Math 146B.001 (Ordinary and Partial Differential Equations)
Homework 05 Solutions

Problem 1 (5.3.1)**.** Determine $\varphi''(x_0)$, $\varphi'''(x_0)$, and $\varphi^{(4)}(x_0)$ for the point x_0 if $y = \varphi(x)$ is a solution to the initial value problem

$$y'' + xy' + y = 0; \qquad y(0) = 1, \quad y'(0) = 0.$$

Solution. Suppose that φ is a solution to the differential equation. Then, with $x_0 = 0$, the initial values become

$$y(0) = \varphi(x_0) = 1 \qquad \text{and} \qquad y'(0) = \varphi'(x_0) = 0.$$

As φ solves the initial value problem, it satisfies the relation

$$\varphi''(x) + x\varphi'(x) + \varphi(x) = 0 \implies \varphi''(x) = -x\varphi'(x) - \varphi(x). \tag{1}$$

To find higher order derivatives, differentiate on both sides of (1) in order to obtain

$$\begin{aligned}
\varphi'''(x) &= -x\varphi''(x) - 2\varphi'(x) \\
\varphi^{(4)}(x) &= -x\varphi'''(x) - 3\varphi''(x) \\
&\ \ \cdots \\
\varphi^{(n+2)}(x) &= -x\varphi^{(n+1)}(x) - (n+1)\varphi^{(n)}(x).
\end{aligned} \tag{2}$$

This last identity follows from an induction argument: suppose that the formula holds for some k. That is, suppose that

$$\varphi^{(k+2)}(x) = -x\varphi^{(k+1)}(x) - (k+1)\varphi^{(k)}(x).$$

Differentiate on both sides in order to obtain

$$\begin{aligned}
\varphi^{(k+3)}(x) &= -x\varphi^{(k+2)}(x) - \varphi^{(k+1)}(x) - (k+1)\varphi^{k+1}(x) \\
&= -x\varphi^{(k+2)}(x) - (k+2)\varphi^{(k+1)}(x).
\end{aligned}$$

This is precisely the claimed identity at (2) for $k+1$, hence by the principle of mathematical induction, (2) holds for all natural numbers n. Values of $\varphi^{(n)}(x_0)$ may be determined by recursively substituting the values obtained from the initial values. As $x_0 = 0$, this reduces to $\varphi^{(n+2)}(x_0) = -(n+1)\varphi^{(n)}(x)$. Thus

$$\begin{aligned}
\varphi(x_0) &= y(0) = 1, \\
\varphi'(x_0) &= y'(0) = 0, \\
\varphi''(x_0) &= -\varphi(x_0) = -1, \\
\varphi'''(x_0) &= -2\varphi'(x_0) = 0, \\
\varphi^{(4)}(x_0) &= -3\varphi''(x_0) = 3, \\
\varphi^{(5)}(x_0) &= -4\varphi'''(x_0) = 0,
\end{aligned}$$

and so on. □

Problem 2 (5.3.2)**.** Determine $\varphi''(x_0)$, $\varphi'''(x_0)$, and $\varphi^{(4)}(x_0)$ for the point x_0 if $y = \varphi(x)$ is a solution to the initial value problem

$$y'' + \sin(x)y' + \cos(x)y = 0; \qquad y(0) = 0, \quad y'(0) = 1.$$

Solution. As in the previous problem, isolate the highest order derivative and take derivatives to obtain

$$\begin{aligned} y'' &= -\sin(x)y' - \cos(x)y, \\ y''' &= -\sin(x)y'' - 2\cos(x)y' + \sin(x)y, \\ y^{(4)} &= -\sin(x)y''' - 3\cos(x)y'' + 3\sin(x)y' + \cos(x)y, \end{aligned}$$

and so on. This process can be continued indefinitely, and there is a reasonable closed form for higher order derivatives in terms of binomial coefficients, however continuing this sequences is not the goal of this exercise, and it is sufficiently tedious and off-topic to bear omission at this time. It would be a good exercise to determine this relation. In any event, if $y = \varphi(x)$ solves the initial value problem, then, with $x_0 = 0$, these identities reduce to

$$\begin{aligned} \varphi(x_0) &= y(0) = 0, \\ \varphi'(x_0) &= y'(0) = 1, \\ \varphi''(x_0) &= -\sin(0)y'(0) - \cos(0)y(0) = 0, \\ \varphi'''(x_0) &= -\sin(0)y''(0) - 2\cos(0)y'(0) + \sin(0)y(0) = -2, \\ \varphi^{(4)}(x_0) &= -\sin(0)y'''(0) - 3\cos(0)y''(0) + 3\sin(0)y'(0) + \cos(0)y(0) = 0. \end{aligned}$$

