Math 131: Linear Algebra

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1.B. Vector spaces

- Memorize the definition of vector spaces.
- Know examples of vector spaces: $\mathbb{F}^n$, P_n, $\mathtt{Mat}_{n\times m}(\mathbb{F})$.
- Know how to use general statements to produce examples.
- Practice proof skills.

2.1 Definition and properties of vector spaces

Definition 1 A *vector space* consists of a set of vectors V and a field $\mathbb{F}$.

- The vectors can be added to yield another vector: if $x, y \in V$, then $x + y \in V$.
- The scalars can be multiplied with the vectors to yield a new vector: if $a \in \mathbb{F}$ and $x \in V$, then $ax \in V$.
- The vector space contains a *zero vector* 0, also known as the *origin* of V.

Addition and scalar multiplication satisfy the following axioms:
For any $x, y, z \in V$ and $a, b \in \mathbb{F}$,

1. The associative law for vectors:
$$x + (y + z) = (x + y) + z.$$
2. The commutative law for vectors:
$$x + y = y + x.$$
3. Addition by 0:
$$x + 0 = x.$$
4. Existence of negative vectors: For each x, we can find y such that
$$x + y = 0.$$
We also give a name to y: $-x$.
5. The associative law for scalar product:
$$a(bx) = (ab)x.$$

6. Multiplication by $1 \in \mathbb{F}$:
$$1x = x.$$

7. The distributive law when vectors are added:
$$a(x + y) = ax + ay.$$

8. The distributive law when scalars are added:
$$(a + b)x = ax + bx.$$

Remark 1

- We won't distinguish $0 \in \mathbb{F}$ and $0 \in V$. You should recognize it from the contexts around it.
- The scalar multiplication in a vector space depends on $\mathbb{F}$. We should say V is a vector space *over* $\mathbb{F}$. A vector space over $\mathbb{R}$ is called a *real vector space*. A vector space over $\mathbb{C}$ is called a *complex vector space*.

Remark 2 The proofs below are tediously long because we should practice writing proofs rigorously. After you finish this training and really know which steps can be skipped, you are allowed to write shorter proofs in this course.

Proposition 1 *In a vector space V, the negative of a vector v is unique.*

Proof: The proof is very similar to the uniqueness of negative in a field. Let u and w be two negatives of v. Then by the definition of negative,

$$v + u = 0, \quad v + w = 0. \tag{1}$$

Then

$$\begin{aligned}
u &= u + 0 \quad \text{(by Addition by 0)} && (2)\\
&= u + (v + w) \quad \text{(by (1))} && (3)\\
&= (u + v) + w \quad \text{(by the associative law for vectors)} && (4)\\
&= (v + u) + w \quad \text{(by the commutative law for vectors)} && (5)\\
&= 0 + w \quad \text{(by (1))} && (6)\\
&= w. \quad \text{(by Addition by 0)} && (7)
\end{aligned}$$

Remark 3 Similar to the field case, after this proposition, we can use the notation "$-v$" freely. Moreover, we define
$$u - v := w + (-v).$$

Proposition 2 *Let V be a vector space over a field $\mathbb{F}$. If $x \in V$ and $a \in \mathbb{F}$, then:*

1. *$0x = 0$. Note that the first 0 is the scalar, and the second 0 is the origin vector.*
2. *$a0 = 0$. Note that both 0s are the origin vector.*
3. *$(-1)x = -x$.*
4. *If $ax = 0$, then either $a = 0 \in \mathbb{F}$ or $x = 0 \in V$.*

Proof:

1.
$$0x + 0x = (0+0)x = 0x, \text{ (by the distributive law for scalar)} \tag{8}$$

$$\begin{aligned}
(0x + 0x) + (-0x) &= 0x + (-0x) \quad \text{(by addition of } -0x\text{)} && (9)\\
0x + (0x + (-0x)) &= 0x + (-0x) \quad \text{(by the associative law for vectors)} && (10)\\
0x &= 0x + (-0x) \quad \text{(by the existence of negative vectors)} && (11)\\
0x &= 0 \quad \text{(by the existence of negative vectors)} && (12)\\
& && (13)
\end{aligned}$$

2.
$$a0 + a0 = a(0+0) = a0, \quad \text{(by the distributive law and Addition by 0)} \tag{14}$$

$$\begin{aligned}
a0 &= a0 + 0 \quad \text{(by Addition by 0)} && (15)\\
&= a0 + (a0 + (-a0)) \quad \text{(by the existence of negative vectors)} && (16)\\
&= (a0 + a0) + (-a0) \quad \text{(by the associative law for vectors)} && (17)\\
&= a0 + (-a0) \quad \text{(by (14))} && (18)\\
&= 0 \quad \text{(by the existence of negative vectors)} && (19)
\end{aligned}$$

3. *If we can check that*
$$(-1)x + x = 0,$$
then by the uniqueness of negative vector, we have $(-1)x = -x$.
Now we check that
$$(-1)x + x = (-1)x + 1x = (-1+1)x = 0x = 0,$$
by part 1.

