# Math 131: Linear Algebra

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## 1.C. Subspaces

- Know the definition of subspaces and know how to check it.
- Intersection of subspaces
- Sum of subspaces
- Direct Sum (postponed)

### 3.1 Subspace and Criterion of subspace

**Definition 1** A subset $U$ of $V$ is called a *subspace* of $V$ if $U$ is also a vector space (using the same addition and scalar multiplication as on $V$).

To check a subset to be a subspace, we have the following criteria.

**Proposition 1** *A nonempty subset $U \subset V$ of a vector space $V$ is a subspace if and only if it is closed under addition and scalar multiplication:*

$$x, y \in U \Rightarrow x + y \in U,$$

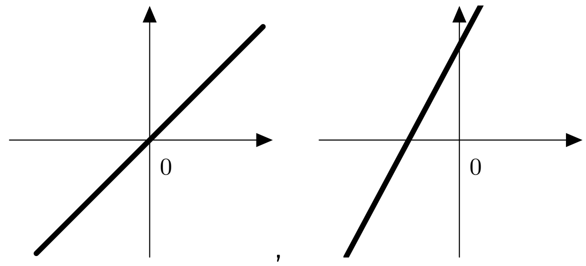
$$c \in \mathbb{F}, x \in U \Rightarrow cx \in U.$$

*We also say that $U$ is closed under vector addition and multiplication by scalars.*

*Sketch of the proof: It is clear that a subspace is closed under addition and scalar multiplication. Conversely, Assume that $U$ is closed under scalar multiplication. Then the addition and scalar multiplication make sense on $U$. We just need to check that they do satisfy all properties of vector spaces. For instance, take $c = 0$ and any $v \in U$, we deduce that the origin $0 = 0v$ is also in $U$. Similarly, we take $c = -1$, then $-x = (-1)x$ is also in $U$. The other parts of the definition of a vector space, such as associativity and commutativity, are automatically satisfied for $U$ because they hold on the larger space $V$.*

In particular, a subspace is closed under the multiplication by the scalars 0 and $-1$. That means the zero element and $-v$ should be in the subspace $U$ for any $v \in U$. We can use this property to rule out many counter-examples.

**Example 1 (Geometric)** The left is a subspace of $\mathbb{R}^2$. The right is NOT a subspace of $\mathbb{R}^2$.




*The right is not a subspace because for most points on the line, its additive inverses is not on the line.*

**Remark 1**

1. Geometrically, the subspaces of $\mathbb{R}^2$ are precisely $\{0\}$, $\mathbb{R}^2$, and all lines in $\mathbb{R}^2$ through the origin $0$.

2. Similarly, the subspaces of $\mathbb{R}^3$ are precisely $\{0\}$, $\mathbb{R}^3$, all lines in $\mathbb{R}^3$ through the origin, and all planes in $\mathbb{R}^3$ through the origin.

3. Clearly $\{0\}$ is the smallest subspace of $V$ and $V$ itself is the largest subspace of $V$. The empty set is not a subspace of $V$ because a subspace must be a vector space and hence must contain at least one element, namely, an additive identity, origin.

**Example 2 (Algebraic)**

*Let $H_1$ be the plane with equation $2x + 4y + 5z = 0$ and $H_2$ the plane with equation $2x + 4y + 5z = 1$. Which plane is a subspace of $\mathbb{R}^3$?*

**Solution:**

*$H_1$ is a subspace. We prove it by using Proposition 1.*

*Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ be two points on the plane $H_1$, which implies that*

$$2x_1 + 4y_1 + 5z_1 = 0 \text{ and } 2x_2 + 4y_2 + 5z_2 = 0$$

*Hence*

$$2x_1 + 4y_1 + 5z_1 + 2x_2 + 4y_2 + 5z_2 = 0$$

$$2(x_1 + x_2) + 4(y_1 + y_2) + 5(z_1 + z_2) = 0$$

*Hence $u + v = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$ is also in $H_1$.*

*Similarly, for any $c \in \mathbb{F}$ and any $u = (x_1, y_1, z_1)$ on $H_1$, we can check that*

$$2(cx_1) + 4(cy_1) + 5(cz_1) = c(2x_1 + 4y_1 + 5z_1) = 0.$$

*Hence $cu = (cx_1, cy_1, cz_1)$ is also on $H_1$. $H_1$ is closed under addition and scalar multiplication. Proposition 1 implies that $H_1$ is a subspace.*

*$H_2$ is not a subspace for many reasons. For instance, since it is not closed under scalar multiplication. For example, $(1, 1, -1)$ is a point on $H_2$, but the scalar multiplication $2(1, 1, -1) = (2, 2, -2)$ is not on $H_2$ since*

$$2 \cdot 2 + 4 \cdot 2 + 5 \cdot (-2) = 2 \neq 1.$$

*Also, if $H_2$ is a subspace, it should contain the origin as the zero vector. But clearly*

$$2 \cdot 0 + 4 \cdot 0 + 5 \cdot 0 = 0 \neq 1.$$

Algebraically, given a plane equation with coefficients in $\mathbb{F}$, it is a subspace if and only if the equation is homogeneous, i,e., it is in the form $Ax + By + Cz = 0$.

