MATH 131—**HOMEWORK** 5

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"$\trianglelefteq$" := "Subspace"

Q1 Suppose U and W are both five-dimensional subspaces of $\mathbb{R}^9$. Prove that $U \cap W \neq \{0\}$.

Proof. Want to show $U \cap W \neq \{0\}$.
$\forall V \trianglelefteq \mathbb{R}^9$: $\forall W \trianglelefteq \mathbb{R}^9$:
$V + W \trianglelefteq \mathbb{R}^9$ (by Proposition 3 in 1C) $\Rightarrow$ dim $V + W \leq 9$ (by Proposition 1 in 2C)
$\Rightarrow 9 \geq$ dim $V + W$= dim V + dim W − dim $V \cap W$ (by Theorem 2 in 2C)
$\Rightarrow 9 \geq$ dim $V + W$= $5 + 5 -$ dim $V \cap W$
$\Rightarrow 9 \geq$ dim $V + W$= $10 -$ dim $V \cap W$ (0)
Suppose, $U \cap W = \{0\}$.
$\Rightarrow$ dim $U \cap W =$ dim $\{0\}$ (1)
Check the dependence test equation $a0 = 0$,
by the zero factors theorem $\exists a \in \mathbb{R} : a \neq 0$.
$\Rightarrow$ 0 the only element of $\{0\}$ is not linearly independent.
So, the number of linearly independent vectors in $\{0\}$ is 0.
Therefore, $\emptyset$ is the set that contains all of the linearly independent vectors in $\{0\}$.
So, $\emptyset$ is linearly independent and Span $\emptyset = \{0\}$ (by Definition 2 in 2A)
It follows that $\emptyset$ is a basis for $\{0\}$ (by Definition 1 in 2B)
$\Rightarrow$ dim $\{0\} = |\emptyset| = 0$ (by Definition 1 in 2C)
(1) $\Rightarrow$ dim $U \cap W = 0$.
(0) $\Rightarrow 9 \geq$ dim $V + W = 10 -$ dim $V \cap W$= $10 - 0 = 10$
$\Rightarrow 9 \geq 10 \Rightarrow$ False
By, contradiction to our assumption $U \cap W \neq \{0\}$

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Q2 Let $U = \{p \in P_4(\mathbb{R}) : p''(4) = 0\}$.

(a) Show that U is a subspace of $P_4(\mathbb{R})$.

(b) Find a basis of U.

(c) Extend the basis in part (b) to a basis of $P_4(\mathbb{R})$.

Proof. Want to show $U \trianglelefteq P_4(\mathbb{R})$
$\forall p, q \in U : \forall x \in \mathbb{R} :$
$(p + q)(x) = p(x) + q(x)$
$\Rightarrow (p + q)''(4) = ((p(4) + q(4))')' = (p'(4) + q'(4))' = p''(4) + q''(4) = 0 + 0 = 0$
$\Rightarrow (p + q)''(x) \in U$ (0)
$\forall r \in U : \forall x, a \in \mathbb{R}$:
$(ar)(x) = a(r(x))$
$\Rightarrow (ar)''(4) = ((a(r(4)))')' = (a(r'(4)))' = a(r''(4)) = a0 = 0$
$\Rightarrow (ar)(x) \in U$ (1)
(0) and (1) $\Rightarrow$ U is closed under vector addition and scalar multiplication.
$U \trianglelefteq P_4(\mathbb{R})$

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For part (b): $\forall p(t) \in P_4(\mathbb{R}) = \text{Span}\,(1, t, t^2, t^3, t^4) : \exists a_0, a_1, a_2, a_3, a_4 \in \mathbb{R} :$
$p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$
$\Rightarrow p'(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3$
$\Rightarrow p''(t) = 2a_2 + 6a_3 t + 12a_4 t^2$
If $p(t) \in U$, then $p''(4) = 0$
$\Rightarrow p''(4) = 2a_2 + 6a_3 4 + 12a_4(4)^2 = 2a_2 + 24a_3 + 192a_4 = 0$
$\Rightarrow a_2 = -12a_3 - 96a_4$
Let $p(t) \in U \Rightarrow p(t) = a_0 + a_1 t + (-12a_3 - 96a_4)t^2 + a_3 t^3 + a_4 t^4$
If $a_0 = 1$ and $a_1 = a_3 = a_4 = 0$, then $p_0(t) = 1$.
If $a_1 = 1$ and $a_0 = a_3 = a_4 = 0$, then $p_1(t) = t$.
If $a_3 = 1$ and $a_0 = a_1 = a_4 = 0$, then $p_3(t) = -12t^2 + t^3$.
If $a_4 = 1$ and $a_0 = a_1 = a_3 = 0$, then $p_4(t) = -96t^2 + t^4$.
Check, if $B = \{p_0, p_1, p_3, p_4\}$ is linearly independent by setting up the dependence test equation:
$b_0 1 + b_1 t + b_3(-12t^2 + t^3) + b_4(-96t^2 + t^4) = 0$
$\Rightarrow b_0 + b_1 t - 12b_3 t^2 + b_3 t^3 - 96b_4 t^2 + b_4 t^4 = 0$
$\Rightarrow b_0 + b_1 t - 12b_3 t^2 - 96b_4 t^2 + b_3 t^3 + b_4 t^4 = 0$
$\Rightarrow b_0 + b_1 t + (-12b_3 - 96b_4)t^2 + b_3 t^3 + b_4 t^4 = 0$
$\{1, t, t^2, t^3, t^4\}$ is linearly independent $\Rightarrow b_0 = b_1 = b_3 = b_4 = 0$
$\Rightarrow B$ is linearly independent
Since, we got the set B by expressing the dependent variable for an arbitrary vector in U in terms of free variables, the set B is still a spanning set of U.
So, $B = \{1, t, -12t^2 + t^3, -96t^2 + t^4\}$ is a basis of U (by Definition 1 in 2B)

