Math 131: Linear Algebra

Zhixian (Susan) Zhu

3.E. Product, Direct Sum and Quotients

Today we study how to construct new vector spaces from given vector spaces. These construction can be generalized to many algebraic objects which we study in MATH 171 and 172. For example, we have similar construction for groups, rings and modules. Essentially, the same idea also works for topology and geometry classes.

- Product (optional)
- Direct sum
- Quotient (optional)

Definition 1 Given two sets V and W, we can form the (Cartesian) product

$$V \times W = \{(v, w) \mid v \in V, w \in W\}.$$

If V and W are vector spaces over the same field $\mathbb{F}$, then the product also has a vector space structure over $\mathbb{F}$. We defined scalar product and addition by

$$a(v, w) = (av, aw),$$
$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2).$$

We leave it as an exercise to check that $V \times W$ is a vector space.

Similarly, we can define products of finitely many vectors spaces $V_1, \ldots, V_m$.
Consider the (Cartesian) product

$$V_1 \times \ldots \times V_m = \{(v_1, \ldots, v_m) \mid v_i \in V_i\}.$$

What is the natual vector space structure on the (Cartesian) product? i.e., how do we defined the addition and scalar multiplication?

Now we consider how to compute the dimension of the product space.

Example 1 *Check that* $\dim \mathbb{R}^2 \times \mathbb{R}^3 = 5$.

Solution: *As a set* $\mathbb{R}^2 \times \mathbb{R}^3 = \{((x, y), (u, v, w)) \mid (x, y) \in \mathbb{R}^2, (u, v, w) \in \mathbb{R}^3\}$. *Note that* $\mathbb{R}^2$ *has the standard basis* $\{e_1 = (1, 0), e_2 = (0, 1)\}$. $\mathbb{R}^3$ *has the standard basis* $\{f_1 = (1, 0, 0), f_2 = (0, 1, 0), f_3 = (0, 0, 1)\}$. *(We can't use* e_i *for the basis of* $\mathbb{R}^3$ *since we have used this notation for the basis of* $\mathbb{R}^2$*.) We need to check that* $\{(e_1, 0), (e_2, 0), (0, f_1), (0, f_2), (0, f_3)\}$ *is a basis of the product space* $\mathbb{R}^2 \times \mathbb{R}^3$. *The reason is that every element* $((x, y), (u, v, w))$ *can be uniquely written as the linear* *combination of* $\{(e_1, 0), (e_2, 0), (0, f_1), (0, f_2), (0, f_3)\}$*:*

$$((x, y), (u, v, w)) = x(e_1, 0) + y(e_2, 0) + u(0, f_1) + v(0, f_2) + w(0, f_3).$$

On the other hand, we can also understand it by constructing an isomorphism between $\mathbb{R}^2 \times \mathbb{R}^3$ *and* $\mathbb{R}^5$*. We define*

$$L : \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^5$$
$$((x, y), (u, v, w)) \to (x, y, u, v, w)$$

We leave it as an exercise to check that this is an isomorphism, i.e., it is linear, injective and surjective.

Proposition 1 *Let V and W be finite-dimensional vectors spaces. Then*

$$\dim(V \times W) = \dim V + \dim W.$$

The proof is similar to the previous example. We use bases of V and W to construct a basis of $V \times W$.

There are two important linear maps related to the product $V \times W$. Let

$$p_1 : V \times W \to V$$

be the projection map to the first factor, which maps (v, w) to v. Let

$$p_2 : V \times W \to W$$

be the projection map to the second factor, which maps (v, w) to w. In fact the product $V \times W$ is described by these two maps. The following theorem is optional.(it won't be on the test.)

Theorem 1 *(Universal property of product) Let V, W be any vector space and p_i the projections defined as above. For any vector space U and linear maps $f : U \to V$ and $g : U \to W$, there exists a unique linear map from $\Phi : U \to V \times W$ such that the following diagrams commute, i.e., $f = p_1 \circ \Phi$ and $g = p_2 \circ \Phi$.*

$$\begin{tikzcd}
V & V \times W \arrow[l, two heads, "p_1"'] \arrow[r, two heads, "p_2"] & W \\
& U \arrow[ul, "f"] \arrow[u, dotted, "\Phi"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Sketch of the proof:

The map Φ is defined by $\Phi(u) := (f(u), g(u))$. We leave it as an exercise to check that the map Φ is linear and it is the unique linear map making the diagrams commutes.

