all $k > j$, $a_k = 0$. Hence

$$a_1 v_1 + \ldots + a_j v_j = 0.$$

Then we can solve v_j, as a linear combination of $v_1, \ldots, v_{j-1}$. This completes the proof of part (a).

The proof of part (b) is similar to the problem 4 in Worksheet 3.

You probably have noticed that in $\mathbb{R}^3$, the span of any nonzero element v gives us a line passing through origin and the point v. We need at least two elements to span a plane in $\mathbb{R}^3$ and we need at least 3 elements to span the entire $\mathbb{R}^3$. There are lower bounds of the number of element of a spanning set.

Proposition 2 *In a finite-dimensional vector space, the length of every linearly independent set is smaller than or equal to the length of every spanning set.*

Sketch of the proof: Let $\{v_1, \ldots, v_n\}$ be a linearly independent set and $\{w_1, \ldots, w_m\}$ spans V. Then we shall show that $n \le m$.

The idea is that we will replace some elements w_{i_1} by v_1 for some i_1. Then we replace another w_{i_2} by v_2. After replacing w_{i_j} by v_j. we get $v_1, \ldots, v_n$ together with the rest of w's spans V and $n \le m$.

Now we explain the replacement. Since $\{w_1, \ldots, w_m\}$ spans V. v_1 is a linear combination of $\{w_1, \ldots, w_m\}$. There exists $a_1, \ldots, a_m$ such that

$$v_1 = a_1 w_1 + \cdots a_m w_m.$$

Since $v_1 \neq 0$, hence at least one of a_j is not equal to 0. Without loss of generality, we may assume that $a_1 \neq 0$ after reordering the w's. We can check that

$$\mathtt{Span}(v_1, w_2, \ldots, w_m) = \mathtt{Span}(v_1, , w_1, w_2, \ldots, w_m) = \mathtt{Span}(w_1, w_2, \ldots, w_m) = V.$$

Here the first equality holds because we can write w_1 as a linear combination of v_1 and $w_2, \ldots, w_m$. This also implies that v_2 is a linear combination of $\{v_1, w_2, \ldots, w_m\}$. Hence there are $b_i \in \mathbb{F}$, such that

$$v_2 = b_1 v_1 + b_2 w_2 + \cdots b_m w_m.$$

We claim that one of the $b_2, b_3, \cdots, b_m$ is not zero. Otherwise, $v_2 = b_1 v_1$, violates with assumption that $\{v_1, \ldots, v_n\}$ is linearly independent. By reordering the index of $w_2, \ldots, w_m$, we may assume that $b_2 \neq 0$. We replace w_2 by v_2 and consider the set $\{v_1, v_2, w_3 \ldots, w_m\}$. By a similar argument, we have $\mathtt{Span}(v_1, w_2, \ldots, w_m) = V$. We start to move the v_3 into the set and so on. This algorithm stops after n times. This means we can replace n different elements in $\{w_1, \ldots, w_m\}$. Hence $n \le m$.

Lemma 3 *Suppose $\{v_1, \ldots, v_m\}$ is linearly independent in V and $w \in V$. Show that $\{v_1, \ldots, v_m, w\}$ is linearly independent if and only if $v \notin \mathtt{Span}(v_1, \ldots, v_m)$.*

Proof: This is the problem 2 in worksheet 4.

This lemma provides us a new method to show linearly independent in practice.

Example 9 Let $S = \left\{f_1 = 1, f_2 = t, f_3 = t^2\right\}$ be a subset of P_3.

1. $f_1 \neq 0$.
2. f_2 is not in $\mathtt{Span}\,\{f_1\} = \mathbb{F} f_1 = \{a : a \in \mathbb{F}\}$.
3. f_3 is not in $\mathtt{Span}\,\{f_1, f_2\} = \mathbb{F} f_1 + \mathbb{F} f_2 = \{a + bt : a, b \in \mathbb{F}\}$.

So S is linearly independent.

Proposition 3 *Every subspace of a finite-dimensional vector space is finite-dimensional.*

Sketch of the proof: Let W be a subspace of a finite-dimensional vector space V. Since V is of finite-dimensional. We may assume that $V = \mathtt{Span}\{v_1, \ldots, v_n\}$. We show that W can be spanned by a set with less than or equal to n elements by the following algorithm.
Case 1. If $W = \{0\}$, then W is spanned by the empty set. In particular, W is of finitely-dimensional.

Case 2. If $W \neq \{0\}$, then take a nonzero element $w_1 \in W$. Clearly $\{w_1\}$ is linearly independent.

Case 2.a. If $W = \mathtt{Span}\, w_1$, then W is of finite-dimensional. We are done.

Case 2.b. Take $w_2 \in W \setminus \mathtt{Span}(w_1)$, i.e, w_2 is an element in W, but not in $\mathtt{Span}\, w_1$. By Lemma 3, we know that $\{w_1, w_2\}$ is linearly independent.

Similarly, we will consider whether $W = \mathtt{Span}(w_1, w_2)$ or not. If $W = \mathtt{Span}(w_1, w_2)$, then we are done. If $W \neq \mathtt{Span}(w_1, w_2)$, then we should take the element w_3 in W but not in $\mathtt{Span}(w_1, w_2)$. Then the Lemma 3 implies that the new set $\{w_1, w_2, w_3\}$ is still linearly independent. We should repeat this construction.

This construction will terminates because the length of the linearly independent element should be less than or equal to n. When it terminates after m steps, we have

$$W = \mathtt{Span}(w_1, \ldots, w_m)$$

and $m \leq n$.

Example 10 $P(\mathbb{F})$ are infinite-dimensional.

Proof 1: We prove it by contradiction. Assume that $P(\mathbb{F})$ is finite-dimensional, i.e., it is spanned by finitely many elements $f_1(t), \ldots, f_n(t)$. Let m denote the highest degree of these polynomials. Then every polynomial in $P(\mathbb{F})$, as a linear combination of $f_1(t), \ldots, f_n(t)$, has degree at most m. Thus $P(\mathbb{F}) \subseteq P_m(\mathbb{F})$. But any polynomial of degree $m+1$ is not in $P_m(\mathbb{F})$, which is a contradiction. Thus $P(\mathbb{F})$ is infinite-dimensional.

Proof 2: We prove it by contradiction. Assume that $P(\mathbb{F})$ is finite-dimensional, i.e., it is spanned by finitely many elements $f_1(t), \ldots, f_n(t)$. However, for any $m \geq n$, we have seen that the set $\{1, t, t^2, \ldots, t^m\}$ is linearly independent. By Proposition 2, we have $m + 1 < n$, which is a contradiction.

Example 11 $\mathbb{R}^{\mathbb{R}}$ are infinite-dimensional.

Solution: We can check that $P(\mathbb{R})$ is a subspace of $\mathbb{R}^{\mathbb{R}}$. We leave this part to the reader. Then by Proposition 3 and the last example that $P(\mathbb{F})$ is infinite-dimensional, we deduce that the larger space $\mathbb{R}^{\mathbb{R}}$ is also infinite-dimensional.