MATH 131—**HOMEWORK** 7

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Q1 Suppose U,V and W are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim(\text{range } ST) \leq \min\{\dim(\text{range } S), \dim(\text{range } T)\}$$

Pf .

Let U,V and W be finite-dimensional vector spaces spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$.

$ST : U \to W$ defined by $ST(u) = S(T(u))$, $u \in U$

range ST = $ST(U) = S(T(U)) = S(\text{range } T)$

$\Rightarrow$ range $ST = \{\text{w} \in W | \exists \text{v} \in \text{range } T : \text{w} = S(v)\}$

$\Rightarrow$ range $ST \trianglelefteq$ range $S \Rightarrow \dim(\text{range } ST) \leq \dim(\text{range } S)$ (1)

Let $B_{T(U)} = \{v_i\}_{i=1}^{n}$ be a basis for range T,

then $B_{S(T(U))} = \{S(v_i)\}_{i=1}^{n}$ spans range ST (by Q2 HW6)

If T is surjective, then range $T = V$,

so range $ST = S(V) =$ range S

If T is not surjective, then either $B_{S(T(U))}$ is linearly independent or not.

If $B_{S(T(U))}$ is linearly independent, then $\dim(\text{range } ST) = n = \dim(\text{ range } T)$

If $B_{S(T(U))}$ is not linearly independent, then we can reduce it to a basis of range ST,

accordingly $\dim(\text{range } ST) < n = \dim(\text{ range } T)$

In both cases, $\dim(\text{range } ST) \leq \dim(\text{ range } T)$ (2)

If $\dim(\text{range } S) < \dim(\text{range } T)$, then $\dim(\text{range } ST) \leq \dim(\text{range } S)$, which is always true by (1).

If $\dim(\text{range } T) \leq \dim(\text{range } S)$, then $\dim(\text{range } ST) \leq \dim(\text{range } T)$, by (2).

So, $\dim(\text{range } ST) \leq min\{ \dim(\text{ range } T), \dim(\text{range } S)\}$.

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Q2 Pf .

■

Q3 Pf .

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Q4 Pf .

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Q5 Pf.

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Q6 Pf.

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