§ 10.1 and 10.5. Separation of Variables and Boundary Conditions.

Problem 1. (Separation of Variables).

Use separation of variables to reduce the following equation into 2 ODEs:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & 0 < x < L, \quad t > 0. \quad (1) \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(L,t) = 0, & t > 0. \quad (2). \end{cases}$$

Solution. Set $u(x,t) = X(x)\,T(t)$.

In (1). $X(x)\,T'(t) = X''(x)\,T(t) \Rightarrow \dfrac{X''}{X} = \dfrac{T'}{T} = -\lambda$

where λ is a constant.

So $X''(x) + \lambda X(x) = 0$. and $T'(t) + \lambda T(t) = 0$.

In (2). $X'(0)\,T(t) = X'(L)\,T(t) = 0. \Rightarrow X'(0) = X'(L) = 0.$

So. $\begin{cases} X''(x) + \lambda X(x) = 0. \\ X'(0) = X'(L) = 0. \end{cases}$ and $T'(t) + \lambda T(t) = 0.$

Problem 2. (Boundary Condition).

Find eigenvalues (λ) and eigenfunctions (solutions) for.

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = X'(L) = 0. \end{cases} \quad \text{where } 0 < x < L.$$

Solution. Characteristic equation $r^2+\lambda=0$. $\Rightarrow$ $r=\pm\sqrt{-\lambda}$.

Case 1°. $\lambda<0$, $r_{1,2}=\pm\sqrt{-\lambda}$. $X(x)=c_1e^{\sqrt{-\lambda}x}+c_2e^{-\sqrt{-\lambda}x}$.

Then $\begin{cases} X'(0)=c_1\sqrt{-\lambda}+c_2(-\sqrt{-\lambda})=0. \\ X'(L)=c_1\sqrt{-\lambda}e^{\sqrt{-\lambda}L}+c_2(-\sqrt{-\lambda})e^{-\sqrt{-\lambda}L}=0. \end{cases}$

$\Rightarrow$ $c_1=c_2=0$. trivial solution.

Case 2° $\lambda=0$. $r_{1,2}=0$. $X(x)=c_1+c_2x$

Then $\begin{cases} X'(0)=c_2=0. \\ X'(L)=c_2=0. \end{cases}$ $\Rightarrow$ $X(x)=c_1$ nontrivial solution.

So $\lambda_0=0$ with $X_0(x)=1$ eigenvalue & eigenfunction.

Case 3°. $\lambda>0$. $r_{1,2}=\pm\sqrt{\lambda}\,i$. $X(x)=c_1\cos(\sqrt{\lambda}x)+c_2\sin(\sqrt{\lambda}x)$.

Then $\begin{cases} X'(0)=-\sqrt{\lambda}c_1\sin(\sqrt{\lambda}\cdot 0)+\sqrt{\lambda}c_2\cos(\sqrt{\lambda}\cdot 0)=0. \\ X'(L)=-\sqrt{\lambda}c_1\sin(\sqrt{\lambda}\cdot L)+\sqrt{\lambda}c_2\cos(\sqrt{\lambda}\cdot L)=0. \end{cases}$

$\Rightarrow$ $c_2=0$ and $\sin(\sqrt{\lambda}\cdot L)=0$. $\Rightarrow$ $\sqrt{\lambda}\cdot L=n\pi$, $n=1,2,\cdots$

So $\lambda_n=\left(\frac{n\pi}{L}\right)^2$ with $X_n(x)=\cos\left(\frac{n\pi x}{L}\right)$.
eigenvalue & eigenfunction.

Combine 2° & 3°: $\lambda_n=\left(\frac{n\pi}{L}\right)^2$ and $X_n(x)=\cos\left(\frac{n\pi x}{L}\right)$.
for $n=0,1,2,\cdots$

Problem 3 (Heat Equation).

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & 0<x<\pi, \quad t>0. \quad (1) \\ u(0,t) = u(\pi,t) = 0. & t>0. \quad (2) \\ u(x,0) = \sin(x) - 4\sin(3x). & 0<x<\pi. \quad (3). \end{cases}$$

Solution. In this PDE. $L=\pi$.

From lecture, (1) + (2) have general solution :

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin(nx). \qquad 0<x<\pi, \ t>0.$$

Use (3) and take $t=0$:

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(nx) = \sin(x) - 4\sin(3x).$$

By comparing the coefficients : $a_1 = 1$ and $a_3 = -4$

others : $a_n = 0$. $(n \neq 1, n \neq 3)$.

So $u(x,t) = e^{-t}\sin(x) - 4e^{-9t}\sin(3t)$.