# Worksheet 2 Solutions

**Problem 1.** *Prove or disprove: The complex field $\mathbb{C}$ is a vector space over $\mathbb{R}$, with addition and scalar multiplication defined as follows: For any*

$$(a+bi),(c+di) \in \mathbb{C} \text{ and } r \in \mathbb{R}$$

*define*

$$(a+bi)+(c+di) := (a+c)+(b+d)i$$

$$r(a+bi) = (ra)+(rb)i$$

*Proof.* We need to show a lot of properties hold. In words, we need to show that the addition (defined above) is associative, commutative, and admits an identity and inverses. We also need to show that the scalar multiplication defined above is associative and that it distributes over addition. All of this work is for general elements $x, y, z \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{R}$.

Because $x, y, z \in \mathbb{C}$ there exist real numbers $a, b, c, d, e, f$ so that

$$x = a+bi$$

$$y = c+di$$

$$z = e+fi$$

- Addition is associative: show $x+(y+z) = (x+y)+z$ First, we will use the defined addition to compute $(x+(y+z))$.

  $$x+(y+z) = (a+bi)+((c+di)+(e+fi)) = (a+bi)+((c+e)+(d+f)i)$$

  $$(a+(c+e))+(b+(d+f))i$$

  Therefore $x+(y+z) = (a+(c+e))+(b+(d+f))i$. Now we will use the definition of addition to compute $(x+y)+z$.

  $$(x+y)+z = ((a+bi)+(c+di))+(e+fi) = ((a+c)+(b+d)i)+(e+fi)$$

  $$= ((a+c)+e)+((b+d)+f)i$$

  Therefore $(x+y)+z = ((a+c)+e)+((b+d)+f)i$. Now we need to show why the two are equal.

We know that $a, c$, and $e$ are real numbers. In particular, we know that $a+(c+e) = (a+c)+e$. Therefore the real components of $x+(y+z)$ and $(x+y)+z$ are equal.

Likewise, because $b, d, f \in \mathbb{R}, b + (d + f) = (b + d) + f$. Therefore the imaginary components of $x + (y + z)$ and $(x + y) + z$ are equal. Because both the real parts and the imaginary parts are equal,

$$x + (y + z) = (x + y) + z$$

- Addition is commutatuve: show $x + y = y + x$

  $$x + y = (a + bi) + (c + di) = (a + c) + (b + d)i$$

  Because $a, b, c, d \in \mathbb{R}$, we know that they commute under addition. Therefore

  $$a + c = c + a$$
  $$b + d = d + b$$

  Now we have that

  $$(a + c) + (b + d)i = (c + a) + (d + b)i$$
  $$= (c + di) + (a + bi) = y + x$$

  Thus $x + y = y + x$.

- Addition has an identity: show there is a 0 element in $\mathbb{C}$.

  Define $0_{\mathbb{C}}$ to be

  $$0_{\mathbb{C}} = 0 + 0i$$

  We need to show that $0_{\mathbb{C}} + x = x + 0_{\mathbb{C}} = x$

  $$0_{\mathbb{C}} + x = (0 + 0i) + (a + bi) = (0 + a) + (0 + b)i = a + bi = x$$
  $$x + 0_{\mathbb{C}} = (a + bi) + (0 + 0i) = (a + 0) + (b + 0)i = a + bi = x$$

  Thus $0_{\mathbb{C}}$ is an identity for addition as defined in $\mathbb{C}$.

- Additive inverses: show that an element $-x$ exists so that

  $$x + (-x) = (-x) + x = 0_{\mathbb{C}}$$

  Define $(-x)$ to be the element $(-a) + (-b)i$. We need to show it cancels $x$ under addition.

  $$x+(-x) = (a+bi)+((-a)+(-b)i) = (a+(-a))+(b+(-b))i = 0+0i = 0_{\mathbb{C}}$$
  $$(-x)+x = ((-a)+(-b)i)+(a+bi) = ((-a)+a)+((-b)+b)i = 0+0i = 0_{\mathbb{C}}$$

- Scalar associativity: show that $\alpha(\beta x) = (\alpha\beta)x$

$$\alpha(\beta x) = \alpha(\beta(a + bi)) = \alpha((\beta a) + (\beta b)i) = (\alpha(\beta a)) + (\alpha(\beta b))i$$

$$= ((\alpha\beta)a) + ((\alpha\beta)b)i = (\alpha\beta)(a + bi) = (\alpha\beta)x$$

- Distribution over addition in $\mathbb{R}$: show that $(\alpha + \beta)x = (\alpha x) + (\beta x)$

$$(\alpha + \beta)x = (\alpha + \beta)(a + bi) = ((\alpha + \beta)a) + ((\alpha + \beta)b)i$$

$$= ((\alpha a) + (\beta a)) + ((\alpha b) + (\beta b))i = ((\alpha a) + (\alpha b)i) + ((\beta a) + (\beta b)i)$$
$$= (\alpha(a + bi)) + (\beta(a + bi)) = (\alpha x) + (\beta x)$$