11.1 Direct sum

Definition 2 Let U and W be subspaces of V. Recall that $U + W$ is the smallest subspaces of V which contains U and W. If $U \cap W = \{0\}$, then we say $U + W$ is the *direct sum* of U and W. We denote it by

$$U \oplus W.$$

U and W is called a direct summand of $U \oplus W$.

Using the dimension formula, we have the following criteria that when a sum is a direct sum. When it is indeed a direct sum, then we can get a basis of the direct sum from the union of bases of each direct summand.

Lemma 1 *$U + W$ is a direct sum if and only if $\dim(U + W) = \dim U + \dim V$.*

Sketch of the proof:

Combine the dimension formula

$$\dim U + \dim W = \dim(U \cap W) + \dim(U + W).$$

and the fact that $\dim(U \cap W) = 0$ if and only if $U \cap W = \{0\}$.

Proposition 2 *Let $\{u_1, \ldots, u_n\}$ be a basis of U and $\{w_1, \ldots, w_m\}$ a basis of W. If $U + W$ is a direct sum, then $\{u_1, \ldots, u_n, w_1, \ldots, w_m\}$ is a basis of $U \oplus W$.*

This is the problem 3, worksheet 10. It follows the proof of the dimension formula.

There are two important linear maps related to the direct sum $V \oplus W$. Let

$$l_1 : V \hookrightarrow V \oplus W$$

be the embedding map of the first factor, which maps v to v. Let

$$l_2 : W \hookrightarrow V \oplus W$$

be the embedding of the second factor, which maps w to w. In fact, the direct sum is described by these two maps. The following theorem is optional.(it won't be on the test.)

Theorem 2 *(Universal property of direct sum) Let V, W be any vector spaces and l_i the embedding maps defined as above. For any vector space U and linear maps $f : V \to U$ and $g : W \to U$, there exists a unique linear map from $\Phi : V \oplus W \to U$ such that the following diagrams commute, i.e., $f = \Phi \circ l_1$ and $g = \Phi \circ l_2$.*

$$\begin{tikzcd}
V \arrow[r, hook, "l_1"] \arrow[dr, "f"'] & V \oplus W \arrow[d, dotted, "\Phi"] & W \arrow[l, hook', "l_2"'] \arrow[dl, "g"] \\
& U &
\end{tikzcd}$$

Sketch of the proof:

The map Φ is defined by $\Phi(v + w) := f(v) + g(w)$ for any $u \in U$ and $w \in W$. Here every element in $V \oplus W$ is uniquely written as a sum of the form $v + w$ for $v \in V$ and $w \in W$. We leave it as an exercise to check that the map Φ is linear and it is the unique linear map making the diagrams commute.

In the homework 9, we will show that we can construct many direct sums from basis. More explicitly, suppose V is finite-dimensional, with $\dim V = n \geq 1$. Prove that there exist 1–dimensional subspaces $U_1, \ldots, U_n$ of V such that

$$V = U_1 \oplus \cdots \oplus U_n.$$

The idea of the proof will be revealed in the Worksheet 10, problem 1.

Now let review the proof of fundamental theorem of linear maps and restate it in terms of direct sums.

Theorem 3 (Fundamental Theorem of Linear Maps, revisit) *Let V be a finite-dimensional vector space over $\mathbb{F}$ and $L : V \to W$ a linear map. Then $\texttt{range}(L)$ is finite-dimensional and there exists a subspace U of V such that*

$$V = U \oplus \texttt{null}(L)$$

and the restriction map $L|_U : U \to W$ maps U isomorphically to $\texttt{range}(L)$. As a consequence, we have

$$\dim_{\mathbb{F}} V = \dim_{\mathbb{F}} \texttt{null}(L) + \dim_{\mathbb{F}} \texttt{range}(L).$$

Sketch of the proof:

Recall the proof of the dimension formula for linear maps. We choose a basis of $\texttt{null}(L)$, say $S = \{w_1, \ldots, w_m\}$. Here $m = \dim \texttt{null}(L)$. Then we extend it to a basis of V by adding $u_1, \ldots, u_n$. Hence $\dim V = m + n$. We take $U = \texttt{Span}(u_1, \ldots, u_n)$. Since $\{w_1, \ldots, w_m, u_1, \ldots, u_n\}$ is a basis of V. It implies that $V = U + \texttt{null}\, L$. By the dimension formula, then $U \cap \texttt{null}(L) = \{0\}$. Hence $V = U \oplus \texttt{null}(L)$.