For part (c):
Let $B_0 = (1, t, -12t^2 + t^3, -96t^2 + t^4, 1, t, t^2, t^3, t^4)$ an ordered list:
We've seen the first 4 are already non-zero and linearly independent,
For the fifth, is just equal to the first, so we delete it.
$\Rightarrow B_1 = (1, t, -12t^2 + t^3, -96t^2 + t^4, t, t^2, t^3, t^4)$
For the fifth in B_1, is just equal to the second, so we delete it.
$\Rightarrow B_2 = (1, t, -12t^2 + t^3, -96t^2 + t^4, t^2, t^3, t^4)$
Check the dependence test equation:
$d_1 1 + d_2 t + d_3(-12t^2 + t^3) + d_4(-96t^2 + t^4) + d_5 t^2 = 0$
$\Rightarrow d_1 1 + d_2 t - 12d_3 t^2 + d_3 t^3 - 96d_4 t^2 + d_4 t^4 + d_5 t^2 =$
$d_1 + d_2 t - 12d_3 t^2 - 96d_4 t^2 + d_5 t^2 + d_3 t^3 + d_4 t^4 =$
$d_1 + d_2 t + (-12d_3 - 96d_4 + d_5)t^2 + d_3 t^3 + d_4 t^4 = 0$
$\{1, t, t^2, t^3, t^4\}$ is linearly independent $\Rightarrow d_1 = d_2 = d_3 = d_4 = 0$ and $-12d_3 - 96d_4 + d_5 = 0$
$\Rightarrow -12d_3 - 96d_4 + d_5 = -12(0) - 96(0) + d_5 = 0 + 0 + d_5 = d_5 = 0$
$\Rightarrow B_3 = \{1, t, -12t^2 + t^3, -96t^2 + t^4, t^2\}$ is linearly independent
Since $|B_3| = 5 = \dim P_4(\mathbb{R})$, B_3 is a basis of $P_4(\mathbb{R})$ (by Proposition 2 in 2C).

Q3 $\mathbb{C}^n$ can be thought of as $\mathbb{C}$–vector space and $\mathbb{R}$–vector space. Find the bases of $\mathbb{C}^n$ over $\mathbb{C}$ and $\mathbb{R}$ respectively. Compute $\dim_{\mathbb{C}}\mathbb{C}^n$ and $\dim_{\mathbb{R}}\mathbb{C}^n$.

If we think about $\mathbb{C}^n$ as a $\mathbb{C}$-vector space $\mathbb{C}^n = \{(\alpha_1, ..., \alpha_n)|\alpha_i \in \mathbb{C}\}$.
It's easy to see that $B = \{e_1, ..., e_n\}$ the standard basis, is a basis for $\mathbb{C}^n$. B is linearly independent by construction. And, since Span $B = \sum_{i=1}^{n} \beta_i e_i, \beta_i \in \mathbb{C}$. Every vector in $\mathbb{C}^n$ is in Span B, because every possible complex number gets sent to every possible entry in every vector in $\mathbb{C}^n$. So B is a linearly independent, spanning set of $\mathbb{C}^n$. So B is a basis of $\mathbb{C}^n$.Therefore, dim ${}_{\mathbb{C}}\mathbb{C}^n = |B| = n$.

