MATH 131—**HOMEWORK** 4

Ricardo J. Acuña
(862079740)

Q1 Suppose $v_1, ..., v_m$ is linearly independent in V and $w \in V$.
Prove that if $v_1 + w, ..., v_m + w$ is linearly dependent, then $w \in Span(v_1, ..., v_m)$.

pf.
Suppose $v_1, ..., v_m$ is linearly independent in V and $w \in V$.
Also, suppose $v_1 + w, ..., v_m + w$ is linearly dependent.
$\Rightarrow a_1(v_1 + w) + ... + a_m(v_m + w) = 0$ and $\exists a_k \in \{a_i\}_1^m \subset \mathbb{F} : a_k \neq 0_{\mathbb{F}}$
Add the additive inverse $-a_m(v_m + w)$ to both sides.
$\Rightarrow b_1(v_1 + w) + ... + b_{m-1}(v_{m-1} + w) = -a_k(v_k + w)$ where $\{b_j\}_1^{m-1} = \{a_i\}_1^m \setminus \{a_k\}$
Here the b_js represent a re-indexing of the a_is with a_k removed.

By the distributive property of scalars over sums of vectors,
$\Rightarrow b_1v_1 + b_1w + ... + b_{m-1}v_{m-1} + b_{m-1}w = -a_kv_k - a_kw$
Add a_kv_k to both sides,
$\Rightarrow b_1v_1 + b_1w + ... + b_{m-1}v_{m-1} + b_{m-1}w + a_kv_k = -a_kw$
Add every $-b_jw$ to both sides,
$\Rightarrow b_1v_1 + ... + b_{m-1}v_{m-1} + a_kv_k = b_1w + ... + b_{m-1}w - a_kw$
By distributive property of vectors over sums of scalars,
$\Rightarrow b_1v_1 + ... + b_{m-1}v_{m-1} + a_kv_k = (b_1 + ... + b_{m-1} - a_k)w$ (1)

Assume $b_1 + ... + b_{m-1} - a_k = 0$
$\Rightarrow 0w = 0$
$\Rightarrow b_1v_1 + ... + b_{m-1}v_{m-1} + a_kv_k = 0$ (2)
Since $a_1(v_1 + w) + ... + a_m(v_m + w) = 0$, was the dependence test equation, and we solved for a_k and then re-indexed the remaining terms with the b_js.
Before adding back the a_kv_k to the left-hand side of the equation the left hand side didn't contain any multiple of v_k.
Now, equation (2) becomes the dependence test equation, for $v_1, ..., v_m$. Since, we know $v_1, ..., v_m$ is linearly independent, this equation has solutions only if all the scalars are 0, in particular $a_k \neq 0$.
$\Rightarrow$ (1) is false—i.e: not equal to zero.
So, by contradiction our assumption is false.
$\Rightarrow b_1 + ... + b_{m-1} - a_k \neq 0$
$\Rightarrow \exists \alpha^{-1} \in \mathbb{F} : \alpha^{-1}(b_1 + ... + b_{m-1} - a_k) = 1_{\mathbb{F}}$.
Multiply by α^{-1} on both sides of (1),
$\Rightarrow \alpha^{-1}(b_1v_1 + ... + b_{m-1}v_{m-1} + a_kv_k) = \alpha^{-1}(b_1 + ... + b_{m-1} - a_k)w = 1w = 1$
By the distributive property of scalars over sums of vectors,
$\Rightarrow \alpha^{-1}b_1v_1 + ... + \alpha^{-1}b_{m-1}v_{m-1} + \alpha^{-1}a_kv_k = w$
Let $\alpha^{-1}b_jv_j = d_lv_l$, $1 \leq l \leq m - 1$, and $\alpha^{-1}a_kv_k = d_mv_m$:
$\Rightarrow d_1v_1 + ... + d_mv_m = w$
$\Rightarrow w \in \text{Span}(v_1, ..., v_m)$

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Q2 Let $V=\{A\in\text{Mat}_{2\times 2}(\mathbb{F})|\ \ \text{tr } A=0\}$. Find a basis of V.

