Math 146B.010 (Ordinary and Partial Differential Equations)
Homework 03 Solutions

Problem 1 (10.1.1)**.** Either solve the boundary value problem

$$y'' + y = 0; \qquad y(0) = 0, \quad y'(\pi) = 1$$

or show that it has no solution.

Solution. A general solution to the differential equation $y'' + y = 0$ is given by

$$y(x) = k_1 \cos(x) + k_2 \sin(x),$$

where k_1 and k_1 are constants. The first boundary value implies that

$$0 = y(0) = k_1 \cos(0) + k_2 \sin(0) = k_1.$$

Thus $k_1 = 0$. Substituting this into the solution and applying the information from the second boundary value gives

$$1 = y'(\pi) = k_2 \cos(\pi) = -k_2.$$

Therefore the boundary value problem is solved by

$$y(x) = -\sin(x).$$

□

Problem 2 (10.1.2)**.** Either solve the boundary value problem

$$y'' + 2y = 0; \qquad y'(0) = 1, \quad y'(\pi) = 0$$

or show that it has no solution.

Solution. A general solution to the differential equation $y'' + 2y = 0$ is given by

$$y(x) = k_1 \cos(\sqrt{2}x) + k_2 \sin(\sqrt{2}x)$$

where k_1 and k_2 are constants. Note that

$$y'(x) = -k_1\sqrt{2}\sin(\sqrt{2}x) + k_2\sqrt{2}\cos(\sqrt{2}x). \tag{1}$$

Substituting the first boundary value into the above gives

$$1 = y'(0) = k_2\sqrt{2} \implies k_2 = \frac{1}{\sqrt{2}}.$$

Substituting this and the second boundary value into (1) gives

$$0 = -k_1\sqrt{2}\sin(\sqrt{2}\pi) + \cos(\sqrt{2}\pi) \implies k_1 = \frac{\cos(\sqrt{2}\pi)}{\sqrt{2}\sin(\sqrt{2}\pi)} = \frac{1}{\sqrt{2}}\cot(\sqrt{2}\pi)$$

Substituting the values of k_1 and k_2 into the general solution to the differential equation gives a solution to the boundary value problem:

$$y(x) = \frac{1}{\sqrt{2}} \cot(\sqrt{2}\pi) \cos(\sqrt{2}x) + \frac{1}{\sqrt{2}} \sin(\sqrt{2}x).$$

Note that $\cos(\sqrt{2}\pi)$ is "just" a constant. It isn't pretty, but, at the end of the day, it is just some number which has to be carried around in future computations. $\square$

Problem 3 (10.1.14)**.** Find the eigenvalues and eigenfunctions of the boundary value problem

$$y'' + \lambda y = 0; \qquad y(0) = 0, \quad y'(\pi) = 0.$$

Assume that all eigenvalues are real.

Solution. The eigenvalues of the boundary value problem are, by definition, the values of λ such that nontrivial solutions to the problem exist. The eigenfunctions are the associated nontrivial solutions.

Note that, in this case, a general solution to the differential equation depends on the sign of λ. There are three cases to consider:

(i) $\lambda = 0$. If $\lambda = 0$, then $y'' = 0$, which has a general solution of the form

$$y(x) = k_1 x + k_2.$$

The boundary conditions imply that

$$0 = k_2 \qquad \text{and} \qquad 0 = k_1.$$

Thus if $\lambda = 0$, then the only solution is the trivial solution. This implies that $\lambda = 0$ is not an eigenvalue of the boundary value problem.

(ii) $\lambda < 0$. If $\lambda > 0$, then $y'' + \lambda y = 0$ has a general solution of the form

$$y(x) = k_1 e^{\lambda x} + k_2 e^{-\lambda x}.$$

The boundary conditions imply that

$$0 = k_1 + k_2 \qquad \text{and} \qquad 0 = k_1 e^{\lambda\pi} - k_2 e^{-\lambda\pi}.$$

The first equation implies that $k_1 = -k_2$, which, when substituted into the second equation, gives

$$2k_1 \left(e^{\lambda\pi} + e^{-\lambda\pi} \right) = 0.$$

This is possible if and only if $k_1 = 0$, as the exponential term never vanishes. As this corresponds to the trivial solution, no negative value of λ can be a eigenvalue of this problem.