4. *If* $ax = 0$, *assume* $a \neq 0$. *Then* a *has an inverse* $a^{-1} \in \mathbb{F}$ *since* $\mathbb{F}$ *is a field. So*

$$\begin{aligned}
x &= 1x \quad \text{(by multiplication by } 1 \in \mathbb{F}\text{)} && (20)\\
&= (a^{-1}a)x \quad \text{(by the existence of inverses in a field } \mathbb{F}\text{)} && (21)\\
&= a^{-1}(ax) \quad \text{(by the associative law for scalar product)} && (22)\\
&= a^{-1}0 \quad \text{(by the assumption)} && (23)\\
&= 0 \quad \text{(by (2) above)} && (24)
\end{aligned}$$

Therefore, either $a = 0$ *or* $x = 0$.

Example 1 Trivial vector space: $V = \{0\}$. Scalar product: $a0 = 0, \forall a \in \mathbb{F}$. Addition: $0 + 0 = 0$.

Example 2 Here let us work out a few examples. We know that $\mathbb{F}^n$ is always a vector space. Here $\mathbb{F}$ refers to a field, and n refers to an integer. They can be any fields and any integers. So let's just choose any as you wish.

1. $\mathbb{F} = \mathbb{R}$, $n = 2$. Then we get $\mathbb{R}^2$. It is the real plane which you should be very familiar with. Now we practice to write elements in column vectors. Then its

elements look like $\begin{bmatrix} a \\ b \end{bmatrix}$ and you know how to do addition and scalar product.

2. $\mathbb{F} = \mathbb{C}$, $n = 2$. Then we get $\mathbb{C}^2$. It is the complex plane. Its elements look like $(a+bi, c+di)$ or $\begin{bmatrix} a+bi \\ c+di \end{bmatrix}$.

By changing $\mathbb{F}$ and n, we can get infinite many examples we want. And the previous proof guarantee that all we get are vector spaces.

Example 3 We can apply theorems or propositions to these examples. For example, we can apply Proposition 2 to all examples we get above. In Proposition 2 (1), we know that $0x = 0$ for any $x \in V$. Therefore we have

1. In $\mathbb{R}^2$, choose any vector. For example, $\begin{bmatrix} -100 \\ 1000 \end{bmatrix}$. Then we have
$$0 \cdot \begin{bmatrix} -100 \\ 1000 \end{bmatrix} = 0.$$

2. In $\mathbb{C}^2$, choose any vector. For example, $\begin{bmatrix} -100i \\ 0.111 \end{bmatrix}$. Then we have
$$0 \cdot \begin{bmatrix} -100i \\ 0.111 \end{bmatrix} = 0.$$

You have to try to construct examples by yourselves from those general statements.

2.2 Pool of examples of vector spaces

We will see a lot of examples of vector spaces. Besides the stardard vector spaces $\mathbb{F}^n$, we list some of them here. In later sections, we will revisit these examples in different contexts.

Example 4 The set of polynomials whose coefficients lie in the field $\mathbb{F}$:
$$\mathbb{F}[t] = \{p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_k t^k : k \in \mathbb{N}, a_0, a_1, \ldots, a_k \in \mathbb{F}\}$$
is a vector space. In our book, they use the notation $P(\mathbb{F})$, instead of $\mathbb{F}[t]$.

Example 5 $\mathtt{Mat}_{n\times m}(\mathbb{F})$: the set of all matrices of size $n \times m$ whose entries are numbers in the field $\mathbb{F}$. Addition and scalar product are defined in the obvious way. The proof of $\mathtt{Mat}_{n\times m}(\mathbb{F})$ being a vector space is very standard. We leave it as an exercise to the reader.

Example 6 $\mathbb{F}^S$ is a vector space.

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let S be a set, then $\mathbb{F}^S$ denotes the set of functions from S to $\mathbb{F}$. We now define the addition and scalar multiplication on $\mathbb{F}^S$.

For $f, g \in \mathbb{F}^S$, the sum $f + g \in \mathbb{F}^S$ is the function defined by
$$(f+g)(x) := f(x) + g(x)$$

for all $x \in S$.

For $\alpha \in \mathbb{F}$ and $f \in \mathbb{F}^S$, the scalar multiplication αf is the function defined by

$$(\alpha f)(x) := \alpha f(x)$$

for all $x \in S$.

In the vector space $\mathbb{F}^S$, the zero element is the constant function with value 0.

Example 7

- If $S = [0, 1]$, then $\mathbb{F}^S$ means the vector space of all functions on $[0, 1]$ with values in $\mathbb{F}$.
- Let $S = \{1, 2, \ldots, n\}$. Then we can identify $\mathbb{F}^S$ with $\mathbb{F}^n$.
- If $S = \mathbb{N}$ the set of natural numbers, we also write $\mathbb{F}^S = \mathbb{F}^\infty$.