**Example 3** *The set of polynomials whose coefficients lie in the field $\mathbb{F}$:*

$$P(\mathbb{F}) = \mathbb{F}[t] = \{p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_k t^k : k \in \mathbb{N}, a_0, a_1, \ldots, a_k \in \mathbb{F}\}$$

*is a vector space. We can consider the subset of polynomials with degree no higher than $n$. That is*

$$P_n = \{p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_k t^k : k \in \mathbb{N}, k \leq n, a_0, a_1, \ldots, a_k \in \mathbb{F}\}.$$

*$P_n$ is a subspace of $\mathbb{F}[t]$.*

**Example 4** *The set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbb{R}^{[0,1]}$.*

*The set of continuous real-valued function $f : [0, 1] \to \mathbb{R}^1$ with $f(0) = 0$ is a subspace of $\mathbb{R}^{[0,1]}$. We leave the proof to the discussion section.*

*The set of continuous real-valued function $f : [0, 1] \to \mathbb{R}^1$ with $f(0) = 1$ is NOT a subspace of $\mathbb{R}^{[0,1]}$. The reason is that this section doesn't have the zero function as the zero element.*

## 3.2 Intersection and Sum of two subspaces

**Proposition 2** *Suppose $U_1$ and $U_2$ are subspaces of $V$. Prove that the intersection $U_1 \cap U_2$ is a subspace of $V$.*

*Proof: Homework*

**Example 5** *Find an example of two subspace $U_1$ and $U_2$ and show that $U_1 \cup U_2$ is not a subspace of $V$.*

**Solution:** *There are a lot of such examples. For example, we take $U_1$ and $U_2$ as the $x$-axis and $y$-axis on the $xy$-plane $V$. Check that the union is not closed under addition.*

**Definition 2** Suppose $U_1, \ldots, U_m$ are subsets of $V$. The *sum* of $U_1, \ldots, U_m$, denoted by $U_1 + \cdots + U_m$, is the set of all possible sums of elements of $U_1, \ldots, U_m$. More precisely,

$$U_1 + \cdots + U_m := \{u_1 + \cdots + u_m \mid u_1 \in U_1, \ldots, u_m \in U_m\}.$$

We first focus on the sum of only two terms, i.e., $m = 2$, we consider $U_1 + U_2$:

$$U_1 + U_2 := \{u_1 + u_2 \mid u_1 \in U_1, u_2 \in U_2\}.$$

**Example 6** *Suppose $U$ is the $x$-axis of $\mathbb{R}^3$, and $W$ is the $y$-axis of $\mathbb{R}^3$. I.e.*

$$U = \{(a, 0, 0) \mid a \in \mathbb{R}\}$$

*and*

$$W = \{(0, b, 0) \mid b \in \mathbb{R}\}$$

Then

$$U + W = \{(a, b, 0) \mid a, b \in \mathbb{R}\},$$

is the $xy$ plane.

$$U \cap W = \{(0, 0, 0)\} = \{0\}.$$

**Example 7** Suppose $U$ is the $(y = x)$-line of $\mathbb{R}^3$, and $W$ is the $y$-axis of $\mathbb{R}^3$. I.e.

$$U = \{(a, a, 0) \mid a \in \mathbb{R}\}$$

and

$$W = \{(0, b, 0) \mid b \in \mathbb{R}\}$$

Then

$$U + W = \{(a, a + b, 0) \in \mathbb{R}^3 \mid a, b \in \mathbb{R}\} = \{(a, b', 0) \in \mathbb{R}^3 | a, b' \in \mathbb{R}\}$$

is also the $xy$ plane.

$$U \cap W = \{(0, 0, 0)\} = \{0\}.$$

This is because for any $a$ and $b$ in $\mathbb{R}$, if $(a, a, 0) = (0, b, 0)$, then $a = 0$ and $a = b$. Hence $b = 0$ too. Hence the only vector in $U \cap W$ is $(0, 0, 0)$.

**Remark 2** Hence different subspaces may give the same sum.

**Example 8** Suppose $U = \{(a, b, 0, 0) \in \mathbb{R}^4 \mid a, b \in \mathbb{R}\}$, and $W = \{(0, c, d, 0) \in \mathbb{R}^4 \mid c, d \in \mathbb{R}\}$. I.e. Then

$$U + W = \{(a, b + c, d, 0) \mid a, b, c, d \in \mathbb{R}\} = \{(a, e, d, 0) \mid a, e, d \in \mathbb{R}\}.$$

$$U \cap W = \{(0, b, 0, 0) \mid b \in \mathbb{R}\}.$$

**Proposition 3** *Suppose $U$ and $W$ are subspaces of $V$. Then $U + W$ is the smallest subspace of $V$ containing $U$ and $W$.*

*Sketch of the proof: there are 2 things we should prove.*

1. *$U + W$ is a subspace of $V$.*
2. *Let $V'$ be a subspace of $V$ which contains $U$ and $W$. we need to show that $U+W \subseteq V'$. This is ture because $V'$ is closed under addition. Hence $U+W$ is the smallest subspace which contains $U$ and $W$.*