(iii) $\lambda > 0$. If $\lambda < 0$, then $y'' + \lambda y = 0$ has a general solution of the form

$$y(x) = k_1 \cos(\sqrt{-\lambda} x) + k_2 \sin(\sqrt{-\lambda} x). \tag{2}$$

The boundary conditions imply that

$$0 = k_1 \cos(0) + k_2 \sin(0) = k_1 \qquad \text{and} \qquad 0 = k_2 \cos(\sqrt{-\lambda}\pi).$$

If $k_2 = 0$, then the solution is the trivial one, so assume that $k_2 \neq 0$. Then it must be the case that

$$\cos(\sqrt{-\lambda}\pi) = 0,$$

which implies that $\sqrt{-\lambda}$ is an odd multiple of $1/2$. That is,

$$\sqrt{-\lambda} \in \left\{ \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \dots \right\} = \left\{ \frac{2n+1}{2} \;\middle|\; n \in \mathbb{Z} \right\}.$$

Isolating λ, this implies that

$$\lambda \in \left\{ \frac{(2n+1)^2}{4} \;\middle|\; n \in \mathbb{Z} \right\} = \left\{ \frac{(2n+1)^2}{4} \;\middle|\; n \in \mathbb{N}_0 \right\},$$

where $\mathbb{N}_0$ denotes the set of natural numbers (including zero). With $k_1 = 0$, any value of k_2 will give rise to an eigenfunction in (2). For the sake of convenience, choose $k_2 = -1$.

The eigenpairs (λ, y_λ) (e.g. the collection of eigenvalues λ with their corresponding eigenfunctions y_λ) of the given boundary value problem are, therefore,

$$\left\{ \left(\lambda_n = \frac{(2n+1)^2}{4}, y_n(x) = \sin\left(\frac{2n+1}{2} x \right) \right) \;\middle|\; n \in \mathbb{N}_0 \right\}.$$

□

Problem 4 (10.1.17)**.** Find the eigenvalues and eigenfunctions of the boundary value problem

$$y'' + \lambda y = 0; \qquad y'(0) = 0, \quad y(L) = 0.$$

Assume that all eigenvalues are real.

Solution. Again, there are three cases to consider:

(i) $\lambda = 0$. In this case, the differential equation devolves to $y'' = 0$, which has the general solution

$$y(x) = k_1 x + k_2,$$

where k_1 and k_2 are constants. The first boundary condition implies that $k_1 = 0$, while the second boundary condition implies that $k_2 = 0$. Thus if $\lambda = 0$, then the only solution is the trivial solution. Hence zero is not an eigenvalue of the boundary value problem.

(ii) $\lambda < 0$. In this case, the differential equation has general solution

$$y(x) = k_1 e^{\sqrt{-\lambda}x} + k_2 e^{-\sqrt{-\lambda}x},$$

where k_1 and k_2 are constants. The first boundary condition gives

$$0 = y'(0) = k_1\sqrt{-\lambda} - k_2\sqrt{-\lambda} \implies k_1 = k_2.$$

This, combined with the second boundary condition, implies that

$$0 = y(L) = k_1\left(e^{\sqrt{-\lambda}L} + e^{-\sqrt{-\lambda}L}\right) = k_1 e^{\sqrt{-\lambda}L}\left(e^{2\sqrt{-\lambda}L} + 1\right) \implies k_1 = 0.$$

Thus if $\lambda < 0$, then the only solution is the trivial solution. Therefore no negative number is an eigenvalue of the boundary value problem.

