MATH 131—**HOMEWORK** 2

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Q1 For each of the following subsets of $\mathbb{F}^3$, determine whether it is a subspace of $\mathbb{F}^3$:

(a) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 | x_1 + 2x_2 + 3x_3 = 0\}$

It is!

pf.
Let $W = \{(x_1, x_2, x_3) \in \mathbb{F}^3 | x_1 + 2x_2 + 3x_3 = 0\}$

$\forall u, v \in W$: $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$
$\Rightarrow u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$
$\Rightarrow (u_1 + v_1) + 2(u_2 + v_2) + 3(u_3 + v_3) =$
$u_1 + v_1 + 2u_2 + 2v_2 + 3u_3 + 3v_3 =$
$(u_1 + 2u_2 + 3u_3) + (v_1 + 2v_2 + 3v_3) =$
$0 + 0 = 0$ Since $u, v \in W$.
$\Rightarrow u + v \in W$, since $u + v$ satisfies the defining equation of the set.

$\forall k \in \mathbb{F} : ku = (ku_1, ku_2, ku_3)$
$0 = k0 = k(u_1 + 2u_2 + 3u_3) = ku_1 + k2u_2 + k3u_3 = ku_1 + 2ku_2 + 3ku_3$
$\Rightarrow ku \in W$ Finally, $0 = 0 + 2(0) + 3(0) \Rightarrow (0, 0, 0) \in W$.

W is a non-empty subset of $\mathbb{F}^3$, closed under addition and multiplication. So, W is a subspace of $\mathbb{F}^3$. ■

(b) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 | x_1 + 2x_2 + 3x_3 = 4\}$

It is not, since $0 + 2(0) + 3(0) = 0 \neq 4$, so (0,0,0) is not in the set. That means for some $v \neq 0$ in the set, we have $0v = (0, 0, 0)$ since $0 \in \mathbb{F}$ the set can't be closed under scalar multiplication.

(b) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 | x_1 x_2 x_3 = 0\}$

Nope! $\forall a, b, c \in \mathbb{F} : a > 0$ and $b > 0$, and $c > 0, 0 \in \mathbb{F}$. $(a, b, 0)$ is in the set since $ab0 = 0$, and $(0, 0, c)$ is also in the set, since $0^2 c = 0$. But, since $(a, b, 0) + (0, 0, c) = (a, b, c)$, (a, b, c) is not in the set because $abc > 0$. So the set is not closed under addition.

Q2 Prove or give a counterexample: The intersection of any two subspaces of V is a subspace.

pf.
$\forall W_1, W_2 : W_1, W_2$ are subspaces of V:
$W_1 \cap W_2 = \{w | w \in W_1 \text{and} w \in W_2\}$

W_1 is a subspace of V$\Rightarrow 0 \in W_1 \Rightarrow 0 \in W_1 \cap W_2$
$\Rightarrow W_1 \cap W_2$ is not empty

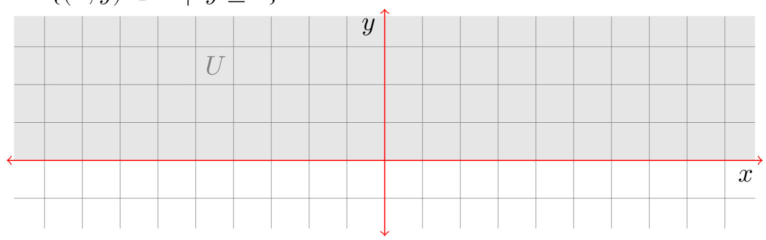
$\forall k \in \mathbb{F}$:Let $u, v \in W_1 \cap W_2$:
$\Rightarrow u \in W_1$ and $v \in W_1 \Rightarrow k(u+v) \in W_1$ Since W_1 is a subspace of V
$\Rightarrow u \in W_2$ and $v \in W_2 \Rightarrow k(u+v) \in W_2$ Since W_2 is a subspace of V
$k(u+v) \in W_1$ and $k(u+v) \in W_2 \Rightarrow k(u+v) \in W_1 \cap W_2$
$\Rightarrow W_1 \cap W_2$ is closed under addition and multiplication.
So, $W_1 \cap W_2$ is a subspace of V ■

Q3 Prove or give a counterexample: If U_1 , U_2 and W are subspaces of V such that $U_1 + W = U_2 + W$, then $U_1 = U_2$.

