Midterm

Math 131, section 01

Winter 2019

By filling out the box below, I pledge to neither give nor receive help on this exam.

First name (please write as legibly as possible within the boxes)

Susan

Last name

Student ID

Instructions:

- Show ALL your work to receive credit! Unless otherwise specified, an answer without a full explanation might receive no credit.
- Solve each problem on its own page. If you need extra space, use the extra pages at the end, but clearly indicate that you did so.
- No notes, books, or any electronic devices can be used during the exam.
- You have 50 minutes to complete this exam.

Question:	1	2	3	Total
Points:	10	10	10	30

Question 1 (10 points)

(a) (4 points) Show that the subset

$$U = \{(x,y,z) \in \mathbb{R}^3 \mid y + 2z = 0\}$$

is a subspace of $\mathbb{R}^3$.

Pf: · $U \neq \phi$ b/c $(0,0,0) \in U$, This is because $(0,0,0)$ satisfies $y + 2z = 0$

· U is closed under addition

let (x_1, y_1, z_1) and (x_2, y_2, z_2) be any vectors in U.

$\Rightarrow$ $y_1 + 2z_1 = 0$ and $y_2 + 2z_2 = 0$

Consider $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1+x_2, y_1+y_2, z_1+z_2)$

$(y_1+y_2) + 2(z_1+z_2) = (y_1 + 2z_1) + (y_2 + 2z_2)$

$= 0 + 0 = 0$

$\Rightarrow$ $(x_1+x_2, y_1+y_2, z_1+z_2) \in U$.

· U is closed under scalar multiplication.

For any $c \in \mathbb{R}$, $(x,y,z) \in U$. $\Rightarrow$ $y + 2z = 0$.

$c \cdot (x,y,z) = (cx, cy, cz)$

$cy + 2(cz) = c(y+2z) = c \cdot 0 = 0$.

$c(x,y,z) \in U$.

$\therefore$ U is closed under addition and scalar multiplication $\Rightarrow$ it's a subspace

(b) (4 points) Let $U = \{(x,y,z) \in \mathbb{R}^3 \mid y+2z=0\}$. Find a set of basis of U and state the dimension of U. Briefly explain why the set you find is a basis.

Pf. For any $(x,y,z) \in U \Rightarrow y+2z=0 \Rightarrow y=-2z$

$$(x,y,z) = (x,-2z,z) = x(1,0,0) + z(0,-2,1) \quad (*)$$

Claim: $\{(1,0,0), (0,-2,1)\}$ is a basis of U.

Clearly $(*) \Rightarrow \mathrm{span}((1,0,0),(0,-2,1)) = U$.

We only need to check Lin. Ind. let $c, d \in \mathbb{R}$ s.t

$$c(1,0,0) + d(0,-2,1) = 0 = (0,0,0)$$

$$\Rightarrow (c,-2d,d) = (0,0,0) \Rightarrow c=d=0$$

$\dim(U) = 2$.

(c) (2 points) Extend the basis of U you have in part (b) to a basis of $\mathbb{R}^3$. Briefly explain why the new set is a basis.

Sol. Since $\mathbb{R}^3$ has dim 3. We only need to add one vector which isn't in U to span $\mathbb{R}^3$.

Eg $(0,1,0) \notin U$. b/c $1+2\cdot 0 \neq 0$.

Then $\{(1,0,0), (0,-2,1), (0,1,0)\}$ is a basis of $\mathbb{R}^3$.

Question 2 (10 points) Let U and W be subspaces of V.

(a) (3 points) State what it means that "the set of vectors $\{u_1, \ldots, u_n\}$ spans U".

Sol. "the set of vectors $\{u_1, \cdots, u_n\}$ spans U" means.

For any $u \in U$, $\exists a_i \in \mathbb{F}$ s.t. $u = \sum_{i=1}^{n} a_i u_i$

(b) (7 points) If $\{u_1, \ldots, u_n\}$ spans U and $\{w_1, \ldots, w_m\}$ spans W. Using the definitions of span and sum of spaces to show that

$$\{u_1, \ldots, u_n, w_1, \ldots, w_m\}$$

spans $U + W$.

Pf. $\{u_1, \cdots, u_n\}$ spans $U \Rightarrow U = \mathrm{span}(u_1, \cdots, u_n)$

$$= \left\{ \sum_{i=1}^{n} a_i u_i \mid \forall a_i \in \mathbb{F} \right\}$$

$\{w_1, \cdots, w_m\}$ spans $W \Rightarrow W = \mathrm{Span}(w_1, \cdots, w_n)$

$$= \left\{ \sum_{j=1}^{m} b_j w_j \mid \forall b_j \in \mathbb{F} \right\}.$$

$\forall$ element in $U+W$ is in of form $u \in U$, $w \in W$.