(iii) $\lambda > 0$. In this case, the differential equation has the general solution

$$y(x) = k_1 \cos(\sqrt{\lambda}x) + k_2 \sin(\sqrt{\lambda}x). \tag{3}$$

The first boundary value implies that

$$0 = y'(0) = -k_1\sqrt{\lambda}\sin(0) + k_2\sqrt{\lambda}\cos(0) = k_2\sqrt{\lambda} \implies k_2 = 0.$$

This, combined with the second boundary value, implies that

$$0 = y(L) = k_1 \cos(\sqrt{\lambda}L)$$

Assuming that $k_2 \neq 0$ (which is necessary if the boundary value problem is to have any non-trivial solutions), it follows that

$$\begin{aligned} \cos(\sqrt{\lambda}L) = 0 &\implies \sqrt{\lambda}L = (2n+1)\pi \qquad (n \in \mathbb{Z}) \\ &\implies \lambda = \left(\frac{(2n+1)\pi}{L}\right)^2. \end{aligned}$$

The set of eigenvalues is therefore given by

$$\left\{ \lambda_n = \left(\frac{(2n+1)\pi}{L}\right)^2 \,\middle|\, n \in \mathbb{N}_0 \right\},$$

where $\mathbb{N}_0$ denote the set of natural numbers (including zero). Note that negative n need not be considered as, after squaring each negative n corresponds to an eigenvalue obtained from a nonnegative integers. The corresponding eigenfunctions can be found by substituting these eigenvalues into the general solution at (**??**).

The eigenpairs (λ, y_λ) (e.g. the collection of eigenvalues λ with their corresponding eigenfunctions y_λ) of the given boundary value problem are, therefore,

$$\left\{ \left(\lambda_n = \left(\frac{(2n+1)\pi}{L}\right)^2, y_n = \cos\left(\frac{(2n+1)\pi}{L}x\right)\right) \,\middle|\, n \in \mathbb{N}_0 \right\},$$

□

Problem 5 (10.2.2)**.** Is the function defined by the formula

$$f(x) = \cos(2\pi x)$$

periodic? If so, what is its fundamental period?

Solution. Yes, this function is periodic with fundamental period $T = 1$. Observe that if $x \in \mathbb{R}$, then

$$\begin{aligned} f(x+1) &= \cos(2\pi(x+1)) \\ &= \cos(2\pi x + 2\pi) \\ &= \cos(2\pi x)\cos(2\pi) - \sin(2\pi x)\sin(2\pi) && \text{(angle addition formula)} \\ &= \cos(2\pi x) && (\cos(2\pi) = 1;\ \sin(2\pi) = 0) \\ &= f(x). \end{aligned}$$

Thus f is 1-periodic. To see that this is the fundamental period, suppose for contradiction that there is some $0 < T < 1$ such that f is T-periodic. Then for any $x \in \mathbb{R}$,

$$\begin{aligned} \cos(2\pi x) &= f(x) \\ &= f(x+T) \\ &= \cos(2\pi(x+T)) \\ &= \cos(2\pi x + 2\pi T) \\ &= \cos(2\pi x)\cos(2\pi T) - \sin(2\pi x)\sin(2\pi T). \end{aligned} \tag{4}$$

As this must be true for any $x \in \mathbb{R}$, it must be true when $x = 1/2$. Hence

$$\cos(\pi) = \cos(\pi)\cos(2\pi T) - \sin(\pi)\sin(2\pi T) = -\sin(2\pi T).$$

By assumption, f is T-periodic, and so $-\sin(2\pi T) = 0$. This implies that T is a half-integer (i.e. $T = n/2$ for some $n \in \mathbb{Z}$). But $0 < T < 1$, and so it must be that $T = 1/2$.

