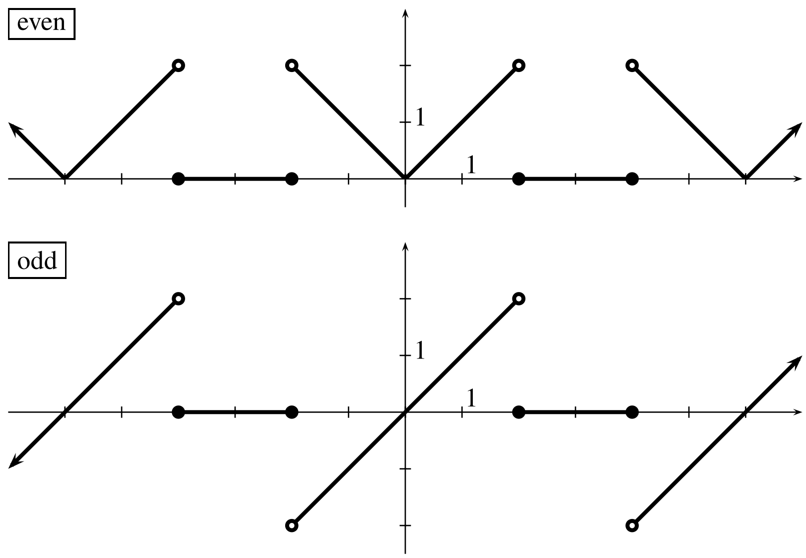
Math 146B.001 (Ordinary and Partial Differential Equations)
Homework 08 Solutions

Problem 1 (10.4.7)**.** Define $f : [0,3] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 2, \text{ and} \\ 0 & \text{if } 2 \leq x < 3. \end{cases}$$

Sketch the graphs of the even and odd periodic extensions of f.

Solution. The graphs are given by



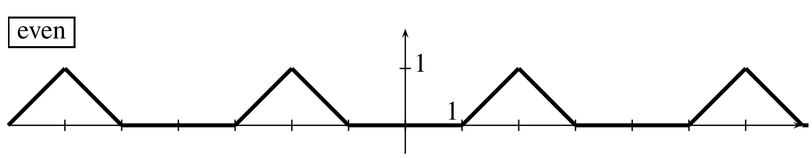
□

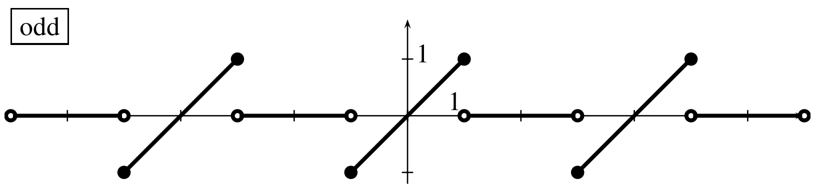
Problem 2 (10.4.8)**.** Define $f : [0,2] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \text{ and} \\ x - 1 & \text{if } 1 \leq x < 2. \end{cases}$$

Sketch the graphs of the even and odd periodic extensions of f.

Solution. The graphs are given by





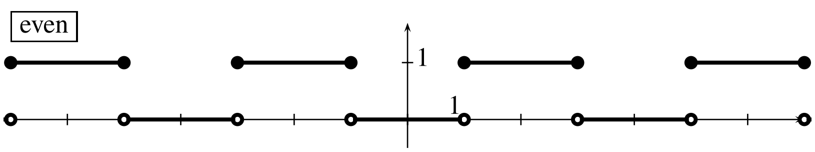
□

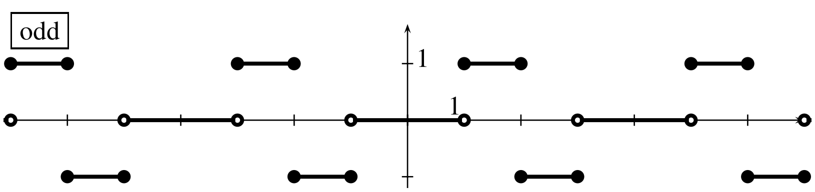
Problem 3 (10.4.11)**.** Define $f : [0, 2] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \text{ and} \\ 1 & \text{if } 1 \leq x < 2. \end{cases}$$

Sketch the graphs of the even and odd periodic extensions of f.

Solution. The graphs are given by





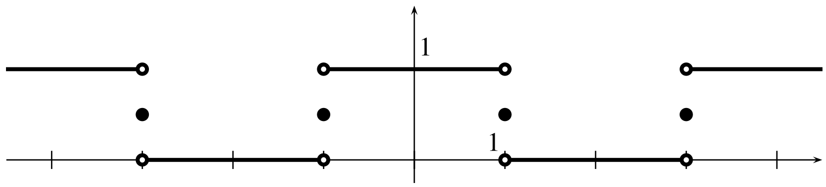
□

Problem 4 (10.4.15)**.** Define $f : [0, 2] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \text{ and} \\ 0 & \text{if } 1 < x < 2. \end{cases}$$

Sketch a graph of the even, 4-periodic extension of f. Determine the Fourier series for this even extension (in terms of cosines).

