Math 146B. Midterm Review.

§4.1 General theory:

$$y^{(n)} + p_1(t)\, y^{(n-1)} + p_2(t)\, y^{(n-2)} + \cdots + p_{n-1}(t)\, y' + p_n(t)\, y = g(t) \qquad \langle 1 \rangle$$

with initial conditions:

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \cdots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)} \qquad \langle 2 \rangle.$$

Theorem (Existence & Uniqueness).

If $p_1, \cdots, p_n$ and g are continuous, there exists exactly one solution $y = y(t)$ of $\langle 1 \rangle$ satisfying $\langle 2 \rangle$.

Consider $\langle 1 \rangle$ only. If Y_1 and Y_2 are solutions, then $Y_1 - Y_2$ is a solution of

$$y^{(n)} + p_1(t)\, y^{(n-1)} + \cdots + p_{n-1}(t)\, y' + p_n(t)\, y = 0. \qquad \langle 3 \rangle.$$

$\langle 3 \rangle$ is called the associated homogeneous equation of $\langle 1 \rangle$.

So it reduces to find general solution y_c of $\langle 3 \rangle$ and a particular solution Y_p of $\langle 1 \rangle$. The structure is:

$$y = y_c + Y_p. \qquad (\text{general solution of } \langle 1 \rangle).$$

For $\langle 3 \rangle$, we expect to have n linear independent solutions $y_1, y_2, \cdots, y_n$.

Def. $f_1, f_2, \cdots, f_n$ are said to be linearly dependent on I if there exists a set of constants $k_1, \cdots k_n$, not all zero, such that

$$k_1 f_1(t) + \cdots + k_n f_n(t) = 0 \qquad \text{for all } t \text{ in } I.$$

Otherwise, they are linearly independent.

Example. $f_1(t) = 2t-3$, $f_2(t) = 2t^2+1$, $f_3(t) = 3t^2+t$.

Set $k_1 f_1(t) + k_2 f_2(t) + k_3 f_3(t) = 0$. for all t.

Then $k_1(2t-3) + k_2(2t^2+1) + k_3(3t^2+t) = 0$.

$\Rightarrow (2k_2+3k_3)t^2 + (2k_1+k_3)t + (-3k_1+k_2) = 0$.

So $\begin{cases} 2k_2+3k_3=0 \\ 2k_1+k_3=0 \\ -3k_1+k_2=0 \end{cases} \Rightarrow \begin{cases} k_2 = 3k_1 \\ k_3 = -2k_1 \end{cases}$ $(2k_2+3k_3 = 6k_1 - 6k_1 = 0)$ automatic

There exists non zero k_1, k_2, k_3. So f_1, f_2, f_3 are linearly dependent.

Example. $f_1(t) = t-1$, $f_2(t) = t^2+2$, $f_3(t) = 2t^2+t$.

Set. $k_1 f_1(t) + k_2 f_2(t) + k_3 f_3(t) = 0$ for all t.

Then. $k_1(t-1) + k_2(t^2+2) + k_3(2t^2+t) = 0$

$\Rightarrow (k_2+2k_3)t^2 + (k_1+k_3)t + (-k_1+2k_2) = 0$.

So $\begin{cases} k_2+2k_3=0 \\ k_1+k_3=0. \\ -k_1+2k_2=0. \end{cases} \Rightarrow \begin{cases} k_2 = -2k_3. \\ k_1 = -k_3 \end{cases}$ $-k_1+2k_2 = k_3 - 4k_3 = 0$. $\Rightarrow \begin{cases} k_1=0 \\ k_2=0 \\ k_3=0. \end{cases}$

So f_1, f_2, f_3 are linearly independent.

Fundamental set of solutions:

Let $y_1, y_2, \cdots, y_n$ be solutions of ⟨3⟩. Define the Wronskian

$$W(y_1, y_2, \cdots, y_n) = \det \begin{pmatrix} y_1 & y_2 & & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{pmatrix}$$

If $W(y_1, \cdots, y_n) \neq 0$ for at least one point in I, then $\{y_1, y_2, \cdots, y_n\}$ is a fundamental set of solutions of ⟨3⟩.

As a consequence, the general solution y_c of <3> can be expressed

$$y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n ;$$

also $y_1, y_2, \cdots, y_n$ are linearly independent in I.

§4.2. Constant Coefficients:

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0. \quad <4>.$$

where $a_0 \neq 0, a_1, a_2, \cdots, a_n$ are real constants.

