Math 146B.001 (Ordinary and Partial Differential Equations)
Homework 01 Solutions

The first two problems are meant to demonstrate applications of Theorem 4.1.1, which states

Theorem 1. *Suppose that*

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1} y' + p_n y = g(t), \tag{1}$$

subject to the initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y'_0, \qquad \ldots, \qquad y^{(n-1)}(t_0) = y_0^{(n-1)}. \tag{2}$$

If the functions p_1, p_2, ..., p_n, and g are continuous on the open interval I, then there exists exactly one solution $y = \phi(t)$ of the differential equation (1) that also satisfies the initial condition (2), where t_0 is any point in I. This solution exists throughout the interval I.

Problem 1 (4.1.3)**.** Determine intervals in which solutions to the differential equation

$$t(t-1)y^{(4)} + \mathrm{e}^t y'' + 4t^2 y = 0 \tag{3}$$

are sure to exist.

Solution. The hypotheses of Theorem 1 assume that the coefficient of the highest order term is 1. Rewrite 3 as the equivalent differential equation

$$y^{(4)} + \frac{\mathrm{e}^t}{t(t-1)} y'' + \frac{4t^2}{t(t-1)} y = 0.$$

In the notation of (1), the p_j and g are defined by

$$p_1(t) = 0, \qquad p_2(t) = \frac{\mathrm{e}^t}{t(t-1)}, \qquad p_3(t) = \frac{4t^2}{t(t-1)}, \qquad \text{and} \qquad g(t) = 0.$$

Both p_1 and g are continuous on all of $\mathbb{R}$, while p_2 and p_3 are continuous on any interval not containing either 0 or 1. Hence by Theorem 1, there are three intervals on which the equation (3) must have exactly one solution (for any set of initial conditions):

$$I_1 = (-\infty, 0), \qquad I_2 = (0, 1), \qquad \text{and} \qquad I_3 = (1, \infty).$$

□

Problem 2 (4.1.6)**.** Determine intervals in which solutions to the differential equation

$$(x^2 - 4))y^{(6)} + x^2 y''' + 9y = 0 \tag{4}$$

are sure to exist.

Solution. The independent variable in (4) is x, rather than t. However, this labeling is not important. As in the previous problem, rewrite the differential equation so that the leading coefficient is 1:

$$y^{(6)} + \frac{x^2}{x^2 - 4} y''' + \frac{9}{x^2 - 4} y = 0.$$

Then, in the notation of 1, the non-trivial (i.e. the nonzero) coefficient functions are given by

$$p_3(x) = \frac{x^2}{x^2 - 4}, \qquad p_5(x) = \frac{9}{x^2 - 4}, \qquad \text{and} \qquad g(x) = 0.$$

The function g is continuous on $\mathbb{R}$, while both p_3 and p_5 are continuous everywhere except $x = \pm 2$. Hence by Theorem 1, there are three intervals on which the equation (4) must have exactly one solution (for any set of initial conditions):

$$I_1 = (-\infty, -2), \qquad I_2 = (-2, 2), \qquad \text{and} \qquad I_3 = (2, \infty).$$

$\square$

The next two problems are about the linear independence of sets of functions. The main tool used in these problems is the Wronskian. The basic idea is that if the Wronskian matrix is non-singular, then the functions involved are linearly independent. This is good to know, as Theorem 4.1.3 of our text implies that this condition is sufficient to verify the existence of a fundamental set of solutions to a linear differential equation.

Definition 2. Let $\{f_1, f_2, \ldots, f_n\}$ be a set functions. Then the *Wronskian* is defined by

$$W(f_1, f_2, \ldots, f_n) := \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

The following proposition is not presented as an independent result in the text. However, it is discussed on page 223–5. It is given here for the sake of completeness.

Proposition 3. *Let $\{f_1, f_2, \ldots, f_n\}$ be a set functions. This set is linearly independent if and only if*

$$W(f_1, f_2, \ldots, f_n) \neq 0.$$

Problem 3 (4.1.8)**.** Determine whether the functions f_1, f_2, and f_3, defined by

$$f_1(t) = 2t - 3, \qquad f_2(t) = 2t^2 + 1, \qquad \text{and} \qquad f_3(t) = 3t^2 + t,$$

are linear dependent or linearly independent. If they are linearly dependent, find a linear relation among them.

