Math 131: Linear Algebra

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2.B. Bases

- Know the definition and criterion for bases
- examples of basis
- reduce a basis from a spanning list
- extend a basis from a linearly independent list

Definition 1 A **basis** of V is a set of vectors in V that is linearly independent and spans V.

Example 1 Consider $\mathbb{R}^2$. $S = \{v_1 = (1,0), v_2 = (0,1)\}$. S is a basis of $\mathbb{R}^2$.

- We have seen that for any vector $v = (a, b)$ in $\mathbb{R}^2$, we have
$$v = av_1 + bv_2.$$
Hence S spans $\mathbb{R}^2$.
- The set S is also linearly independent. For any $a, b \in \mathbb{R}$ such that $av_1 + bv_2 = 0$, then $(a, b) = av_1 + bv_2 = (0, 0)$. Hence $a = b = 0$.

Proposition 1 *A set $\{v_1, \ldots, v_m\}$ of vectors in V is a basis of V if and only if every $v \in V$ can be written UNIQUELY in the form*
$$v = a_1v_1 + \ldots + a_mv_m.$$

Sketch of the proof:

$\Rightarrow$. *We assume that $\{v_1, \ldots, v_m\}$ is a basis of V and show that every element $v \in V$ can be written uniquely in the form $v = a_1v_1 + \ldots + a_mv_m$. Since* $\mathrm{Span}(v_1, \ldots, v_n) = V$. *Hence $v = a_1v_1 + \ldots + a_mv_m$ for some $a_i \in \mathbb{F}$. We only need to show the uniqueness. Assume that*
$$v = a_1v_1 + \ldots + a_mv_m = b_1v_1 + \ldots + b_mv_m,$$
for some $a_i \in \mathbb{F}$ and $b_i \in \mathbb{F}$. Then we have
$$(a_1v_1 + \ldots + a_mv_m) - (b_1v_1 + \ldots + b_mv_m) = 0.$$
Equivalently,
$$(a_1 - b_1)v_1 + \ldots + (a_m - b_m)v_m = 0.$$
Since $\{v_!, \cdots, v_m\}$ is linearly independent. Hence for every i, $a_i - b_i = 0$. Hence we express v as a linear combination of $\{v_1, \ldots, v_m\}$ uniquely.

$\Leftarrow$. *The idea is similar to the above argument. Every $v \in V$ can be written in the form*

$$v = a_1 v_1 + \ldots + a_m v_m$$

means that $\{v_1, \ldots, v_m\}$ spans S. Now take $v = 0$ and consider to write

$$0 = a_i v_1 + \ldots + a_m v_m.$$

By the uniqueness of such expression and the fact that $0 = 0 \cdot v_1 + \ldots + 0 \cdot v_m$. We know that all a_i has to be zero. This implies the linearly independence.

5.1 Examples in $\mathbb{R}^2$

Example 2 We revisit the above example. Consider $S = \{v_1 = (1,0), v_2 = (0,1)\} \subset \mathbb{R}^2$. For any vector $v = (a, b)$ in $\mathbb{R}^2$, we have

$$v = av_1 + bv_2.$$

and the expression is unique. Therefore S forms a basis of $\mathbb{R}^2$.

Example 3 In $\mathbb{R}^2$. $S = \{v_1 = (1,1), v_2 = (0,1)\}$ is a basis. We use Proposition 1 to check it. For any vector $x = (x_1, x_2)$ in $\mathbb{R}^2$, we write

$$x = av_1 + bv_2 = (a, a+b) = (x_1, x_2).$$

We can check that $a = x_1$ and $b = x_2 - x_1$ is the unique solution of the above equation. Hence $\{v_1 . v_2\}$ is also a basis of $\mathbb{R}^2$.

5.2 Examples of bases in other spaces

Example 4 $\mathtt{Mat}_{n\times m}(\mathbb{F})$ has a natural basis $\{E_{ij}\}$ of nm elements, where E_{ij} is the matrix that is zero in every entry except the (i, j)-th position, where it is 1.

