Midterm Question #4: $y'' - x^2 y = 0$

$$\begin{cases} P(x) = 1, \quad Q(x) = 0, \quad R(x) = -x^2 & \text{polynomials} \\ P(x_0) = P(0) = 1 \neq 0 & \text{ordinary pt} \end{cases}$$

Let $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

So, $y'' - x^2 y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x^2 \sum_{n=0}^{\infty} a_n x^n = 0$

$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$

$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} - \sum_{n=2}^{\infty} a_{n-2} x^n = 0$

$\Rightarrow 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_{n-2}\right] x^n = 0$

$\Rightarrow a_2 = 0, \ a_3 = 0, \quad (n+2)(n+1) a_{n+2} - a_{n-2} = 0$ for $n = 2, 3, \ldots$

$\Rightarrow a_{n+2} = \frac{1}{(n+2)(n+1)} a_{n-2} \quad \Rightarrow \quad a_n = \frac{1}{n(n-1)} a_{n-4}$

For $n = 4, 8, 12, \ldots$ or $n = 4k$, $k = 1, 2, \ldots$

$a_4 = \frac{1}{4 \cdot 3} a_0$, $a_8 = \frac{1}{8 \cdot 7} a_4 = \frac{1}{8 \cdot 7 \cdot 4 \cdot 3} a_0$, $\ldots$, $a_{4k} = \frac{a_0}{3 \cdot 4 \cdots (4k-1)(4k)}$, $k = 1, 2, \ldots$

For $n = 5, 9, 13, \ldots$ or $n = 4k+1$, $k = 1, 2, \ldots$

$a_5 = \frac{1}{5 \cdot 4} a_1$, $a_9 = \frac{1}{9 \cdot 8} a_5 = \frac{1}{9 \cdot 8 \cdot 5 \cdot 4} a_1, \ldots$, $a_{4k+1} = \frac{a_1}{4 \cdot 5 \cdots (4k)(4k+1)}$, $k = 1, 2, \ldots$

And
$a_2 = a_6 = \cdots = a_{4k+2} = 0$, $\quad a_3 = a_7 = \cdots = a_{4k+3} = 0$

Thus,

$$y = \left(\sum_{k=1}^{\infty} \frac{a_0}{3 \cdot 4 \cdots (4k-1)(4k)} x^{4k} + a_0 \right) + \left(\sum_{k=1}^{\infty} \frac{a_1}{4 \cdot 5 \cdots (4k)(4k+1)} x^{4k+1} + a_1 x \right) = a_0 y_1 + a_1 y_2$$