Variation of Parameters

Ex: Find a particular sol of $y''+4y=3\csc(t)$

• sol:

1) homog: $r^2+4=0 \Rightarrow r=\pm 2i \Rightarrow y_c = C_1\cos(2t)+C_2\sin(2t)$

Let $y_1=\cos(2t)$, $y_2=\sin(2t)$

2) Let $y_p = u_1(t)y_1(t)+u_2(t)y_2(t)$
$= u_1(t)\cos(2t)+u_2(t)\sin(2t)$

$\boxed{\text{Need to solve for } u_1, u_2}$

Set up system of 2 equations:

$$\begin{cases} u_1'(t)\cos(2t)+u_2'(t)\sin(2t)=0 \\ u_1'(t)(-2\sin(2t))+u_2'(t)(2\cos(2t)) = 3\csc(t) \end{cases}$$

Solving this:

$$u_1'(t) = \frac{g(t)W_1(t)}{W(t)} = \frac{3\csc(t)\cdot(-\sin(2t))}{2}$$

$$u_2'(t) = \frac{g(t)W_2(t)}{W(t)} = \frac{3\csc(t)(\cos(2t))}{2}$$

Since

$$W(t) = \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix} = 2\cos^2(2t)+2\sin^2(2t)=2$$

$$W_1(t) = \begin{vmatrix} 0 & \sin(2t) \\ 1 & 2\cos(2t) \end{vmatrix} = -\sin(2t)$$

$$W_2(t) = \begin{vmatrix} \cos(2t) & 0 \\ -2\sin(2t) & 1 \end{vmatrix} = \cos(2t)$$

$\rightarrow$

Then,

$$u_1 = \int u_1'(t)\,dt = \int -\frac{3\csc(t)\sin(2t)}{2}\,dt$$

using $\sin(2t) = 2\sin t\cos t$

$$= -\frac{3}{2}\int \frac{1}{\sin(t)} \cdot 2\sin(t)\cos(t)\,dt$$

$$= -3\int \cos(t)\,dt$$

$$= -3\sin(t) + C_1$$

$$u_2 = \int u_2'(t)\,dt = \int \frac{3\csc t\cos(2t)}{2}\,dt$$

using $\cos(2t) = 1 - 2\sin^2 t$

$$= \frac{3}{2}\int \csc t\,(1 - 2\sin^2 t)\,dt$$

$$= \frac{3}{2}\int \csc t - 3\int \sin t\,dt$$

double-check this integral

$$= -\frac{3}{2}\ln|\csc t + \cot(t)| + 3\cos t + C_2$$

So,

$$y_p = u_1(t)y_1(t) + u_2(t)y_2(t)$$

$$\Rightarrow y_p = (-3\sin t + C_1)\cos(2t) + \left(-\frac{3}{2}\ln|\csc t + \cot(t)| + 3\cos t + C_2\right)\sin(2t)$$

$$\Rightarrow \boxed{y_p = -3\sin t\cos(2t) - \frac{3}{2}\ln|\csc t + \cot(t)|\sin(2t) + 3\cos t\sin(2t) + C_1\cos(2t) + C_2\sin(2t)}$$

Power Series

- <u>Ex:</u> Assume $\sum_{n=2}^{\infty} a_n \cdot n(n-1)x^{n-2} = -\sum_{n=0}^{\infty} a_n x^n$.

Determine a_n.

- <u>Sol:</u>

First, get indices to match:

$$\sum_{n=2}^{\infty} a_n \cdot n(n-1)x^{n-2} = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n$$

So,

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n = \sum_{n=0}^{\infty} (-a_n)x^n$$

$$\Rightarrow a_{n+2}(n+2)(n+1) = -a_n$$

Multiplying both sides by $n!$:

$$a_{n+2}(n+2)! = (-1)n!\,a_n$$

$$\Rightarrow a_{n+2} = \frac{(-1)n!}{(n+2)!}a_n \quad \ldots (\star)$$

Since the a_{n+2} term is in terms of a_n, need to look at even and odd terms separately:

- <u>Let $n = 2m$ (even terms):</u> $m = 0, 1, 2, \ldots$

$$a_n = a_{2m} = \frac{(-1)(2m-2)!}{(2m-2+2)!}a_{2m-2} = \frac{(-1)(2m-2)!}{(2m-2+2)!} \cdot \frac{(-1)(2m-4)!}{(2m-4+2)!}a_{2m-4}$$

by using $(\star)$

$\longrightarrow$

So

$$a_{2m} = \frac{(-1)(2m-2)!}{(2m-2+2)!}a_{2m-2} = \frac{(-1)\cancel{(2m-2)!}}{(2m-2+2)!}\cdot\frac{(-1)(2m-4)!}{\cancel{(2m-4+2)!}}a_{2m-4} = \cdots = \frac{(-1)^m\, 0!}{(2m)!}a_0$$

by using $(\star)$ m times

Thus, for even we get $a_{2m} = \dfrac{(-1)^m}{(2m)!}a_0$

- Similarly, we can find an expression for odd:

$$a_n = a_{2m+1} = \frac{(-1)(2m-1)!}{(2m+1)!}a_{2m-1} = \frac{(-1)(2m-1)!}{(2m+1)!}\cdot\frac{(-1)(2m-3)!}{(2m-1)!}a_{2m-3}$$

by using $(\star)$

So,

$$a_{2m+1} = \frac{(-1)(2m-1)!}{(2m+1)!}a_{2m-1} = \frac{(-1)\cancel{(2m-1)!}}{(2m+1)!}\cdot\frac{(-1)(2m-3)!}{\cancel{(2m-1)!}}a_{2m-3} = \cdots = \frac{(-1)^m\, 1!}{(2m+1)!}a_1$$

by using $(\star)$ m times

Thus, for odd we get $a_{2m+1} = \dfrac{(-1)^m}{(2m+1)!}a_1$

Therefore,

$$\boxed{a_n = \begin{cases} \dfrac{(-1)^m}{(2m)!}a_0, & n = 2m \\[2ex] \dfrac{(-1)^m}{(2m+1)!}a_1, & n = 2m+1 \end{cases} \qquad m = 0, 1, 2, \ldots}$$