Math 131: Linear Algebra

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3.D Isomorphism

- Isomorphisms
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10.1 Isomorphisms

Definition 1 We say that a linear map $L : V \to W$ is an *isomorphism* if we can find a linear map $K : W \to V$ such that $LK = 1_W$ and $KL = 1_V$.

In this case, we say L is *invertible*. The linear map K, characterized by the identities $LK = 1_W$ and $KL = 1_V$, is unique. It is called the *inverse* of L. We often denote it by L^{-1}.

Definition 2 Two vector spaces V and W over $\mathbb{F}$ are said to be *isomorphic* if we can find an isomorphism $L : V \to W$. It is denoted by $V \simeq W$, or $V \cong W$.

Lemma 1 *V and W are isomorphic if and only if there is a bijective linear map $L : V \to W$.*

Sketch of the proof:

$\Rightarrow$: *If V and W are isomorphic, then there exists linear maps $L : V \to W$ and $K : W \to V$ such that $LK = 1_W$ and $KL = 1_V$. We need to show that L is bijective.*

- *Assume $L(x) = L(y)$ for some $x, y \in V$. Then*

$$x = 1_V(x) = K(L(x)) = K(L(y)) = 1_V(y) = y.$$

So L is injective.

- *For any $w \in W$, choose $v = K(w) \in V$. Then*

$$L(v) = L(K(w)) = 1_W(w) = w.$$

So L is surjective. Therefore L is a linear bijective map. We illustrate these by the following *diagram. When we study algebra, it is useful to draw many diagrams to keep track elements.*

$$\left\{ \begin{array}{ccc} V & \longrightarrow & W \\ & & \\ x = KL(x) = KL(y) = y & \longleftrightarrow & L(x) = L(y) \\ & & \\ v = K(w) & \longleftrightarrow & w = L(K(w)) = L(v) \end{array} \right\}.$$

$\Leftarrow$: *If L is linear and bijective, then we construct a map $K : W \to V$ in the following way. Let $w \in W$. Then since L is surjective, $\exists v \in V$ such that $w = L(v)$. Since L is injective, this v is unique. Therefore we let $K(w) = v$.*

We need to prove that K defined in this way is linear and satisfies $KL = 1_V$ and $LK = 1_W$.

- *(Linearity): For any $x, y \in W$, $K(x + y)$ satisfies $L(K(x + y)) = x + y$, $K(x)$ satisfies $L(K(x)) = x$ and $K(y)$ satisfies $L(K(y)) = y$. Therefore $L(K(x + y)) = L(K(x)) + L(K(y)) = L(K(x) + K(y))$. Since L is injective, $K(x + y) = K(x) + K(y)$.*

  For any $x \in W$, $a \in \mathbb{F}$, $K(x)$ satisfies $L(K(x)) = x$ and $K(ax)$ satisfies $L(K(ax)) = ax$. Then $L(aK(x)) = aL(K(x)) = ax = L(K(ax))$. Since L is injective, $aK(x) = K(ax)$.

  Therefore K is linear.

- $(KL = 1_V)$*: For any $v \in V$. $K(L(v))$ is the element satisfies $L(K(L(v))) = L(v)$. Since L is injective, $K(L(v)) = v$.* $(LK = 1_W)$*: This directly comes from the definition of K.*

To sum up, we get a linear map $K : W \to V$ such that $LK = 1_W$ and $KL = 1_V$. Therefore W and V are isomorphic to each other.

There are two important examples of isomorphisms.

Example 1 Let V be a finite-dimensional vector space. Choose a basis $S = \{v_1, \ldots, v_n\}$ of V. Then there exists an isomorphism such that $V \simeq \mathbb{F}^n$.