Example 5 Show that $\{1, t, \ldots, t^n\}$ forms a basis for P_n.

Solution: By definition, $P_n = \{a_0 + a_1 t + \ldots + a_n t^n \mid a_0, \ldots, a_n \in \mathbb{F}\}$. So for any $p(t) \in P_n$, there exists $a_0, \ldots, a_n \in \mathbb{F}$ such that $p(t) = a_0 + a_1 t + \ldots + a_n t^n$. Thus any polynomial in P_n is a linear combination of $1, t, \ldots, t^n$.

For $p(t) \in P_n$, assume that there exists two linear combinations

$$a_0 + a_1 t + \ldots + a_n t^n = b_0 + b_1 t + \ldots + b_n t^n.$$

Then we have $(a_0 - b_0) + (a_1 - b_1)t + \ldots + (a_n - b_n)t^n = 0$, as the zero polynomial. This implies $a_0 = b_0$, $a_1 = b_1, \ldots, a_n = b_n$. Therefore the expression is unique. Then $1, t, \ldots, t^n$ is a basis.

The same argument can be applied to the polynomial ring $\mathbb{R}[x]$.

Example 6 (Optional) Show that $\{1, t, \ldots, t^n, \ldots\}$ form a basis for $\mathbb{R}[x]$.

Now let us list some examples which are not bases.

Example 7 Consider $\mathbb{R}^2$. $S = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. For any vector $x = \begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$, we have

$$x = av_1 + bv_2 + 0v_3,$$

so any vector can be written as a linear combination of vectors in S. But the expression is not unique. For example,

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = v_1 + v_2 + 0v_3,$$

but also

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0v_1 + 0v_2 + v_3.$$

So the expression is NOT unique. Therefore S doesn't form a basis of $\mathbb{R}^2$.

Example 8 Consider $\mathbb{R}^2$. $S = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$. For any vector $x = \begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$, we cannot guarantee it can be written as a linear combination of vectors in S. For example, let $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Set up the equation $c_1 v_1 + c_2 v_2 = x$:

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The equation doesn't have solutions. So x cannot be written as linear combinations of vectors in S. Therefore S doesn't form a basis of $\mathbb{R}^2$. We can also check that S is not linearly independent neither.

We will learn two methods to obtain a new basis from the given data. We mainly focus on two cases:

- S is a spanning set, but not linearly independent. In this case, we will delete some elements in S to obtain a basis.
- S is a linearly independent set. But it doesn't span the entire vector space. We will add more elements to obtain a basis.

5.3 Construct a basis from a spanning set

Example 9 Consider the set $S = \{v_1 = (1, 1), v_2 = (2, 2), v_3 = (1, -1)\}$. Show that S spans $\mathbb{R}^2$, but not linearly independent. Construct a new basis from the set S.

Solution: We leave it as as exercise to check the span and linearly dependence. There are two methods to obtain a basis.

Method 1: we construct it by deleting elements in order.

(1) Consider the element $v_1 = (1, 1)$. Since it is not the zero element. We keep it.

(2) Consider the element $v_2 = (2, 2)$. Since $v_2 = 2v_1$. It is in $\mathrm{Span}(v_1)$. We should delete v_2 from the spanning set. We can check that $\mathrm{Span}(v_1, v_2) = \mathrm{Span}(v_1)$. This implies that $\mathrm{Span}(v_1, v_3) = \mathrm{Span}(v_1, v_2, v_3) = V$.

(3) Consider the element v_3. We should check that $v_3 \notin \mathtt{Span}(v_1, v_2)$. Then we will keep v_3.

Now we claim $\{v_1, v_3\}$ is a basis of $\mathbb{R}^2$. CLearly

$$\mathtt{Span}(v_1, v_3) = \mathtt{Span}(v_1, v_2, v_3) = V$$

implies that $\{v_1, v_3\}$ spans $\mathbb{R}^2$. It is linearly independent because of the linearly dependent lemma. We prove it by contradiction. Assume it is linearly dependent, then v_3 must be in the span of v_1, which implies that v_3 is in the span of v_1 and v_2. We will get a contradiction.

