Math 131: Linear Algebra

Zhixian (Susan) Zhu

1.A. $\mathbb{R}^n$ and $\mathbb{C}^n$

- Recall set theory notation.
- Know examples of fields: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, etc.
- Definition and properties of field. (optional)
- Study $\mathbb{F}^n$.
- Practice proof skills.

1.1 Set theory

Notation 1 Let S and T be sets.

- If x is an element in S, we denote by $x \in S$.
- If T is a subset of S, then we denote by $T \subseteq S$. Given two subsets $T_1 \subseteq S$ and $T_2 \subseteq S$, we say $T_1 = T_2$ if and only if they have exactly the same elements. If we want to prove that $T_1 = T_2$, we often prove that $T_1 \subseteq T_2$ and $T_2 \subseteq T_1$.
- The expression
$$T = \{x \in S \mid P(x)\}$$
means the subset T which collecting elements x in S such that the statement $P(x)$ is true.
- $\exists$ means there exists an element, $\exists!$ means there exists one and only one element, $\forall$ means for every element.
- Let T_1 and T_2 be two subsets of S. We define the union $T_1 \cup T_2$ and the intersection $T_1 \cap T_2$ as follows:
$$T_1 \cup T_2 := \{x \in S \mid x \in T_1 \text{ or } x \in T_2\};$$
$$T_1 \cap T_2 := \{x \in S \mid x \in T_1 \text{ and } x \in T_2\};$$
- A map $f : S \to T$ is said to be *injective or one-to-one* if for any two elements $s_1 \neq s_2 \in S$, $f(s_1) \neq f(s_2)$.
- A map $f : S \to T$ is said to be *surjective or onto* if for any element $t \in T$, there exists some $s \in S$ such that $t = f(s)$.

Example 1

- *We can check that*

$$\{x \in \mathbb{R} \mid |x| < 1\} = \{x \in \mathbb{R} \mid -1 < x < 1\} = (-1, 1),$$

  representing the subset of the real numbers which is larger than -1 and less than 1. Here $(-1, 1)$ means the open interval. In this class, $(-1, 1)$ has a different meaning. It usually represents a vector on the xy plane.

- *Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2$. We can check that f is injective, but not surjective.*

1.2 Complex numbers

Notation 2

- $\mathbb{Q}$ is the field of rational numbers.
- $\mathbb{R}$ is the field of real numbers.
- $\mathbb{C}$ is the field of complex numbers, which is defined below.

$\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are examples of what are called *fields*. In this class, the letter $\mathbb{F}$ stands for either $\mathbb{R}$ or $\mathbb{C}$. Most of the results we learn in this class also hold for an arbitrary field F.

There are two operations defined over $\mathbb{C}$ (more generally, over any field): **addition** and **multiplication**. We will assume that everyone know how to add and multiply real numbers and apply these rules to study the complex numbers.

Definition 1 $\mathbb{C}$ is the field of all complex numbers. A complex number looks like

$$a + bi \quad \text{where } a, b \in \mathbb{R}.$$

The algebra of two complex numbers are

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

These two operations satisfy many arithmetic properties, which makes $\mathbb{C}$ to be a field. We leave it as an exercise to check all the properties in Defintion 2 for $\mathbb{C}$.

Remark 1 It might be your first time to learn complex numbers. So if you cannot understand it right now, don't be afraid and take your time to practice. The key is to understand the identity

$$i^2 = -1.$$

For instance, we compute the multiplication as following:

$$\begin{aligned}(a+bi)(c+di) &= ac+adi+bic+bidi && (1)\\ &= ac+adi+bci+bdi^2 && (2)\\ &= ac+(ad+bc)i-bd && (3)\\ &= (ac-bd)+(ad+bc)i. && (4)\end{aligned}$$

We will practice more calculations in the homework §1. Complex numbers are ESSENTIAL in MATH132. .

1.3 Field

$\mathbb{R}$ and $\mathbb{C}$ are examples of the algebraic object, which is called field. We list the definition of field here. You may skip this subsection for the first time.