- Distribution over addition in $\mathbb{C}$: show that $\alpha(x + y) = (\alpha x) + (\alpha y)$

$$\alpha(x+y) = \alpha((a+bi)+(c+di)) = \alpha((a+c)+(b+d)i) = (\alpha(a+c))+(\alpha(b+d))i$$

$$= ((\alpha a) + (\alpha c)) + ((\alpha b) + (\alpha d))i = ((\alpha a) + (\alpha b)i) + ((\alpha c) + (\alpha d)i)$$
$$= (\alpha(a + bi)) + (\alpha(c + di)) = (\alpha x) + (\alpha y)$$

□

**Problem 2.** *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $S$ be a set, and denote by $\mathbb{F}^S$ the set of all functions from $S$ to $\mathbb{F}$. That is,*

$$\mathbb{F}^S = \{f : S \to \mathbb{F}\}$$

*Note that if we have two elements $f, g \in \mathbb{F}^S$, we have two functions from $S$ to $\mathbb{F}$. The way we test if $f = g$ is by plugging in all values of $S$ and checking if the outputs are equal. That is,*

$$f = g \iff \forall s \in S, f(s) = g(s)$$

*Define addition like so: for $f, g \in \mathbb{F}^S$, the element $f + g$ is the function such that, for all $s \in S$,*

$$(f + g)(s) = f(s) + g(s)$$

*For $f \in \mathbb{F}^S$ and $\alpha \in \mathbb{F}$, the element $\alpha \cdot f$ is the function such that for all $s \in S$,*

$$(\alpha \cdot f)(s) = \alpha \cdot f(s)$$

*Show that these operations make $\mathbb{F}^S$ a vector space over $\mathbb{F}$.*

*Proof.* We need to check that the addition and scalar multiplication satisfy all the vector space axioms. This work is for general $f, g, h \in \mathbb{F}^S$ and general $\alpha, \beta \in \mathbb{F}$.

- Addition is associative: check that $f + (g + h) = (f + g) + h$

  Let $s \in S$. Then

  $$(f + (g + h))(s) = f(s) + (g + h)(s) = f(s) + (g(s) + h(s))$$

  $$= (f(s) + g(s)) + h(s) = (f + g)(s) + h(s) = ((f + g) + h)(s)$$

  Because the functions $(f + (g + h))$ and $((f + g) + h)$ give the same output for every $s \in S$, they are equal.

- Addition is commutative: check that $f + g = g + f$ Let $s \in S$. Then

  $$(f + g)(s) = f(s) + g(s) = g(s) + f(s) = (g + f)(s)$$

  Because the functions $(f + g)$ and $(g + f)$ give the same output for every $s \in S$, the two functinos are equal.

- Additive identity: find an additive identity in $\mathbb{F}^S$

  Let the function $0_{\mathbb{F}}$ be defined as $0_{\mathbb{F}}(s) = 0$ for all $s \in S$. We need to check that $f + 0_{\mathbb{F}} = f$ and $0_{\mathbb{F}} + f = f$.

  Let $s \in S$. Then

  $$(f + 0_{\mathbb{F}})(s) = f(s) + 0_{\mathbb{F}}(s) = f(s) + 0 = f(s)$$

  Therefore $f + 0_{\mathbb{F}} = f$. Likewise,

  $$(0_{\mathbb{F}} + f)(s) = 0_{\mathbb{F}}(s) + f(s) = 0 + f(s) = f(s)$$

  Therefore $0_{\mathbb{F}} + f = f$. Thus $0_{\mathbb{F}}$ is an additive identity in $\mathbb{F}^S$.

- Additive inverses: show negatives exist

  Let the function $(-f) : S \to \mathbb{F}$ be defined as, for $s \in S$,

  $$(-f)(s) = -(f(s))$$

  Then we need to check that $f + (-f) = (-f) + f = 0_{\mathbb{F}}$. Let $s \in S$.

  $$(f + (-f))(s) = f(s) + (-f)(s) = f(s) + (-(f(s)) = 0 = 0_{\mathbb{F}}(s)$$

  Because $f + (-f)$ and $0_{\mathbb{F}}$ give the same output for all $s \in S$, they are equal in $\mathbb{F}^S$.

  Likewise,

  $$((-f) + f)(s) = (-f)(s) + f(s) = -(f(s)) + f(s) = 0 = 0_{\mathbb{F}}(s)$$

  Thus $((-f) + f)$ and $0_{\mathbb{F}}$ are also equal.

- Scalar associativity: show $\alpha \cdot (\beta \cdot f) = (\alpha\beta) \cdot f$

  Let $s \in S$. Then

$$(\alpha(\beta \cdot f))(s) = \alpha \cdot ((\beta \cdot f)(s)) = \alpha \cdot (\beta \cdot (f(s))$$
$$= (\alpha\beta) \cdot f(s) = ((\alpha\beta) \cdot f)(s)$$

  Because $\alpha \cdot (\beta \cdot f)$ and $(\alpha\beta) \cdot f$ give the same output for every $s \in S$, they are eqaul functions.