The associated characteristic equation is $a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$. <5>

Case 1°. <5> has distinct real roots: $r_1, r_2, \cdots, r_n$.

$\{ e^{r_1 t}, e^{r_2 t}, \cdots, e^{r_n t} \}$ is a fundamental set of solutions.

Case 2°. <5> has complex roots: (in pairs). $\lambda \pm i\mu$.

$e^{\lambda t}\cos(\mu t)$, $e^{\lambda t}\sin(\mu t)$ are the solutions to these roots.

Case 3°. <5> has repeated roots: r_1 (real, s times).

$e^{r_1 t}, t e^{r_1 t}, \cdots, t^{s-1} e^{r_1 t}$ are the solutions.

$r_{1,2} = \lambda \pm i\mu$. (in pairs, each s times).

$e^{\lambda t}\cos(\mu t), e^{\lambda t}\sin(\mu t), \cdots, t^{s-1} e^{\lambda t}\cos(\mu t), t^{s-1} e^{\lambda t}\sin(\mu t)$ are solutions.

Example. $y^{(4)} - y = 0$.

$r^4 - 1 = 0. \Rightarrow (r-1)(r+1)(r^2+1) = 0. \quad r = 1, -1, i, -i$.

So $y = c_1 e^{t} + c_2 e^{-t} + c_3 \cos(t) + c_4 \sin(t)$.

Example. 1° $y'' + 4y' + 4y = 0$.

2°. $y'' + 4y' + 3y = 0$.

3°. $y'' + 4y' + 5y = 0$.

1°. $r^2 + 4r + 4 = 0$. $\Rightarrow r = -2$ (multiplicity 2).

$$y = c_1 e^{-2t} + c_2 t e^{-2t}.$$

2°. $r^2 + 4r + 3 = 0 \Rightarrow r_1 = -3$ and $r_2 = -1$.

$$y = c_1 e^{-3t} + c_2 e^{-t}.$$

3°. $r^2 + 4r + 5 = 0$. $\Rightarrow r_{1,2} = -2 \pm i$.

$$y = c_1 e^{-2t} \cos(t) + c_2 e^{-2t} \sin(t).$$

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§ 4.3 Undetermined Coefficients

Focus on: $$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t). \qquad \langle 6 \rangle$$

where $a_0 \neq 0, a_1, \cdots, a_n$ are constants (real). and $g(t)$ is of an appropriate form: polynomial / exponential / sine & cosine

1° When $g(t)$ is polynomial $k_n t^n + \cdots + k_1 t + k_0$, then

$$Y_p = A_n t^n + \cdots + A_1 t + A_0.$$

2° When $g(t)$ is exponential $b_1 e^{k_1 t} + \cdots + b_n e^{k_n t}$, then

$$Y_p = B_1 e^{k_1 t} + \cdots + B_n e^{k_n t}$$

3° When $g(t)$ is sine / cosine / both: $b \sin(kt)$, $b \cos(kt)$, $b_1 \cos(kt) + b_2 \sin(kt)$

$$Y_p = B_1 \cos(kt) + B_2 \sin(kt).$$

Example. Find a particular solution Y_p using undetermined coefficients:

$$y'' - y = g(t).$$

1° $g(t) = 2t^2+1$ 2° $g(t) = e^{4t}$. 3° $g(t) = \cos(3t)$.

Homogeneous equation: $y'' - y = 0$. char. equation $r^2 - 1 = 0$.

$\Rightarrow$ $r = \pm 1$. $y_c = c_1 e^t + c_2 e^{-t}$.

1° ~ 3° $g(t)$ does not involve any terms of y_c.

1°. Let $Y_p = A_2 t^2 + A_1 t + A_0$

Then $Y_p' = 2A_2 t + A_1$ and $Y_p'' = 2A_2$

So $Y_p'' - Y_p = -A_2 t^2 - A_1 t + (2A_2 - A_0) = g(t) = 2t^2 + 1$.

$$\Rightarrow \begin{cases} -A_2 = 2. \\ -A_1 = 0. \\ 2A_2 - A_0 = 1 \end{cases} \Rightarrow \begin{cases} A_2 = -2. \\ A_1 = 0. \\ A_0 = -5 \end{cases} \Rightarrow Y_p = -2t^2 - 5.$$

2°. Let $Y_p = Ae^{4t}$. Then $Y_p' = 4Ae^{4t}$ and $Y_p'' = 16Ae^{4t}$.

So $Y_p'' - Y_p = 15Ae^{4t} = g(t) = e^{4t}$.

