MATH 131—**HOMEWORK** n

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"»n«" := "Statement number n"

Q1 Suppose V is finite-dimensional, with dim $V = n \geq 1$. Prove that there exist 1–dimensional subspaces $U_1, ..., U_n$ of V such that

$$V = U_1 \oplus \cdots \oplus U_n.$$

Pf.

$\forall V :$ dim $V = n \geq 1, n \in \mathbb{N}$

Let $\mathcal{U} := \{u_i\}_{i=1}^n$ be a basis of V

$\forall u_i \in \mathcal{U} : U_i :=$ span$(\{u_i\})$

$\{u_i\}$ is a basis for U_i by construction $\Rightarrow$ dim $U_i = |\{u_i\}| = 1$

if $i = j$, then $U_i \cap U_j = U_i$

if $i \neq j$, then $U_i \cap U_j =$ span $(\{u_i\}) \cap$ span $(\{u_j\})$

$= \{v \in V | v \in$span $(\{u_i\})$ and $v \in$ span $(\{u_j\})\}$

$\Rightarrow \exists a, b \in \mathbb{F} : v = au_i = bu_j \Rightarrow au_i - bu_j = 0$

Since, $u_i, u_j \in \mathcal{U}$ and $\mathcal{U}$ is a basis for V.

It follows that, $a = b = 0$ in the dependence test equation above.

$\Rightarrow v = 0 \Rightarrow U_i \cap U_j = \{0\}$

Since the spaces intersect pairwise on $\{0\}$ as shown,

and $\forall i : 0 \in U_i$, because $U_i =$ span $(\{u_i\})$.

It follows $\bigcap_{i=1}^{n} U_i = \{0\} \Rightarrow \sum_{i=1}^{n} U_i = \bigoplus_{i=1}^{n} U_i$

Now, by definition

$\sum_{i=1}^{n} U_i = \{\sum_{i=1}^n s_i | \ s_i \in U_i\}$

$= \{\sum_{i=1}^n s_i | \ s_i \in$ span $(\{u_i\})\} = \{\sum_{i=1}^n a_i u_i | \ a_i \in \mathbb{F}\} =$ span $(\mathcal{U}) = V$

So, to sum up. for any basis, the span of each singleton subset of the basis is one dimensional. The intersection of the generated spans is $\{0\}$, because pairwise it is so, so their sum as subspaces is a direct sum. And, their direct sum spans the ambient space. So, there exist 1-dimensional subspaces of V such that

$$V = U_1 \oplus \cdots \oplus U_n.$$

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Q2 Suppose that U and V are subspaces of $\mathbb{R}^8$ such that dim $U = 3$, dim $W = 5$, and $U + W = \mathbb{R}^8$. Prove that $\mathbb{R}^8 = U \oplus W$.

Pf .

$\forall\, U, V \trianglelefteq \mathbb{R}^8$: dim $U = 3$, dim $W = 5$, and $U + W = \mathbb{R}^8$

Since, dim $(\mathbb{R}^8) = 8$ and $8 \in \mathbb{N}$

dim $U + W =$ dim $U+$ dim $W+$ dim $U \cap W$ (by Theorem 2 in 2C)

$\Rightarrow 8 = 3 + 5+$ dim $U \cap W$

$\Rightarrow 8 = 8+$ dim $U \cap W$

$\Rightarrow 8 - 8 =$ dim $U \cap W$

$\Rightarrow 0 =$ dim $U \cap W$

$\Rightarrow U \cap W = \{0\}$

$\Rightarrow \mathbb{R}^8 = U \oplus W$

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Q3 Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$U_1 \oplus W = U_2 \oplus W,$$

then $U_1 = U_2$.

span $(\{(1, 0)\})+$ span $(\{(0, 1)\}) = \mathbb{R}^2$

and

span $(\{(1, 0)\})\cap$ span $(\{(0, 1)\}) = \{(0, 0)\}$ Clearly, just look at the intersection of the x and the y axes

