MATH 131—HOMEWORK 8

Ricardo J. Acuña

(862079740)

"»n«" := "Statement number n"

Q1 Determine the following linear maps of vector spaces over $\mathbb{R}$ are isomorphism or not. If it is an isomorphism, find its inverse map. (Hint: inverse of matrices.) If it is not an isomorphism, briefly explain why.

(1) (Rotation by 60°)

$$L : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y) \mapsto (\frac{x}{2} - \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x + \frac{1}{2}y)$$

$\mathcal{E} = \{e_1, e_2\} = \mathcal{F} : e_1 = (1, 0), e_2 = (0, 1) :$

$[L]_{\mathcal{F}\leftarrow\mathcal{E}} = \left[\begin{array}{c|c} L(e_1) & L(e_2) \end{array}\right] = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

I know that the transpose will work because, $L(e_i) \perp L(e_2)$, and $|L(e_1)| = |L(e_2)| = 1$— i.e. L is an orthogonal matrix.

$[L]^T_{\mathcal{E}\leftarrow\mathcal{F}} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = [K]_{\mathcal{F}\leftarrow\mathcal{E}}$

Just routine verification will show that $[L][K] = I = [K][L]$

$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} (\frac{1}{2})^2 + (-\frac{\sqrt{3}}{2})^2 & \frac{1}{2}(\frac{\sqrt{3}}{2}) - \frac{\sqrt{3}}{2}(\frac{1}{2}) \\ \frac{\sqrt{3}}{2}(\frac{1}{2}) + \frac{1}{2}(-\frac{\sqrt{3}}{2}) & (\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2 \end{bmatrix}$

$= I =$

$\begin{bmatrix} (\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2 & \frac{1}{2}(-\frac{\sqrt{3}}{2}) + \frac{\sqrt{3}}{2}(\frac{1}{2}) \\ -\frac{\sqrt{3}}{2}(\frac{1}{2}) + \frac{1}{2}(\frac{\sqrt{3}}{2}) & (-\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

Reading back the linear map corresponding to $[K]$ we have, the inverse map K of L:

$K : \mathbb{R}^2 \to \mathbb{R}^2$ defined by
$(x, y) \mapsto (\frac{\sqrt{3}}{2}y + \frac{1}{2}x, \frac{1}{2}y - \frac{\sqrt{3}}{2}x)$

So, L is an isomorphism.

(2) (Reflection about x–axis)

$$L : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y) \mapsto (x, -y)$$

Immediately L is it's own inverse.

$(x, y) \xrightarrow{L} (x, -y) \xrightarrow{L} (x, -(-y)) = (x, y)$

So, L is an isomorphism.

(3)

$$L : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y, z) = (x + 2y + 3z, 4x + 5y + 6z, 7x + 8y + 9z)$$

$\mathcal{E} = \{e_1, e_2, e_3\} = \mathcal{F} : e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1) :$

$[L]_{\mathcal{F}\leftarrow\mathcal{E}} =$

$\left[\begin{array}{c|c|c} L(e_1) & L(e_2) & L(e_3) \end{array}\right] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Gauss-Jordan Elimination:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-7R_1+R_3\mapsto R_3]{-4R_1+R_2\mapsto R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2\mapsto R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

Since, adding a scalar multiple of a row to another doesn't change the determinant, this Gauss-Jordan Matrix in Row Echelon Form (REF), has the same determinant as $[L]$. The determinant of the REF is the trace of the REF which is 0, so the determinant of $[L]$ is 0.

So, L is not invertible.

So, L is not an isomorphism.

Q2 Determine the following spaces are isomorphic or not. If they are isomorphic, give one isomorphism explicitly.

(1) $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^5)$ and $\mathbb{R}^7$.

(by Example 2 in 3D) $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^5) \cong \text{Mat}_{2\times5}(\mathbb{R})$

$\dim(\text{Mat}_{2\times5}(\mathbb{R})) = 2(5) = 10 \neq 7 = \dim(\mathbb{R}^7)$

So, $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^5)$ and $\mathbb{R}^7$ are not isomorphic (by Theorem 1 in 3D)

(2) $\text{Span}((1, 1, 0), (2, 5, 6))$ and $\mathbb{R}^3$

Since, $(1, 1, 0) \neq k(2, 5, 6)$, where $k \in \mathbb{R}$

$\{(1, 1, 0), (2, 5, 6)\}$ is a basis for $\text{Span}((1, 1, 0), (2, 5, 6))$.

So, $\dim(\text{Span}((1, 1, 0), (2, 5, 6))) = |\{(1, 1, 0), (2, 5, 6)\}| = 2 \neq 3 = \dim(\mathbb{R}^3)$

(3) $\{(x,y,z) \in \mathbb{R}^3 | 2x+2y+z=0\}$ and $\mathbb{R}^2$.