Solution. To check linear independence, compute the Wronskian for an arbitrary value of t:

$$\begin{aligned} W(f_1, f_2, f_3)(t) &= \begin{vmatrix} f_1(t) & f_2(t) & f_3(t) \\ f_1'(t) & f_2'(t) & f_3'(t) \\ f_1''(t) & f_2''(t) & f_3''(t) \end{vmatrix} = \begin{vmatrix} 2t-3 & 2t^2+1 & 3t^2+t \\ 2 & 4t & 6t+1 \\ 0 & 4 & 6 \end{vmatrix} \\ &= (2t-3)\begin{vmatrix} 4t & 6t+1 \\ 4 & 6 \end{vmatrix} - 2\begin{vmatrix} 2t^2+1 & 3t^2+t \\ 4 & 6 \end{vmatrix} + 0\begin{vmatrix} 2t^2+1 & 3t^2+t \\ 4t & 6t+1 \end{vmatrix} \\ &= (2t-3)\big(24t - (24t+4)\big) - 2\big((12t^2+6) - (12t^2+4t)\big) \\ &= (-8t+12) - 2(6-4t) = -8t + 12 - 12 + 8t = 0. \end{aligned}$$

As the Wronskian is zero, the functions are not linearly independent, and are therefore linearly dependent.

Finding a linear relation among these functions is a somewhat less trivial task. The goal is to find scalars α_1, α_2, and α_3 such that

$$\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 = 0.$$

One possibility is to attempt to find the α_j by inspection or trial and error. In the case of the given functions, this isn't too difficult: it is possible to "kill" the -3 in f_1 by adding $3f_2$:

$$f_1(t) + 3f_2(t) = (2t - 3) + 3(2t^2 + 1) = 2t - 3 + 6t^2 + 3 = 6t^2 + 2t.$$

But

$$6t^2 + 2t = 2f_3(t).$$

Therefore

$$f_1(t) + 3f_2(t) = 2f_3(t) \implies f_1 + 3f_2 - 2f_3 = 0.$$

However, in more complicated examples, it will be useful to have a better technique for finding an appropriate linear relation. One possibility for this problem is outlined below.

Because differentiation is linear (that is, $(f+g)' = f' + g'$ for any differentiable f and g), it follows that the α_j further satisfy the relation

$$\begin{aligned}\begin{pmatrix}0\\0\\0\end{pmatrix} &= \alpha_1\begin{pmatrix}f_1\\f_1'\\f_1''\end{pmatrix} + \alpha_2\begin{pmatrix}f_2\\f_2'\\f_2''\end{pmatrix} + \alpha_3\begin{pmatrix}f_3\\f_3'\\f_3''\end{pmatrix} = \begin{pmatrix}f_1 & f_2 & f_3\\ f_1' & f_2' & f_3'\\ f_1'' & f_2'' & f_3''\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}\\ &= \begin{pmatrix}2t-3 & 2t^2+1 & 3t^2+t\\ 2 & 4t & 6t+1\\ 0 & 4 & 6\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}.\end{aligned}$$

As this relation must hold for any value of t, a clever choice will simplify the problem. For example, if $t = 0$, then

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}-3 & 1 & 0\\ 2 & 0 & 1\\ 0 & 4 & 6\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}.$$

Using the linear algebra technique of performing row operations on an augmented matrix, this sytem can be solved as follows:

$$\left(\begin{array}{ccc|c}-3 & 1 & 0 & 0\\ 2 & 0 & 1 & 0\\ 0 & 4 & 6 & 0\end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c}1 & -\frac{1}{3} & 0 & 0\\ 0 & \frac{2}{3} & 1 & 0\\ 0 & 4 & 6 & 0\end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c}1 & 0 & \frac{1}{2} & 0\\ 0 & 1 & \frac{3}{2} & 0\\ 0 & 0 & 0 & 0\end{array}\right).$$