$\Rightarrow$ span $(\{(1, 0)\})\oplus$ span $(\{(0, 1)\}) = \mathbb{R}^2$ » -1«

span $(\{(1, -1)\})+$ span $(\{(0, 1)\}) = \mathbb{R}^2$

span $(\{(1, -1)\})\cap$ span $(\{(0, 1)\}) = \{v \in \mathbb{R}^2 |\ v \in$ span $(\{(-1, 1)\})$ and $v \in$ span $(\{(0, 1)\})\}$

$\Rightarrow a, b \in \mathbb{R} : v = b(-1, 1) = a(0, 1)$

$\Rightarrow -b = 0$ and $b = a \Rightarrow v = (0, 0)$

$\Rightarrow$ span $(\{(-1, 1)\})\cap$ span $(\{(0, 1)\}) = \{(0, 0)\}$

$\Rightarrow$ span $(\{(1, -1)\})\oplus$ span $(\{(0, 1)\}) = \mathbb{R}^2$ »1«

» -1« and »1« $\Rightarrow$ span $(\{(1, -1)\})\oplus$span $(\{(0, 1)\}) =$ span $(\{(1, 0)\})\oplus$ span $(\{(0, 1)\})$

$(-1, 1) \notin$ span $(\{(1, 0)\}) \Rightarrow$ span $(\{(1, 0)\}) \neq$ span $(\{(-1, 1)\})$

So, by counterexample Q3 is false

Q4 Suppose $U = \{(x, x, y, y) \in \mathbb{R}^4 | x, y \in \mathbb{R}\}$. Find a subspace W of $\mathbb{R}^4$ such that

$$\mathbb{R}^4 = U \oplus W$$

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For this problem I use facts from Linear Algebra Done Right (LADR), by Sheldon Axler.

Define $< \mathbf{x}, \mathbf{y} >:= \sum_{i=1}^{4} x_i y_i$. We say that if $< \mathbf{x}, \mathbf{y} >= 0$, $\mathbf{x}$ is orthogonal to $\mathbf{y}$ (by 6.11 in LADR).

$U^\perp = \{v \in V| < v, u >= 0 \text{ for every } u \in U\}$ is called the orthogonal complement of U (by 6.45 in LADR)

It's easy to see that, $U = \text{span}(\{(1, 1, 0, 0), (0, 0, 1, 1)\})$

Form the coefficient matrix of U, with the elements of U as row vectors as such:
$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

This coefficient matrix is already in RREF, so I can read it's nullspace.

From column two, I read $(-1, 1, 0, 0)$, and from column 4, I read $(0, 0, -1, 1)$.

I claim $U^\perp = \text{span}(\{(-1, 1, 0, 0), (0, 0, -1, 1)\})$,
since $\{(1, 1, 0, 0), (0, 0, 1, 1)\}$ and $\{(-1, 1, 0, 0), (0, 0, -1, 1)\}$
are bases for U and $U^\perp$ respectively. It's enough to check that:

$< (1, 1, 0, 0), (-1, 1, 0, 0) >= 0$ and
$< (0, 0, 1, 1), (-1, 1, 0, 0) >= 0$ and
$< (1, 1, 0, 0), (0, 0, -1, 1) >= 0$ and
$< (0, 0, 1, 1), (0, 0, -1, 1) >= 0$.

So, $\mathbb{R}^4 = U \oplus U^\perp$ (by 6.47 in LADR)

Q5 Let $U = \{p \in P_4(\mathbb{R}) : p''(4) = 0\}$. (In homework 5, we have computed a basis of U and extended it to a basis of $P_4(\mathbb{R})$). Find a subspace W of $P_4(\mathbb{R})$ such that $P_4(\mathbb{R}) = W \oplus U$. Justify your answer.

