Variation of Parameters (general notes/process)

Trying to solve: $y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_{n-1}(t)y' + p_n(t)y = g(t)$.

- Find general solution of the homogeneous eqn:

$$y_c = C_1 y_1(t) + \dots + C_n y_n(t)$$

where $y_1, \dots, y_n$ is a fundamental set of solutions

- Define a particular sol:

$$y_p(t) = u_1(t)y_1(t) + \dots + u_n(t)y_n(t)$$

☆ Entire goal of variation of parameters is to find $u_1, \dots, u_n$

- Do this by setting up and solving a system of n equations:

$$\begin{cases} u_1'(t)y_1(t) + u_2'(t)y_2(t) + \dots + u_n'(t)y_n(t) = 0 \\ u_1'(t)y_1'(t) + u_2'(t)y_2'(t) + \dots + u_n'(t)y_n'(t) = 0 \\ \vdots \\ u_1'(t)y_1^{(n-1)}(t) + u_2'(t)y_2^{(n-1)}(t) + \dots + u_n'(t)y_n^{(n-1)}(t) = g(t) \end{cases}$$

Solving this system we get

$$u_m'(t) = \frac{g(t)W_m(t)}{W(t)}, \quad m = 1, 2, \dots, n$$

where:

$W(t)$: Wronskian of $y_1, \dots, y_n$

$$W(t) = W(y_1, \dots, y_n)(t) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

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and $W_m(t)$: the Wronskian w/ column m replaced by the column $(0, 0, \dots, 0, 1)$

$$W_m(t) = \begin{vmatrix} y_1 & \cdots & y_{m-1} & 0 & y_{m+1} & \cdots & y_n \\ y_1' & \cdots & y_{m-1}' & 0 & y_{m+1}' & \cdots & y_n' \\ \vdots & & & & & & \\ y_1^{(n-1)} & \cdots & y_{m-1}^{(n-1)} & 1 & y_{m+1}^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Thus,

$$u_m(t) = \int \frac{g(s) W_m(s)}{W(s)} \, ds$$

So, the particular solution is:

$$Y(t) = \sum_{m=1}^{n} y_m(t) u_m(t)$$

$$= \sum_{m=1}^{n} y_m(t) \int \frac{g(s) W_m(s)}{W(s)} \, ds$$

$$= y_1(t) \int \frac{g(s) W_1(s)}{W(s)} \, ds + \cdots + y_n(t) \int \frac{g(s) W_n(s)}{W(s)} \, ds$$