§10.5 and §10.6. Heat Equations

Problem 1. (homogeneous boundary condition).

Solve
$$\begin{cases} \dfrac{\partial u}{\partial t} = 4\dfrac{\partial^2 u}{\partial x^2}, & 0<x<\pi,\ t>0. \\ u(0,t) = u(\pi,t) = 0, & t>0 \\ u(x,0) = x^2 & 0<x<\pi. \end{cases}$$

Solution. $\beta = 4$, $L = \pi$.

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-4n^2 t} \sin(nx).$$

So $u(x,0) = \sum_{n=1}^{\infty} c_n \sin(nx) = x^2.$

$$\Rightarrow. \quad c_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin(nx)\,dx. \qquad n = 1, 2, \cdots$$

$$= \frac{2}{\pi n^3}\int_0^{n\pi} u^2 \sin(u)\,du \qquad \text{subs } \begin{cases} u = nx \\ du = n\,dx \end{cases}$$

$$= \frac{2}{\pi n^3}\left(-u^2\cos(u)\Big|_0^{n\pi} + \int_0^{n\pi} 2u\cos(u)\,du \right).$$

$$= \frac{2}{\pi n^3}\left[-n^2\pi^2\cos(n\pi) + 2\cdot\left(u\sin(u)\Big|_0^{n\pi} - \int_0^{n\pi}\sin(u)\,du \right)\right]$$

$$= \frac{2}{\pi n^3}\left[-n^2\pi^2(-1)^n + 0 + 2\cos(u)\Big|_0^{n\pi} \right].$$

$$= \frac{2}{\pi n^3}\left[-n^2\pi^2(-1)^n + 2\left(\cos(n\pi) - 1\right)\right].$$

$$= \frac{-2\pi(-1)^n}{n} + \frac{4\left[(-1)^n - 1\right]}{\pi n^3}.$$

Hence: $$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{-2\pi(-1)^n}{n} + \frac{4\left[(-1)^n-1\right]}{\pi n^3}\right] e^{-4n^2 t}\sin(nx).$$

Problem 2. (homogeneous boundary condition).

Solve
$$\begin{cases} \dfrac{\partial u}{\partial t} = 3\dfrac{\partial^2 u}{\partial x^2} , & 0<x<\pi ,\ t>0 . \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(\pi,t) = 0 , & t>0 \\ u(x,0) = 1+x , & 0<x<\pi . \end{cases}$$

Solution. $\beta = 3$, $L=\pi$. it's variant boundary condition.

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-3n^2t} \cos(nx) .$$

So $u(x,0) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) = 1+x$.

$$\Rightarrow . \quad a_0 = \frac{2}{\pi}\int_0^{\pi} (1+x)\,dx = \frac{2}{\pi}\cdot\left(x+\tfrac{1}{2}x^2\right)\Big|_0^{\pi} = \pi+2 .$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(1+x)\cos(nx)\,dx \qquad n=1,2,\cdots$$

$$= \frac{2}{\pi}\left(\int_0^{\pi}\cos(nx)dx + \int_0^{\pi} x\cos(nx)\,dx\right).$$

$$= \frac{2}{\pi}\left(\frac{1}{n}\sin(nx)\Big|_0^{\pi} + \frac{1}{n}\sin(nx)\cdot x\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{n}\sin(nx)dx\right).$$

$$= \frac{2}{\pi}\left(0 + 0 - \frac{-1}{n^2}\cos(nx)\Big|_0^{\pi} \right).$$

$$= \frac{2}{\pi}\cdot\frac{1}{n^2}\cdot\left[\cos(n\pi) - \cos(0)\right].$$

$$= \frac{2}{\pi n^2}\left((-1)^n - 1\right).$$

Hence: $u(x,t) = \dfrac{\pi+2}{2} + \sum_{n=1}^{\infty} \dfrac{2}{\pi n^2}\left((-1)^n-1\right) e^{-3n^2t} \cos(nx)$.

Problem 3. (non-homogeneous boundary condition).

Solve
$$\begin{cases} \dfrac{\partial u}{\partial t} = 2\cdot\dfrac{\partial^2 u}{\partial x^2}, & 0<x<1, \quad t>0 \\ u(0,t)=1, \quad u(1,t)=3, & t>0 \\ u(x,0) = 2x+1+\sin(3\pi x) - \sin(5\pi x), & 0<x<1. \end{cases}$$

Solution. $\beta=2$. $L=1$. non-homogeneous boundary:

$$u(x,t) = v(x) + w(x,t).$$

where
$$\begin{cases} v''(x) = 0 \\ v(0)=1 \text{ and } v(1)=3. \end{cases}$$

$$\Rightarrow \quad v(x) = c_1 x + c_2 \quad \text{and} \quad \begin{cases} v(0) = c_2 \\ v(1) = c_1+c_2 \end{cases} \Rightarrow v(x) = 2x+1$$

Then
$$\begin{cases} \dfrac{\partial w}{\partial t} = 2\cdot\dfrac{\partial^2 w}{\partial x^2} \\ w(0,t) = w(1,t) = 0 \\ w(x,0) = \sin(3\pi x) - \sin(5\pi x). \end{cases}$$

So
$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-2n^2\pi^2 t} \sin(n\pi x).$$

and
$$w(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) = \sin(3\pi x) - \sin(5\pi x).$$

$\Rightarrow$. $c_3 = 1$, $c_5 = -1$, and. $c_n = 0$ others.

Hence
$$w(x,t) = e^{-2\cdot 3^2\pi^2 t}\sin(3\pi x) - e^{-2 5^2\pi^2 t}\sin(5\pi x).$$
$$= e^{-18\pi^2 t}\sin(3\pi x) - e^{-50\pi^2 t}\sin(5\pi x).$$

So
$$u(x,t) = 2x+1+e^{-18\pi^2 t}\sin(3\pi x) - e^{-50\pi^2 t}\sin(5\pi x).$$