Definition 2 (optional)

A *field* $\mathbb{F}$ is a set of at least two elements 0 and 1. It has two operations $+$ and $\times$ ($\cdot$ or writing nothing), which satisfy the following properties: for ANY $a, b, c \in \mathbb{F}$,

1. The associative law
$$a+(b+c)=(a+b)+c.$$
2. The commutative law
$$a+b=b+a.$$
3. Addition by 0
$$a+0=a.$$
4. Existence of negative numbers: For each a, we can find b such that
$$a+b=0.$$
We also give a name to b: $-a$.
5. The associative law
$$a(bc)=(ab)c.$$
6. The commutative law
$$ab=ba.$$
7. Multiplication by 1
$$a1=a.$$
8. Existence of inverse numbers: For each $a \neq 0$, we can find b such that
$$ab=1.$$
We also give a name to b: a^{-1}.
9. The distributive law:
$$a(b+c)=ab+ac.$$

Remark 2 We can embed the real number field $\mathbb{R}$ in $\mathbb{C}$ by identifying a with $a + 0i$. Then we can check that $0 = 0 + 0i$ is the additive identity, and $1 = 1 + 0i$ is the multiplicative identity. For an element $\alpha = a + bi \in \mathbb{C}$, the additive inverse is $-\alpha = -a - bi$. If $\alpha \neq 0$, we can show that the multiplicative inverse also exists, see homework set 1, problem 3.

Example 2 *Let $\mathbb{Z}$ be the set of integer numbers. Let $\mathbb{N}$ be the set of natural numbers. $\mathbb{Z}$ or $\mathbb{N}$ are not fields.*

Proof 1 *For $\mathbb{Z}$, $2 \in \mathbb{Z}$ doesn't have an inverse since there are no integers such that it multiplied with 2 gives 1. The same applies to $\mathbb{N}$.*

We list some properties of field here. It is not required to understand the concept "field" in our class. However, the idea of the proofs will be used again in next section.

Proposition 1 *In a field $\mathbb{F}$, the negative of a number a is unique.*

Proof: Suppose b and c are two negatives of a. Then

$$a + b = 0 \quad \textit{(by the definition of negatives)} \tag{5}$$

and

$$a + c = 0 \quad \textit{(by the definition of negatives)} \tag{6}$$

Then

$$\begin{aligned}
b =& b + 0 \quad \textit{(by Addition by 0)} && (7)\\
=& b + (a + c) \quad \textit{(by (6))} && (8)\\
=& (b + a) + c \quad \textit{(by the associative law for addition)} && (9)\\
=& 0 + c \quad \textit{(by (5) and the commutative law for addition)} && (10)\\
=& c \quad \textit{(by Addition by 0 and the commutative law for addition)} && (11)
\end{aligned}$$

Then since $b = c$, the negative of a is unique.

Notation 3 We treat the "$-$" in "$-a$" as an operator to find the negative of a. Therefore the uniqueness of negative is required to use the notation "$-a$". If we don't have this proposition, we are not allowed to use "$-$" notation.

Remark 3 Note that -1 maybe refers to an element called "-1" or to the negative element of element 1. If the negative is unique, there are no real differences between these two notations. But in some places if the negative is not unique you should be careful and understand the meaning of it from contexts.

Proposition 2 *In a field $\mathbb{F}$, the inverse of a number $a \neq 0$ is unique.*

Proof: Suppose b and c are two inverse of a. Then

$$ab = 1 \quad \textit{(by the definition of inverses)} \tag{12}$$

and

$$ac = 1 \quad \textit{(by the definition of inverses)} \tag{13}$$

Then

$$\begin{aligned} b =& b1 \quad \textit{(by Multiplication by 1)} && (14)\\ =& b(ac) \quad \textit{(by (13))} && (15)\\ =& (ba)c \quad \textit{(by the associative law for multiplication)} && (16)\\ =& 1c \quad \textit{(by (12) and the commutative law for multiplication)} && (17)\\ =& c \quad \textit{(by Multiplication by 1 and the commutative law for multiplication)} && (18) \end{aligned}$$

Then since $b = c$, the inverse of a is unique.

Notation 4 Similar to the negative case, now we can use the notation a^{-1} or $\frac{1}{a}$ to talk about the multiplicative inverse of a.

