MATH 131—HOMEWORK 6

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Q1 Prove that there does not exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that

$$\text{range } T = \text{null } T$$

Pf .

$\forall\ T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^5)$:
$5 = \text{dim } (\mathbb{R}^5) = \text{dim } \text{null}(T) + \text{dim } \text{range}(T)$ (by FTLA)
Assume $K = \text{dim } \text{null}(T) = \text{dim } \text{range}(T)$
$\Rightarrow 5 = 2K \Rightarrow K = \frac{5}{2} \notin \mathbb{N}$, but $\text{dim } \text{null}(T) \in \mathbb{N}$
By contradiction $\text{dim } \text{null}(T) \neq \text{dim } \text{range}(T)$
$\Rightarrow \text{range}(T) \neq \text{null}(T)$ (by Proposition 1 in 2C)
$\Rightarrow \nexists\ T : \text{range}(T) = \text{null}(T)$

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Q2 Prove or give a counterexample:
If $T : V \to W$ is a linear map and $v_1, ..., v_n$ spans V. Then $T(v_1), ..., T(v_n)$ spans range(T).

Pf .

range(T) $= \{w \in W |\ \exists\ v \in V : w = T(v)\}$

$v_1, ..., v_n$ spans $V \Rightarrow V =$ span($\{v_1, ..., v_n\}$)

$\forall w \in$ range(T): $\exists\ v \in V : \exists a_1, ..., a_n \in \mathbb{F} : v = \sum_{i=1}^{n} a_i v_i$:

$w = T(v) \Rightarrow w = T(\sum_{i=1}^{n} a_i v_i) = \sum_{i=1}^{n} a_i T(v_i)$

$\Rightarrow w \in$ span ($\{T(v_1), ..., T(v_n)\}$)

So, $T(v_1), ..., T(v_n)$ spans range(T)

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Q3 Prove or give a counterexample: Let $T : V \to W$ and $S : V \to W$ are two linear maps. If null T = null S, then $T = S$

Let $V = W = \mathbb{R}$: Define $T(x) = x$ and $S(x) = 5x$

null $T = \{x \in \mathbb{R} | T(x) = 0\}$

T is the identity map so null $T = \{0\}$

null $S = \{x \in \mathbb{R} | S(x) = 0\}$

S is defined by $x \mapsto 5x$ and $0 = 5(0) \Rightarrow x = 0$

$\Rightarrow$ null S =$\{0\}$ = null T

but, $T \neq S$

Q4 Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if dim $V \leq$ dim W

Pf .

($\Rightarrow$) Assume $\exists T : V \to W$ and T is injective.

(by Proposition 1 in 3B) null $T = \{0\} \Rightarrow$ dim null $T = 0$

(by FTLA) dim V = dim null $T+$ dim range T

$\Rightarrow$ dim $V = 0+$ dim range T

$\Rightarrow$ dim V = dim range T

range $T \trianglelefteq W \Rightarrow$ dim range $T \leq$ dim W

$\Rightarrow$ dim $V \leq$ dim W

($\Leftarrow$) Assume $n =$ dim $V \leq$ dim $W = m$

Suppose $B = \{v_1, ..., v_n\}$ and $B^* = \{w_1, ..., w_m\}$ are bases for V and W respectively.

Consider $R : V \Rightarrow W$ defined by

$R(v_i) = w_i, 1 \leq i \leq n$

$v_i \neq v_j \Rightarrow R(v_i) \neq R(v_j) \Rightarrow w_i \neq w_j$, whenever $i \neq j$

R is injective as long as $n \leq m.$ If $n > m$, then some v_k where $k > m$, has to get mapped to some vector in W that has already been mapped by R.

So, R exists and it is injective.

So, there exists an injective linear map from V to W if and only if dim $V \leq$ dim W

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Q5 Prove that there does not exist a linear map from $\mathbb{F}^5$ to $\mathbb{F}^2$ whose nullspace equals

$$\{(x_1, x_2, x_3, x_4, x_5)|x_1 = 3x_2; x_3 = x_4 = x_5\}$$

Pf .

Assume $R : \mathbb{F}^5 \to \mathbb{F}^2$: null $R = \{(x_1, x_2, x_3, x_4, x_5)|x_1 = 3x_2; x_3 = x_4 = x_5, x_i \in \mathbb{F}\}$

$\forall s \in$ null $R \Rightarrow s = (3x_2, x_2, x_3, x_3, x_3)$

$\Rightarrow s = x_2(3, 1, 0, 0, 0) + x_3(0, 0, 1, 1, 1)$

and $(3, 1, 0, 0, 0) \neq k(0, 0, 1, 1, 1)$, $k \in \mathbb{F}$, because $1 \neq k0 = 0$

$\Rightarrow B = \{(3, 1, 0, 0, 0), (0, 0, 1, 1, 1)\}$ is a basis for null R

$\Rightarrow$ dim null $R = |B| = 2$

dim $\mathbb{F}^5 =$ dim null $R+$ dim range R (By FTLA)

$5 = 2+$ dim range $R \Rightarrow$ dim range $R = 5 - 2 = 3$

range $R \trianglelefteq \mathbb{F}^2 \Rightarrow$ dim range $R \leq$ dim $_{\mathbb{F}}\mathbb{F}^2 = 2 \Rightarrow 3 \leq 2 \Rightarrow False$

So, by contradiction to our assumption R doesn't exist

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Q6 Suppose $T \in \mathcal{L}(V, W)$ is injective and $\{v_1, ..., v_n\}$ is linearly independent in V. Prove or disprove $\{T(v_1), ..., T(v_n)\}$ is linearly independent in W.

Pf .

$\forall T \in \mathcal{L}(V, W)$: T is injective

$\Rightarrow$ null $T = \{0\}$

and $\{v_1, ..., v_n\}$ is linearly independent in V

Set up the dependence test equation as such

$\sum_{i=1}^{n} a_i T(v_i) = 0 : \exists a_1, ..., a_n \in \mathbb{F}$

$\Rightarrow \sum_{i=1}^{n} a_i T(v_i) = T(\sum_{i=1}^{n} a_i v_i) = 0$

$\sum_{i=1}^{n} a_i v_i \in$ null $T \Rightarrow \sum_{i=1}^{n} a_i v_i \in \{0\}$

$\Rightarrow \sum_{i=1}^{n} a_i v_i = 0$

Since $\{v_1, ..., v_n\}$ is linearly independent in V, then $a_1 = ... = a_n = 0$.

So, $\{T(v_1), ..., T(v_n)\}$ is linearly independent in W.

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