Problem 1. Find the Fourier series for $f(x) = \begin{cases} 1, & -L \le x < 0 \\ 0, & 0 \le x < L. \end{cases}$

Solutions.
$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$
$$= \frac{1}{L}\int_{-L}^{0} dx = 1.$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx. \qquad \text{for } n=1,2,\cdots$$
$$= \frac{1}{L}\int_{-L}^{0} \cos\frac{n\pi x}{L}\,dx$$
$$= \frac{1}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{0} = -\frac{1}{n\pi}\sin(-n\pi). = 0.$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx. \qquad \text{for } n=1,2,\ldots$$
$$= \frac{1}{L}\int_{-L}^{0} \sin\frac{n\pi x}{L}\,dx.$$
$$= \frac{1}{n\pi}\left(-\cos\frac{n\pi x}{L}\right)\Big|_{-L}^{0} = \frac{-1+\cos(-n\pi)}{n\pi} = \frac{-1+(-1)^n}{n\pi}.$$

So
$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right].$$
$$= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{-1+(-1)^n}{n\pi}\sin\frac{n\pi x}{L}.$$

or (let $n=2k+1$).
$$= \frac{1}{2} + \sum_{k=0}^{\infty}\frac{-2}{\pi(2k+1)}\sin\frac{(2k+1)\pi x}{L}.$$

Problem 2. Compute the Fourier <u>sine series</u> of

$$f(x) = 1 - \cos(2x) \qquad 0 < x < \pi .$$

Solutions. $f \sim \sum_{n=1}^{\infty} b_n \sin(nx)$

$$b_n = \frac{2}{\pi}\int_0^{\pi}(1-\cos(2x))\sin(nx)\,dx. \qquad \text{for } n=1,2,\cdots$$

$$= \frac{2}{\pi}\int_0^{\pi}\left[\sin(nx) - \cos(2x)\sin(nx)\right]dx.$$

$$= \frac{2}{\pi}\left[\int_0^{\pi}\sin(nx)dx - \int_0^{\pi}\cos(2x)\sin(nx)\,dx\right].$$

$$\boxed{\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]}$$

$$= \frac{2}{\pi}\left[-\frac{\cos(nx)}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{2}\left[\sin(n+2)x + \sin(n-2)x\right]dx\right]$$

$$= \frac{2}{\pi}\left[\frac{-\cos(n\pi)+\cos(0)}{n} - \frac{1}{2}\left(\int_0^{\pi}\sin(n+2)x\,dx + \int_0^{\pi}\sin(n-2)x\,dx\right)\right].$$

$$= \frac{2}{\pi}\left[\frac{-(-1)^n+1}{n} - \frac{1}{2}\left(\frac{-\cos(n+2)x\big|_0^{\pi}}{n+2} + \frac{-\cos(n-2)x\big|_0^{\pi}}{n-2}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{1-(-1)^n}{n} - \frac{1}{2}\left(\frac{1-(-1)^{n+2}}{n+2} + \frac{1-(-1)^{n-2}}{n-2}\right)\right]$$

$$= \frac{2}{\pi}\cdot\left[1-(-1)^n\right]\cdot\left(\frac{1}{n} - \frac{1}{2}\left(\frac{1}{n+2}+\frac{1}{n-2}\right)\right]$$

$$= \frac{-4}{n(n-2)(n+2)}\cdot\frac{2}{\pi}\cdot\left(1-(-1)^n\right). = \begin{cases} 0 & \text{if } n=2k. \\ \frac{-16}{\pi(2k+1)(2k-1)(2k+3)} & \text{if } n=2k+1 \end{cases}$$

So $$f \sim \sum_{n=1}^{\infty}\frac{-8(1-(-1)^n)}{\pi n(n-2)(n+2)}\sin(nx) = \frac{-16}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)x)}{(2k+1)(2k-1)(2k+3)}.$$

Problem. Solve the heat flow problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & 0<x<\pi,\ t>0. & (1) \\ u(0,t) = u(\pi,t) = 0, & t>0. & (2) \\ u(x,0) = f(x) = 1-\cos(2x), & 0<x<\pi. & (3) \end{cases}$$

Solution. Separation of variables: $u(x,t) = X(x)T(t)$.

$(1)+(2) \Rightarrow \dfrac{X''}{X} = \dfrac{T'}{T} = -\lambda$. for some constant λ.

and $X(0) = X(\pi) = 0$.

So $\begin{cases} X''+\lambda X = 0. \\ X(0) = X(\pi) = 0. \end{cases} \Rightarrow \lambda_n = n^2$ for $n=1,2,\cdots$ (eigenvalues).

and $X_n(x) = \sin(nx)$ (eigenfunctions).

So $T' + \lambda_n T = 0. \Rightarrow T_n(t) = e^{-n^2 t}$

Hence. $u_n = X_n \cdot T_n = e^{-n^2 t}\sin(nx)$. for $n=1,2,\cdots$

General solution for (1)+(2) is $u = \sum_{n=1}^{\infty} c_n u_n = \sum_{n=1}^{\infty} c_n e^{-n^2 t}\sin(nx)$.

From (3). $u(x,0) = \sum_{n=1}^{\infty} c_n \sin(nx) = 1-\cos(2x)$.

Previous problem shows: $1-\cos(2x) \sim \sum_{n=1}^{\infty} \dfrac{-8(1-(-1)^n)}{\pi n(n-2)(n+2)}\sin(nx)$.

$\Rightarrow c_n = \dfrac{-8(1-(-1)^n)}{\pi n(n-2)(n+2)}$ for $n=1,2,\ldots$

So $u(x,t) = \sum_{n=1}^{\infty} \dfrac{-8(1-(-1)^n)}{\pi n(n-2)(n+2)} e^{-n^2 t}\sin(nx)$

$\left(\text{or } = \sum_{k=0}^{\infty} \dfrac{-16}{\pi}\cdot\dfrac{\sin((2k+1)x)}{(2k+1)(2k-1)(2k+3)} e^{-(2k+1)^2 t}\right)$.