$\forall A\in V:\text{tr } A=0$
$\Rightarrow a_{11}+a_{22}=0$
$\Rightarrow a_{22}=-a_{11}$
In particular any $A\in V$ has the form: $\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & -a_{11}\end{pmatrix}$

So, by alternatively setting each of the three free variables one can get a basis for V:

$B=\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\}$

Span $(B)=a\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+b\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}+c\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}=\begin{pmatrix}a & b\\ c & -a\end{pmatrix}$

Clearly any $W\in$ Span (B) has tr $W=0$ So, Span $(B)=V$
And,

$$a\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+b\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}+c\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\Rightarrow a=b=c=0$$

$\Rightarrow B$ is linearly independent.
So, B is a basis for V.

Q3 Let U be the subspace of $\mathbb{C}^5$ defined by:
$U=\{(z_1,z_2,z_3,z_4,z_5)\in\mathbb{C}^5:6z_1=z_2$ and $z_3+2z_4+3z_5=0\}$.
(a) Find a basis of U.
(b) Extend the basis in part (a) to a basis of $\mathbb{C}^5$.

(I) Let $z_1=1$ and $z_3=z_4=z_5=0$, then $6(1)=z_2\Rightarrow(1,6,0,0,0)\in U$
(II) Let $z_1=0\Rightarrow z_2=0$:
(α) $z_3=0$ and $z_4=3\Rightarrow z_5=-2\Rightarrow(0,0,0,3,-2)\in U$
(β) $z_4=0$ and $z_3=3\Rightarrow z_5=-1\Rightarrow(0,0,3,0,-1)\in U$
(γ) $z_5=0$ and $z_3=2\Rightarrow z_4=-1\Rightarrow(0,0,2,-1,0)\in U$

$B=\{(1,6,0,0,0),(0,0,0,3,-2),(0,0,3,0,-1),(0,0,2,-1,0)\}$ (a)

$\forall z\in$ Span $(B):\exists\ a,b,c,d\in\mathbb{C}:$
$z=a(1,6,0,0,0)+b(0,0,0,3,-2)+c(0,0,3,0,-1)+d(0,0,2,-1,0)=(a,6a,3c+2d,3b-d,-2b-c)$ (0)

$\Rightarrow z_1=a$, and $z_2=6a\Rightarrow 6z_1=z_2$ (1)

$\Rightarrow z_3=3c+2d$ and $z_4=3b-d$ and $z_5=-2b-c$
$\Rightarrow z_3+2z_4+3z_5=3c+2d+2(3b-d)+3(-2b-c)=3c+2d+6b-2d-6b-3c$
$\Rightarrow z_3+2z_4+3z_5=\cancel{3c}+\cancel{2d}+\cancel{6b}-\cancel{2d}-\cancel{6b}-\cancel{3c}=0$ (2)

(1) and (2) $\Rightarrow$ Span $(B)=U$

Solve for a, b, c, d in equation (0) when $z = 0$:
$a(1,6,0,0,0) + b(0,0,0,3,-2) + c(0,0,3,0,-1) + d(0,0,2,-1,0) = 0$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 3 & 0 & -1 \\ 0 & -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0$$