$z=-2x-2y \Rightarrow (x,y,z) \mapsto (x,y,-2x-2y)$

$\Rightarrow (1,0,0) \mapsto (1,0,-2)$,$(0,1,0) \mapsto (0,1,-2)$, and $(0,0,1) \mapsto (0,0,0)$

$\Rightarrow\{(x,y,z) \in \mathbb{R}^3 | 2x+2y+z=0\} =$ span($\{(1,0,-2),(0,1,-2)\}$)

$\Rightarrow$dim($\{(x,y,z) \in \mathbb{R}^3 | 2x+2y+z=0\}$) $= |\{(1,0,-2),(0,1,-2)\}| = 2 =$ dim($\mathbb{R}^2$)

(by Theorem 1 in 3D) they're isomorphic

$R : \{(x,y,z) \in \mathbb{R}^3 | 2x+2y+z=0\} \to \mathbb{R}^2$ defined by

$(1,0,-2) \mapsto (1,0)$
$(0,1,-2) \mapsto (0,1)$

Extend R to a linear map for any $a_1, a_2 \in \mathbb{R}$ by,
$R(a_1(1,0,-2)+a_2(0,1,-2)) = a_1R(1,0,-2)+a_2R(0,1,-2)$

$J : \mathbb{R}^2 \to \{(x,y,z) \in \mathbb{R}^3 | 2x+2y+z=0\}$ defined by

$(x,y) \mapsto (x,y,-2x-2y)$

$(1,0,-2) \overset{R}{\to} (1,0) \overset{J}{\to} (1,0,-2)$
$(0,1,-2) \overset{R}{\to} (0,1) \overset{J}{\to} (0,1,-2)$

$\Rightarrow JR = id_{\{(x,y,z)\in\mathbb{R}^3|2x+2y+z=0\}}$

$(1,0) \overset{J}{\to} (1,0,-2) \overset{R}{\to} (1,0)$
$(0,1) \overset{J}{\to} (0,1,-2) \overset{R}{\to} (0,1)$

$\Rightarrow JR = id_{\mathbb{R}^2}$

Q3 Suppose $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U,W)$ is invertible and $(ST)^{-1} = T^{-1}S^{-1}$.

Pf .

Suppose $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$ are both invertible linear maps.

T is invertible $\Rightarrow \exists T^{-1} \in \mathcal{L}(V,U) : TT^{-1} = I_{n\times n} = T^{-1}T, n \in \mathbb{N}$ »1«

$\Rightarrow T$ is an isomorphism $\Rightarrow U \cong V \Rightarrow$ dim$(U) =$ dim$(V) := n$ »α«

S is invertible $\Rightarrow \exists S^{-1} \in \mathcal{L}(W,V) : SS^{-1} = I_{m\times m} = S^{-1}S, m \in \mathbb{N}$ »2«

$\Rightarrow S$ is an isomorphism $\Rightarrow V \cong W \Rightarrow$ dim$(V) =$ dim$(W) := m$ »β«

»α« and »β« $\Rightarrow m = n \Rightarrow I_{n\times n} = I_{m\times m} := I$

$STT^{-1}S^{-1} \overset{»1«}{=\!=} SIS^{-1} \xlongequal{multiply\ by\ I} SS^{-1} \overset{»2«}{=\!=} I$ »3«
$T^{-1}S^{-1}ST \overset{»2«}{=\!=} TIT^{-1} \xlongequal{multiply\ by\ I} T^{-1}T \overset{»1«}{=\!=} I$ »4«

»3« and »4« $\Rightarrow (ST)^{-1} = T^{-1}S^{-1}$

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Q4 Suppose V is a finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.

Pf .

$\forall S, T \in \mathcal{L}(V)$:

($\Leftarrow$) Let $V = U = W$ in Q3 of this homework and the desired result is proven. Namely ST is invertible.

($\Rightarrow$) Ass. ST is invertible

Consider the contrapositive statement, of "if ST is invertible, then both S and T are invertible",
which is "if not both S and T are invertible, then ST is not invertible".
By DeMorgan's Theorem it turns into "if S is not invertible or T is not invertible, then ST is not invertible"

So, further assume S is not invertible.

$\Rightarrow$det$([S]) = 0$

(By, Ass.) ST invertible $\Rightarrow$ det$([ST]) \neq 0$ $\quad$ »α«

But, by the properties of the determinant,

det$([ST])$ = det$([S][T])$ = det$([S])$det$([T]) = 0$ det$([T]) = 0$ $\quad$ »β«

»α« contradicts »β« so ST cannot be invertible if S is not invertible.
By the symmetry of the problem, ST cannot be invertible if T is not invertible.
So, the contrapositive statement is proven. This completes the proof.

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Q5 Suppose V is finite-dimensional and dim V > 1. Prove that the set of noninvertible operators on V is not a subspace of $\mathcal{L}(V)$. (Hint: you have seen this example in previous homeworks when dim V= 3.)

Pf .

Let $\mathcal{F} = \{f_i\}_{i=1}^n$ be a basis for V

$\forall T \in \mathcal{L}(V) : \exists!\ [T]_{\mathcal{F}\leftarrow\mathcal{F}} \in$Mat$_{n\times n}(F)$ and $\mathcal{L}(V) \cong$ Mat$_{n\times n}(F)$

Since, T is invertible det$([T]) \neq 0$

So, $X = \{A \in$Mat$_{n\times n}(\mathbb{F})|$ det$A = 0\}$ is isomorphic to the set of noninvertible operators on V.

E_{nn} is the $n \times n$ matrix with zeros everywhere except for a one at the entry on the nth row and nth column.

det$(I_{n\times n} - E_{nn})$ = tr$(I_{n\times n} - E_{nn}) = 1^{n-1}(0) = 0$

$\Rightarrow I - E_{nn} \in X$

det(E_{nn}) = tr$(E_{nn}) = 0^{n-1}(1) = 0$

$\Rightarrow E_{nn} \in X$

$I_{n\times n} = I_{n\times n} + 0_{n\times n} = I_{n\times n} + (E_{nn} - E_{nn}) = (I_{n\times n} - E_{nn}) + E_{nn}$

det$(I_{n\times n}) = 1^n = 1$

So, X is not closed under addition.

$X \ntrianglelefteq$ Mat$_{n\times n}(F)$

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