pf.
$\forall U_1, U_2, W : U_1, U_2, W$ are subspaces of V
$U_1 + W = \{u_1 + w | u_1 \in U_1$ and $w \in W\}$
$U_2 + W = \{u_2 + w | u_2 \in U_2$ and $w \in W\}$
$U_1 + W = U_2 + W$
$\Rightarrow \{u_1 + w | \ u_1 \in U_1$ and $w \in W\} = \{u_2 + w | \ u_2 \in U_2$ and $w \in W\}$
Since W is a subspace of V, $0 \in W$, so I can:
Let $X \subset U_1 + W : X = \{u_1 + 0 | u_1 \in U_1$ and $0 \in W\}$
and $Y \subset U_2 + W : Y = \{u_2 + 0 | u_2 \in U_2$ and $0 \in W\}$
So, restricting both sets to the same element of W, namely $0 \Rightarrow X = Y$.
Now, $\forall u_1 + 0 \in X : u_1 + 0 = u_1 \Rightarrow X = U_1$
, and $\forall u_2 + 0 \in Y : u_2 + 0 = u_2 \Rightarrow Y = U_2$
So, if $U_1 + W = U_2 + W$, then $U_1 = U_2$ ■

Q4 Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under addition, but U is not a subspace of $\mathbb{R}^2$.

$U = \{(x, y) \in \mathbb{R}^2 | \ y \geq 0\}$



Let $(x_1, y_1), (x_2, y_2) \in U$: Since there's no restriction on x, $x_1 + x_2$ is ok, now since y_1 and y_2 are bigger than or equal to 0, $y_1 + y_2$ is greater than or equal to 0. So U is closed under addition. But, $\exists -1 \in \mathbb{F} : -1(x_1, y_1) = (-x_1, -y_1)$ Since y_1 is positive, $-y_1$ is negative. That means $(-x_1, -y_1) \notin U$, so it's not closed under, scalar multiplication.

Q5 Prove or disprove: $U = \{f \in R[x] | f''(x) = 0\}$ is a subspace of $\mathbb{R}[x]$.

pf.

$\forall k \in \mathbb{F} : \forall f, g \in U :$

$f''(x) = 0 \Rightarrow f'(x) = \int 0 \ \ ds = C \in \mathbb{F} \Rightarrow f(x) = \int C dx = Cx + E_1$

Similarly $g(x) = Dx + E_2$, some $D, E_1, E_2 \in \mathbb{F}$.

$\Rightarrow k(f + g)(x) = k(f + g(x)) = k(f(x) + g(x))$

$= k(Cx + E_1 + Dx + E_2) = k(C + D)x + k(E_1 + E_2)$

$\Rightarrow [k(f + g)]'(x) = k(C + D) \in \mathbb{F} \Rightarrow [k(f + g)]''(x) = 0.$

The zero function $0(x) = 0$, has $0''(x) = 0$.

So, U is a non-empty subset of $\mathbb{R}[x]$, with the coefficients of higher powers of x set to 0, that is closed under addition and multiplication.$\blacksquare$

Q6 For each of the following subset of $\text{Mat}_{3\times3}(\mathbb{F})$, determine whether it is a subspace of $\text{Mat}_{3\times3}(\mathbb{F})$. If it is a subspace, prove it. If it is not a subspace, explain why.

(1) $S_1 = \{A \in \text{Mat}_{3\times3}(\mathbb{F}) | \text{ tr}A = 0\}$

pf.:
$\forall A, B \in S_1 : A = (a_{ij})$ and $B = (b_{ij}) \Rightarrow A + B = (a_{ij} + b_{ij})$
$\Rightarrow$ tr $A + B = \sum_{i=1}^{3} a_{ii} + b_{ii} = \sum_{i=1}^{3} a_{ii} + \sum_{i=1}^{3} b_{ii} = 0 + 0 = 0$
$\Rightarrow A + B \in S_1$

$\forall k \in \mathbb{F} : kA = (ka_{ij})$
$\Rightarrow$ tr $kA = \sum_{i=1}^{3} ka_{ii} = k\sum_{i=1}^{3} a_{ii} = k0 = 0$
$\Rightarrow kA \in S_1$

$0_{3x3} \in \text{Mat}_{3x3}(\mathbb{F})$: $0_{3x3} = (z_{ij}) := \forall z_{ij} \in 0_{3x3} : z_{ij} = 0$
$\Rightarrow$ tr $0_{ij} = \sum_{i=1}^{3} z_{ij} = \sum_{i=1}^{3} 0 = 0$
$\Rightarrow 0_{ij} \in S_1$

$\Rightarrow$ S_1 is a non-empty subset of $\text{Mat}_{3\times3}(\mathbb{F})$, that is closed under addition and scalar multiplication.
So, S_1 is a subspace of $\text{Mat}_{3\times3}(\mathbb{F})$ ■

(2) $S_2 = \{A \in \text{Mat}_{3\times3}(\mathbb{F}) | \text{ det}A = 0\}$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

But,
$$\left| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

So, S_2 is not closed under addition.
Note: here |_|:= det (_)