Solution: Let $L : V \to \mathbb{F}^n$ be the map defined in the following way: $\forall x \in V$, write $x = c_1v_1 + \ldots + c_nv_n$. Then L maps x to its coordinates with respect to the basis S, i.e.,

$$L(x) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

First, we need to show that it is linear. We did this in section §3.C. Let us recall it here. Let $x, y \in V$ and $a \in \mathbb{F}$. We have $x = \sum_{i=1}^n c_i v_i$ and $y = \sum_{i=1}^n d_i v_i$.

$$L(x) + L(y) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix} = L(x+y),$$
$$L(ax) = L\left(\sum_{i=1}^n (ac_i)v_i\right) = \begin{bmatrix} ac_1 \\ \vdots \\ ac_n \end{bmatrix} = a\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = aL(x).$$

Then L is linear.

Then we need to show that L is bijective. If $L(x) = L(y)$, then $c_i = d_i$ for any $i = 1, \ldots, n$. So $x = \sum_{i=1}^n c_i v_i = \sum_{i=1}^n d_i v_i = y$. Then L is injective. For any $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, we know that $L(\sum_{i=1}^n c_i v_i) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. Then L is surjective. To sum up, L is bijective. By Lemma 2, L is an isomorphism.

Example 2 Let V, W be two vector spaces. Choose a basis $\mathcal{E} = \{e_1, \ldots, e_n\}$ of V and a basis $\mathcal{F} = \{f_1, \ldots, f_m\}$ of W. Then there exists an isomorphism such that $\mathcal{L}_{\mathbb{F}}(V, W) \simeq \mathtt{Mat}_{m\times n}(\mathbb{F})$.

Solution: Let $\Phi : \mathcal{L}_{\mathbb{F}}(V, W) \to \mathtt{Mat}_{m\times n}(\mathbb{F})$ be the map defined as follows. Let $L \in \mathcal{L}_{\mathbb{F}}(V, W)$. Use the method in section §3.C, we can write it as a matrix $[L]_{\mathcal{F}\leftarrow\mathcal{E}}$. Define

$$\Phi(L) = [L]_{\mathcal{F}\leftarrow\mathcal{E}}.$$

We need to show that

1. Φ is linear.
2. Φ is bijective.

The proof is very similar to the previous example, Hence we omit it here.

10.2 Dimension

Dimension is a numerical invariant of vector space in the sense that it is preserved under isomorphisms.

Theorem 1 *Let U and V are two finite dimensional vector spaces. Then U and V are isomorphic if and only if $\dim U = \dim V$.*

Sketch of the proof:

$\Rightarrow$: *Suppose that there is an isomorphism $L : U \to V$. We show that $\dim U = \dim V$. We apply the Fundamental Theorem of linear maps.*

$$\dim U = \dim \mathtt{null}(L) + \dim \mathtt{range}(L).$$

Now L is an isomorphism. Hence L is injective and surjective. Hence $\mathtt{null}(L) = \{0\}$ *and* $\mathtt{range}(L) = V$. *Hence* $\dim \mathtt{null}(L) = 0$ *and* $\dim \mathtt{range}(L) = \dim V$. *By the fundamental theorem of linear maps, we have* $\dim U = 0 + \dim V = \dim V$.

If we understand the proof of Fundamental Theorem of linear maps. An isomorphism L maps a basis of U to a basis of V.

$\Leftarrow$*: Assume that* $\dim U = \dim V = n$. *We want to construct an isomorphism $L : U \to V$. If such a map L exists, we know that it maps a basis of U to a basis of V.*

Now we take a basis $S = \{u_1, \dots, u_n\}$ of U and $T = \{v_1, \dots, v_n\}$, a basis of V. We define a linear map

$$L : U \to V$$

as follows. The idea is that L maps u_i to v_i for each i. We have seen that any linear map from U to V are given by the images of u_i. More explicitly, let u be a vector of S. Then we can write $u = \sum_{i=1}^{n} a_i u_i$ for unique coefficients $a_i \in \mathbb{F}$. We define

$$L(v) := \sum_{i=1}^{n} a_i v_i.$$

Equivalently, the matrix associated to the above linear map with respect to the basis S and T is the identity matrix. We leave it as an exercise to check that L is an isomorphism.