We only need to show that $L|_U : U \to W$ maps U isomorphic to $\texttt{range}\, L$. This is because $\{L(u_1), \ldots, L(u_n)\}$ is a basis of $\texttt{range}(L)$, i.e., via span and linearly independent. Hence $L|_U$ sends the basis of U to a bais of $\texttt{range}(L)$. Hence $L|_U$ is an isomorphism.

Remark 1 The direct summand U is not unique. Consider the linear map.

$$\begin{aligned} L : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x, y) &\to (x, 0). \end{aligned}$$

We can check that $\texttt{null}(L) = \{(0, y) \mid y \in \mathbb{R}\}$. Hence $\texttt{null}(L)$ is 1–dimensional. Let $\{e_2 = (0, 1)\}$ be a basis of $\texttt{null}(L)$. We extend it to a basis of $\mathbb{R}^2$.

If we add $e_1 = (1, 0)$, then $U_1 = \texttt{Span}(e_1)$ and $U_1 \oplus \texttt{null}(L) \cong \mathbb{R}^2$. We can check that the linear map

$$\begin{aligned} L|_{U_1} : U_1 &\to \texttt{range}(L) \subseteq \mathbb{R}^2 \\ (x, 0) &\to (x, 0). \end{aligned}$$

gives an isomorphism from U_1 to $\texttt{range}(L)$.

If we add $u = (1, 1)$, then $U_2 = \texttt{Span}(u) = \{(x, x) \mid x \in \mathbb{R}\}$ and $U_2 \oplus \texttt{null}(L) \cong \mathbb{R}^2$. We can check that the linear map

$$\begin{aligned} L|_{U_2} : U_2 &\to \texttt{range}(L) \subseteq \mathbb{R}^2 \\ (x, x) &\to (x, 0). \end{aligned}$$

also gives an isomorphism from U_2 to $\texttt{range}(L)$.

Given a subspace W of V. There are many different direct summand. In Math 132, we will knew that there is a canonical choice of U, called orthogonal complement after we post an inner product on V.

The subsection "quotient" is optional. This concept plays an important role in upper devision class. For example, in algebra classes, we use it to construct quotient group, to describe finitely generated communicative ring and more importantly, field extensions. In topology classes, we construct quotient spaces.

11.2 Quotient

Definition 3 An *equivalence relation* on the set X is a binary relation $\sim$ satisfying three properties:

- (reflexivity) For every element a in X, $a \sim a$;
- (symmetry) For every two elements a and b in X, if $a \sim b$, then $b \sim a$;
- (transitivity) For every three elements a, b, and c in X, if $a \sim b$ and $b \sim c$, then $a \sim c$.

The *equivalence class* of an element a is denoted $[a]$, and is defined as the subset of X,

$$[a] = \{x \in X \mid a \sim x\}.$$

We can check that $[a] = [b]$ if and only if $a \sim b$.
The set of all equivalence classes, which is called the *quotient set*, is often denoted as

$$X/\sim := \{[a] \mid a \in X\}.$$

In our class, we use equivalence classes to define quotient spaces. Given a subspace $U \subseteq V$. We can define an equivalent relation $\sim$ on V by stating that

$$x \sim y \text{ if and only if } x - y \in U.$$

In this case, we often use the notation V/U for the quotient set. For every element $v \in V$, we write $v + U$ as the equivalence class of v, instead of $[v]$, which is already used to indicate the column vector of v with respect to a basis.

We can check that $v + U = w + U$ as the same equivalence class if and only if $v - w \in U$. It turns out we can use the addition and scalar multiplication on V to define a vector space structure of V/U.

Definition 4 Suppose U is a subspace of V. Then addition and scalar multiplication are defined on V/U by

$$(v + U) + (w + U) = (v + w) + U,$$

and

$$\lambda(v + U) = (\lambda v) + U,$$

for $v, w \in V$ and $\lambda \in F$.

The biggest problem when we dealing with quotient sets is that whenever we define some operations on the quotient set, we need to make sure that the definition is well-defined, i.e., it doesn't depend on the representatives we use. For example, here we need to check that the above addition and scalar multiplication is well-defined.