From this, conclude that

$$\alpha_1 = -\frac{1}{2}\alpha_3 \qquad \text{and} \qquad \alpha_2 = -\frac{3}{2}\alpha_3,$$

where α_3 may be chosen freely. A convenient choice is $\alpha_3 = -2$, from which

$$\alpha_1 = 1, \qquad \alpha_2 = 3, \qquad \text{and} \qquad \alpha_3 = -2.$$

Therefore a linear relation among the given functions is

$$f_1 + 3f_2 - 2f_3 = 0.$$

Note that this can be quickly verified:

$$\begin{aligned} f_1(t) + 3f_2(t) - 2f_3(t) &= (2t-3) + 3(2t^2+1) - 2(3t^2+t) \\ &= 2t - 3 + 6t^2 + 3 - 6t^2 - 2t \\ &= 0, \end{aligned}$$

and it matches the relation found above by inspection. □

Problem 4 (4.1.10)**.** Determine whether the functions f_1, f_2, f_3, and f_4, defined by

$$f_1(t) = 2t-3, \quad f_2(t) = t^3+1, \quad f_3(t) = 2t^2 - t, \quad \text{and} \quad f_4(t) = t^2+t+1,$$

are linear dependent or linearly independent. If they are linearly dependent, find a linear relation among them.

Solution. To check linear independence, compute the Wronskian for an arbitrary value of t. The computation is somewhat tedious, but is simplified if cofactor expansion starts along the bottom row:

$$\begin{aligned} W(f_1, f_2, f_3, f_4)(t) &= \begin{vmatrix} f_1(t) & f_2(t) & f_3(t) & f_4(t) \\ f_1'(t) & f_2'(t) & f_3'(t) & f_4'(t) \\ f_1''(t) & f_2''(t) & f_3''(t) & f_3''(t) \\ f_1'''(t) & f_2'''(t) & f_3'''(t) & f_3'''(t) \end{vmatrix} \\ &= \begin{vmatrix} 2t-3 & t^3+1 & 2t^2-t & t^2+t+1 \\ 2 & 3t^2 & 4t-1 & 2t+1 \\ 0 & 6t & 4 & 2 \\ 0 & 6 & 0 & 0 \end{vmatrix} \\ &= 6 \begin{vmatrix} 2t-3 & 2t^2-t & t^2+t+1 \\ 2 & 4t-1 & 2t+1 \\ 0 & 4 & 2 \end{vmatrix} \\ &= 6 \left((2t-3) \begin{vmatrix} 4t-1 & 2t+1 \\ 4 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2t^2-t & t^2+t+1 \\ 4 & 2 \end{vmatrix} \right) \\ &= 6 \left((2t-3)\big((8t-2) - (8t+4)\big) - 2\big((4t^2-2t) - (4t^2+4t+4)\big) \right) \\ &= 6\left((2t-3)(-6) - 2(-6t-4)\right) \\ &= 6\left(-12t + 18 + 12t + 8\right) \\ &= 6 \cdot 26 \\ &= 156 \\ &\neq 0. \end{aligned}$$

As the Wronskian is nonzero, the given functions are linearly independent. □

Problem 5 (4.1.12)**.** Verify that that the functions f_1, f_2, f_3, and f_4, given by

$$f_1(t) = 1, \qquad f_2(t) = t, \qquad f_3(t) = \cos(t), \qquad \text{and} \qquad f_4(t) = \sin(t)$$

are solutions to the differential equation

$$y^{(4)} + y'' = 0.$$

Determine $W(f_1, f_2, f_3, f_4)(t)$.

Solution. Verification is a matter of computation:

$$\begin{aligned}
f_1^{(4)}(t) + f_1''(t) &= \frac{\mathrm{d}^4}{\mathrm{d}t^4}1 + \frac{\mathrm{d}^2}{\mathrm{d}t^2}1 = 0 + 0 = 0, && \checkmark\\
f_2^{(4)}(t) + f_2''(t) &= \frac{\mathrm{d}^4}{\mathrm{d}t^4}t + \frac{\mathrm{d}^2}{\mathrm{d}t^2}t = 0 - 0 = 0, && \checkmark\\
f_3^{(4)}(t) + f_3''(t) &= \frac{\mathrm{d}^4}{\mathrm{d}t^4}\cos(t) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\cos(t) = \cos(t) - \cos(t) = 0, && \checkmark\\
f_4^{(4)}(t) + f_4''(t) &= \frac{\mathrm{d}^4}{\mathrm{d}t^4}\sin(t) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\sin(t) = \sin(t) - \sin(t) = 0. && \checkmark
\end{aligned}$$