However, with $T = 1/2$ and $x = 1/8$, the identity at (4) becomes

$$\frac{1}{\sqrt{2}} = \cos\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)\cos(\pi) - \sin\left(\frac{\pi}{4}\right)\sin(\pi) = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$$

This is a contradiction, which implies that T cannot be $1/2$, which further implies that T cannot be less than 1. Therefore the fundamental period of f is 1. □

Problem 6 (10.2.4)**.** Let $L > 0$. Is the function defined by the formula

$$f(x) = \sin\left(\frac{\pi x}{L}\right)$$

periodic? If so, what is its fundamental period?

Solution. Yes, this function is periodic with fundamental period $2L$. This can be justified via an argument identical to that given in Problem 5, changing every instance of $T = 1$ to $T = \pi/L$. ◻

Remark. Problems 5 and 6 suggest a more general theorem: let $a, b \in \mathbb{R}$ with $a > 0$. If a function f is periodic with fundamental period T, then the function g defined by

$$g(x) = f(ax + b)$$

is periodic with fundamental period T/a. To prove this, first observe that if $x \in \mathbb{R}$, then

$$g\left(x + \frac{T}{a}\right) = f\left(a\left(x + \frac{T}{a}\right) + b\right) = f((ax + b) + T) = f(ax + b) = g(x).$$

Therefore g is periodic with period T/a, as claimed. To see that this is the fundamental period of g, suppose for contradiction that there is some $T' > 0$ with $T' < T/a$ such that g is T'-periodic. Then for any $x \in \mathbb{R}$,

$$f(x) = g\left(\frac{1}{a}(x - b)\right) = g\left(\frac{1}{a}(x - b) + T'\right) = f(x + aT').$$

Thus f must be aT'-periodic. But $aT' < T$, which contradicts the hypothesis that the fundamental period of f is T. Therefore the fundamental period of g is T/a, as claimed.

Note that the hypothesis that $a > 0$ is not necessary. If $a < 0$, then g (as defined above) is periodic with fundamental period $T/|a|$. The argument is identical to that given above, with some extra care taken to keep track of signs.

Problem 7 (10.3.1)**.** Define the function $f : [-1, 1) \to \mathbb{R}$ by

$$f(x) = \begin{cases} -1 & \text{if } -1 \leq x < 0, \text{ and} \\ 1 & \text{if } 0 \leq x < 1. \end{cases}$$

Extend this function to $\mathbb{R}$ by periodicity. That is, define f on the remainder of $\mathbb{R}$ so that

$$f(x + 2) = f(x).$$

Sketch a graph of f, and determine the Fourier series expansion of f.

Solution. Recall that the Fourier series expansion of a function g which is L-periodic is given by

$$g(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right),$$

where the Fourier coefficients are given by the integrals

$$a_m = \frac{1}{L}\int_{-L}^{L} g(x) \cos\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x \qquad \text{and} \qquad b_m = \frac{1}{L}\int_{-L}^{L} g(x) \sin\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x.$$

For the given function, $L = 1$, and so the zeroth Fourier coefficient is given by

$$a_0 = \int_{-1}^{1} f(x)\,\mathrm{d}x = -\int_{-1}^{0} \mathrm{d}x + \int_{0}^{1} \mathrm{d}x = (-1) + 1 = 0.$$

The remaining "even" Fourier coefficients are given by

$$\begin{aligned}
a_m &= \int_{-1}^{1} f(x)\cos(m\pi x)\,\mathrm{d}x \\
&= -\int_{-1}^{0} \cos(m\pi x)\,\mathrm{d}x + \int_{0}^{1} \cos(m\pi x)\,\mathrm{d}x \\
&= \left[-\frac{1}{m\pi}\sin(m\pi x)\right]_{x=-1}^{0} + \left[\frac{1}{m\pi}\sin(m\pi x)\right]_{x=0}^{1} \\
&= \left[\frac{1}{m\pi} - 0\right] + \left[0 - \frac{1}{m\pi}\right] \\
&= 0.
\end{aligned}$$

At this moment, it is perhaps worth noting that all of the "even" Fourier coefficients are zero. This is not unexpected: the function f is an odd function, and so there should be no contribution to the Fourier series expansion from the even functions $\cos(m\pi x)$. Indeed, it is a general result that if f is odd, then the coefficients a_m will be zero for all m (and, similarly, if f is odd, then the coefficients b_m will be zero for all m).