$\Rightarrow$ $15A = 1$ $\Rightarrow$ $A = \frac{1}{15}$. $\Rightarrow$ $Y_p = \frac{1}{15}e^{4t}$.

3°. Let $Y_p = A\sin(3t) + B\cos(3t)$.

Then $Y_p' = 3A\cos(3t) - 3B\sin(3t)$. and $Y_p'' = -9A\sin(3t) - 9B\cos(3t)$.

So $Y_p'' - Y_p = -10A\sin(3t) - 10B\cos(3t) = g(t) = \cos(3t)$.

$$\Rightarrow \begin{cases} -10A = 0 \\ -10B = 1. \end{cases} \Rightarrow \begin{cases} A = 0. \\ B = -\frac{1}{10}. \end{cases} \Rightarrow Y_p = -\frac{1}{10}\cos(3t).$$

Exception case 4°: In <6>, when $g(t)$ involves a term that is a solution of <4>, modify $\tilde{Y}_p = t^s Y_p$, where s is the smallest nonnegative integer so that no terms in $\tilde{Y}_p$ is a solution of <4>.

Example: $y^{(3)} - 3y'' + 3y' - y = 4e^t$.

For $y^{(3)} - 3y'' + 3y' - y = 0$. char. eq. is $r^3 - 3r^2 + 3r - 1 = 0$.

$\Rightarrow (r-1)^3 = 0$. i.e. $r = 1$ (3 times). $\{e^t, te^t, t^2e^t\}$.

$g(t) = 4e^t$. is one of the solutions. $Y_p = At^3e^t$.

Then. $Y_p' = A(3t^2+t^3)e^t$, $Y_p'' = A(6t+6t^2+t^3)e^t$, $Y_p^{(3)} = A(6+18t+9t^2+t^3)e^t$.

So $Y_p^{(3)} - 3Y_p'' + 3Y_p' - Y_p = 6A\,e^t = g(t) = 4e^t$. $\Rightarrow 6A = 4$.

i.e. $A = \frac{2}{3}$. So $Y_p = \frac{2}{3}t^3e^t$. and $y = c_1e^t + c_2te^t + c_3t^2e^t + \frac{2}{3}t^3e^t$.

$=$.

§4.4. Variation of parameters.

Focus on $(n=2)$. $y'' + p_1(t)y' + p_2(t)y = g(t)$. <7>.

with a fundamental set of solutions $\{y_1, y_2\}$ of $y'' + p_1y' + p_2y = 0$.

$$Y_p = u_1y_1 + u_2y_2 = -y_1\int \frac{g(t)\,y_2(t)\,dt}{W(y_1,y_2)} + y_2\int \frac{g(t)\,y_1(t)\,dt}{W(y_1,y_2)}.$$

Example. Find a particular solution of $y'' + y = 3\sec(t)$ by variation of parameters

Look at $y'' + y = 0$.. char. eq. $r^2 + 1 = 0$. $\Rightarrow r = \pm i$.

So $y_1 = \cos(t)$ and $y_2 = \sin(t)$. $W(y_1, y_2) = \left|\begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}\right| = 1$.

$$-\int \frac{g(t)\,y_2(t)\,dt}{W(y_1,y_2)} = -\int \frac{3\sec(t)\sin(t)}{1}dt = -3\int \tan(t)\,dt = 3\ln|\cos(t)| + c_1$$

$$\int \frac{g(t)\,y_1(t)\,dt}{W(y_1,y_2)} = \int \frac{3\sec(t)\cos(t)\,dt}{1} = 3\int dt = 3t + C_2.$$

So $Y_p = 3\cos(t)\ln|\cos(t)| + 3t\sin(t) + c_1\cos(t) + c_2\sin(t)$.

§5.1 Power Series:

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n.$$

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$. and $\rho = \frac{1}{L}$. then ρ is the radius of convergence.

the power series converges for all $|x-x_0| < \rho$.

Example. $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n\cdot 2^n}$.

Test: $\lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)2^{n+1}}}{\frac{1}{n\cdot 2^n}}\right| = \lim_{n\to\infty} \frac{n}{n+1}\cdot\frac{1}{2} = \frac{1}{2}$. $\rho = 1/\frac{1}{2} = 2$.

Shift of index:

Example. $\sum_{n=2}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{n+2} x^{n+2}$.

Example. Assume $\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$.

then $\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^n$. $\Rightarrow$ $a_n = (n+1)a_{n+1}$ for $n=0,1,2,\cdots$

So $a_0 = a_1$, $a_1 = 2a_2$, $a_2 = 3a_3, \cdots$, $a_{n-1} = n a_n$.

