MATH 131—**HOMEWORK** 7

Ricardo J. Acuña

(862079740)

Q1 Suppose U,V and W are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\text{dim(range } ST) \leq min\{\text{dim(range } S), \text{dim(range } T)\}$$

Pf .

Let U,V and W be finite-dimensional vector spaces spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$.

$ST : U \to W$ defined by $ST(u) = S(T(u))$, $u \in U$

range ST = $ST(U) = S(T(U)) = S(\text{range } T)$

$\Rightarrow$ range $ST = \{\text{w} \in W | \exists \text{v} \in \text{range } T : \text{w} = S(v)\}$

$\Rightarrow$ range $ST \trianglelefteq$ range $S \Rightarrow$ dim(range ST) $\leq$ dim(range S) (1)

Let $B_{T(U)} = \{v_i\}_{i=1}^{n}$ be a basis for range T,

then $B_{S(T(U))} = \{S(v_i)\}_{i=1}^{n}$ spans range ST (by Q2 HW6)

If T is surjective, then range $T = V$,

so range $ST = S(V) =$ range S

If T is not surjective, then either $B_{S(T(U))}$ is linearly independent or not.

If $B_{S(T(U))}$ is linearly independent, then dim(range ST) $= n =$ dim(range T)

If $B_{S(T(U))}$ is not linearly independent, then we can reduce it to a basis of range ST,

accordingly dim(range ST) $< n =$ dim(range T)

In both cases, dim(range ST) $\leq$ dim(range T) (2)

If dim(range S) $<$ dim(range T), then dim(range ST) $\leq$ dim(range S), which is always true by (1).

If dim(range T) $\leq$ dim(range S), then dim(range ST) $\leq$ dim(range T), by (2).

So, dim(range ST) $\leq min\{$ dim(range T), dim(range S)$\}$.

■

Q2 Suppose that V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V.

Pf .

$\forall V, W : \text{dim}(V) \in \mathbb{N} : T \in \mathcal{L}(V, W)$

$(\Rightarrow)$ Assume T is injective

$\Rightarrow$ null $T = \{0\}$

$\Rightarrow \text{dim}(V) = \text{dim}(\text{null } T) + \text{dim}(\text{range } T)$

$= \text{dim}(\{0\}) + \text{dim}(\text{range } T)$

$= 0 + \text{dim}(\text{range } T) = \text{dim}(\text{range } T)$

$\Rightarrow \text{dim}(V) = m = \text{dim}(\text{range } T)$ (0)

Let $B_0 = \{v_i\}_{i=1}^m$, be a basis for V,

then $B = \{T(v_i) = w_i\}_{i=1}^m$, is a basis for range T (by 0)

One can extend B to a basis $B' = \{w_i\}_{i=1}^n, m \leq n$

Define, $S : W \to V$ by $\begin{cases} S(w_i) = v_i, 1 \leq i \leq m \\ S(w_i) = 0, m < i \leq n \end{cases}$

$ST(v_i) = S(T(v_i)) = S(w_i) = v_i = id_V(v_i), 1 < i < m$

$\Rightarrow \exists S \in \mathcal{L}(W, V) : ST = id_V$

$(\Leftarrow)$ Assume $\exists S \in \mathcal{L}(W, V) : ST = id_V$

$\forall v_x, v_y \in V : T(v_x) = T(v_y)$

$\Rightarrow S(T(v_x)) = S(T(v_y))$

$\Rightarrow ST(v_x) = ST(v_y)$

$\Rightarrow id_V(v_x) = id_V(v_y)$

$\Rightarrow v_x = v_y$

$\Rightarrow T$ is injective

So,

T is injective $\Leftrightarrow \exists S \in \mathcal{L}(W, V) : ST = id_V$

■

Q3 Suppose $T \in L(P_2(\mathbb{R}), P_4(\mathbb{R}))$ is the linear map defined by

$$Tp = x^2 p.$$

(1) Find the matrix of T with respect to the standard basis.

$\mathcal{E} = \{1, x, x^2\}$

$\mathcal{F} = \{1, x, x^2, x^3, x^4\}$

$$T1 = x^2 1 = x^2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{F}} \quad Tx = x^2 x = x^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{F}} \quad Tx^2 = x^2 x^2 = x^4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{F}}$$

$$\Rightarrow [T]_{\mathcal{F} \leftarrow \mathcal{E}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) Verify the fundamental theorem of linear maps.

range $T =$ span $\{x^2, x^3, x^4\}$

null $T =$ span $\{0\}$, since $x^2 p = 0$, has only solution $p = 0, \forall x$

dim $P_2(\mathbb{R}) = |\mathcal{E}| = 3 = 0 + 3 =$ dim $\{0\}+$ dim $\{x^2, x^3, x^4\}$ ✓

Q4 Let $S, T \in \mathcal{L}(V, W)$ and $\lambda \in F$. Let $\mathcal{E} = \{e_1, ..., e_n\}$ be a basis of V, and $\mathcal{F} = \{f_1, ..., f_m\}$ be a basis of W. Show that there are identities of matrices as following:

$$[S+T]_{\mathcal{F}\leftarrow\mathcal{E}} = [S]_{\mathcal{F}\leftarrow\mathcal{E}} + [T]_{\mathcal{F}\leftarrow\mathcal{E}},$$

and

$$[\lambda S]_{\mathcal{F}\leftarrow\mathcal{E}} = \lambda[S]_{\mathcal{F}\leftarrow\mathcal{E}}.$$

Pf .