Again, higher order derivatives can be evaluated at $x_0 = 0$ by continuing in the manner indicated above. □

Problem 3 (5.3.8)**.** Determine a lower bound on the radius of convergence of series solutions about $x_0 = 1$ for the differential equation

$$xy'' + y = 0.$$

Solution. A series solution at $x_0 = 1$ will converge on a disk which has radius at least as large as the distance from x_0 to the nearest singular point of the differential equation. Written in standard form, the differential equation is

$$y'' + \frac{1}{x}y = 0,$$

which (by theorems in Chapter 4) has solutions wherever $x \mapsto 1/x$ is continuous. This function is continuous everywhere in $\mathbb{R}$ except at zero, thus the only singular point of the differential equation is $x = 0$. Since

$$|x_0 - 0| = |1 - 0| = 1,$$

it follows that the radius of convergence is bounded by $\rho \geq 1$. □

Euler's Equation

Let α and β be fixed real constants. Euler's equation is the differential equation

$$x^2 y'' + \alpha x y + \beta y = 0.$$

This equation is singular at zero, but has no other singularities. Because zero is a singular point, we do not expect to find power series solutions near zero. Instead, make the *ansatz* that a solution of the form $y(x) = x^\lambda$ exists, where $x > 0$. Then

$$x^2 y'' + \alpha x y + \beta y = \lambda(\lambda - 1)x^\lambda + \alpha\lambda x^\lambda = \beta x^\lambda = 0.$$

This is possible if and only if

$$0 = \lambda(\lambda - 1) + \alpha\lambda + \beta = \lambda^2 + (\alpha - 1)\lambda + \beta =: F(\lambda).$$

This quadratic polynomial F is called the *indicial polynomial*, and the roots of this polynomial (which correspond to solutions to Euler's equation) are the *Frobenius indices*. There are essentially three cases to consider:

(i) F has two distinct, real roots;

(ii) F has two complex roots, which come in a conjugate pair; or

(iii) F has a real root of multiplicity two.

Distinct Real Roots: In case (i), the indicial polynomial has two roots, λ_1 and λ_2. The assumption was that solutions are of the form $y(x) = x^\lambda$, and so these two roots correspond to the solutions

$$y_1(x) = x^{\lambda_1} \qquad \text{and} \qquad y_2(x) = x^{\lambda_2}.$$

As Euer's equation is linear and these two solutions are linearly independent, it follows that a general solution is given by

$$y(x) = k_1 x^{\lambda_1} + k_2 x^{\lambda_2},$$

where k_1 and k_2 are arbitrary constants.

Complex Roots: In case (ii), the two Frobenius indices occur in a conjuget pair, and are therefore of the form

$$\lambda_1 = \alpha + i\beta \qquad \text{and} \qquad \lambda_2 = \alpha - i\beta,$$

where α and β are real numbers. Hence there are two linearly independent solutions

$$\tilde{y}_1(x) = x^{\alpha + i\beta} \qquad \text{and} \qquad \tilde{y}_2(x) = x^{\alpha - i\beta}.$$

However, as real solutions are expected, linear combinations of these two solutions are taken to obtain real-valued functions. The complex exponential of a real number x is defined by

$$x^{\alpha + i\beta} = \mathrm{e}^{\log(x)(\alpha + i\beta)}.$$

Expanding the right hand side gives

$$e^{\log(x)(\alpha+i\beta)} = e^{\alpha\log(x)}e^{i\beta\log(x)} = x^\alpha e^{i\beta\log(x)}.$$

Therefore define

$$\begin{aligned}
y_1(x) &:= \frac{1}{2}(\tilde{y}_1(x) + \tilde{y}_2(x)) \\
&= \frac{1}{2}\left(x^{\alpha+i\beta} + x^{\alpha-i\beta}\right) \\
&= x^\alpha\left(\frac{e^{i\beta\log(x)} + e^{-i\beta\log(x)}}{2}\right) \\
&= x^\alpha\cos(\beta\log(x)).
\end{aligned}$$

By a similar computation, take

$$y_2(x) := \frac{1}{i2}(\tilde{y}_1(x) - \tilde{y}_2(x)) = x^\alpha\sin(\beta\log(x)).$$

Hence a general solution is given by

$$y(x) = k_1|x|^\alpha\cos(\beta\log|x|) + k_2|x|^\alpha\sin(\beta\log|x|),$$

where k_1 and k_2 are arbitrary constants.