Multiply all. $\Rightarrow$ $a_0 \cdot a_1 \cdots a_{n-1} = a_1 \cdots a_{n-1} a_n \cdot (1\cdot 2\cdot 3 \cdots n)$.

$\Rightarrow$ $a_0 = a_n \cdot n!$

$\Rightarrow$ $a_n = \frac{a_0}{n!}$

So $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x$.

§5.2/5.3. Series solutions. (ordinary point).

Mainly focus on $P(x)y'' + Q(x)y' + R(x)y = 0$. $\langle 8 \rangle$

where P, Q, R are polynomials.

Then $P(x_0) \neq 0 \Rightarrow x_0$ is an ordinary point of $\langle 8 \rangle$.

Otherwise, x_0 is a singular point.

Existence Theorem:

If x_0 is an ordinary point of $\langle 8 \rangle$, then the general solution of $\langle 8 \rangle$ is

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 y_1 + a_1 y_2 = a_0\left(1 + b_2(x-x_0)^2 + \cdots\right) + a_1\left((x-x_0) + c_2(x-x_0)^2 + \cdots\right)$$

where $\{y_1, y_2\}$ is a fundamental set of solutions. The radius of convergence of y is at least the min of radii of convergence of $\frac{Q}{P}$ and $\frac{R}{P}$.

Example. (Lengendre equation). Near $x_0 = 0$.

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

where α is constant.

$P(x) = 1 - x^2$, $Q(x) = -2x$, $R(x) = \alpha(\alpha+1)$. $P(x_0) = P(0) \neq 0$.

Zeros of $P(x)$ are $x = \pm 1$. distance from x_0 to nearest zero is 1. So radius of convergence $\rho \geq 1$.

Example. (Airy's equation). $y'' - xy = 0$. near $x_0 = 0$.

$P(x) \equiv 1$, $Q(x) \equiv 0$, $R(x) = -x$. $P(x_0) = P(0) \neq 0$.

P does not have zeros. $\Rightarrow \rho = \infty$.

Let $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

Plug back in: $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=0}^{\infty} a_n x^n = 0.$

Shift index: $\sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$

(Separate n=0 term): $2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$

$$\Rightarrow \quad 2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_{n-1}\right] x^n = 0.$$

So $a_2 = 0$ and $(n+2)(n+1) a_{n+2} = a_{n-1}$ for $n = 1, 2, 3, \cdots$

Take $n = 3, 6, 9, \cdots \Rightarrow 0 = a_2 = a_5 = a_8 = \cdots = a_{3k+2} = \cdots \quad k = 0, 1, 2, \cdots$

Take $n = 1, 4, 7, \cdots \Rightarrow a_3 = \frac{a_0}{2\cdot 3}, \quad a_6 = \frac{a_3}{5\cdot 6} = \frac{a_0}{2\cdot3\cdot5\cdot6}, \quad \cdots$

Take $n = 2, 5, 8, \cdots \Rightarrow a_4 = \frac{a_1}{3\cdot 4}, \quad a_7 = \frac{a_4}{6\cdot 7} = \frac{a_1}{3\cdot4\cdot6\cdot7}, \quad \cdots$

In general: $a_{3k} = \dfrac{a_0}{2\cdot3\cdot5\cdot6\cdots(3k-1)3k}, \quad k = 1, 2, \cdots$

and $a_{3k+1} = \dfrac{a_1}{3\cdot4\cdot\cdots\cdot 3k(3k+1)}, \quad k = 1, 2, \cdots.$

So
$$y = \sum_{n=0}^{\infty} a_n x^n = a_0\left(1 + \frac{x^3}{2\cdot3} + \cdots + \frac{x^{3k}}{2\cdot3\cdots(3k-1)3k} + \cdots\right) + a_1\left(x + \frac{x^4}{3\cdot4} + \cdots + \frac{x^{3k+1}}{3\cdot4\cdots3k(3k+1)} + \cdots\right)$$
$$= a_0\left(\sum_{k=1}^{\infty} \frac{x^{3k}}{2\cdot3\cdots(3k-1)3k} + 1\right) + a_1\left(x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{3\cdot4\cdots3k(3k+1)}\right).$$
$$= a_0 y_1 + a_1 y_2.$$

Note: $y_1(0) = 1$ and $y_1'(0) = 0$
$y_2(0) = 0$ and $y_2'(0) = 1$. $\Rightarrow W(y_1, y_2)(0) = \det\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 1$

So $\{y_1, y_2\}$ is a fundamental set of solutions.