Now $u \in U \Rightarrow \exists a_i \quad u = \sum_{i=1}^{n} a_i u_i$

$w \in W \Rightarrow \exists b_j \quad w = \sum_{j=1}^{m} b_j w_j$

$\therefore u + w = \sum_i a_i u_i + \sum_j b_j w_j$ is a linear combination of $\{u_1, \cdots, u_n, w_1, \cdots, w_m\}$

$\therefore \{u_1, \cdots, u_n, w_1, \cdots, w_m\}$ spans $U + W$.

$\boxtimes$

Question 3 (10 points) Circle TRUE or FALSE for each statement. You don't need to give any explanation. If it is false and you give a correct counter-example, then you gain one bonus point.

(a) (2 points) TRUE or FALSE :

Let U_1, U_2, U_3 be subspaces of V, then their intersection $U_1 \cap U_2 \cap U_3$ is also a subspace of V.

We can prove it by checking $U_1 \cap U_2 \cap U_3$ is closed under addition and closed under scalar multiplication.

$U_1 \cap U_2 \cap U_3 \ni 0 \Rightarrow U_1 \cap U_2 \cap U_3 \neq \phi$.

For example: closed under addition.

$\forall x, y \in U_1 \cap U_2 \cap U_3 \Rightarrow x \in U_1$ & $x \in U_2$ & $x \in U_3$.
$y \in U_1$ & $y \in U_2$ & $y \in U_3$

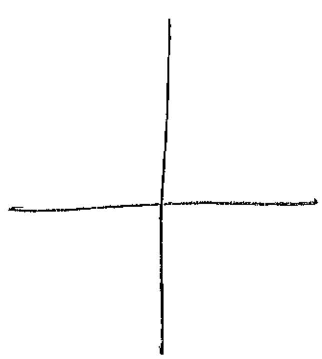
Since U_i is a subspace $\Rightarrow x+y \in U_i$ for any i.

$\Rightarrow x+y \in U_1 \cap U_2 \cap U_3$.

(b) (2 points) TRUE or FALSE :

Let W and U are two subspaces of V and $x \in V$. If $x \notin W$ and $x \notin U$, then $x \notin W + U$.

E.g.



$V = \mathbb{R}^2$

$U =$ x-axis $\Rightarrow U + W = \mathbb{R}^2$

$W =$ y-axis

$x = (1,1) \notin U$

$x \notin W$.

but $x \in W + U$

(c) (2 points) TRUE or FALSE :

If $\{v_1, v_2, \ldots, v_n\}$ is a linearly independent subset of V and $\lambda \neq 0$, then $\{\lambda v_1, \lambda v_2, \ldots, \lambda v_n\}$ is linearly independent.

Consider any a_i s.t. $\sum a_i (\lambda v_i) = 0$.

$$\sum a_i (\lambda v_i) = \lambda \left(\sum a_i v_i \right) = 0$$

$$\lambda \neq 0 \Rightarrow \exists \lambda^{-1} \Rightarrow \lambda^{-1} \lambda \left(\sum a_i v_i \right) = 0$$

$$\Rightarrow \sum a_i v_i = 0 \Rightarrow \forall a_i = 0.$$

$\{v_1, \ldots, v_n\}$ Lin Ind

(d) (2 points) TRUE or FALSE

$\dim \mathtt{Mat}_{n \times m}(\mathbb{R})$ is $n + m$. Here $\mathtt{Mat}_{n \times m}$ stands for the space of matrices of size n by m.

$\dim \mathrm{Mat}_{n \times m}(\mathbb{R}) = nm$ usually $nm \neq n + m$

E.g. $n = 3$, $m = 2$.

$nm = 6 \neq 2 + 3$.

(e) (2 points) TRUE or FALSE :
If $\{u, v\}$ be a linearly dependent set in a vector space V over the field $\mathbb{F}$, then there exists some $a \in \mathbb{F}$ such that $u = av$.

$\{u, v\}$ Lin dep $\Rightarrow \exists\, \alpha, \beta \in \mathbb{F}$. $\alpha u + \beta v = 0$ (*)

α or $\beta \neq 0$

If $\alpha \neq 0$, then $u = -\frac{\beta}{\alpha} v$. OK.

If $\alpha = 0$, then $\beta \neq 0 \Rightarrow$ (*) means $\beta \cdot v = 0 \Rightarrow v = 0$.

$\boxed{\text{When } v = 0,\ u \neq 0,\ \nexists\, a \text{ s.t. } u = av}$

$\{u, v\}$ LIN DEP $\Rightarrow u = av$ or $v = bu$ for some $a, b \in \mathbb{F}$

SCRATCH PAPER

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