Let $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$. Let $\mathcal{E} = \{e_1, ..., e_n\}$ be a basis of V, and $\mathcal{F} = \{f_1, ..., f_m\}$be a basis of W.

$\exists a_{ji} \in \mathbb{F} : S(e_i) = \sum_{j=1}^{m} a_{ji} f_j \Rightarrow [S]_{\mathcal{F}\leftarrow\mathcal{E}} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$

and

$\exists b_{ji} \in \mathbb{F} : T(e_i) = \sum_{j=1}^{m} b_{ji} f_j \Rightarrow [T]_{\mathcal{F}\leftarrow\mathcal{E}} = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}$

$\Rightarrow [S]_{\mathcal{F}\leftarrow\mathcal{E}} + [T]_{\mathcal{F}\leftarrow\mathcal{E}} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$

$(S+T)(e_i) := S(e_i) + T(e_i) = \sum_{j=1}^{m} a_{j,i} f_j + \sum_{j=1}^{m} b_{j,i} f_j = \sum_{j=1}^{m} (a_{j,i} + b_{j,i}) f_j$

$\Rightarrow [S+T]_{\mathcal{F}\leftarrow\mathcal{E}} = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$

$\Rightarrow [S+T]_{\mathcal{F}\leftarrow\mathcal{E}} = [S]_{\mathcal{F}\leftarrow\mathcal{E}} + [T]_{\mathcal{F}\leftarrow\mathcal{E}}$

$\lambda S(e_i) = \lambda \sum_{j=1}^{m} a_{j,i} f_j = \sum_{j=1}^{m} \lambda a_{j,i} f_j$

Also,

$\Rightarrow [S]_{\mathcal{F}\leftarrow\mathcal{E}} = \begin{bmatrix} \lambda a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ \lambda a_{m1} & \cdots & \lambda a_{mn} \end{bmatrix}$

and

$[S]_{\mathcal{F}\leftarrow\mathcal{E}} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \Rightarrow \lambda[S]_{\mathcal{F}\leftarrow\mathcal{E}} = \lambda \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ \lambda a_{m1} & \cdots & \lambda a_{mn} \end{bmatrix}$

$\Rightarrow [\lambda S]_{\mathcal{F}\leftarrow\mathcal{E}} = \lambda[S]_{\mathcal{F}\leftarrow\mathcal{E}}$

■

Q5 Suppose $D \in \mathcal{L}(P_3(\mathbb{R}), P_2(\mathbb{R}))$ is the differential map defined by

$$Dp = p'$$

Find a basis $\mathcal{E}$ of $P_3(\mathcal{R})$ and a basis $\mathcal{F}$ of $P_2(\mathcal{R})$ such that the matrix of D with respect to these bases is

$$[D]_{\mathcal{F}\leftarrow\mathcal{E}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$Dx = 1$, $D\frac{1}{2}x^2 = x$, $D\frac{1}{3}x^3 = x^2$, $D1 = 0$

So, $\mathcal{E} = \{x, \frac{1}{2}x^2, \frac{1}{3}x^3, 1\}$, and $\mathcal{F} = \{1, x, x^2\}$, or $\mathcal{E} = \{x, x^2, x^3, 1\}$, and $\mathcal{F} = \{1, 2x, 3x^2\}$,

or $\mathcal{E} = \{2000 + x, 2000 + x^2, 2000 + x^3, 5\}$, and $\mathcal{F} = \{1, 2x, 3x^2\}$

Really, bunch, infinitely many.

Q6 Find linear maps $S, T \in \mathcal{L}(\mathbb{R}^2)$ such that $ST \neq TS$.

$S, T \in \mathcal{L}(\mathbb{R}^2) : \{e_1, e_2\}$ is the standard basis for $\mathbb{R}^2$

$S : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$S(e_1) = e_1$

$S(e_2) = 2e_2$

$T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$T(e_1) = e_2$

$T(e_2) = 2e_1$

$\Rightarrow ST(e_1) = S(T(e_1)) = S(e_2) = 2e_2$ and $ST(e_2) = S(T(e_2)) = S(2e_1) = 2S(e_1) = 2e_1$

$\Rightarrow TS(e_1) = T(S(e_1)) = T(e_1) = e_2$ and $TS(e_2) = T(S(e_2)) = T(2e_2) = 2T(e_2) = 2(2e_1) = 4e_1$

$\Rightarrow ST \neq TS$