Repeated Roots: Finally, in case (iii), there is a single repeated Frobenius root λ, giving rise to a single solution $y_1(x) = x^\lambda$. For reasons which will remain mysterious for the time being, a second solution is given by $y_2(x) = \log(x)x^\lambda$. A technique for obtaining this solution is discussed on page 273 of Boyce and DiPrima. Another method for maintaining a second solution is via a *reduction of order* argument. This argument will give a slightly different solution, but the two methods give the same solution space. In either event, a general solution is given by

$$y(x) = k_1x^\lambda + k_2\log(x)x^\lambda,$$

where k_1 and k_2 are arbitrary constants.

A final important point is that the three types of solutions above are valid only for $x > 0$. If $x < 0$, then many of the terms involved are ill-defined (for example, x^λ has not been defined for most negative values of x). The "fix" to this problem is to take absolute values.

Summary: Let α and β be real numbers and suppose that

$$x^2y'' + \alpha xy + \beta y = 0. \tag{3}$$

Let λ_1 and λ_2 denote Frobenius indices, i.e. the two roots of the indicial polynomial

$$F(\lambda) = \lambda^2 + (\alpha-1)\lambda + \beta.$$

(i) If $\lambda_1 \neq \lambda_2$ are real, then a general solution to (3) is given by

$$y(x) = k_1|x|^{\lambda_1} + k_2|x|^{\lambda_2}.$$

(ii) If $\lambda_{1,2} = \alpha \pm i\beta$ are non-real complex conjugates, then a general solution to (3) is given by

$$y(x) = k_1|x|^{\alpha}\cos(\beta\log|x|) + k_2|x|^{\alpha}\sin(\beta\log|x|),$$

(iii) If $\lambda_1 = \lambda_2$ are real, then a general solution to (3) is given by

$$y(x) = k_1x^{\lambda} + k_2\log(x)x^{\lambda},$$

In each case, k_1 and k_2 are arbitrary constants.

Problem 4 (5.4.1)**.** Determine a general solution to the differential equation

$$x^2y'' + 4xy' + 2y = 0$$

which is valid on any interval not containing a singular point.

Solution. In this problem, $\alpha = 4$ and $\beta = 2$. Thus the Frobenius polynomial is given by

$$F(\lambda) = \lambda^2 + (\alpha - 1)\lambda + \beta = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2).$$

The Frobenius indices are given by $\lambda_1 = -1$ and $\lambda_2 = -2$. Therefore, by application of the formula given for case (i), above, a general solution to the differential equation on the domain $\mathbb{R} \setminus \{0\}$ is given by

$$y(x) = k_1|x|^{-1} + k_2|x|^{-2}$$

where k_1 and k_2 are arbitrary real constants. □

Problem 5 (5.4.4)**.** Determine a general solution to the differential equation

$$x^2y'' + 3xy' + 5y = 0$$

which is valid on any interval not containing a singular point.

Solution. In this problem, $\alpha = 3$ and $\beta = 5$. Thus the Frobenius polynomial is given by

$$F(\lambda) = \lambda^2 + (\alpha - 1)\lambda + \beta = \lambda^2 + 2\lambda + 5.$$

By application of the quadratic formula, the Frobenius indices are given by

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} = -1 \pm i2.$$

Therefore, by application of the formula given for case (ii), above, a general solution to the differential equation on the domain $\mathbb{R} \setminus \{0\}$ is given by

$$y(x) = k_1|x|^{-1}\cos(2\log|x|) + k_2|x|^{-1}\sin(2\log|x|)$$

where k_1 and k_2 are arbitrary real constants. □

Problem 6 (5.4.5)**.** Determine a general solution to the differential equation

$$x^2y'' - xy' + y = 0$$

which is valid on any interval not containing a singular point.