Do Gaussian-Elimination:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & -2 & -1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[-\frac{R_1}{6}+R_2 \mapsto R_2]{R_4 \rightleftharpoons R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & -2 & -1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{2}{3}R_3+R_5 \mapsto R_5} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & -1 & -2/3 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{3}R_4+R_5 \mapsto R_5} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[\frac{1}{3}R_4 \mapsto R_4]{\frac{1}{3}R_3 \mapsto R_4} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[R_3 \rightleftharpoons R_2]{R_4 \rightleftharpoons R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow (0,0,2,-1,0) = -\frac{1}{3}(0,0,0,3,-2) + \frac{2}{3}(0,0,3,0,-1)$
$\Rightarrow (0,0,2,-1,0) = (0,0,0,-1,+\frac{2}{3}) + (0,0,2,0,-\frac{2}{3})$
Since, the augmented matrix is in Reduced Row Echelon Form, I've shown that the first three vectors, of B are linearly independent, and that Span $B = U$. It follows that $B' = B \backslash \{(0,0,2,-1)\} = \{(1,6,0,0,0),(0,0,0,3,-2),(0,0,3,0,-1)\}$ is a basis for U.

Now for part (b):
Have $B'' = B \cup \{e_i\}$, where e_i are the standard basis vectors for $\mathbb{C}^5$ thinking of $\mathbb{C}$ as a vector space over itself—i.e the generator of $\mathbb{C}$ over $\mathbb{C}$ is 1. There, are 5 e_i vectors, and they span $\mathbb{C}^5$, and they're linearly independent, so they form a basis and the dimension of $\mathbb{C}^5$ over $\mathbb{C}$ is 5. The dimension of U is 3. So, we only need to find 2 linearly independent vectors.

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -2 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\frac{1}{6}R_1 + R_2 \mapsto R_2} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -2 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\frac{2}{3}R_4 + R_5 \mapsto R_5} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2/3 & 1 \end{pmatrix} \xrightarrow{3R_5 + R_3 \mapsto R_3} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2/3 & 1 \end{pmatrix}$$

$$\xrightarrow[R_2 \rightharpoondown R_4, R_3 \rightharpoondown R_5]{R_4 \rightharpoonup R_2, R_5 \rightharpoonup R_3} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2/3 & 1 \\ 0 & 0 & 0 & -6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{pmatrix} \xrightarrow{\frac{1}{6}R_5 + R_1 \mapsto R_1} \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2/3 & 1 \\ 0 & 0 & 0 & -6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{pmatrix}$$

Immediately from the intermediate reduction we can see, that e_1 and e_3, are linearly independent with respect to B'. So, my algorithm stops because I've hit the right dimension.
Now $B''' = \{(1,6,0,0,0), (0,0,0,3,-2), (0,0,3,0,-1), (1,0,0,0,0), (0,0,1,0,0)\}$ is indeed a basis for $\mathbb{C}^5$.

Q4 Prove or disprove: there exists a basis $p_0, p_1, p_2, p_3 of P_3(\mathbb{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

pf.

Well, consider the standard basis for $P_3(\mathbb{F})$, namely $\{1, t, t^2, t^3\}$. Now, take 4 linearly independent vectors in the span of that basis. For simplicity, one can see $t^3 + 2t^2$ and $t^3 + t^2$ are not scalar multiples of each other, so they're pairwise linearly independent, both are not degree 2. Notice $t^3+2t^2-(t^3+t^2) = t^2$, and $2(t^3+t^2)-(t^3+2t^2) = t^3$. So the Span $(t^3 + 2t^2, t^3 + t^2)$ = Span (t^2, t^3). Now, we only need to extend it to a basis for $P_3(\mathbb{F})$, we add 1 and t, which are not of degree 2. The original $t^3 + 2t^2$ and $t^3 + t^2$ are already linearly independent, and 1 is not a scalar multiple of them, since there's no scalar that can reduce a power, so we keep 1. Similarly we keep t, t is not a scalar multiple of any sum of them, because scalar multiplication cannot either increase or reduce a power. So, we reach the dimension of the ambient space, and we get a linearly independent set $B = \{t^3 + 2t^2, t^3 + t^2, 1, t\}$ that spans the space. Because we can write combinations of the first two as t^2 and t^3 respectively and take combinations of them with the two other vectors 1 and t to span all of $P_3(\mathbb{F})$. So B is a basis, and no element of B has degree 2.