Example 3

1. Is 0.5 in $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$? Yes, Yes, Yes
2. Is π in $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$? No, Yes, Yes
3. Is $\sqrt{-2}$ in $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$? No, No, Yes ($\sqrt{-2} = \sqrt{2}i$)

Definition 3 (Complex Conjugation) let $\alpha = a + bi$ be a complex number, where $a, b \in \mathbb{R}$. Then

$$\overline{\alpha} = a - bi$$

is called the *complex conjugate* of α.

Example 4 Let $\alpha = 1 + 2i \neq 0$ be an element of $\mathbb{C}$, we calculate the additive inverse and multiplicative inverse of α.

Solution: The additive inverse is easy:

$$-\alpha = -1 - 2i;$$

We use the complex conjugation to compute the multiplicative inverse. Consider $\overline{\alpha} = 1 - 2i$. Then $\alpha\overline{\alpha} = (1 + 2i)(1 - 2i) = 1 + 2 \cdot 2 = 5$.

Hence we can check that

$$\alpha^{-1} = \frac{\overline{\alpha}}{5} = \frac{1 - 2i}{5}$$

is the multiplicative inverse.

1.4 $\mathbb{R}^n$ and $\mathbb{C}^n$

In Math 10A, we have learned the vector space

$$\mathbb{R}^n = \{\vec{a} = (a_1, a_2, \ldots, a_n) : \ a_i \in \mathbb{R}\}.$$

Here $(a_1, a_2, \dots, a_n)$ is an ordered collection of n elements in $\mathbb{R}$. In particular, we have studied the geometry of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Proposition 3 *Two vectors are equal if and only if all their components are equal.*

Example 5

$$(1, 2, 3) = (1, 2, 3)$$
$$(1, 3, 2) \neq (1, 2, 3)$$

Definition 4 There are two operations on the vector space $\mathbb{R}^3$.

- **Scalar multiplication**: $\alpha\vec{a} = \alpha(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, \alpha a_3)$;
- **Addition**: $\vec{a} + \vec{b} = (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$;

Example 6 $\vec{a}=(1,2)$, $\vec{b}=(3,4)$, compute $\vec{a}+\vec{b}$, $\vec{a}-\frac{1}{2}\vec{b}$.

Solution:

$$\vec{a} + \vec{b} = (1 + 3, 2 + 4) = (4, 6)$$
$$\vec{a} - \frac{1}{2}\vec{b} = (1, 2) - (\frac{3}{2}, 2) = (-\frac{1}{2}, 0).$$

Vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ are geometric objects. They are line segments in $\mathbb{R}^2$ or $\mathbb{R}^3$, represented as an arrow begin at a point (tail) to an end (head).

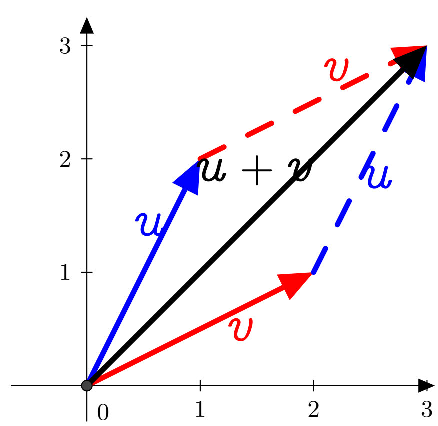
Example 7 (Addition):

Algebraically, vectors can be added component by component. Geometrically, vectors can be added using the parallelogram law.