Lemma 2 *(1) For any $v, v', w, w \in V$ such that $v + U = v' + U$ and $w + U = w' + U$, we have*

$$(v + w) + U = (v' + w') + U.$$

(2) For any $v, v' \in V$ such that $v + U = v' + U$ and any $\lambda \in \mathbb{F}$, we have

$$\lambda(v + U) = \lambda(v' + U).$$

(3) V/U with the above addition and scalar multiplication is a vector space.

Sketch of the proof: we will prove part (1) and leave part (2) and (3) as an exercise to the readers. The identity $v + U = v' + U$ implies that $v - v' \in U$. Similarly, $w + U = w' + U$ implies that $w - w' \in U$. Since U is a subspace of V, then U is closed under addition. Hence

$$(v + w) - (v' + w') = (v - v') + (w - w') \in U.$$

This implies that

$$(v + w) + U = (v' + w') + U.$$

There is a canonical linear map from V to the quotient space.

Definition 5 The *canonical quotient map* $\pi : V \to V/U$ is defined by

$$\pi(v) = v + U.$$

We can check that π is a linear map. For any $v, w \in V$. we can check that π preserves addition.

$$\pi(v + w) = (v + w) + U = (v + U) + (w + U) = \pi(v) + \pi(w).$$

For any $v \in V$ and $\lambda \in \mathbb{F}$. we can check that π preserves scalar multiplication.

$$\pi(\lambda v) = (\lambda v) + U = \lambda(v + U) = \lambda\pi(v).$$

Remark 2

- We can check that π is surjective because for any equivalence class $v + U$ in the quotient space, it is the image of v.
- We claim that $\texttt{null}(\pi) = U$ as the subspace of V. We first check that $U \subseteq \texttt{null}(\pi)$. Let $u \in U$. Then $\pi(u) = u + U = 0 + U$. Hence $u \in \texttt{null}(\pi)$. Now we prove that $\texttt{null}(\pi) \subseteq U$. Let $v \in \texttt{null}(\pi)$. Then $\pi(v) = v + U = 0 + U$. This implies that $v \in U$. Hence $\texttt{null}(\pi) = U$.
- Consider the fundamental theorem of linear map for the map π. It implies that

  $$\dim V = \dim U + \dim(V/U).$$

  More explicitly, we choose a basis of U, say $\{u_1, \ldots, u_n\}$. We add more elements $v_1, \ldots, v_m$ to obtain a basis V. Then $\pi(v_1), \ldots, \pi(v_m)$ is a basis of V/U. In other words, $v_1 + U, \ldots, v_m + U$ is a basis of V/U.

Although the direct summands are not unique, but they are isomorphic to each other.

Lemma 3 *If $V = W_1 \oplus U = W_2 \oplus U$, then $W_1 \cong W_2$.*

Sketch of the proof: Let π be the projection from V to V/U. Theorem 3 implies that $\pi|_{W_1} : W_1 \to V/U$ is an isomorphism. Similarly, $\pi|_{W_2} : W_2 \to V/U$ is also an isomorphism. Hence $W_1 \cong V/U \cong W_2$.

The most important property of the map π is the following universal property.

Theorem 4 *(Universal property of quotient) Let V, W be vector spaces over field $\mathbb{F}$, $U \subseteq V$ a subspace. Then for every linear transformation $L : V \to W$ so that $U \subseteq \texttt{null}(L)$, there exists a unique linear map $L' : V/U \to W$ so that $L' \circ \pi = L$, where $\pi : V \to V/U$.*

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "\pi"'] & W \\ V/U \arrow[ur, dotted, "L'"'] & \end{tikzcd}$$

Sketch of the proof: We define $L' : V/U \to W$ by $L'(v + U) = L(v) \in W$. We need to check that this is well-defined. That means for any v, v' such that $v + U = v' + U$, we should check that $L'(v + U) = L'(v' + U) \in W$.

This is because $v - v' \in U$. Let $u = v - v'$. The inclusion $U \subseteq \texttt{null}(L)$ implies that $L(u) = 0$. Hence we have $L(v) = L(v' + u) = L(v') + L(u) = L(v')$. Hence $L'(v + U) = L'(v' + U)$ in W.

We leave it as an exercise to check that the above L' is the unique linear map satisfying the desired property.