- Distributivity over addition in $\mathbb{F}$: show $(\alpha + \beta) \cdot f = (\alpha \cdot f) + (\beta \cdot f)$

  Let $s \in S$. Then

$$((\alpha + \beta) \cdot f)(s) = (\alpha + \beta) \cdot (f(s)) = \alpha \cdot f(s) + \beta \cdot f(s)$$
$$= (\alpha \cdot f)(s) + (\beta \cdot f)(s)$$

  Because the two functions evaluate to the same element for every $s \in S$, they are equal.

- Distributivity over addition in $\mathbb{F}^S$: show $\alpha \cdot (f + g) = (\alpha \cdot f) + (\alpha \cdot g)$

  Let $s \in S$. Then

$$(\alpha \cdot (f + g))(s) = \alpha \cdot ((f + g)(s)) = \alpha \cdot (f(s) + g(s))$$
$$\alpha \cdot f(s) + \alpha \cdot g(s) = (\alpha \cdot f)(s) + (\alpha \cdot g)(s) = ((\alpha \cdot f) + (\alpha \cdot g))(s)$$

  That is what we needed to show.

□

**Problem 3.** *In the previous problem we showed that the set of functions from* $[0, 1]$ *to* $\mathbb{R}$ *is a vector space (*$S = [0, 1]$ *and* $\mathbb{F} = \mathbb{R}$*). Define the set* $A \subset \mathbb{R}^{[0,1]}$ *as*

$$A = \{f \in \mathbb{R}^{[0,1]} | f \textit{ is continuous and } f(0) = 0\}$$

*Show that* $A$ *is a vector subspace of* $\mathbb{R}^{[0,1]}$.

*Proof.* All we need to show is that $A$ is nonempty and closed under addition and scalar multiplication. To show that $A$ is nonempty, let's show that $0_{\mathbb{F}}$, the constant 0 function, is in $A$. We know by definition of $0_{\mathbb{F}}$ that $0_{\mathbb{F}}$ takes anything to 0. In particular, $0_{\mathbb{F}}(0) = 0$. Also, the constant 0 function is continuous. Therefore $0_{\mathbb{F}} \in A$. Therefore $A$ is nonempty.

Now suppose that $f, g \in A$. Then $f$ and $g$ are continous functions. The sum of continuous functions is continuous, so $f + g$ is continuous. Also,

$$(f + g)(0) = f(0) + g(0)$$

Because $f, g \in A$, we know that $f(0) = g(0) = 0$. Therefore $(f + g)(0) = 0 + 0 = 0$ as well. That means $f + g \in A$.

Now suppose that $f \in A$ and $r \in \mathbb{R}$. We need to show that $r \cdot f \in A$ as well.

Because $f \in A$, $f$ is continuous. We know that the scalar multiple of a continuous function is again continuous, so $r \cdot f$ is continuous. Now we need to check that it evaluates to 0 at 0. Remember that $f \in A$ so we already know $f(0) = 0$.

$$(r \cdot f)(0) = r \cdot f(0) = r \cdot 0 = 0$$

Thus $(r \cdot f)(0) = 0$, so we have shown that $r \cdot f \in A$. Thus $A$ is a nonempty subset of $\mathbb{R}^{[0,1]}$ which is closed under addition and scalar multiplication, so $A$ is a vector subspace of $\mathbb{R}^{[0,1]}$. □

**Problem 4.** *Give an example of a nonempty subset $U \subseteq \mathbb{R}^2$ which is closed under scalar multiplication but not addition.*

*Proof.* Let $U$ be the union of the (closed) first and third quadrants, i.e.

$$U = \{(x, y) \in \mathbb{R}^2 | xy \geq 0\}$$

We need to show that $U$ is nonempty, that $U$ is closed under scalar multiplication, and that $U$ is not closed under addition.

- $U$ is nonempty:

  The point $(1, 1)$ is in $U$ because $1 \cdot 1 = 1$ and $1 \geq 0$.

- $U$ is closed under scalar multiplication:

  Let $(x, y) \in U$. Then $xy \geq 0$ by definition of $U$. Let $r \in \mathbb{R}$. Then

  $$r \cdot (x, y) = (rx, ry)$$

  To see that $r \cdot (x, y) \in U$, we just need to show

  $$rxry \geq 0$$

  We can rewrite $rxry$ as $r^2xy$. Now $r^2$ is a nonnegative number because it is a square, and $xy$ is nonnegative by assumption. The product of two nonnegative numbers is nonnegative, so $r^2xy \geq 0$. Thus $U$ is closed under scalar multiplication.

- $U$ is not closed under addition:

  As we saw already, the point $(1, 1) \in U$. Also, the point $(-2, -\frac{1}{2}) \in U$ because

  $$(-2)\left(-\frac{1}{2}\right) = 1 \geq 0$$

  However, their sum is

  $$(1, 1) + (-2, -\frac{1}{2}) = (-\frac{3}{2}, \frac{1}{2})$$

This vector is not in $U$ because

$$\left(-\frac{3}{2}\right)\left(\frac{1}{2}\right) = -\frac{3}{4} < 0$$

Thus $U$ is not closed under addition.

$\square$