2^{nd}- order ODE with constant coefficients

Problem 1. (Initial Value Problem).

$$y'' + 2y' + 5y = 0 \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 3.$$

Characteristic equation: $r^2 + 2r + 5 = 0$.

Note: $r = \dfrac{-2 \pm \sqrt{2^2 - 4 \times 5}}{2} = -1 \pm 2i$ (no real solution).

$\lambda = -1$ and $\mu = 2$. $\Rightarrow$ $y_1 = e^{-t}\cos(2t)$ and $y_2 = e^{-t}\sin(2t)$.

General solution $y = c_1 y_1 + c_2 y_2 = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$.

Plug in initial values:

$$\begin{cases} y(0) = c_1 = 1 \\ y'(0) = -c_1 + 2c_2 = 3. \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = 2. \end{cases}$$

So $y = e^{-t}\cos(2t) + 2e^{-t}\sin(2t)$.

Problem 2. (Non-homogeneous equation)

Find the general solution of $y'' + 2y' - 3y = \sin(2t)$. $(*)$

First look at $y'' + 2y' - 3y = 0$. with characteristic equation

$r^2 + 2r - 3 = 0$. $\Rightarrow$ $(r+3)(r-1) = 0$. $r_1 = 1$, $r_2 = -3$

So $y_1 = e^{t}$ and $y_2 = e^{-3t}$.

Then $y_c = c_1 y_1 + c_2 y_2 = c_1 e^{t} + c_2 e^{-3t}$.

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$g(t) = \sin(2t)$ does not involve y_1 or y_2.

So a particular solution of $(*)$ is of the form:

$$Y(t) = A\sin(2t) + B\cos(2t).$$

Compute: $Y'(t) = 2A\cos(2t) - 2B\sin(2t)$ and.

$$Y''(t) = -4A\sin(2t) - 4B\cos(2t).$$

Plug in $(*)$ $\Rightarrow$.

$$Y'' + 2Y' - 3Y = [-4A\sin(2t) - 4B\cos(2t)] + 2[2A\cos(2t) - 2B\sin(2t)]$$
$$- 3\,(A\sin(2t) + B\cos(2t)).$$
$$= (-7A - 4B)\sin(2t) + (-7B + 4A)\cos(2t).$$
$$= g(t) = \sin(2t).$$

$$\Rightarrow \begin{cases} -7A - 4B = 1 \\ -7B + 4A = 0 \end{cases} \Rightarrow \begin{cases} A = -\frac{7}{65} \\ B = -\frac{4}{65} \end{cases}$$

So $Y(t) = -\frac{7}{65}\sin(2t) - \frac{4}{65}\cos(2t)$.

and $y = y_c + Y = c_1 e^{t} + c_2 e^{-3t} - \frac{7}{65}\sin(2t) - \frac{4}{65}\cos(2t)$.

Problem 3. (linear dependence).

Determine whether the given functions are linearly dependent or not:

$$f_1(t)=2t-3, \quad f_2(t)=2t^2+1, \quad f_3(t)=3t^2+t.$$

Set. $k_1 f_1(t) + k_2 f_2(t) + k_3 f_3(t) = 0.$ for all t.

$\Leftrightarrow \quad k_1(2t-3) + k_2(2t^2+1) + k_3(3t^2+t) = 0.$

So $(2k_2+3k_3)t^2 + (2k_1+k_3)t + (-3k_1+k_2) = 0.$ for all t.

$$\Rightarrow \begin{cases} 2k_2+3k_3 = 0. & (1) \\ 2k_1+k_3 = 0. & (2) \\ -3k_1+k_2 = 0 & (3). \end{cases}$$

from (2) & (3) $\Rightarrow \begin{cases} k_2 = 3k_1 \\ k_3 = -2k_1 \end{cases}$

It satisfies (1) for arbitrary k_1. For example, take $k_1=1$. then $k_2=3$ and $k_3=-2$.

So $(2t-3) + 3(2t^2+1) - 2(3t^2+t) \equiv 0.$

i.e. $f_1(t) + 3f_2(t) - 2f_3(t) = 0.$

f_1, f_2, f_3 are linearly dependent.