Math 131: Linear Algebra

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2.C. Dimension

- Any bases have the same length
- Dimension counting
- Dimension formula of sum space

6.1 Some properties

Theorem 1 *Any two bases of a finite-dimensional vector space have the same length.*

Sketch of the proof: Let B_1 and B_2 are two bases of a finite-dimensional vector space V. By the symmetry, it is enough to show that the length of B_1 is smaller than the length of B_2. This is true since B_1 is linearly independent and B_2 spans V and the fact that any linearly independent set has less element than any spanning set.

Definition 1 The number of elements in a basis is defined to be the *dimension* of V over $\mathbb{F}$, denoted by $\dim V$.

Remark 1 Note that the basis and the dimension of a vector space highly depends on the field. For example, $V = \mathbb{C}$ can be treated as a vector space over $\mathbb{C}$, over $\mathbb{R}$, or $\mathbb{Q}$, or some other fields. $\dim_{\mathbb{C}} V = 1$, $\dim_{\mathbb{R}} V = 2$, and $\dim_{\mathbb{Q}} V = \infty$.

Example 1 $V = \mathbb{C}$ with complex number addition and real number scalar product forms a $\mathbb{R}$-vector space. Show that V is of dimension 2.

Solution: All complex numbers have the form $a + bi$. Choose $v_1 = 1$ and $v_2 = i$. Try to solve equation for arbitrary complex number $a + bi$:

$$c_1 v_1 + c_2 v_2 = a + bi.$$

No matter what a and b are, $c_1 = a$ and $c_2 = b$ is the only solution. Therefore $\{1, i\}$ forms a basis of V. Then the dimension of V is 2.

Example 2 $V = \mathbb{C}$ with complex number addition and complex number scalar product forms a $\mathbb{C}$-vector space. Show that V is of dimension 1.

Solution: Choose $v_1 = 1$. Try to solve equation for arbitrary complex number c:

$$c_1 v_1 = c.$$

No matter what c is, $c_1 = c$ is the only solution of the equation. Therefore $\{1\}$

forms a basis of V. Then the dimension of V is 1.

Remark 2 Because of the above examples, we call the first dimension $\mathbb{R}$-*dimension* and the second $\mathbb{C}$-*dimension*. You can image other $\mathbb{F}$-dimensions depending on the base fields of the vector spaces.

6.2 Properties of dimension

Proposition 1 *If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$. Furthermore, $U = V$ if and only if $\dim U = \dim V$.*

Sketch of the proof: The idea is that a basis of U is also linearly independent in V. We can extend it to a basis of U. Hence $\dim U \leq \dim V$. If $\dim U = \dim V$, then we don't need to add any new element to the basis of U. Hence it spans V and it is a basis of U.

Proposition 2 *(Criterion for basis I)*
Suppose V is finite-dimensional. Then every linearly independent list of vectors in V with length $\dim V$ is a basis of V.

Proposition 3 *(Criterion for basis II)*
Suppose V is finite-dimensional. Then every spanning list of vectors in V with length $\dim V$ is a basis of V.

6.3 Dimension formula for sum of two vector spaces

Theorem 2 *If U_1 and U_2 are subspaces of a finite-dimensional vector space, then*

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

Sketch of the proof: We first take a basis of $U_1 \cap U_2$, say $\{u_1, \ldots, u_r\}$. Then we extend it to bases of U_1 and U_2 by adding elements $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$, respectively. Clearly, the union $\{u_1, \ldots, u_r, v_1, \ldots, v_m, w_1, \ldots, w_n$ spans $U_1 + U_2$.

If we can prove that $\{u_1, \ldots, u_r, v_1, \ldots, v_m, w_1, \ldots, w_n$ is linearly independent, then

$$\dim(U_1 + U_2) = r + m + n = (r + m) + (r + n) - r = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

The key is to show linearly independence. Read the book for a detailed proof.

We list a couple of examples of dimension counting.

Example 3

1. $\dim \mathbb{F}^n = n$.
2. $\dim P_n(\mathbb{F}) = n + 1$.
3. $\dim \mathtt{Mat}_{k\times m}(\mathbb{F}) = km$.
4. Let $U = \{p(t) \in \mathbb{R}[t] \mid p''(t) = 0\} = 2$ and $V = \mathtt{Span}(1, t^2.t^3)$ in $\mathbb{R}[t]$. Then we

can verify the dimension formula:

$$U + V = P_3(\mathbb{R}) = \mathtt{Span}(1, t, t^2, t^3),$$

and $U \cap V = P_0(\mathbb{R}) = \mathtt{Span}(1)$. Hence we can verify the dimension formula by checking that $\dim U + V = 4, \dim U = 2, \dim V = 3, \dim U \cap V = 1$.