If we think about $\mathbb{C}^n$ as a $\mathbb{R}$-vector space $\mathbb{C}^n = \{(1 + b_1\mathrm{i}, ..., a_n + b_n\mathrm{i})|a_j, b_j \in \mathbb{R}\}$
Let $B^* = \{e_1, ..., e_n\} \cup \{\mathrm{i}e_1, ..., \mathrm{i}e_n\}$ where the left side of the union is the standard basis, and the right hand side is every element of the standard basis times i, we want to show it is a basis for $\mathbb{C}^n$. B is linearly independent by construction, so we need only check whether we should keep the rest of the elements. Since, $\mathrm{i}e_j$ is the vector with i in the ${}_j$th place, and zeros elsewhere, the only candidate where a dependency would show up is e_j. Now, recall $\mathrm{i}^2 = -1$ has no solutions in $\mathbb{R}$, it follows i is not a real number. So, $\mathrm{i}e_j \neq ke_j$ some $k \in \mathbb{R}$. So, every $\mathrm{i}e_j$ is independent from vectors in B. All that's left to check is that the right hand side of the union is linearly independent, it is by construction, since it inherits its independence from B. So, B^* is linearly independent.
Now we can express B^* as $B^* = \{e_1, ..., e_n, \mathrm{i}e_{n+1}, ..., \mathrm{i}e_{2n}\}$.
And, since Span $B^* = \sum_{j=1}^{2n} a_j e_j + b_j \mathrm{i}e_j = \sum_{j=1}^{2n}(a_j + b_j\mathrm{i})e_j$, $a_j, b_j \in \mathbb{R}$. Every vector in $\mathbb{C}^n$ is in Span B, because every possible real number gets sent to every possible complex entry in every vector in $\mathbb{C}^n$. So B is a linearly independent, spanning set of $\mathbb{C}^n$. So B^* is a basis of $\mathbb{C}^n$.Therefore, dim ${}_{\mathbb{R}}\mathbb{C}^n = |B^*| = 2n$.

Q4 Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (2x - 4y + 2z + b, 6x + cxyz)$$

Show that T is linear if and only if $b = c = 0$.

Proof. Want to show that T is linear if and only if $b = c = 0$.
($\Rightarrow$) Assume T is linear
$\forall (s_1, s_2, s_3), (t_1, t_2, t_3) \in \mathbb{R}^3 : \forall r \in \mathbb{R} :$
$\Rightarrow T((s_1, s_2, s_3) + (t_1, t_2, t_3)) = T(s_1, s_2, s_3) + T(t_1, t_2, t_3)$
$\Rightarrow T(s_1+t_1, s_2+t_2, s_3+t_3) = (2(s_1+t_1)-4(s_2+t_2)+2(s_3+t_3)+b, 6(s_1+s_2)+c(s_1+t_1)(s_2+t_2)(s_3+t_3))$
$\Rightarrow T(s_1, s_2, s_3) = (2s_1 - 4s_2 + 2s_3 + b, 6s_1 + cs_1s_2s_3)$
$\Rightarrow T(t_1, t_2, t_3) = (2t_1 - 4t_2 + 2t_3 + b, 6t_1 + ct_1t_2t_3)$
$\Rightarrow T(s_1, s_2, s_3) + T(t_1, t_2, t_3) = (2(s_1 + t_1) - 4(s_2 + t_2) + 2(s_3 + t_3) + 2b, 6(s_1 + s_2) + c(s_1s_2s_3 + t_1t_2t_3))$
$\Rightarrow$
$(2(s_1 + t_1) - 4(s_2 + t_2) + 2(s_3 + t_3) + b, 6(s_1 + s_2) + c(s_1 + t_1)(s_2 + t_2)(s_3 + t_3)) =$
$(2(s_1 + t_1) - 4(s_2 + t_2) + 2(s_3 + t_3) + 2b, 6(s_1 + s_2) + c(s_1s_2s_3 + t_1t_2t_3))$
$\Rightarrow b = 2b$ and $c(s_1 + t_1)(s_2 + t_2)(s_3 + t_3) = c(s_1s_2s_3 + t_1t_2t_3)$
$\Rightarrow b = 0$
$c(s_1 + t_1)(s_2 + t_2)(s_3 + t_3) = c(s_1s_2s_3 + t_1t_2t_3) \Rightarrow c((s_1 + t_1)(s_2 + t_2)(s_3 + t_3) - (s_1s_2s_3 + t_1t_2t_3)) = 0$
$\Rightarrow (s_1 + t_1)(s_2 + t_2)(s_3 + t_3) - (s_1s_2s_3 + t_1t_2t_3) = 0$ or $c = 0$
$\Rightarrow$
$(s_1 + t_1)(s_2 + t_2)(s_3 + t_3) - (s_1s_2s_3 + t_1t_2t_3) =$
$(s_1s_2 + 2t_1s_2 + t_1t_2)(s_3 + t_3) - (s_1s_2s_3 + t_1t_2t_3) =$
$(s_1s_2s_3 + 2t_1s_2s_3 + t_1t_2s_3) + (s_1s_2t_3 + 2t_1s_2t_3 + t_1t_2t_3) - (s_1s_2s_3 + t_1t_2t_3) =$
$(\cancel{s_1s_2s_3} + 2t_1s_2s_3 + t_1t_2s_3) + (s_1s_2t_3 + 2t_1s_2t_3 + \cancel{t_1t_2t_3}) - (\cancel{s_1s_2s_3} + \cancel{t_1t_2t_3}) =$
$2t_1s_2s_3 + t_1t_2s_3 + s_1s_2t_3 + 2t_1s_2t_3 = 0$
Since, T is a map it must be defined for all elements of its domain. So, for T to be linear on $\mathbb{R}^3$ there must be no restriction on (s_1, s_2, s_3) and (t_1, t_2, t_3). So, c must be 0. And we already have that $b = 0$. So, $b = c = 0$.
With that condition on T verify the second linearity property:
$T(r(s_1, s_2, s_3)) = T(rs_1, rs_2, rs_3) = (2rs_1 - 4rs_2 + 2rs_3, 6rs_1)$
and
$rT(s_1, s_2, s_3) = r(2s_1 - 4s_2 + 2s_3, 6s_1) = (r(2s_1 - 4s_2 + 2s_3), r6s_1) = (2rs_1 - 4rs_2 + 2rs_3, 6rs_1)$
$\Rightarrow T(r(s_1, s_2, s_3)) = rT(s_1, s_2, s_3) \quad (0)$