Returning to the computation at hand, the "odd" coefficients are given by

$$\begin{aligned}
b_m &= \int_{-1}^{1} f(x)\sin(m\pi x)\,\mathrm{d}x \\
&= -\int_{-1}^{0} \sin(m\pi x)\,\mathrm{d}x + \int_{0}^{1} \sin(m\pi x)\,\mathrm{d}x \\
&= \int_{1}^{0} \sin(-m\pi t)\,\mathrm{d}t + \int_{0}^{1} \sin(m\pi x)\,\mathrm{d}x && \text{(change of variables: } t = -x\text{)} \\
&= -\int_{1}^{0} \sin(m\pi t)\,\mathrm{d}t + \int_{0}^{1} \sin(m\pi x)\,\mathrm{d}x && \text{(sin is an odd function)} \\
&= \int_{0}^{1} \sin(m\pi t)\,\mathrm{d}t + \int_{0}^{1} \sin(m\pi x)\,\mathrm{d}x \\
&= 2\int_{0}^{1} \sin(m\pi x)\,\mathrm{d}x \\
&= \frac{2}{m\pi}\Big[-\cos(m\pi x)\Big]_{x=0}^{1} \\
&= \frac{2}{m\pi}\left[(-1)^{m+1} - (-1)\right]. && (\cos(m\pi) = (-1)^m \text{ for } m \in \mathbb{N})
\end{aligned}$$

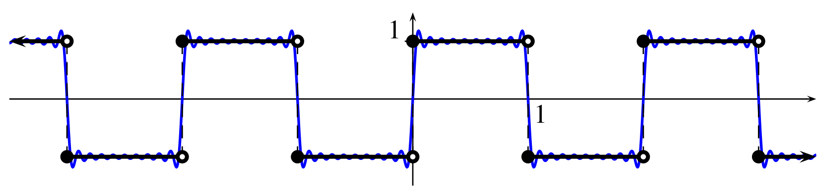
Observe that $[(-1)^{m+1} + 1]$ is 2 when m is odd, and is zero otherwise. Hence

$$b_m = \begin{cases} \dfrac{4}{m\pi} & \text{if } m \text{ is odd, and} \\ 0 & \text{if } m \text{ is even.} \end{cases}$$

Therefore the Fourier series expansion of f is given by

$$\begin{aligned} f(x) &= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin\bigl((2k-1)\pi x\bigr) \\ &= \frac{4}{\pi} \left[\sin(\pi x) + \frac{1}{3}\sin(3\pi x) + \frac{1}{5}\sin(5\pi x) + \frac{1}{7}\sin(7\pi x) + \cdots \right]. \end{aligned}$$

A graph of the function (in black), as well as a Fourier series approximation with ten terms (in blue), is given by



The blue curve matches the graph of the function fairly well away from the jump discontinuities, but seems to oscillate with increasing amplitude near the jumps. This is a general artifact of Fourier series expansions, first described by Henry Wilbraham in 1848, then later re-discovered (and named for) J. Willard Gibbs in 1899. The interested reader can learn more by searching the internet for the "Gibbs phenomenon". □

Problem 8 (10.3.2)**.** Let $L > 0$ and define the function $f : [-\pi, \pi) \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } -\pi \le x < 0, \text{ and} \\ x & \text{if } 0 \le x < \pi. \end{cases}$$

Extend this function to $\mathbb{R}$ by periodicity. That is, define f on the remainder of $\mathbb{R}$ so that

$$f(x + 2\pi) = f(x).$$

Sketch a graph of f, and determine the Fourier series expansion of f.