I dont' like my answer, so I'm using the bases Susan used in class:

$\mathcal{U} = \{1, t - 4, (t - 4)^3, (t - 4)^4\}$ a basis for U
$\mathcal{P} = \{1, t - 4, (t - 4)^2, (t - 4)^3, (t - 4)^4\}$ a basis for $P_4(\mathbb{R})$

Let $\mathcal{W} = \mathcal{P} \backslash \mathcal{U} = \{(t - 4)^2\}$: $W = \text{span } \mathcal{W}$

$W \cap U = \{p \in P_4(\mathbb{R}) | p \in W \text{ and } p \in U\}$
$\Rightarrow \exists a_i \in \mathbb{R} : p = a_2(t - 4)^2 = a_0(1) + a_1(t - 4) + a_3(t - 4)^3 + a_4(t - 4)^4$
$\Rightarrow 0 = -a_2(t - 4)^2 + a_0(1) + a_1(t - 4) + a_3(t - 4)^3 + a_4(t - 4)^4$

Since, $\mathcal{P}$ is a basis for $P_4(\mathbb{R})$, all a_i must be 0 in the dependence test equation above.

So, $p = 0 \Rightarrow W \cap U = \{0\}$ »23«

Since, adding linear combinations of vectors in $\mathcal{W}$ to linear combinations of vectors in $\mathcal{U}$,
you get linear combinations of vectors in $\mathcal{P}$.
$W + U = \text{span } \mathcal{W} + \text{ span } \mathcal{U} = P_4(\mathbb{R})$ »32«

»23« and »32« $\Rightarrow P_4(\mathbb{R}) = W \oplus U$

Q6 Suppose $\phi \in \mathcal{L}(V, \mathbb{F})$. Suppose $u \in V$ is not in Null ϕ. Let $U = \text{Span}(u)$. Prove that

$$V = \text{Null } \phi \oplus U$$

Pf.:

Suppose $\phi \in \mathcal{L}(V, \mathbb{F})$. Suppose $u \in V$ is not in Null ϕ. Let $U = \text{Span}(u)$.

Null $\phi \cap U = \{v \in V | v \in \text{Null}\phi \text{ and } v \in U\}$

Let $b, c \in \mathbb{F}$, and $u' \in$ Null $\phi : v = bu' = cu$
$\Rightarrow \phi(bu') = \phi(cu)$
$\Rightarrow b\phi(u') = b0 = 0 = c\phi(u)$
$u \notin$ Null $\phi \Rightarrow \phi(u) \neq 0 \Rightarrow c = 0 \Rightarrow v = 0$

Null $\phi \cap U = \{0\}$

Range $\phi = \phi(V) = \{x \in \mathbb{F} | \phi(v) = x, v \in V\}$

Either $v = au$ or $v \neq au$, some $a \in \mathbb{F}$
If $v = au$, then $\phi(v) = \phi(au) = a\phi(u) \neq 0$, whenever $a \neq 0$, since $u \notin$ Null ϕ
If $v \neq au$, then $\phi(v) = 0$, since $v \in$ Null ϕ
So, Range $\phi = \phi(V) = \text{span}(\{\phi(u)\}) \Rightarrow \phi^{-1}(\mathbb{F}) = U$
$\Rightarrow$ dim(Range ϕ) $=$ dim(U)

dim$(V) =$ dim(Null ϕ)$+$ dim(Range ϕ) (by FTLA)
$\Rightarrow$ dim$(V) =$ dim(Null ϕ)$+$ dim(U)

dim(Null $\phi + U$) $=$ dim(Null ϕ)$+$ dim$(U)$$+$ dim(Null $\phi \cap U$)
$=$ dim(Null ϕ)$+$ dim$(U)$$+$ dim$(\{0\})$
$=$ dim(Null ϕ)$+$ dim$(U) + 0$
$=$ dim(Nullϕ)$+$dim(U)

$\Rightarrow$ dim$(V) =$ dim(Null $\phi + U$)

$\Rightarrow V =$ Null $\phi + U$

$\Rightarrow V =$ Null $\phi \oplus U$

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