MATH 131—**HOMEWORK** 3

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Q1 Find the plane equation of the subspace $Span((1,1,0),(0,0,1))$ in $\mathbb{R}^3$:

Since $(1,1,0)\cdot(0,0,1) = 0$, we can see $(1,1,0) \neq k(0,0,1)\forall k \in \mathbb{R}$. So, $B = (1,1,0),(0,0,1)$ is a basis for the subspace. The plane containing the subspace can be characterized by its normal vector. Since $(1,1,0)$ and $(0,0,1)$, are two non-zero vectors in the plane,

$$(1,1,0)\times(0,0,1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (1,-1,0).$$

So, we get the family of planes $\pi_d := x-y+z+d = 0$ generated by the normal vector $(1,-1,0)$, we know that $Span(B)$ is a subspace of $\mathbb{R}^3$, so $(0,0,0)$ must be a point in the plane. So, we can solve for d, by setting $x = y = z = 0 \Rightarrow d = 0$. Thus, the plane equation becomes $x - y + z = 0$.

Q2 Show that if $v_1, ..., v_m$ and $w_1, ..., w_n$ are vectors in V , then $Span(v_1, ..., v_m) + Span(w_1, ..., w_n) = Span(v_1, ..., v_m, w_1, ..., w_n)$.

pf.:
$\forall v \in Span(v_1, ..., v_m) : v = \sum_{i=1}^{m} a_i v_i, \{a_i\} \subset \mathbb{F}, i \in [1,m] \subset \mathbb{N}$
and
$\forall w \in Span(w_1, ..., w_n) : w = \sum_{i=1}^{n} b_i w_i, \{b_i\} \subset \mathbb{F}, i \in [1,n] \subset \mathbb{N}$
$\Rightarrow v + w = \sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{n} b_i w_i$
So, any arbitrary $v + w$ is in the $Span(v_1, ..., v_m, w_1, ..., w_n)$, because it is a linear combination of the $vectors\{v_1, ..., v_m, w_1, ..., w_n\}$.
$\Rightarrow lhs \subseteq rhs$

$\forall s \in Span(v_1, ..., v_m, w_1, ..., w_n) : s = \sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{n} b_i w_i$
Chose, $\{a_i\} = \{0\} \Rightarrow s_1 = 0 + \sum_{i=1}^{n} b_i w_i = \sum_{i=1}^{n} b_i w_i \in Span(w_1, ..., w_n)$, so we can always chose vectors type s_1 in the $Span(w_1, ..., w_n)$, for some $\{b_i\}_{i=1}^{n} \subset \mathbb{F}$. Similarly, choosing $\{b_i\} = \{0\}$ gives vectors type $s_2 \in Span(v_1, ..., v_m)$, for some $\{a_i\}_{i=1}^{m} \subset \mathbb{F}$.
So,
we can always find vectors type $s_3 = s_1 + s_2 \in Span(v_1, ..., v_n) + Span(w_1, ..., w_n)$.
$\Rightarrow rhs \subseteq lhs$
$\Rightarrow lhs = rhs$

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Q3 Explain why no set of four polynomials spans $(P_4\mathbb{F})$.

$P_4(\mathbb{F}) = \{p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 | \ i \in [0, 4] \subset \mathbb{N}; \{a_i\} \in \mathbb{F}\}$

pf .
So, a natural basis for $P_4(\mathbb{F})$ is $B = \{1, t, t^2, t^3, t^4\}$,
that is $Span(B) = \sum_{i=0}^{4} a_i t^i = P_4(\mathbb{F})$. This fact can be found on '2A.pdf', Example 5. Notice, that the number of elements in B is equal to 5. So, any set of polynomials $p(t)$, with less that 5 elements cannot Span $P_4(\mathbb{F})$. Because, by Proposition 2 on the same file which reads 'In a finite-dimensional vector space, the length of every linearly independent set is smaller than or equal to the length of every spanning set'. And $4 < 5$ so, no set of four polynomials can span $P_4(\mathbb{F})$, because if the set is linearly independent, then it cannot span it. If the set is not linearly dependent, one of the polynomials can be expressed as a linear combination of the others, so it can be removed without changing the span of the set, so the new set B' of 3 elements also cannot $span P_4(\mathbb{F})$, and doesn't even have four elements.

