week 4

§5.4. Euler Equations and Regular Singular Points.

Problem 1. (distinct real roots). $y = x^r$

Find the general solution of the Euler equation

$$x^2y'' - 4xy' + 4y = 0. \qquad (x>0).$$

Solution. The indicial equation is $r(r-1) - 4r + 4 = 0$.

$\Rightarrow \quad r^2 - 5r + 4 = 0, \quad (r-1)(r-4) = 0.$

So $\quad r_1 = 4, \quad r_2 = 1.$

Then $\quad y_1 = x^4$ and $y_2 = x$.

The general solution is $y = c_1x^4 + c_2x$.

Problem 2. (repeated real roots).

Find the general solution of the Euler equation:

$$x^2y'' - 5xy' + 9y = 0. \qquad (x>0).$$

Solution. The indicial equation is $r(r-1) - 5r + 9 = 0$.

$\Rightarrow \quad r^2 - 6r + 9 = 0, \quad (r-3)^2 = 0.$

So $\quad r_1 = r_2 = 3.$

Then $\quad y_1 = x^3$ and $y_2 = x^3 \ln(x)$.

The general solution is $y = c_1x^3 + c_2x^3\ln(x)$.

hw 2:
3.6 #3, 5, 7
5.1 #5, 7, 15, 18, 28
5.2 #1, 2, 10
5.3 #6

Problem 3. (complex roots).

Find the general solution of the Euler equation:

$$2x^2y'' - 4xy' + 6y = 0. \qquad (x>0)$$

Solution. The indicial equation is $2r(r-1) - 4r + 6 = 0$.

$$\Rightarrow \quad r^2 - 3r + 3 = 0.$$

$$r_{1,2} = \frac{3 \pm \sqrt{3^2 - 4\cdot 3}}{2} = \frac{3}{2} \pm \frac{\sqrt{3}}{2} i.$$

i.e. $\lambda = \frac{3}{2}$ and $\mu = \frac{\sqrt{3}}{2}$.

Then $y_1 = x^{\frac{3}{2}} \cos\left(\frac{\sqrt{3}}{2} \ln(x)\right)$, $y_2 = x^{\frac{3}{2}} \sin\left(\frac{\sqrt{3}}{2} \ln(x)\right)$.

The general solution is $y = c_1 x^{\frac{3}{2}} \cos\left(\frac{\sqrt{3}}{2} \ln(x)\right) + c_2 x^{\frac{3}{2}} \sin\left(\frac{\sqrt{3}}{2} \ln(x)\right)$.

Problem 4. (singular points).

Find all singular points and determine which one is regular.

$$(x+2)^2(x-1)\, y'' + 3(x-1) y' - 2(x+2) y = 0.$$

Solution. $P(x) = (x+2)^2(x-1)$, $Q(x) = 3(x-1)$, $R(x) = -2(x+2)$.

polynomials. Only check $P(x) = 0$. $\Rightarrow$ $x = 1, -2$.

So $x_0 = 1$, $x_0 = -2$ are singular points.

At $x_0 = 1$:

$$(x - x_0)\frac{Q(x)}{P(x)} = (x-1)\frac{3(x-1)}{(x+2)^2(x-1)} = \frac{3(x-1)}{(x+2)^2} \quad \text{analytic at } x_0 = 1$$

$$(x - x_0)^2 \frac{R(x)}{P(x)} = (x-1)^2 \cdot \frac{-2(x+2)}{(x+2)^2(x-1)} = \frac{-2(x-1)}{(x+2)} \quad \text{analytic at } x_0 = 1.$$

So $x_0 = 1$ is a regular singular point.

At $x_0 = -2$:

$$(x - x_0)\frac{Q(x)}{P(x)} = (x+2)\cdot \frac{3(x-1)}{(x+2)^2(x-1)} = \frac{3}{x+2}. \quad \text{not analytic at } x_0 = -2.$$

So $x_0 = -2$ is irregular.