The Wronskian is given by

$$\begin{aligned}
W(f_1, f_2, f_3)(t) &= \begin{vmatrix} f_1(t) & f_2(t) & f_3(t) & f_4(t)\\ f_1'(t) & f_2'(t) & f_3'(t) & f_4'(t)\\ f_1''(t) & f_2''(t) & f_3''(t) & f_4''(t)\\ f_1'''(t) & f_2'''(t) & f_4'''(t) & f_4'''(t) \end{vmatrix} = \begin{vmatrix} 1 & t & \cos(t) & \sin(t)\\ 0 & 1 & -\sin(t) & \cos(t)\\ 0 & 0 & -\cos(t) & -\sin(t)\\ 0 & 0 & \sin(t) & -\cos(t) \end{vmatrix}\\
&= \begin{vmatrix} 1 & -\sin(t) & \cos(t)\\ 0 & -\cos(t) & -\sin(t)\\ 0 & \sin(t) & -\cos(t) \end{vmatrix} = \begin{vmatrix} -\cos(t) & -\sin(t)\\ \sin(t) & -\cos(t) \end{vmatrix} = \cos(t)^2 + \sin(t)^2 = 1.
\end{aligned}$$

Note that the Wronskian is nonzero, hence these four functions are a fundamental set of solutions to the differential equation, see Theorem 4.1.3 in the text. □

Problem 6 (4.1.12)**.** Verify that that the functions f_1, f_2, f_3, and f_4, given by

$$f_1(t) = 1, \qquad f_2(t) = t, \qquad f_3(t) = \mathrm{e}^{-t}, \qquad \text{and} \qquad f_4(t) = t\mathrm{e}^{-t}$$

are solutions to the differential equation

$$y^{(4)} + 2y''' + y'' = 0.$$

Determine $W(f_1, f_2, f_3, f_4)(t)$.

Solution. Again, verification of the first three functions is a matter of straightforward computation:

$$\begin{aligned}
f_1^{(4)}(t) + 2f_1'''(t) + f_1''(t) &= \frac{\mathrm{d}^4}{\mathrm{d}t^4}1 + 2\frac{\mathrm{d}^3}{\mathrm{d}t^3}1 + \frac{\mathrm{d}^2}{\mathrm{d}t^2}1 = 0 + 2\cdot 0 + 0 = 0, && \checkmark \\
f_2^{(4)}(t) + 2f_2'''(t) + f_2''(t) &= \frac{\mathrm{d}^4}{\mathrm{d}t^4}t + 2\frac{\mathrm{d}^3}{\mathrm{d}t^3}t + \frac{\mathrm{d}^2}{\mathrm{d}t^2}t = 0 + 2\cdot 0 + 0 = 0, && \checkmark \\
f_3^{(4)}(t) + 2f_3'''(t) + f_3''(t) &= \frac{\mathrm{d}^4}{\mathrm{d}t^4}\mathrm{e}^{-t} + 2\frac{\mathrm{d}^3}{\mathrm{d}t^3}\mathrm{e}^{-t} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{e}^{-t} = \mathrm{e}^{-t} - 2\mathrm{e}^{-t} + \mathrm{e}^{-t} = 0. && \checkmark
\end{aligned}$$

For the remaining function, it might be helpful to observe that

$$f_4^{(n)}(t) = (-1)^{n+1}\left(n\mathrm{e}^{-t} - t\mathrm{e}^{-t}\right).$$

While these derivatives may be computed by hand, the formula should also prove useful in computing the Wronskian, below. The formula can be shown via a quick induction argument: the relation holds when $n = 0$, and if it holds for $n = k$ then

$$\begin{aligned}
f_4^{(k+1)}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}f_4^{(k)}(t) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left((-1)^{k+1}\left(k\mathrm{e}^{-t} - f_4(t)\right)\right) && \text{(induction hypothesis)} \\
&= (-1)^{k+1}\left(-k\mathrm{e}^{-t} - \frac{\mathrm{d}}{\mathrm{d}t}t\mathrm{e}^{-t}\right) \\
&= (-1)^{k+1}\left(-k\mathrm{e}^{-t} - \mathrm{e}^{-t} + t\mathrm{e}^{t}\right) \\
&= (-1)^{k+1}(-1)\left((k+1)\mathrm{e}^{-t} - t\mathrm{e}^{t}\right) \\
&= (-1)^{(k+1)+1}\left((k+1)\mathrm{e}^{-t} - t\mathrm{e}^{-t}\right),
\end{aligned}$$

which is the desired result. Then

$$\begin{aligned}
&f_4^{(4)}(t) + 2f_4^{(3)}(t) + f_4^{(2)}(t) \\
&\quad = (-1)^5\left(4\mathrm{e}^{-t} - t\mathrm{e}^{t}\right) + 2(-1)^4\left(3\mathrm{e}^{-t} - t\mathrm{e}^{t}\right) + (-1)^3\left(2\mathrm{e}^{-t} - t\mathrm{e}^{t}\right) \\
&\quad = \left(-4\mathrm{e}^{-t} + t\mathrm{e}^{t}\right) + \left(6\mathrm{e}^{-t} - 2\mathrm{e}^{-t}\right) + \left(-2\mathrm{e}^{-t} + t\mathrm{e}^{-t}\right) \\
&\quad = 0. && \checkmark
\end{aligned}$$