Solution. The graph of the function to which the Fourier series converges is given below. Note that at the "jumps", the series converges to the mean of the left-hand and right-hand limits, i.e. $1/2$.



The corresponding Fourier cosine series is

$$f(x) \equiv \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right),$$

where the Fourier coefficients are given by

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)\, \mathrm{d}x = \int_0^1 \cos\left(\frac{n\pi x}{2}\right)\, \mathrm{d}x.$$

When $n = 0$, this integral reduces to $a_0 = \int_0^1 \mathrm{d}x = 1$. For other values of n,

$$a_n = \int_0^1 \cos\left(\frac{n\pi x}{2}\right)\, \mathrm{d}x = \left[\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]_{x=0}^{1} = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right).$$

The sine term evaluates to zero when n is even, and evaluates to $(-1)^{2k-1}$ when $n = 2k-1$ is odd. The Fourier cosine series is therefore given by

$$f(x) \equiv \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{2k-1}}{2k-1}\cos\left(\frac{(2k-1)\pi x}{2}\right).$$

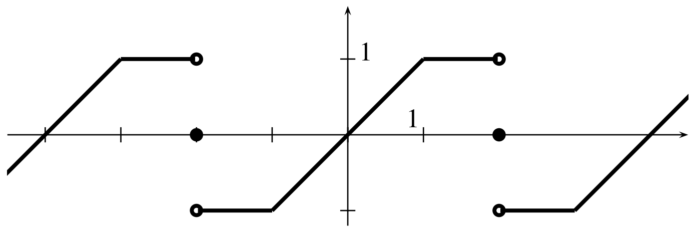
Note that the notation $\equiv$ here is meant to emphasize that the (value of the) function on the left and the series on the right are not equal *everywhere*, but are related. Specifically, for any x where the function f is defined (i.e. away from the jumps) the series converges pointwise to $f(x)$. However, at the jumps, the function is not defined, and the series converges to $1/2$. □

Problem 5 (10.4.16)**.** Define $f : [0,2] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 1, \text{ and} \\ 1 & \text{if } 1 \le x < 2. \end{cases}$$

Sketch a graph of the odd, 4-periodic extension of f. Determine the Fourier series for this even extension (in terms of sines).

Solution. The graph of the function to which the Fourier series converges is shown below.



The Fourier sine series is given by

$$f(x) \equiv \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

where the Fourier coefficients are given by

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)\,\mathrm{d}x = \int_0^1 x\sin\left(\frac{n\pi x}{2}\right)\,\mathrm{d}x + \int_1^2 \sin\left(\frac{n\pi x}{2}\right)\,\mathrm{d}x.$$

Tackling the integrals individually, the first integral simplifies to

$$\begin{aligned}
\int_0^1 x\sin\left(\frac{n\pi x}{2}\right)\,\mathrm{d}x &= \left[-\frac{2x}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right]_{x=0}^{1} + \frac{2}{n\pi}\int_0^1 \cos\left(\frac{n\pi x}{2}\right)\,\mathrm{d}x \\
&= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \left[\frac{4}{(n\pi)^2}\sin\left(\frac{n\pi x}{2}\right)\right]_{x=0}^{1} \\
&= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right). \qquad (1)
\end{aligned}$$

The second integral is

$$\int_1^2 \sin\left(\frac{n\pi x}{2}\right)\,\mathrm{d}x = \left[-\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_{x=1}^{2} = \frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right). \qquad (2)$$

The Fourier coefficients are then obtained by adding the quantities in (1) and (1). Thus

$$\begin{aligned}
b_n &= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right) \\
&= \frac{2}{n\pi}\left(\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right). \qquad (3)
\end{aligned}$$

This expression further simplifies, depending on the equivalence class of n modulo 4:

$$b_n = \begin{cases} -\dfrac{2}{n\pi} & \text{if } n = 4k \text{ for some } k \in \mathbb{N}, \\ \dfrac{4}{(n\pi)^2} + \dfrac{2}{n\pi} & \text{if } n = 4k+1 \text{ for some } k \in \mathbb{N}, \\ -\dfrac{2}{n\pi} & \text{if } n = 4k+2 \text{ for some } k \in \mathbb{N}, \text{ and} \\ -\dfrac{4}{(n\pi)^2} + \dfrac{2}{n\pi} & \text{if } n = 4k+3 \text{ for some } k \in \mathbb{N}. \end{cases}$$

