146B — 2
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Section 3.6

In each of Problems 1 through 4, use the method of variation of parameters to find a particular solution of the given differential equation. Then check your answer by using the method of undetermined coefficients.

3 $y'' + 2y' + y = 3e^{-t}$

slu.

$r^2 + 2r + 1 = 0 \implies r = \frac{-2 \pm \sqrt{4-4}}{2} = -1$

$\implies \{e^{-t}, te^{-t}\}$ is a fundamental set of solutions.

$W[e^{-t}, te^{-t}] = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & (1-t)e^{-t} \end{vmatrix} = (1-t)e^{-2t} - (-te^{-2t}) = e^{-2t}$

$W_1[e^{-t}, te^{-t}] = \begin{vmatrix} 0 & te^{-t} \\ 1 & (1-t)e^{-t} \end{vmatrix} = -te^{-t}$

$W_2[e^{-t}, te^{-t}] = \begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & 1 \end{vmatrix} = e^{-t}$

So, by Alysha's notes:

$u_1(t) = \int \frac{3e^{-t}(-te^{-t})}{e^{-2t}} dt = -3 \int \frac{te^{-2t}}{e^{-2t}} dt = -3 \int t dt$

$u_1(t) = -\frac{3}{2}t^2 + C$

$u_2(t) = \int \frac{3e^{-t}(e^{-t})}{e^{-2t}} dt = 3 \int 1 dt = 3t + C$

The C's give the homogeneous solution so they don't add anything.

$$\begin{aligned} y_p(t) &= (-\frac{3}{2}t^2)e^{-t} + (3t)te^{-t} \\ &= \frac{3}{2}t^2 e^{-t} \end{aligned}$$

The method of undetermined coefficients yields,

$Y = At^2e^{-t} \implies Y' = A(2te^{-t} - t^2e^{-t}) \implies Y'' = A(2e^{-t} - 4te^{-t} + t^2e^{-t})$

$\implies L[Y] = 2Ae^{-t} = 3e^{-t} \implies A = \frac{3}{2} \implies y_p(t) = \frac{3}{2}t^2e^{-t} \quad \Diamond$

In each of Problems 5 through 12, find the general solution of the given differential equation.

5 $y' + y = \text{tan}(t)$, $0 < t < \pi/2$. slu.

$r^2 + 1 = 0 \implies r = \pm i \implies \phi := \{\text{cos}(t), \text{sin}(t)\}$ is a fundamental set of solutions. $W[\phi](t) = 1$.

Then it follows that,

$$\begin{aligned} u_1(t) &= \int \text{tan}(t)(-\text{sin}(t))dt \\ &= -\frac{1}{2}\,\log\left(\sin\left(t\right) + 1\right) + \frac{1}{2}\,\log\left(\sin\left(t\right) - 1\right) + \sin\left(t\right) \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \text{tan}(t)\,\text{cos}(t)dt \\ &= -\cos\left(t\right) \end{aligned}$$

$$\begin{aligned} y_p(t) &= \text{cos}(t)(-\frac{1}{2}\,\log\left(\sin\left(t\right) + 1\right) + \frac{1}{2}\,\log\left(\sin\left(t\right) - 1\right) + \sin\left(t\right)\,) + \text{sin}(t)(-\cos\left(t\right)\,) \\ &= -\frac{1}{2}\left(\log\left(\sin\left(t\right) + 1\right) - \log\left(\sin\left(t\right) - 1\right) - 2\,\sin\left(t\right)\right)\cos\left(t\right)\; + -\cos\left(t\right)\sin\left(t\right) \\ &= -\frac{1}{2}\,\cos\left(t\right)\log\left(\sin\left(t\right) + 1\right) + \frac{1}{2}\,\cos\left(t\right)\log\left(\sin\left(t\right) - 1\right) \end{aligned}$$

$y(t) = A\,\text{cos}(t) + B\,\text{sin}(t) - \frac{1}{2}\,\cos\left(t\right)\log\left(\sin\left(t\right) + 1\right) + \frac{1}{2}\,\cos\left(t\right)\log\left(\sin\left(t\right) - 1\right)$ ♦

7 $y'' + 4y' + 4y = t^{-2}e^{-2t}$.

slu.

$r^2 + 4r^2 + 4 = 0 \implies r = \frac{-4 \pm \sqrt{16-16}}{2} = -2 \implies \phi := \{e^{-2t}, te^{-2t}\}$ is a fundamental set of solutions for 7.

$W[\phi](t) = -\left(2\,te^{(-2\,t)} - e^{(-2\,t)}\right)e^{(-2\,t)} + 2\,te^{(-4\,t)} = e^{(-4\,t)}$

$$\begin{aligned} u_1(t) &= \int \frac{\frac{e^{(-2\,t)}}{t^2}(-te^{(-2\,t)})}{e^{(-4\,t)}}dt \\ &= -\log\left(t\right)\, + C \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{\frac{e^{(-2\,t)}}{t^2}(e^{(-2\,t)})}{e^{(-4\,t)}}dt \\ &= -\frac{1}{t} + C \end{aligned}$$

Then,

$$\begin{aligned} y_p(t) &= e^{(-2\,t)} - \log\left(t\right)\, + te^{(-2\,t)} - \frac{1}{t} \\ &= -e^{(-2\,t)}\log\left(t\right) - e^{(-2\,t)} \end{aligned}$$

Since, $-e^{-t}$ is in ϕ, it follows it gets absorbed by the constant in the general solution.

$$y(t) = Ae^{(-2\,t)} + Bte^{(-2\,t)} - e^{(-2\,t)}\log\left(t\right) \quad ♦$$

Section 5.1

In each of the Problems 1 through 8, determine the radius of convergence of the given power series.