10.3 Injectivity is equivalent to surjectivity in finite dimensional

In section §2.C, that if a set is of the right size, that is equal to the dimension, then it is linearly independent is equivalent to the statement that it spans the space. Hence we only need to check one of the condition.

Between finite-dimensional vector spaces of the same dimension, in order to check isomorphism, we also only need to check injectivity or surjectivity.

Theorem 2 *Suppose $T : U \to V$ is a linear map between two finite dimensional vector spaces. If* $\dim U = \dim V$, *then the following are equivalent:*

(a) T is an isomorphism;

(b) T is injective;

(c) T is surjective.

Sketch of the proof:

It is clear that (a) is equivalent bijectivitiy, which is equivalent to (b) and (c). In particular, (a) implies (b) and (c).

Now we prove that (b) implies that (a) and (c). In fact, we only need to prove (b) implies (a). By the Fundamental Theorem of linear maps, we have

$$\dim U = \dim \mathtt{null}(T) + \dim \mathtt{range}(T).$$

Now (b) T is injective implies that $\dim \mathtt{null}(T) = 0$. *Hence* $\dim \mathtt{range}(T) = \dim U = \dim V$. $\mathtt{range}(L)$ *as a subspace of V has the same dimension of V. Hence* $\mathtt{range}(T) = V$. *Hence T is surjective.*

Now we prove that (c) surjectivity implies (a) and (b). It is suffice to show that (c) surjectivity implies (b). T is surjective means that $\texttt{range}(T) = V$, which implies that $\dim \texttt{range}(T) = \dim V = \dim U$. *Hence* $\dim \texttt{null}(T) = 0$. *This means T is injective. Hence T is injective. Plus T is already surjective. We deduce that T is an isomorphism.*

In general, the above theorem fail when U or V is of infinitely dimensional. For example, the derivative map or integration map on the function space $\mathbb{R}[x]$.

In practice, injectivity is usually easier to check. Then we can use it to show surjectivity. For instance, we can show that the following ODE has solutions without solving it. Also we are dealing with infinite-dimensional space, we can restrict the map to finite dimensional vector space to verify the desired conclusion.

Example 3 Show that for every polynomial $q(t) \in \mathbb{R}[t]$, there exists a polynomial $p(t) \in \mathbb{R}[t]$ such that

$$((t^2 + 5t + 7)p(t))'' = q(t).$$

Solution: Let $m = \deg q(t)$. It is enough to show that there exist a polynomial $p(t)$ of degree (less than or) equal to m such that $((t^2 + 5t + 7)p(t))'' = q(t)$.

Consider the map

$$\begin{aligned} T : P_m(\mathbb{R}) &\to P_m(\mathbb{R}) \\ p(t) &\to ((t^2 + 5t + 7)p(t))'' \end{aligned}$$

We have seen that T is a linear operator on $P_m(\mathbb{R})$. Once we show that T is injective, then it is surjective. Then there exists a polynomial $p(t)$ as a solution to the given ODE equation. Now we show that $\texttt{null}(T) = \{0\}$. Let $f(t) = \sum_{i=0}^{n} a_i t^i$ be polynomial of degree n for some n. Then

$$f''(t) = \sum_{i=2}^{n} a_i \cdot i \cdot (i-1)t^{i-2}.$$

Hence if $f''(t) = 0$ then $a_i = 0$ for any $i \geq 2$. Equivalently, $f(t) = a_0 + a_1 t$. Hence $\deg f(t) \leq 1$. Hence $((t^2 + 5t + 7)p(t))'' = 0$ means $\deg((t^2 + 5t + 7)p(t)) \leq 1$. This means $p(t) = 0$. (Otherwise, $\deg((t^2 + 5t + 7)p(t)) = 2 + \deg p(t) > 1$, we get a contradiction.)