Solution. The zeroth Fourier coefficient is given by

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, \mathrm{d}x = \int_0^{\pi} x\, \mathrm{d}x = \frac{\pi^2}{2}.$$

The remaining "even" coefficients are

$$\begin{aligned}
a_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{0}^{\pi} x\cos(mx)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\frac{x}{m}\sin(mx)\right]_{x=0}^{\pi} - \frac{1}{m\pi}\int_0^{\pi}\sin(mx)\,\mathrm{d}x \qquad \text{(integration by parts)} \\
&= 0 - \frac{1}{m\pi}\left[-\frac{1}{m}\cos(mx)\right]_{x=0}^{\pi} \\
&= \frac{1}{m^2\pi}\big(\cos(mx) - 1\big) \\
&= \frac{1}{m^2\pi}\big((-1)^m - 1\big) \\
&= \begin{cases} -\dfrac{2}{m^2\pi} & \text{if } m \text{ is odd, and} \\ 0 & \text{if } m \text{ is even.} \end{cases}
\end{aligned}$$

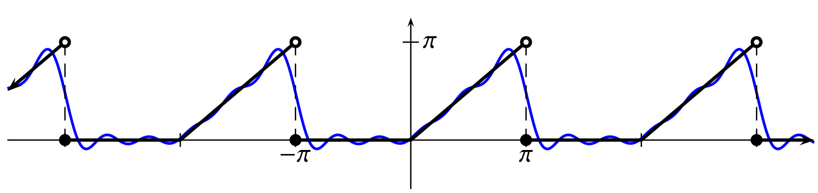
Finally, the "odd" coefficients are given by

$$\begin{aligned}
b_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{0}^{\pi} x\sin(mx)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[-\frac{x}{m}\cos(mx)\right]_{x=0}^{\pi} + \frac{1}{m\pi}\int_0^{\pi}\cos(mx)\,\mathrm{d}x \\
&= -\frac{1}{m}\cos(m\pi) + \left[\frac{1}{m^2}\sin(mx)\right]_{x=0}^{\pi} \\
&= \frac{(-1)^{m+1}}{m}.
\end{aligned}$$

Therefore the Fourier series corresponding to f is given by

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos((2m-1)x) - \sum_{m=1}^{\infty}\frac{(-1)^m}{m}\sin(mx).$$

The graph of this function is given below (in black), as well as a Fourier series approximation (in blue) with three cosine terms and five sine terms (this gives an approximation with all cosine and sine terms of the form $\cos(mx)$ and $\sin(mx)$ with $m \le 5$).



□

Problem 9 (10.3.3)**.** Fix $L > 0$ and define the function $f : [-L, L) \to \mathbb{R}$ by

$$f(x) = \begin{cases} L + x & \text{if } -L \leq x < 0, \text{ and} \\ L - x & \text{if } 0 \leq x < L. \end{cases}$$

Extend this function to $\mathbb{R}$ by periodicity. That is, define f on the remainder of $\mathbb{R}$ so that

$$f(x + 2L) = f(x).$$

Sketch a graph of f, and determine the Fourier series expansion of f.

Solution. Observe that for any $x \in (0, L)$,

$$f(-x) = L + (-x) = L - x = f(x),$$

hence f is an even function. This implies that the coefficients of the sine terms in the Fourier series expansion will be zero (that is, $b_m = 0$ for all m). The zeroth Fourier coefficient is given by

$$\begin{aligned} a_0 &= \frac{1}{L} \int_{-L}^{L} f(x)\,\mathrm{d}x \\ &= \frac{1}{L} \int_{-L}^{0} L + x\,\mathrm{d}x + \int_{0}^{L} L - x\,\mathrm{d}x \\ &= \frac{1}{L} \left[Lx + \frac{x^2}{2} \right]_{x=-L}^{0} + \frac{1}{L} \left[Lx - \frac{x^2}{2} \right]_{x=0}^{L} \\ &= \frac{1}{L} \left[L^2 - \frac{L^2}{2} \right] + \frac{1}{L} \left[L^2 - \frac{L^2}{2} \right] \\ &= L. \end{aligned}$$