The Wronskian is given by

$$W(f_1, f_2, f_3, f_4)(t) = \begin{vmatrix} 1 & t & e^{-t} & te^{-t} \\ 0 & 1 & -e^{-t} & e^{-t} - te^{-t} \\ 0 & 0 & e^{-t} & -2e^{-t} + te^{-t} \\ 0 & 0 & -e^{-t} & 3e^{-t} - te^{-t} \end{vmatrix} = \begin{vmatrix} e^{-t} & -2e^{-t} + te^{-t} \\ -e^{-t} & 3e^{-t} - te^{-t} \end{vmatrix}$$

$$= \left(3e^{-2t} - te^{-2t}\right) - \left(2e^{-2t} - te^{-2t}\right) = e^{-2t}.$$

Again, the Wronskian is nonzero (for any t), hence these functions are a fundamental set of solutions to the differential equation on the real line. □

Problem 7 (4.1.17)**.** Show that $W(5, \sin(t)^2, \cos(2t)) = 0$ for all t. Can you establish this result without direct evaluation of the Wronskian?

Proof. Computing the Wronskian directly, we get

$$\begin{aligned}
&W(5, \sin(t)^2, \cos(2t)) \\
&\qquad = \begin{vmatrix} 5 & \sin(t)^2 & \cos(2t) \\ 0 & 2\sin(t)\cos(t) & -2\sin(2t) \\ 0 & 2(\cos(t)^2 - \sin(t)^2) & -4\cos(2t) \end{vmatrix} \\
&\qquad = 5\left((2\sin(t)\cos(t))(-4\cos(2t)) - 2(\cos(t)^2 - \sin(t)^2)(-2\sin(2t))\right) \\
&\qquad = 5\left(-8\sin(t)\cos(t)\cos(2t) + 4(\cos(t)^2 - \sin(t)^2)\sin(2t)\right) \\
&\qquad = 5\left(-4\sin(2t)\cos(2t) + 4(\cos(t)^2 - \sin(t)^2)\sin(2t)\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\text{since } \sin(2t) = 2\cos(t)\sin(t)) \\
&\qquad = 5\left(-4\sin(2t)\cos(2t) + 4\cos(2t)\sin(2t)\right) \quad (\text{since } \cos(2t) = \cos(t)^2 - \sin(t)^2) \\
&\qquad = 0.
\end{aligned}$$

Alternatively, we can show that the Wronskian is zero by showing that the three functions given are linearly dependent, which can be done by finding an explicit linear relation among these functions. Recall that

$$\cos(2t) = \cos(t)^2 - \sin(t)^2.$$

The Pythagorean identity states that $\cos(t)^2 + \sin(t)^2 = 1$, so we may add $2\sin(t)^2$ to both sides of this equation in order to obtain

$$2\sin(t)^2 + \cos(2t) = 2\sin(t)^2 + \cos(t)^2 - \sin(t)^2 = 1.$$

But $1 = \frac{1}{5} \cdot 5$, from which it follows that

$$\frac{1}{5} \cdot (5) + 2 \cdot \sin(t)^2 + \cos(2t) = 0,$$

which gives the desired linear relation. □

Problem 8 (4.1.18)**.** Verify that the differential operator defined by

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y$$

is a linear differential operator. That is, show that

$$L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2],$$

where y_1 and y_2 are n-times-differentiable functions, and c_1 and c_2 are arbitrary constants. Hence, show that if y_1, y_2, ..., y_n are solutions of $L[y] = 0$, then the linear combination $c_1y_1 + \cdots c_ny_n$ is also a solution of $L[y] = 0$.