$P(x)y'' + Q(x)y' + R(x)y = 0$

- Def: A point x_0 is an ordinary point if $P(x_0) \neq 0$
- Def: A point x_0 is a singular point if $P(x_0) = 0$
- Euler equation: $x^2 y'' + \alpha x y' + \beta y = 0 \quad (x>0)$
  - $x_0 = 0$ is only singular pt
  - let $y = x^r$ be a sol. Then,

$$x^2\left(r(r-1)x^{r-2}\right) + \alpha x\left(r x^{r-1}\right) + \beta x^r = 0$$

$$\Rightarrow x^r\left[r(r-1) + \alpha r + \beta\right] = 0$$

$$\Rightarrow r^2 + (\alpha - 1)r + \beta = 0 \leftarrow \text{indicial equation}$$

- case 1: $r_1 \neq r_2$ real $\Rightarrow y = c_1 x^{r_1} + c_2 x^{r_2}$
- case 2: $r_1 = r_2$ real $\Rightarrow y = c_1 x^{r_1} + c_2 (\ln x) x^{r_1}$
- case 3: $r = \lambda \pm i\mu$ $\Rightarrow y = c_1 x^{\lambda}\cos(\mu \ln x) + c_2 x^{\lambda}\sin(\mu \ln x)$

- Def: A singular pt x_0 of a DE is called a regular singular point if

1) $\lim_{x \to x_0} (x - x_0)\dfrac{Q(x)}{P(x)} < \infty$ or $(x - x_0)\dfrac{Q(x)}{P(x)}$ is analytic at $x = x_0$

2) $\lim_{x \to x_0} (x - x_0)^2\dfrac{R(x)}{P(x)} < \infty$ or $(x - x_0)^2\dfrac{R(x)}{P(x)}$ is analytic at $x = x_0$

for polynomials | in general

5.2 #1 $y''-y=0\ ,\ x_0=0$

a) Let $y=\sum_{n=0}^{\infty} a_n x^n$

Then, $y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$, $y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$

So,

$$y''-y=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow (n+2)(n+1)a_{n+2} - a_n = 0$$

$$\Rightarrow \boxed{a_{n+2} = \frac{a_n}{(n+2)(n+1)}} \leftarrow \text{recurrence relation}$$

Rewriting this by shifting indices: $a_n = \dfrac{a_{n-2}}{n(n-1)}$ .....(✩)

We can see that the even terms will rely on the previous even term and the odd terms will rely on the previous odd terms. Thus, we will look at each case separately.

- $n=2,4,6,\ldots$ or $n=2k$ for $k=1,2,\ldots$

$$a_{2k} = \frac{a_{2k-2}}{(2k)(2k-1)} = \frac{a_{2k-4}}{(2k)(2k-1)(2k-2)(2k-3)} = \frac{a_{2k-6}}{(2k)(2k-1)(2k-2)(2k-3)(2k-4)(2k-5)}$$

$$= \cdots = \frac{a_0}{(2k)!} \text{ using (✩)}$$

- $n=3,5,7,\ldots$ or $n=2k+1$ for $k=1,2,\ldots$

$$a_{2k+1} = \frac{a_{2k-1}}{(2k+1)(2k)} = \frac{a_{2k-3}}{(2k+1)(2k)(2k-1)(2k-2)} = \cdots = \frac{a_1}{(2k+1)!}$$

Thus, $a_n = \begin{cases} \dfrac{a_0}{n!}, & n = 2k \\ \dfrac{a_1}{n!}, & n = 2k+1 \end{cases}$ $\quad k = 0, 1, 2, \ldots$

b)

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \frac{a_0}{2!}x^2 + \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 + \cdots$$

$$= a_0 \underbrace{\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right)}_{y_1} + a_1 \underbrace{\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right)}_{y_2}$$

So,

$$\boxed{\begin{aligned} y_1(x) &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \\ y_2(x) &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \end{aligned}}$$

c). $W(y_1, y_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$ so Linearly independent

d) $y_1(x) = 1 + \dfrac{x^2}{2!} + \cdots = \boxed{\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}}$ which is the Taylor series of $\underline{\cosh(x)}$

$y_2(x) = x + \dfrac{x^3}{3!} + \cdots = \boxed{\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}}$ which is the Taylor series of $\underline{\sinh(x)}$

5.2 #2 $y'' - xy' - y = 0 \quad , \quad x_0 = 0$

a) Let $y = \sum_{n=0}^{\infty} a_n x^n \; , \; y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \; y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

so

$$y'' - xy' - y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow (2)(1) a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} -n a_n x^n - a_0 + \sum_{n=1}^{\infty} -a_n x^n = 0$$