With some clever algebra, this can be further simplified. First, observe that the expression

$$1 - (-1)^n$$

is zero if n is even, and is two otherwise. This expression can be used to eliminate the $4/(n\pi)^2$ terms from the coefficients with even indices. To obtain the alternating signs, the expression

$$(-1)^{\lfloor n/2 \rfloor}$$

is 1 if $n = 4k + 1$ for some integer k, and is -1 if $n = 4k + 3$. Combining these two expressions gives

$$\frac{1}{2}\big(1 - (-1)^n\big)(-1)^{\lfloor n/2 \rfloor} = \begin{cases} 0 & \text{if } n = 4k \text{ for some } k \in \mathbb{N}, \\ 1 & \text{if } n = 4k + 1 \text{ for some } k \in \mathbb{N}, \\ 0 & \text{if } n = 4k + 2 \text{ for some } k \in \mathbb{N}, \text{ and} \\ -1 & \text{if } n = 4k + 3 \text{ for some } k \in \mathbb{N}. \end{cases}$$

While this isn't particular pretty or enlightening, it can be used to obtain the Fourier series expansion

$$f(x) \equiv \sum_{n=1}^{\infty} \left[\frac{2\big(1 - (-1)^n\big)(-1)^{\lfloor n/2 \rfloor}}{(n\pi)^2} + \frac{2(-1)^n}{n\pi} \right] \sin\!\left(\frac{n\pi x}{2}\right).$$

Your text offers an alternative form of this series expansion in terms of the coefficients given at (3). ◻

Problem 6 (10.4.33)**.** Prove that the derivative of an even function is odd, and that the derivative of an odd function is even.

Proof. Recall that if a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at some point $x \in \mathbb{R}$, then the derivative of f at x, denoted $f'(x)$, is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Proving the desired results is an application of this definition:

- Suppose that f is even, i.e. $f(-x) = f(x)$ for all $x \in \mathbb{R}$. Then for any $x \in \mathbb{R}$,

$$\begin{aligned} f'(-x) &= \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} && \text{(defn. of the derivative)} \\ &= \lim_{h \to 0} \frac{f(x-h) - f(x)}{h} && \text{(since } f \text{ is even)} \\ &= \lim_{k \to 0} \frac{f(x+k) - f(x)}{-k} && \text{(substitute } k = -h\text{)} \\ &= -f'(x). \end{aligned}$$

  Therefore the derivative of an even function is odd.

- Similarly, suppose that f is odd, i.e. $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Then for any $x \in \mathbb{R}$

  $$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{k\to 0} \frac{f(x+k) - f(x)}{k} = f'(x).$$

  Therefore the derivative of an odd function is even.

$\square$

Problem 7 (10.4.34)**.** Let

$$F(x) = \int_0^x f(t)\,\mathrm{d}t.$$

Show that if f is even, then F is odd, and that if f is odd, then F is even.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be any integrable function, and define

$$F(x) = \int_0^x f(t)\,\mathrm{d}t.$$

Then for any $x \in \mathbb{R}$, make the change of variables $t \mapsto -t$ in order to obtain

$$F(-x) = \int_0^{-x} f(t)\,\mathrm{d}t = -\int_0^x f(-t)\,\mathrm{d}t.$$

If f is even, then $f(-t) = f(t)$, and so

$$F(-x) = -\int_0^x f(-t)\,\mathrm{d}t = -\int_0^x f(t)\,\mathrm{d}t = -F(x).$$

Hence if f is even, then it has an odd antiderivative. On the other hand, if f is odd, then $f(-t) = -f(t)$, and so

$$F(-x) = -\int_0^x f(-t)\,\mathrm{d}t = -\int_0^x -f(t)\,\mathrm{d}t = F(x).$$

Hence if f is odd, then it has an even antiderivative. In either case, the desired results hold. $\square$

Problem 8 (10.4.35)**.** From the Fourier series for the square wave in Example 1 of Section 10.3, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

Proof. In Example 10.3.1, it is shown that if $f : (-L, L) \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0 & \text{if } -L < x < 0, \text{ and} \\ L & \text{if } 0 < x < L, \end{cases}$$

then the Fourier series for f is given by

$$f(x) \equiv \frac{L}{2} + \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left(\frac{(2n-1)\pi x}{L}\right).$$

This series converges pointwise to L when $x \neq 2kL$ for some $k \in \mathbb{Z}$, and converges pointwise to $L/2$ otherwise. Thus, with $L = 2$ and $x = 1$, this becomes

$$2 = f(1) = 1 + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left(\frac{(2n-1)\pi}{2}\right) \implies \frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{1}{2n-1}(-1)^n,$$

which is the desired result. $\square$