Proof. The linearity of L follows almost immediately from the linearity of the derivative, that is

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}(c_1y_1 + c_2y_2) = c_1\frac{\mathrm{d}^k}{\mathrm{d}t^k}y_1 + c_2\frac{\mathrm{d}^k}{\mathrm{d}t^k}y_2 = c_1y_1^{(k)} + c_2y_2^{(k)}$$

for any integer k, any n-times-differentiable functions y_1 and y_2, and any constants c_1 and c_2. More explicitly, if y_1 and y_2 are two n-times-differentiable functions and c_1 and c_2 are constants, then

$$\begin{aligned}
&L[c_1y_1 + c_2y_2] \\
&\quad = \frac{\mathrm{d}^n}{\mathrm{d}t^n}(c_1y_1 + c_2y_2) + p_1(t)\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}(c_1y_1 + c_2y_2) + \cdots + p_n(t)(c_1y_1 + c_2y_2) \\
&\quad = \left(c_1y_1^{(n)} + c_2y_2^{(n)}\right) + p_1(t)\left(c_1y_1^{(n-1)} + c_2y_2^{(n-1)}\right) + \ldots + p_n(t)(c_1y_1 + c_2y_2) \\
&\hspace{20em}\text{(by the linearity of differentiation)} \\
&\quad = \left(c_1y_1^{(n)} + p_1(t)c_1y_1^{(n-1)} + \cdots + p_n(t)c_1y_1\right) \\
&\qquad\qquad + \left(c_2y_2^{(n)} + p_1(t)c_2y_2^{(n-1)} + \cdots + p_n(t)c_2y_2\right) \\
&\hspace{20em}\text{(by the commutativity of addition)} \\
&\quad = c_1\left(y_1^{(n)} + p_1(t)y_1^{(n-1)} + \cdots + p_n(t)y_1\right) + c_2\left(y_2^{(n)} + p_1(t)y_2^{(n-1)} + \cdots + p_n(t)y_2\right) \\
&\hspace{20em}\text{(factor out } c_j\text{)} \\
&\quad = c_1L[y_1] + c_2L[y_2],
\end{aligned}$$

which is the desired result.

Let m be a fixed positive integer and suppose that $\{y_j\}_{j=1}^m$ is a set of solutions to the equation $L[y] = 0$. That is, for each $j = 1, 2, \ldots, m$, suppose that y_j is an n-times differentiable function such that

$$y_j^{(n)} + p_1(t)y_j^{(n-1)} + \cdots + p_n(t)y_j = 0.$$

Further suppose that $\{c_j\}_{j=1}^m$ is a set of arbitrary constants, and define the function

$$y = c_1y_1 + c_2y_2 + \cdots + c_my_m = \sum_{j=1}^m c_jy_j.$$

By the linearity of L, we have

$$\begin{aligned}
L[y] &= L\left[\sum_{j=1}^{m} c_j y_j\right] \\
&= L\left[\left(\sum_{j=1}^{m-1} c_j y_j\right) + c_m y_m\right] \\
&= L\left[\sum_{j=1}^{m-1} c_j y_j\right] + c_m L[y_m] \\
&= L\left[\sum_{j=1}^{m-2} c_j y_j\right] + c_{m-1} L[y_{m-1}] + c_m L[y_m] \\
&\dots \\
&= c_1 L[y_1] + c_2 L[y_2] + \cdots + c_{m-1} L[y_{m-1}] + c_m L[y_m] \\
&= \sum_{j=1}^{m} c_j L[y_j]
\end{aligned}$$

This computation is a little informal, but can be made rigorous via an induction argument in the "obvious" manner. By hypothesis, $L[y_j] = 0$ for each j, from which it follows that

$$\sum_{j=1}^{m} c_j L[y_j] = \sum_{j=1}^{m} c_j \cdot 0 = 0.$$

Therefore $L[y] = 0$, which is the desired result.

Note that the result here is *stronger* than what was required. The result holds for any set of solutions $\{y_j\}_{j=1}^{m}$, where m is an arbitrary positive integer. That is, the size m of the collection of solutions needn't be equal to the order n of the differential equation.

If $m \leq n$, all this result says is that any linear combination of solutions is also a solution. This is an important observation, but isn't really earth shattering.

On the other hand, if $m > n$, then Theorem 4.1.3 implies that the set of solutions will be linearly dependent, and so it will be possible to find a linear relation among the m solutions. Such a linear relation may be used to eliminate any "extra" solutions beyond the n required to construct a fundamental solutions. This implies that *every* solution to the differential equation can be expressed as a linear combination of functions from a fundamental set of solutions. □