Solution. In this problem, $\alpha = -11$ and $\beta = 1$. Thus the Frobenius polynomial is given by

$$F(\lambda) = \lambda^2 + (\alpha - 1)\lambda + \beta = \lambda^2 + 2\lambda - 1 = (\lambda - 1)^2.$$

The Frobenius indices are given by $\lambda_{1,2} = 1$. With $\lambda = 1$, the formula given for case (iii), above, gives the general solution to the differential equation on the domain $\mathbb{R} \setminus \{0\}$ as

$$y(x) = k_1|x| + k_2 \log(|x|)|x|,$$

where k_1 and k_2 are arbitrary real constants. □

Problem 7 (5.4.9)**.** Determine a general solution to the differential equation

$$x^2y'' - 5xy' + 9y = 0$$

which is valid on any interval not containing a singular point.

Solution. In this problem, $\alpha = -5$ and $\beta = 9$. Thus the Frobenius polynomial is given by

$$F(\lambda) = \lambda^2 + (\alpha - 1)\lambda + \beta = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$$

The Frobenius indices are given by $\lambda_{1,2} = 3$. With $\lambda = -1$, the formula given for case (iii), above, gives the general solution to the differential equation on the domain $\mathbb{R} \setminus \{0\}$ as

$$y(x) = k_1|x|^3 + k_2 \log(|x|)|x|^3,$$

where k_1 and k_2 are arbitrary real constants. □

Definition 1. Suppose that

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

A point x_0 such that $P(x_0) = 0$ is a *singular point* of this equation. If x_0 is a singular point and both

$$\lim_{x \to x_0} \frac{Q(x)}{P(x)}(x - x_0) \qquad \text{and} \qquad \lim_{x \to x_0} \frac{R(x)}{P(x)}(x - x_0)^2$$

exist, then x_0 is called a *regular singular point*. Otherwise, it is called an *irregular singular point*.

Problem 8 (5.4.19)**.** Find all singular points of the equation

$$x^2(1 - x)y'' + (x - 2)y' - 3xy = 0.$$

Classify each singular point as either regular or irregular.

Solution. In this problem,

$$P(x) = x^2(1 - x), \qquad Q(x) = (x - 2), \qquad \text{and} \qquad R(x) = -3x.$$

The roots of P are $x = 0$ and $x = 1$, hence the equation has two singular points.

Observe that

$$\lim_{x\to 0} \frac{Q(x)}{P(x)}(x - 0) = \lim_{x\to 0} \frac{x - 2}{x^2(1 - x)}x = \lim_{x\to 0} \frac{x - 2}{x(1 - x)}.$$

This limit does not exit, hence the singular point $x_0 = 0$ is an irregular singular point. On the other hand

$$\lim_{x\to 1} \frac{Q(x)}{P(x)}(x - 1) = \lim_{x\to 1} \frac{x - 2}{x^2(1 - x)}(x - 1) = 1,$$

and

$$\lim_{x\to 1} \frac{R(x)}{P(x)}(x - 1)^2 = \lim_{x\to 1} \frac{-3x}{x^2(1 - x)}(x - 1)^2 = 0.$$

As both limits exist, the singular point $x_0 = 1$ is a regular singular point. □

Problem 9 (5.4.22)**.** Find all singular points of the equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0.$$

This equation is called *Bessel's equation*. Classify each singular point as either regular or irregular.

Solution. In this problem,

$$P(x) = x^2, \qquad Q(x) = x, \qquad \text{and} \qquad R(x) = x^2 - \nu^2.$$

The leading coefficient function P has only one root, at zero. Thus the only singular point of Bessel's equation is $x_0 = 0$. Taking the appropriate limits,

$$\lim_{x\to 0} \frac{Q(x)}{P(x)}x = \lim_{x\to 0} \frac{x}{x^2}x = 1,$$

and

$$\lim_{x\to 0} \frac{R(x)}{P(x)}x^2 = \lim_{x\to 0} \frac{x^2 - \nu^2}{x^2}x^2 = -\nu^2.$$

As both limits exist, $x_0 = 0$ is a regular singular point. □

Problem 10 (5.5.1)**.**

Problem 11 (5.5.3)**.**