· Problem 3 (Airy's equation)

Find the series general solution of $y'' - xy = 0$ at $x_0 = 0$.

- sol: For this equation, $P(x) \equiv 1$, $Q(x) \equiv 0$, $R(x) \equiv -x$.

Notice $P(x)$ has no zeros so the radius of convergence is $\rho = \infty$.

Now, since we are centered at $x_0 = 0$, let

$$y = \sum_{n=0}^{\infty} a_n x^n$$

So,

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Plugging these back in:

$$y'' - xy = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0$$

Shifting indices:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Want both series to have $x^n$ so take the $n=0$ term out of the first series and shift the second series to start at $n=1$ as well:

$$\underbrace{2a_2}_{n=0 \text{ term from}} + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

We need this to be $= 0$ for all $x$ so:

$$a_2 = 0 \quad \text{and} \quad (n+2)(n+1) a_{n+2} - a_{n-1} = 0 \quad \text{for } n = 1, 2, \ldots$$

$\rightarrow$

Now, let's rewrite $(n+2)(n+1)a_{n+2} - a_{n-1} = 0$ to get a formula we can use to find a pattern in the $a_n$'s:

$$a_{n+2} = \frac{1}{(n+2)(n+1)} a_{n-1} \xrightarrow[\substack{\text{shifting} \\ \text{indices}}]{\text{rewriting by}} a_n = \frac{1}{n(n-1)} a_{n-3} \quad \text{(✫)}$$

From (✫) we can see that the n'th term relies on the n-3 term so we need to look at 3 different cases:

- $n = 3, 6, 9, \ldots$ or $n = 3K$ for $K = 1, 2, \ldots$

$n=3$: $a_3 = \frac{1}{3\cdot 2} a_0$

$n=6$: $a_6 = \frac{1}{6\cdot 5} a_3 = \frac{1}{6\cdot 5} \cdot \frac{1}{3\cdot 2} a_0$

$n=9$: $a_9 = \frac{1}{9\cdot 8} \cdot a_6 = \frac{1}{9\cdot 8} \cdot \frac{1}{6\cdot 5} \cdot \frac{1}{3\cdot 2} a_0$

$\vdots$

so for $n = 3K$

$$a_{3K} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9\cdots(3K-1)(3K)}, \quad K = 1, 2, \ldots$$

- $n = 4, 7, 10, \ldots$ or $n = 3K+1$ for $K = 1, 2, \ldots$

$n=4$: $a_4 = \frac{1}{4\cdot 3} a_1$

$n=7$: $a_7 = \frac{1}{7\cdot 6} a_4 = \frac{1}{7\cdot 6} \cdot \frac{1}{4\cdot 3} a_1$

$\vdots$

so for $n = 3K+1$

$$a_{3K+1} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdots(3K)(3K+1)}, \quad K = 1, 2, \ldots$$

- $n = 5, 8, 11, \ldots$ or $n = 3K+2$ for $K = 1, 2, \ldots$

$a_5 = \frac{1}{5\cdot 4} a_2 = 0$ since $a_2 = 0$

So $0 = a_2 = a_5 = a_8 = \cdots = a_{3K+2}$

$\longrightarrow$

Combining all of this, we get:

$$a_n = \begin{cases} \dfrac{a_0}{2\cdot 3\cdot 5\cdot 6\cdots(3k-1)(3k)}, & n=3k \\ \dfrac{a_1}{3\cdot 4\cdot 6\cdot 7\cdots(3k)(3k+1)}, & n=3k+1 \qquad k=1,2,\ldots \\ 0, & n=3k+2 \end{cases}$$

Plugging this back into $y=\sum_{n=0}^{\infty} a_n x^n$:

$$y = a_0 + a_1 x + 0 + \frac{1}{2\cdot 3} a_0 x^3 + \frac{1}{3\cdot 4} a_1 x^4 + 0 + \cdots$$

$$= a_0\left(1+\frac{x^3}{2\cdot 3}+\cdots+\frac{x^{3k}}{2\cdot 3\cdots(3k-1)(3k)}+\cdots\right)$$

$$+ a_1\left(x+\frac{x^4}{3\cdot 4}+\cdots+\frac{x^{3k+1}}{3\cdot 4\cdots(3k)(3k+1)}+\cdots\right)$$

$$= a_0 y_1 + a_1 y_2$$

Where $\{y_1, y_2\}$ is a fundamental set of solutions

since $W(y_1,y_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$ $\boxed{\text{Kh}}$

This problem is example 2 in section 5.2 (pg 259)