5 $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n^2}$

slu. $\frac{1}{\rho} = \lim_{n\to\infty} \left|\frac{n^2}{(n+1)^2}\right| = \lim_{n\to\infty} \left|\frac{n^2}{n^2+2n+1}\right| = 1$

$\implies \rho = 1 \quad \Diamond$

7 $\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (x+2)^n}{3^n}$

slu. $\frac{1}{\rho} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n n^2}{3^n}\right|} = \lim_{n\to\infty} \frac{n^{\frac{2}{n}}}{3} = \frac{1}{3}$

$\implies \rho = 3 \quad \Diamond$

In each of Problems 9 through 16, determine the Taylor series about the point x_0 for the given function. Also determine the radius of convergence of the series.

15 $\frac{1}{1-x} \quad , \quad x_0 = 0$

slu.

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, by taking the limit of the geometric series as $n \to \infty$.

$\rho = 1$, because otherwise the series diverges $\quad \Diamond$

18 Given that $y = \sum_{n=0}^{\infty} a_n x^n$ compute y' and y'' and write out the first four terms of each series, as well as the coefficient of x^n in the general term. Show that if $y'' = y$, then the coefficients a_0 and a_1 are arbitrary, and determine a_2 and a_3 in terms of a_0 and a_1. Show that $a_{n+2} = \frac{a_n}{(n+2)(n+1)}, n = 0, 1, 2, 3, ...$

pf.

$\quad y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + ...$

$\implies y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + ...$

$\implies y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + ...$

Let $k = n - 1 \implies n = k + 1 \quad \implies y' = \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k$

Let $s = n - 2 \implies n = s + 2 \quad \implies y'' = \sum_{s=0}^{\infty} (s+2)(s+1) a_{s+2} x^s$

Let $s = n$, and ass. $y'' = y$, then,

Looking at the computed coefficients we can see,

$a_0 = 2a_2 \implies a_2 = \frac{a_0}{2}$ and $a_1 = 6a_3 \implies a_3 = \frac{a_1}{6}$ Then notice that in the general terms,

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\implies a_n = (n+2)(n+1) a_{n+2}$$

$$\implies a_{n+2} = \frac{a_n}{(n+2)(n+1)}$$

So, at a glance all the even terms depend on a_0, and the odd terms on a_1 $\quad \blacksquare$

28 Determine a_n so that the equation

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

is satisfied. Try to identify the function represented by $\sum_{n=0}^{\infty} a_n x^n$.

pf.

$$\begin{aligned}
\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n &= 0 \\
\sum_{n=1}^{\infty} n a_n x^{n-1} &= -2\sum_{n=0}^{\infty} a_n x^n \\
\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n &= -2\sum_{n=0}^{\infty} a_n x^n \\
\implies (n+1)a_{n+1} &= -2a_n \\
\implies a_{n+1} &= -\frac{2a_n}{n+1}
\end{aligned}$$

$$\begin{aligned}
n=0 \quad a_1 &= -\frac{2a_0}{1} = -\frac{2a_0}{1} \\
n=1 \quad a_2 &= -\frac{2a_1}{2} = -\frac{2a_1}{2} = -\frac{2(-2a_0)}{2\cdot 1} = (-1)^2\frac{2^2 a_0}{2\cdot 1} \\
n=2 \quad a_3 &= -\frac{2a_2}{3} = (-1)^3\frac{2^3 a_0}{3\cdot 2\cdot 1} \\
&\vdots \\
n=k-1 \quad a_{k+1} &= \frac{(-1)^{k+1}2^{k+1}a_0}{k!} = -2a_0\frac{(-1)^k 2^k}{k!}
\end{aligned}$$

So, the function is $-2a_0 e^{-2x}$ ■

Section 5.2

1 $y'' - y = 0, x_0 = 0$

slu.

We can see that $a_{n+2} = \frac{a_n}{(n+2)(n+1)}$, from problem **18**.

$$\begin{aligned}
n=0 \quad a_2 &= \frac{a_0}{2} & n=1 \quad a_3 &= \frac{a_1}{6} \\
n=2 \quad a_4 &= \frac{a_2}{(4)(3)} = \frac{a_0}{4!} & n=3 \quad a_5 &= \frac{a_3}{(5)(4)} = \frac{a_0}{5!} \\
&\vdots & &\vdots \\
n=2k \quad a_{2k} &= \frac{a_0}{(2k)!} & n=2k+1 \quad a_{2k+1} &= \frac{a_1}{(2k+1)!}
\end{aligned}$$

So, ⋯ I don't know I didn't have time to finish... Probably e^t, and e^{-t}.