Method 2: When we prove that $\{v_1 = (1, 1), v_2 = (2, 2), v_3 = (1, -1)\}$ is linear dependent, we get a linear relation:

$$2v_1 - v_2 = 0.$$

In this case we can delete v_1 or v_2 from S and test that the linearly dependence relations for the rest.

For example, we delete v_1. Since $v_1 = \frac{1}{2}v_2$, then $\mathtt{Span}(v_2, v_3) = \mathtt{Span}(v_1, v_2, v_3)$. The new set $\{v_2, v_3\}$ is still a spanning set. We test whether it is linearly independent or not. It turns out to be linearly independent. Then $\{v_2, v_3\}$ is a basis. Similarly, if we delete v_2, then we can check that $\{v_1, v_3\}$ is also a basis.

Proposition 2 *Every finite spanning set S of a finite-dimensional vector space V can be reduced to a basis of the vector space.*

Sketch of the proof:

Method 1: This is the same proof in the book. We add some details to prove that the new set we get still spans the vector space V.

We apply the procedure of the example. Let $S = \{v_1, \ldots, v_n\}$ be a spanning set of V.

(1) *If $v_1 \neq 0$, then we don't change the set S and call it S_1. Otherwise, $v_1 = 0$, we delete it from S and name the new set as S_1. Note that no matter in which case,*

$$V = \mathtt{Span}(S) = \mathtt{Span}(S_1).$$

(2) *Consider the element v_2. If $v_2 \notin \mathtt{Span}(v_1)$, then we should keep v_2 in S_1. Then we take $S_2 = S_1$. Otherwise, we delete v_2 from the set S_1 and get $S_2 = S_1 \setminus \{v_2\}$. We can check that*

$$V = \mathtt{Span}(S_1) = \mathtt{Span}(S_2),$$

since v_2 is a multiple of v_1.

(3) *Consider the element v_3. If $v_3 \notin \mathtt{Span}(v_1, v_2)$, then we should keep v_3 in S_2. Then we take $S_3 = S_2$. Otherwise, we delete v_3 from the set S_2 and get $S_3 = S_2 \setminus \{v_3\}$. We can check that in both cases*

$$V = \mathtt{Span}(S_2) = \mathtt{Span}(S_3).$$

This is because v_3 is a linear combination of v_1 and v_2. Hence v_3 is spanned by the elements in $\{v_1, v_2\} \cap S_2$.

Then we repeat this argument for every element in S until the last step. To make it clear, we also write the last step. We have constructed the subset S_{n-1} and checked that

$$V = \texttt{Span}(S_1) = \cdots = \texttt{Span}(S_{n-1}).$$

(n) *Consider the element v_n. If $v_n \notin \texttt{Span}(v_1, \ldots, v_{n-1})$, then we should keep v_n and take $S_n = S_{n-1}$. Otherwise, we delete v_n from the set S_{n-1} to get S_n and check that*

$$V = \texttt{Span}(S_{n-1}) = \texttt{Span}(S_n).$$

The last equality is because v_n is a linear combination of previously elements

$$\{v_1, v_2, \ldots, v_n\}.$$

Note that every deleted element is a linear combination of the previous elements. We can conclude that v_n is a linear combnation of elements in $\{v_1, v_2, \ldots, v_n\} \cap S_{n-1}$.

Now we only need to check that S_n is a linearly independent set. We prove it by contradiction. If S_n is not linearly independent. Then by the Linear Dependence Lemma, we get some j, v_j is spaned by the previous element in S_n. Then in Step j, we should have deleted v_j. Thus we have a contradiction.

