Solutions

Math 131 Section 01 Worksheet 3: §1.C, § 2.A

Name: ______________________ Jan 22, 2017

(1) Determine whether the subset

$$\mathbb{R}^2 = \{(a,b) \in \mathbb{C}^2 \mid a, b \in \mathbb{R}\} \subseteq \mathbb{C}^2$$

is a subspace of $\mathbb{C}^2$ over the field $\mathbb{C}$.

No. Not closed over imaginary scalars.

Consider nonzero $(a,b) \in \mathbb{R}^2$. Then $i \cdot (a,b) = (ia, ib) \notin \mathbb{R}^2$.

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(2) Suppose $\{v_1, v_2, v_3, v_4\}$ spans V. Prove that the subset

$$\{v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\}$$

also spans V.

Let $x \in V$ be arbitrary.

Since $\{v_1, v_2, v_3, v_4\}$ spans V, $\exists$ scalars $a_1, a_2, a_3, a_4 \in \mathbb{F}$

such that $x = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4$.

Consider the linear combination

$$a_1 \cdot (v_1 - v_2) + (a_2 + a_1) \cdot (v_2 - v_3) + (a_3 + a_2 + a_1) \cdot (v_3 - v_4) + (a_4 + a_3 + a_2 + a_1)(v_4)$$

$$= (a_1) v_1 + (a_2) v_2 + (a_3) \cdot v_3 + (a_4) v_4.$$

Therefore, $x \in \mathrm{Span}\{v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\}$

$\Rightarrow \mathrm{Span}\{v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\} = V$ since x is arbitrary.

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(3) Show that the following 4 matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

spans the linear space $\mathtt{Mat}_{2\times 2}(\mathbb{F})$

Let $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in Mat_{2\times 2}(\mathbb{F})$ be arbitrary.

Then $a_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \left(\frac{1}{2}a_{12}\right)\cdot\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} + \left(\frac{1}{3}a_{21}\right)\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} + \left(\frac{1}{4}a_{22}\right)\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$\Rightarrow \operatorname{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}\right\} = Mat_{2\times 2}(\mathbb{F})$ //

(4) Let V be a vector space over $\mathbb{F}$ and let $y, x_1, x_2 \in V$. Show that if $y \in \mathtt{Span}(x_1, x_2)$, then $\mathtt{Span}(y, x_1, x_2) = \mathtt{Span}(x_1, x_2)$.

Clearly, $\operatorname{Span}(y, x_1, x_2) \supseteq \operatorname{Span}(x_1, x_2)$.

To show that $\operatorname{Span}(y, x_1, x_2) \subseteq \operatorname{Span}(x_1, x_2)$,

let $a_y, a_1, a_2 \in \mathbb{F}$ and consider

$a_y \cdot y + a_1 x_1 + a_2 \cdot x_2 \in \operatorname{Span}(y, x_1, x_2)$.

Since $y \in \operatorname{Span}(x_1, x_2)$ by assumption,

$\exists\, b_1, b_2 \in \mathbb{F}$ such that $y = b_1 x_1 + b_2 x_2$.

Then,

$$\begin{aligned} a_y y + a_1 x_1 + a_2 x_2 &= a_y(b_1 x_1 + b_2 x_2) + a_1 x_1 + a_2 x_2 \\ &= a_y b_1 x_1 + a_y b_2 x_2 + a_1 x_1 + a_2 x_2 \\ &= (a_y b_1 + a_1)x_1 + (a_y b_2 + a_2)x_2 \\ &\in \operatorname{Span}(x_1, x_2). \end{aligned}$$

Therefore, the claim follows. ///