An alternative derivation is to note that the graph of f on the interval is a triangle with base $2L$ and height L. This triangle has area L^2, which implies that

$$\int_{-L}^{L} f(x)\,\mathrm{d}x = L^2.$$

It then follows that $a_0 = \frac{1}{L} L^2 = L$, which matches the result from the more complicated computation. Finally, the remaining Fourier coefficients are given by

$$\begin{aligned} a_m &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{m\pi x}{L} \right)\,\mathrm{d}x \\ &= \frac{1}{L} \int_{-L}^{0} (L + x) \cos\left(\frac{m\pi x}{L} \right)\,\mathrm{d}x + \frac{1}{L} \int_{0}^{L} (L - x) \cos\left(\frac{m\pi x}{L} \right)\,\mathrm{d}x. \end{aligned}$$

Making the change of variables $x \mapsto -x$, the first integral becomes

$$\frac{1}{L} \int_{-L}^{0} (L + x) \cos\left(\frac{m\pi x}{L} \right)\,\mathrm{d}x = \frac{1}{L} \int_{0}^{L} (L - x) \cos\left(\frac{m\pi x}{L} \right)\,\mathrm{d}x,$$

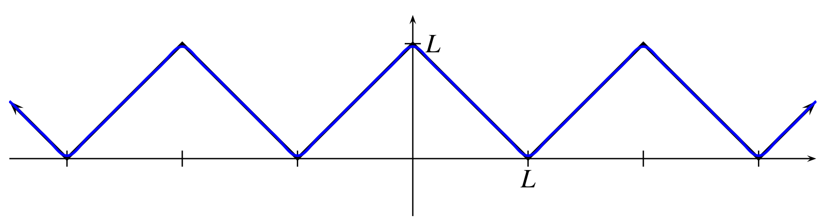
which is identical to the second integral. Thus

$$
\begin{aligned}
a_m &= \frac{2}{L}\int_0^L (L-x)\cos\!\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x \\
&= 2\int_0^L \cos\!\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x - \frac{2}{L}\int_0^L x\cos\!\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x \\
&= 2\left[\frac{L}{m\pi}\sin\!\left(\frac{m\pi x}{L}\right)\right]_{x=0}^{L} - \frac{2}{L}\left(\left[\frac{Lx}{m\pi}\sin\!\left(\frac{m\pi x}{L}\right)\right]_{x=0}^{L} - \frac{L}{m\pi}\int_0^{\pi}\sin\!\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x\right) \\
&= 0 - \frac{2}{L}\left(0 - \frac{L}{m\pi}\left[-\frac{L}{m\pi}\cos\!\left(\frac{m\pi x}{L}\right)\right]_{x=0}^{L}\right) \\
&= -\frac{2L}{(m\pi)^2}\left[(-1)^m - 1\right] \\
&= \begin{cases} \dfrac{4L}{(m\pi)^2} & \text{if } m \text{ is odd, and} \\ 0 & \text{if } m \text{ is even.} \end{cases}
\end{aligned}
$$

Therefore the Fourier series expansion of f is given by

$$
\begin{aligned}
f(x) &= \frac{L}{2} + \frac{4L}{\pi^2}\sum_{m=1} \frac{1}{(2m-1)^2}\cos\!\left(\frac{(2m-1)\pi x}{L}\right) \\
&= \frac{L}{2} + \frac{4L}{\pi^2}\left[\cos\!\left(\frac{\pi x}{L}\right) + \frac{1}{9}\cos\!\left(\frac{3\pi x}{L}\right) + \frac{1}{25}\cos\!\left(\frac{5\pi x}{L}\right) + \frac{1}{49}\cos\!\left(\frac{7\pi x}{L}\right) + \cdots\right].
\end{aligned}
$$