Method 2: If $S = \{v_1, \ldots, v_n\}$ is already linearly independent, then we are done since S is a basis. Otherwise, let

$$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$$

be any nontrivial linearly dependence relation, i.e., $a_i \neq 0$ for some i.

We fix such a v_i, delete v_i from S. Let $S' = S \setminus \{v_i\}$. Since v_i can be solved as a linear combination of elements in S'. Hence

$$\texttt{Span}(S') = \texttt{Span}(v_1, v_2, \ldots, v_n) = V.$$

Hence the subset S' also spans V. We repeat the same argument to S'. This gives an algorithm to delete elements from S. Since the number of elements in S decreases by one each time. The algorithm has to stop after finitely many steps. The set we obtained at the last step is linearly independent. Hence it is a basis.

Corollary 1 *Every finitely-dimensional vector spaces has a basis.*

In fact, arbitrary vector space has a basis by Zorn's Lemma. We do not require the proof in our class.

Proposition 3 *Every linearly independent set of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.*

Sketch of the proofs.
Method 1: Let $S = \{u_1, \ldots, u_n\}$ be the linearly independent set. We take $T = \{w_1, \ldots, w_m\}$ be any spanning set of V. Then

$$S \cup T = \{u_1, \ldots, u_n, w_1, \ldots, w_m\}$$

is also a spanning set. Then we follow the concrete construction of the Method 1 of Propo-

sition 2 to delete elements in $S \cup T$. Since S is linearly independent, we won't delete any element from S and only delete elements in T if necessary.

Method 2: *Given a linearly independent set S of vectors, we only need to add an element $w \in V$ which is not in S. Then the new set $S' = S \cup \{w\}$is also linearly independent. If the set S' spans V, then we are done since it is a basis. Otherwise, the set S' is not a basis, i.e., it doesn't span V, then we repeat this procedure. Since the length of every linearly independent vectors is less than or equal to the length of every spanning vectors. Hence we can only add finitely many element. Then the set we obtain at the last step is a basis.*

These proofs are essentially the same thing. In practice, the second method may be easier once we know the concept of dimension in $2.C$, which tells us exactly how many element we need to add. We illustrate the proof of Proposition 2 and Proposition 3 by the following example.

Example 10 Let $S = \{v_1 = (1,1,1,0), v_2 = (3,1,1,0)\} \subset \mathbb{R}^4$. Check that S is linearly independent. Extend S to be a basis of $\mathbb{R}^4$.

Solution: Note that v_1 is not propositional to v_2, vice versa. This implies that S is linearly independent. Now we start to add elements to S. The potential candidates are the standard basis $T = \{e_1, e_2, e_3, e_4\}$.

We first consider $e_1 = (1,0,0,0)$. We can check that

$$e_1 = \frac{1}{2}(v_2 - v_1) = \frac{1}{2}v_2 - \frac{1}{2}v_1.$$

Hence e_1 is in the span of v_1 and v_2. We should not add it to the set S.

Next, we consider $e_2 = (0,1,0,0)$. We leave it as an exercise to check that

$$e_2 \notin \mathtt{Span}(v_1, v_2).$$

Hence we should add it to S. Now we may update the set S as the new set $\{v_1, v_2, e_2\}$.

Now we consider $e_3 = (0,0,1,0)$. Hence

$$e_3 = v_1 - e_1 - e_2 = v_1 - (\frac{1}{2}(v_2 - v_1)) - e_2 = \frac{3}{2}v_1 - \frac{1}{2}v_2 - e_2.$$

Since we can write e_3 as a linear combination of the elements in the updated set S. We should not add e_3.

At last, we leave it as an We leave it as an exercise to check that

$$e_4 \notin \mathtt{Span}(v_1, v_2, e_2).$$

The hint is that the last component of v_1, v_2 and e_2 are always 0. We should add it to the set S. We update the set S again and obtain a basis

$$\{v_1, v_2, e_2, e_4\}.$$

In section §2.C., we will see that $\mathbb{R}^4$ is of 4-dimensional. It implies that the length of any basis should be 4.