(Also, recommended practice: P263 of textbook. §5.2 Problem 1 and 2).

Answers: Prob. 1. $y = a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$

Prob. 2. $y = a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!} + a_1 \sum_{k=0}^{\infty} \frac{2^k k!\, x^{2k+1}}{(2k+1)!} \leftarrow \left(\text{or } a_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!!}\right)$

§5.4. Euler Equations.

$$x^2y'' + \alpha xy' + \beta y = 0. \quad \text{for } x>0. \quad (x_0=0). \quad \langle 9\rangle.$$

$P(x) = x^2$, $Q(x) = \alpha x$, $R(x) = \beta$. $\quad P(x_0) = P(0) = 0$. Singular point.

Indicial equation : $\quad r^2 + (\alpha-1)r + \beta = 0. \quad \langle 10\rangle.$

Case 1°. $\langle 10\rangle$ has distinct real roots: r_1, r_2.

Then $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$.

Case 2°. $\langle 10\rangle$ has equal roots: $r = r_1 = r_2$.

Then $y_1 = x^{r_1}$ and $y_2 = x^{r_1}\ln(x)$

Case 3° $\langle 10\rangle$ has complex roots: $r = \lambda \pm i\mu$. $(\mu>0)$.

Then $y_1 = x^{\lambda}\cos(\mu\ln(x))$ and $y_2 = x^{\lambda}\sin(\mu\ln(x))$.

Example. 1°. $2x^2y'' + 3xy' - y = 0. \quad (x>0).$

$$r^2 + \left(\frac{3}{2}-1\right)r - \frac{1}{2} = 0. \Rightarrow 2r^2 + r - 1 = 0 \Rightarrow (2r-1)(r+1) = 0.$$

$r_1 = \frac{1}{2}$, $r_2 = -1$. So $\quad y = c_1x^{\frac{1}{2}} + c_2x^{-1}. \ (x>0).$

2°. $x^2y'' + 5xy' + 4y = 0 \quad (x>0).$

$r^2 + 4r + 4 = 0. \Rightarrow (r+2)^2 = 0. \quad r = r_1 = r_2 = -2.$

So $\quad y = c_1x^{-2} + c_2x^{-2}\ln(x). \quad (x>0).$

3°. $x^2y'' + xy' + y = 0 \quad (x>0).$

$r^2 + 1 = 0. \Rightarrow r = \pm i.$

So $\quad y = c_1\cos(\ln(x)) + c_2\sin(\ln(x)), \quad x>0.$

General 2^{nd} order equation:

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0. \quad \langle 8 \rangle. \quad \text{near } x_0$$

where. $P(x_0)=0$. (P, Q, R are polynomials).

Then. x_0 is a singular point of $\langle 8 \rangle$.

If. both. $(x-x_0)\dfrac{Q(x)}{P(x)}$ and $(x-x_0)^2\dfrac{R(x)}{P(x)}$ are analytic at x_0,

then x_0 is a regular singular point of $\langle 8 \rangle$.

Otherwise, it is an irregular singular point.

Example. (Legendre equation) $(1-x^2)\,y'' - 2xy' + \alpha(\alpha+1)\,y = 0$.

$P(x)=0$. $\Rightarrow$. $x_0 = \pm 1$. Singular points.

Look at. $x_0 = -1$:

$$(x+1)\,\frac{Q(x)}{P(x)} = (x+1)\,\frac{-2x}{1-x^2} = \frac{2x}{x-1} \quad \text{analytic at } x_0=-1.$$

$$(x+1)^2 \cdot \frac{R(x)}{P(x)} = (x+1)^2\,\frac{\alpha(\alpha+1)}{1-x^2} = \frac{\alpha(\alpha+1)(x+1)}{1-x} \quad \text{analytic at } x_0=-1$$

So $x_0=-1$ is a regular singular point.

Example. $2x(x-2)^2\,y'' + 3x\,y' + (x-2)\,y = 0$ at $x_0 = 2$.

$P(x_0) = P(2) = 0$. $\Rightarrow$. $x_0 = 2$ singular.

Check: $$(x-2)\cdot\frac{Q(x)}{P(x)} = (x-2)\cdot\frac{3x}{2x(x-2)^2} = \frac{3}{2(x-2)}. \quad \text{not analytic at } x_0=2.$$

So $x_0=2$ is irregular.