Series solutions near an ordinary point (§5.2 & 5.3)

Problem 1. (radius of convergence).

Expand (1). $\frac{1}{x^2-2x+2} = \sum_{n=0}^{\infty} a_n x^n$ ($x_0=0$). find the radius of convergence. (2). $\frac{1}{x^2-2x+2} = \sum_{n=0}^{\infty} b_n (x-1)^n$ ($x_0=1$). find the radius of convergence without computing $\{a_n\}$ and $\{b_n\}$.

Solution. The denominator is the polynomial x^2-2x+2. with zeros: $x = 1 \pm i$.

So (1). center $x_0=0$, so $\rho_1 = |0-(1+i)| = \sqrt{2}$.

(2) center $x_0=1$, so $\rho_2 = |1-(1+i)| = 1$.

Problem 2. (radius of convergence lower bound).

Determine the lower bound for the radius of convergence of the series solution for $(1+x^2)y'' + 2xy' + 4x^2y = 0$.

(1). center $x_0=0$. (2) center $x_0 = -\frac{1}{2}$.

Solution. $P(x)=1+x^2$, $Q(x)=2x$, $R(x)=4x^2$ polynomials.

$P(x)$ has zeros: $x = \pm i$.

(1). Center $x_0=0$, so $\rho_1 = |0-i| = 1$.

The series solution has radius of convergence at least ($\geq$) $\rho_1=1$.

(2) center $x_0 = -\frac{1}{2}$, so $\rho_2 = |-\frac{1}{2}-i| = \sqrt{5}/2$.

The series solution has radius of convergence at least ($\geq$) $\rho_2 = \frac{\sqrt{5}}{2}$.

Problem 3. (Airy's equation).

Find the series general solution of $y'' - xy = 0$. at $x_0 = 0$.

Solution. $P(x) \equiv 1$, $Q(x) \equiv 0$, $R(x) = -x$. polynomials.

$P(x)$ has no zeros. so radius of convergence $\rho = \infty$.

Let $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

Plug back in : $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \cdot \sum_{n=0}^{\infty} a_n x^n = 0$.

Shift index : $\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$.

$$2 \cdot a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

$\Rightarrow$. $a_2 = 0$ and. $(n+2)(n+1) a_{n+2} - a_{n-1} = 0$. for $n = 1, 2, 3, \cdots$

Take. $n = 3, 6, 9, \cdots$. $0 = a_2 = a_5 = a_8 = \cdots = a_{3k+2}$ $(k = 0, 1, 2, \cdots)$.

Take $n = 1, 4, 7, \cdots$. $a_3 = \frac{a_0}{2 \cdot 3}$, $a_6 = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6}$, $\cdots$

Take $n = 2, 5, 8, \cdots$ $a_4 = \frac{a_1}{3 \cdot 4}$, $a_7 = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7}$ $\cdots$

In general
$$\begin{cases} a_{3k} = \dfrac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3k-1) 3k}, & k = 1, 2, \cdots \\ a_{3k+1} = \dfrac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots 3k(3k+1)}, & k = 1, 2, \cdots \end{cases}$$

where a_0, a_1 are arbitrary constants.

So
$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 \left(1 + \frac{x^3}{2 \cdot 3} + \cdots + \frac{x^{3k}}{2 \cdot 3 \cdots (3k-1) 3k} + \cdots\right) + a_1 \left(x + \frac{x^4}{3 \cdot 4} + \cdots + \frac{x^{3k+1}}{3 \cdot 4 \cdots 3k(3k+1)} + \cdots\right).$$

$$= a_0 y_1 + a_1 y_2.$$

where $\{y_1, y_2\}$ is a fundamental set of solutions.

(Since $W(y_1, y_2)(0) = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq 0$).