$\Rightarrow 2a_2 - a_0 = 0$ and $(n+2)(n+1) a_{n+2} - n a_n - a_n = 0$

$\Rightarrow a_2 = \frac{1}{2} a_0$ and $(n+2)(n+1) a_{n+2} - (n+1) a_n = 0$

$\Rightarrow \boxed{a_{n+2} = \frac{a_n}{n+2}}$ ← recurrence relation

Rewriting: $a_n = \frac{a_{n-2}}{n}$ ·····($\star$)

Skips by 2 so look at evens and odds:

- $n = 2k \; , \; k = 0, 1, 2, \ldots$

$$a_{2k} = \frac{a_{2k-2}}{2k} = \frac{a_{2k-4}}{(2k)(2k-2)} = \frac{a_{2k-6}}{(2k)(2k-2)(2k-4)} = \cdots = \frac{a_0}{2 \cdot 4 \cdot 6 \cdots (2k-2)(2k)}$$

- $n = 2k+1 \; , \; k = 0, 1, 2, \ldots$

$$a_{2k+1} = \frac{a_{2k-1}}{(2k+1)} = \frac{a_{2k-3}}{(2k+1)(2k-1)} = \frac{a_{2k-5}}{(2k+1)(2k-1)(2k-3)} = \cdots = \frac{a_1}{3 \cdot 5 \cdot 7 \cdots (2k-1)(2k+1)}$$

Thus,

$$a_n = \begin{cases} \dfrac{a_0}{2\cdot4\cdot6\cdots n}, & n=2k \\ \dfrac{a_1}{3\cdot5\cdot7\cdots n}, & n=2k+1 \end{cases} \qquad k=0,1,2,\dots$$

b) $$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \frac{a_0}{2}x^2 + \frac{a_1}{3}x^3 + \frac{a_0}{2\cdot4}x^4 + \frac{a_1}{3\cdot5}x^5 + \dots$$

$$= a_0\underbrace{\left(1 + \frac{x^2}{2} + \frac{x^4}{2\cdot4} + \frac{x^6}{2\cdot4\cdot6} + \dots\right)}_{y_1} + a_1\underbrace{\left(x + \frac{x^3}{3} + \frac{x^5}{3\cdot5} + \frac{x^7}{3\cdot5\cdot7} + \dots\right)}_{y_2}$$

so,

$$\boxed{\begin{aligned} y_1(x) &= 1 + \frac{x^2}{2} + \frac{x^4}{2\cdot4} + \frac{x^6}{2\cdot4\cdot6} + \dots \\ y_2(x) &= x + \frac{x^3}{3} + \frac{x^5}{3\cdot5} + \frac{x^7}{3\cdot5\cdot7} + \dots \end{aligned}}$$

c) $W(y_1,y_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$ so linearly independent

d) $$y_1(x) = 1 + \frac{x^2}{2} + \frac{x^4}{2\cdot4} + \frac{x^6}{2\cdot4\cdot6} + \dots = x^0 + \frac{x^2}{2(1)} + \frac{x^4}{2(1)\cdot2(2)} + \frac{x^6}{2(1)\cdot2(2)\cdot2(3)} + \dots$$

$$= x^0 + \frac{x^2}{2(1)} + \frac{x^4}{2^2(1\cdot2)} + \frac{x^6}{2^3(1\cdot2\cdot3)} + \dots = \boxed{\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}}$$

$$y_2(x) = x + \frac{x^3}{3} + \frac{x^5}{3\cdot5} + \frac{x^7}{3\cdot5\cdot7} + \dots = x + \frac{x^3}{\frac{1}{2\cdot1}(3\cdot2\cdot1)} + \frac{x^5}{\frac{1}{4\cdot(2\cdot1)}(5\cdot4\cdot3\cdot2\cdot1)} + \frac{x^7}{\frac{1}{8\cdot(3\cdot2\cdot1)}(7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot}$$

$$= x + \frac{2(1)x^3}{3!} + \frac{4(2\cdot1)x^5}{5!} + \frac{8(3\cdot2\cdot1)x^7}{7!} + \dots = \boxed{\sum_{n=0}^{\infty} \frac{2^n n!\, x^{2n+1}}{(2n+1)!}}$$