($\Leftarrow$) Assume $b = c = 0$
$\Rightarrow T(x, y, z) = (2x - 4y + 2z, 6x)$
The second linearity property holds by equation (0) when $b = c = 0$.
$T(s_1, s_2, s_3) + T(t_1, t_2, t_3) =$
$(2s_1 - 4s_2 + 2s_3, 6s_1) + (2t_1 - 4t_2 + 2t_3, 6t_1) =$
$(2(s_1 + t_1) - 4(s_2 + t_2) + 2(s_3 + t_3), 6(s_1 + s_2) =$
$T(s_1 + t_1, s_2 + t_2, s_3 + t_3) =$
$T((s_1, s_2, s_3) + (t_1, t_2, t_3))$
So, T is linear if and only if $b = c = 0$. ■

Q5 Suppose $T \in T(V, W)$ and $v_1, ..., v_m$ is a list of vectors in V such that $T(v_1), ..., T(v_m)$ is a linearly independent list in W. Prove or give a counterexample: $v_1, ..., v_m$ is linearly independent.

Proof. Want to show if $T \in T(V, W)$ and for $v_1, ..., v_m$ a list of vectors in V, $T(v_1), ..., T(v_m)$ is a linearly independent list of vectors in W, then $v_1, ..., v_m$ is linearly independent.
$\forall T \in T(V, W) : \; \exists \; a_i \in \mathbb{F} :$
For some list $v_1, ..., v_n$ of vectors in V, $T(v_1), ..., T(v_m)$ is a linearly independent list of vectors in W
$\Rightarrow$ ($a_1 T(v_1) + ... + a_m T(v_m) = 0 \Rightarrow \forall i : a_i = 0$)
$T \in T(V, W) \Rightarrow T$ is linear $\Rightarrow$
$a_1 T(v_1) + ... + a_m T(v_m) = \sum_{i=1}^m a_i T(v_i) = \sum_{i=1}^m T(a_i v_i) = T(\sum_{i=1}^m a_i v_i) = 0$
$\forall v \in V : T(v) = 0 \Rightarrow T(v) = 0T(v) = T(0v) = T(0) = 0 \Rightarrow v = 0$
$\Rightarrow \sum_{i=1}^m a_i v_i = a_1 v_1 + ... + a_m v_m = 0$
Recall $\forall i : a_i = 0 \Rightarrow v_1, ..., v_m$ is linearly independent.

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Q6 Give an example of a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\phi(av) = a\phi(v)$$

for all $a \in \mathbb{R}$ and all $v \in \mathbb{R}^2$ but ϕ is not linear.

$(x, y) \mapsto \sqrt{xy}$
$a(x, y) = (ax, ay) \mapsto \sqrt{axay} = \sqrt{a^2 xy} = a\sqrt{xy}$
Consider $(1, 1) \mapsto 1$ and $(4, 1) \mapsto 2$,
$(1, 1) + (4, 1) = (5, 2) \mapsto \sqrt{10}$
$1 + 2 = 3 = \sqrt{10} \Rightarrow 3^2 = 9 = 10 \Rightarrow false$
$\Rightarrow (x, y) \mapsto \sqrt{xy}$ is not linear.