The graph of this function is given below (in black), as well as a Fourier series approximation with five terms (in blue), is given by



This series converges quite quickly, thus with only five terms, it is already almost impossible to distinguish the graph of the function from the graph of the Fourier series approximation at this level of zoom. The most "obvious" points of disagreement are where the function is not differentiable (i.e. at the tops and bottoms of the triangle wave). □

Problem 10 (10.3.4)**.** Define the function $f : [-1,1) \to \mathbb{R}$ by

$$f(x) = 1 - x^2$$

Extend this function to $\mathbb{R}$ by periodicity. That is, define f on the remainder of $\mathbb{R}$ so that

$$f(x+2) = f(x).$$

Sketch a graph of f, and determine the Fourier series expansion of f.

Solution. Observe that this function is even: for any $x \in [-1,1]$,

$$f(-x) = 1 - (-x)^2 = 1 - x^2 = f(x).$$

The periodic extension of the function to the entire real line preserves this parity, hence f is even. This implies that the "odd" Fourier coefficients all vanish; that is, $b_m = 0$ for all m. The zeroth Fourier coefficient is given by

$$a_0 = \int_{-1}^{1} 1 - x^2 \,\mathrm{d}x = \left[x - \frac{1}{3}x^3 \right]_{x=-1}^{1} = \left[1 - \frac{1}{3}\right] - \left[-1 + \frac{1}{3}\right] = \frac{4}{3}.$$

The remaining "even" terms are given by

$$\begin{aligned}
a_m &= \int_{-1}^{1} (1-x^2)\cos(m\pi x)\,\mathrm{d}x \\
&= 2\int_0^1 \cos(m\pi x)\,\mathrm{d}x - 2\int_0^1 x^2\cos(m\pi x)\,\mathrm{d}x \qquad (5)\\
&= \left[\frac{2}{m\pi}\sin(m\pi x)\right]_0^1 - 2\left(\left[\frac{x^2}{m\pi}\sin(m\pi x)\right]_{x=0}^1 - \frac{2}{m\pi}\int_0^1 x\sin(m\pi x)\,\mathrm{d}x\right) \\
&= 0 - 2\left(0 - \frac{2}{m\pi}\left(\left[-\frac{x}{m\pi}\cos(m\pi x)\right]_{x=0}^1 + \frac{1}{m\pi}\int_0^1 \cos(m\pi x)\,\mathrm{d}x\right)\right) \\
&= \left[-\frac{4x}{(m\pi)^2}\cos(m\pi x)\right]_{x=0}^1 + \frac{4}{(m\pi)^2}\int_0^1 \cos(m\pi x)\,\mathrm{d}x \\
&= -\frac{4}{(m\pi)^2}(-1)^m + 0 \\
&= (-1)^{m+1}\frac{4}{(m\pi^2)}.
\end{aligned}$$

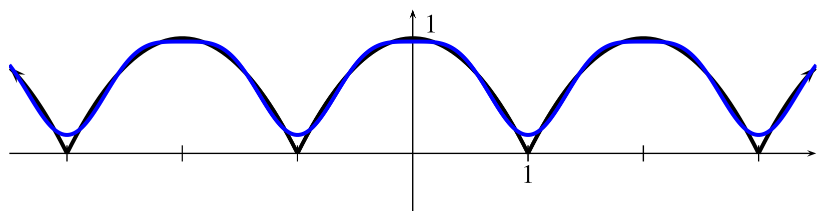
At (5), we are taking advantage of a property of even functions. Specifically, if g is an even function, then

$$\int_{-a}^{a} g(x)\,\mathrm{d}x = 2\int_0^a g(x)\,\mathrm{d}x.$$

The Fourier series is given by

$$f(x) = \frac{2}{3} + \frac{4}{\pi^2}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^2}\cos(m\pi x).$$

A graph of the function (in black), as well as a Fourier series approximation with (just!) three terms (in blue), is given by



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