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Q4 Prove or give a counterexample: Let W and U are two subspaces of V and $x \in V$. If $x \notin W$ and $x \notin U$, then $x \notin W + U$.

Consider the contrapositive statement:
If not($x \notin W + U$), then not ($x \notin W$ and $x \notin U$)
Now, Double Negation:
If $x \in W + U$, then not ($x \notin W$ and $x \notin U$)
by DeMorgan's Law, the statement becomes:
If $x \in W + U$, then not ($x \notin W$) or not ($x \notin U$)
Which becomes by two applications of Double Negation:
If $x \in W + U$, then $x \in W$ or $x \in U$

Suppose $V = \mathbb{R}^3$, $W\{= (k, 0, 0) \in \mathbb{R}^3 | k \in \mathbb{R}\}$ and $U\{= (0, s, 0) \in \mathbb{R}^3 | s \in \mathbb{R}\}$.
So, $W + U = \{(k, s, 0) \in \mathbb{R}^3 | k, s \in \mathbb{R}\}$.

So, clearly this is false, take $(1, 1, 0) \in W + U$, that is not in W, and it is also not in U. So, the original statement can't be true, since $(1, 1, 0) \notin W$ and $(1, 1, 0) \notin U$, but is actually in $W + U$.

Q5 Suppose $\{v_1, v_2, v_3, v_4\}$ is linearly independent in V. Prove or give a counterexample: the subset $\{v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\}$ is also linearly independent.

pf .
$\{v_1, v_2, v_3, v_4\}$ is linearly independent in V $\Rightarrow c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$ (1)
, has only the trivial solution $c_1 = c_2 = c_3 = c_4 = 0$, for some scalars $c_1, c_2, c_3, c_4 \in \mathbb{F}$.

Now, set up the dependence test equation, for some scalars $b_1, b_2, b_3, b_4 \in \mathbb{F}$:
$b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4v_4 = 0$
$\Rightarrow b_1v_1 - b_1v_2 + b_2v_2 - b_2v_3 + b_3v_3 - b_3v_4 + b_4v_4 = 0$
, By the distributive law when vectors are added in Definition 1 property 7 in file '1B.pdf'.
$\Rightarrow b_1v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4 = 0$ (2)
, Since addition is commutative in the field $\mathbb{F}$, and by the distributive law when scalars are added in Definition 1 property 8 in file '1B.pdf'.
Notice, equation (1) implies, equation (2) has the same values for the scalars:
$b_1 = c_1 = 0, b_2 - b_1 = c_2 = 0, b_3 - b_2 = c_3 = 0, b_4 - b_3 = c_4 = 0.$
$\Rightarrow b_2 - b_1 = b_2 - 0 = b_2 = 0$
and $b_3 - b_2 = b_3 - 0 = b_3 = 0$
and $b_4 - b_3 = b_4 - 0 = b_4 = 0$
$\Rightarrow \{v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\}$ is also linearly independent.

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Q6 Prove or give a counterexample: If $\{v_1, ..., v_m\}$ and $\{w_1, ..., w_m\}$ are linearly independent subsets of vectors in V , then $v1 + w1, ..., vm + wm$ is linearly independent.

Take $V = \mathbb{R}^2$, $\{(1, 0)\}$ and $\{(-1, 0)\}$, are both linearly independent, because they are singleton sets, and their element is not the 0 vector. $\{(1, 0) + (-1, 0)\} = \{(0, 0)\}$. Is not linearly independent, since $a(0, 0) = (0, 0)$, has infinitely many non zero solutions $a \in \mathbb{R}$, in particular $a = 1$.