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Q5 Let $p_0 = 1 + x, p_1 = 1 + 3x + x^2, p_2 = 2x + x^2, p_3 = 1 + x + x^2 \in \mathbb{R}[x]$.
(a) Show that p_0, p_1, p_2, p_3 spans the vector space $P_2(\mathbb{R})$.
(b) Reduce the list p_0, p_1, p_2, p_3 to a basis of $P_2(\mathbb{R})$.

pf.

For part (a):
$a(1 + x) + b(1 + 3x + x^2) + c(2x + x^2) + d(1 + x + x^2) = (a + b + d)1 + (a + 3b + 2c + d)x + (b + c + d)x^2$
So, one can always choose a, b, c, d ranging over the reals, such that p_0, p_1, p_2, p_3 span all of $P_2(\mathbb{R})$.

For part (b):
$kp_0 \neq p_1$ because the degree of p_0 is less than that of p_1. And neither are equal to zero so we keep them.
$p_1 - p_0 = 1 + 3x + x^2 - (1 + x) = 2x + x^2 = p_2$, so we delete p_2.
Now, consider linear combinations of p_0 and p_1 and compare them with p_3. That is suppose that p_3 is a linear combination of them:
$a(1 + x) + b(1 + 3x + x^2) = 1 + x + x^2 \Rightarrow (a + b)1 + (a + 3b)x + bx^2 = 1 + x + x^2$

$$\Rightarrow \begin{cases} a + b = 1 \\ a + 3b = 1 \\ b = 1 \end{cases} \Rightarrow \begin{cases} a + b - b = 1 - 1 \\ a + 3b = 1 \\ b = 1 \end{cases} \Rightarrow \begin{cases} a = 0 \\ 0 + 3b = 1 \\ b = 1 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = \frac{1}{3} \\ b = 1 \end{cases}$$

So, b has to equal two different numbers, so by contradiction p_3 is not a linear combination of p_0 and p_1.
Since, $\{p_0, p_1, p_2\}$ is a linearly independent spanning set it is a basis for $P_2(\mathbb{R})$.

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Q6 Suppose v_1, v_2, v_3, v_4 is a basis of V . Prove that $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is also a basis of V.

pf.

$k_1(v_1 + v_2) + K_2(v_2 + v_3) = 0 \Rightarrow k_1 v_1 + (k_1 + k_2)v_2 + k_2 v_3 = 0$ Since, v_1, v_2, v_3 are linearly independent $k_1 = k_1 + k_2 = k_2 = 0$, in particular both k_1 and k_2 are 0. So $v_1 + v_2$ is independent from $v_2 + v_3$.
Now compare them with $v_3 + v_4$:
$m_1(v_1 + v_2) + m_2(v_2 + v_3) + m_3(v_3 + v_4) = m_1 v_1 + (m_1 + m_2)v_2 + m_2 v_3 + m_3(v_3 + v_4)$
$= m_1 v_1 + (m_1 + m_2)v_2 + (m_2 + m_3)v_3 + m_3 v_4 = 0$
So, by the same reasons m_1 and m_3 are 0. So, $0 + (0 + m_2)v_2 + (m_2 + 0)v_3 + 0 = 0$, so again by the independence of the v_is m_2 must be 0. So, $v_3 + v_4$ is independent of the initial 2.
Finally, $n_1(v_1 + v_2) + n_2(v_2 + v_3) + n_3(v_3 + v_4) + n_4 v_4 = n_1 v_1 + (n_1 + n_2)v_2 + n_2 v_3 + n_3(v_3 + v_4) + n_4 v_4$
$= n_1 v_1 + (n_1 + n_2)v_2 + (n_2 + n_3)v_3 + (n_3 + n_4)v_4 = 0$
Clearly, $n_1 = 0 \Rightarrow 0 + n_2 = 0 \Rightarrow 0 + n_3 = 0 \Rightarrow 0 + n_4 = 0$

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