Let $\vec{u} = (1, 2)$, $\vec{v} = (2, 1)$. Then algebraically

$$\vec{u} + \vec{v} = (1, 2) + (2, 1) = (3, 3),$$

and geometrically

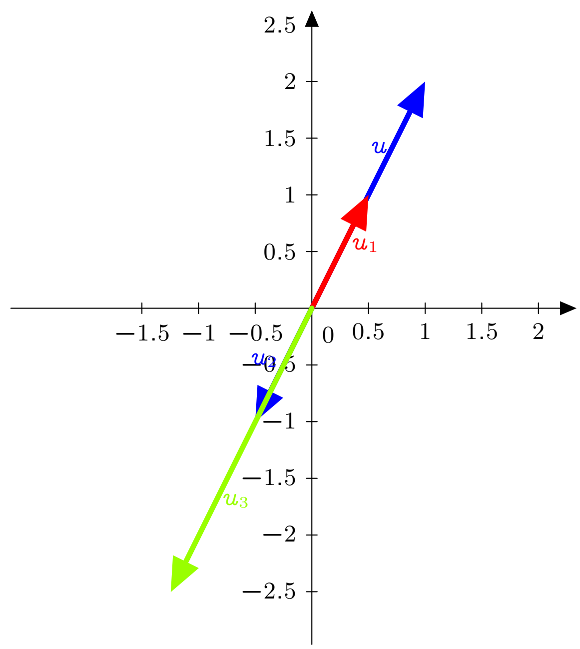


Example 8 (Scalar multiplication): *Algebraically a scalar times a vector is to multiply the scalar to each of the component of the vector. Geometrically, given a vector $\vec{u}$ and $\alpha \in \mathbb{R}$, if $\alpha > 0$, then the scalar multiplication $\alpha\vec{u}$ is to keep the direction of the vector and to change the length by the scalar α. If $\alpha < 0$, then the scalar product takes the opposite direction of the vector and to change the length by the scalar $|\alpha|$.*

Let $\vec{u} = (1, 2)$, $\vec{u}_1 = 0.5\vec{u}$, $\vec{u}_2 = -0.5\vec{u}$, and $\vec{u}_3 = -1.25\vec{u}$. Then algebraically

$$\vec{u}_1 = (0.5, 1), \quad \vec{u}_2 = -0.5(1, 2) = (-0.5, -1), \quad \vec{u}_3 = -1.25(1, 2) = (-1.25, -2.5),$$

and geometrically



We can generalize the definition of $\mathbb{R}^n$ to $\mathbb{F}^n$ for any field $F = \mathbb{R}$ or $F = \mathbb{C}$. On $\mathbb{F}^n$, we have a similar definition of addition and scalar multiplication.

Definition 5

- $\mathbb{F}^n = \{\vec{a} = (a_1, a_2, \ldots, a_n) : \ a_i \in \mathbb{F}\}$.
- Given any
$$\vec{a} = (a_1, a_2, \ldots, a_n), \vec{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{F}^n,$$
any $\alpha \in F$, we have
$$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$
$$\alpha\vec{a} = (\alpha a_1, \alpha a_2, \ldots, \alpha a_n)$$

Remark 4 In this class, we will drop the arrow and write u instead of $\vec{u}$. To distinguish from the real number zero, in MATH $10A$, we usually write the zero vector $\vec{0} = (0, 0, \ldots, 0)$ as $\vec{0}$. In our class, we will use 0.

We can check that $u + 0 = u$ for any $u = (u_1, \ldots, u_n) \in \mathbb{F}^n$. Similarly, we have

$$-u = (-1)u = (-u_1, \ldots, -u_n)$$

as the additive inverse of u, we call it the negative of u.

In some textbook, one use column vectors instead of row vectors. We also introduce the notation here. In our class, we switch between the row and column vector notations as a practice. In Chapter 3, we will prefer column vectors since they works better with the matrix representation of linear maps.

Example 9 $\mathbb{F}^n$. *It is the set of*

$$\left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, a_1, \ldots, a_n \in \mathbb{F} \right\}.$$

Scalar multiplication:

$$c \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}.$$

Addition:

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}.$$

To prove it is a vector space, we need to prove all eight axioms. Here we just prove the associative law for vectors. All other laws can be proved similarly.

Associative law for vectors: *for any three vectors* $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ *and* $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, *want to show that*

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \left(\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} + \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \right) = \left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right) + \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

In fact,

$$\begin{aligned}
\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \left(\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} + \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \right) &= \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 + c_1 \\ \vdots \\ b_n + c_n \end{bmatrix} && \text{(by the definition of addition)} \\
&= \begin{bmatrix} a_1 + (b_1 + c_1) \\ \vdots \\ a_n + (b_n + c_n) \end{bmatrix} && \text{(by the definition of addition)} \\
&= \begin{bmatrix} (a_1 + b_1) + c_1 \\ \vdots \\ (a_n + b_n) + c_n \end{bmatrix} && \text{(by the associative law of } \mathbb{F}\text{)} \\
&= \left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right) + \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} && \text{(by the definition of addition)}
\end{aligned}$$

Therefore the associative law for vectors holds.