Math 146B.001 (Ordinary and Partial Differential Equations)
Homework 07 Solutions

Problem 1 (10.1.1)**.** Either solve the boundary value problem

$$y'' + y = 0; \qquad y(0) = 0, \quad y'(\pi) = 1$$

or show that it has no solution.

Solution. A general solution to the differential equation $y'' + y = 0$ is given by

$$y(x) = k_1 \cos(x) + k_2 \sin(x),$$

where k_1 and k_1 are constants. The first boundary value implies that

$$0 = y(0) = k_1 \cos(0) + k_2 \sin(0) = k_1.$$

Thus $k_1 = 0$. Substituting this into the solution and applying the information from the second boundary value gives

$$1 = y'(\pi) = k_2 \cos(\pi) = -k_2.$$

Therefore the boundary value problem is solved by

$$y(x) = -\sin(x).$$

□

Problem 2 (10.1.3)**.** Either solve the boundary value problem

$$y'' + y = 0; \qquad y(0) = 0, \quad y(L) = 0$$

or show that it has no solution.

Solution. A general solution to the differential equation $y'' + y = 0$ is given by

$$y(x) = k_1 \cos(x) + k_2 \sin(x),$$

where k_1 and k_1 are constants. The first boundary value implies that

$$0 = y(0) = k_1 \cos(0) + k_2 \sin(0) = k_1.$$

Thus $k_1 = 0$. Substituting this into the solution and applying the information from the second boundary value gives

$$0 = y(L) = k_2 \sin(L).$$

This is possible if either $k_2 = 0$ (giving rise to the trivial solution) or if $L = n\pi$ for some integer n. Solutions to the boundary value problem therefore come in two forms: either

$$y(x) = 0$$

for all x; or, in the case that L is an integer multiple of π,

$$y(x) = k \sin(x),$$

where k is an arbitrary real constant. □

Problem 3 (10.1.7)**.** Either solve the boundary value problem

$$y'' + 4y = \cos(x); \qquad y(0) = 0, \quad y(\pi) = 1$$

or show that it has no solution.

Solution. A general solution to the homogeneous differential equation $y'' + 4y = 0$ is given by

$$y_h(x) = k_1 \cos(2x) + k_2 \sin(2x).$$

Using the method of undetermined coefficients, we guess that a particular solution to the inhomogeneous equation is of the form

$$y_p(x) = A\cos(x) + B\sin(x).$$

Substituting this into the original differential equation gives

$$\cos(x) = y_p''(x)+4y(x) = -A\cos(x)-B\sin(x)+4A\cos(x)+4B\sin(x) = 3A\cos(x)+3B\sin(x).$$

Thus $A = \frac{1}{3}$ and $B = 0$, which implies that a particular solution is given by

$$y_p(x) = \frac{1}{3}\cos(x).$$

A general solution is therefore given by

$$y(x) = y_h(x) + y_p(x) = k_1\cos(2x) + k_2\sin(2x) + \frac{1}{3}\cos(x),$$

where k_1 and k_2 are arbitrary real constants. The first boundary condition implies that

$$0 = y(0) = k_1\cos(0) + k_2\sin(0) + \frac{1}{3}\cos(0) = k_1 + \frac{1}{3} \implies k_1 = -\frac{1}{3}.$$

The second boundary condition implies that

$$1 = y(\pi) = -\frac{1}{3}\cos(2\pi) + k_2\sin(2\pi) + \frac{1}{3}\cos(\pi) = -\frac{2}{3},$$

which is a contradiction. Therefore the original boundary value problem has no solutions. □

Problem 4 (10.1.11)**.** Either solve the boundary value problem

$$x^2y'' - 2xy' + 2y = 0; \qquad y(1) = -1, \quad y(2) = 1$$

or show that it has no solution.

Solution. This equation is an Euler equation. Making the *ansatz* that $y(x) = x^\lambda$, the differential equation becomes

$$\begin{aligned}0 &= x^2 y''(x) - 2xy(x) + 2y(x) \\ &= x^2\lambda(\lambda-1)x^{\lambda-2} - 2x\lambda x^{\lambda-1} + 2x^\lambda \\ &= x^\lambda\left(\lambda(\lambda-1) - 2\lambda + 2\right).\end{aligned}$$

This equation has two solutions: $\lambda_1 = 1$ and $\lambda_2 = 2$. Hence the differential equation has a general solution of the form

$$y(x) = k_1 x + k_2 x^2,$$

where k_1 and k_2 are constants. Evaluating this at the boundaries gives

$$\begin{cases} -1 = k_1 + k_2, \\ \ \ 1 = 2k_1 + 4k_2 \end{cases} \implies \begin{cases} k_1 = -\frac{5}{2}, \\ k_2 = \frac{3}{2}. \end{cases}$$

Thus the solution to the original boundary value problem is

$$y(x) = -\frac{5}{2}x + \frac{3}{2}x^2.$$

□

Problem 5 (10.1.14)**.** Find the eigenvalues and eigenfunctions of the boundary value problem

$$y'' + \lambda y = 0; \qquad y(0) = 0, \quad y'(\pi) = 0.$$

Assume that all eigenvalues are real.

Solution. The eigenvalues of the boundary value problem are, by definition, the values of λ such that nontrivial solutions to the problem exist. The eigenfunctions are the associated nontrivial solutions.

Note that, in this case, a general solution to the differential equation depends on the sign of λ. There are three cases to consider:

(i) $\lambda = 0$. If $\lambda = 0$, then $y'' = 0$, which has a general solution of the form

$$y(x) = k_1 x + k_2.$$

The boundary conditions imply that

$$0 = k_2 \qquad \text{and} \qquad 0 = k_1.$$

Thus if $\lambda = 0$, then the only solution is the trivial solution. This implies that $\lambda = 0$ is not an eigenvalue of the boundary value problem.

(ii) $\lambda < 0$. If $\lambda > 0$, then $y'' + \lambda y = 0$ has a general solution of the form

$$y(x) = k_1 e^{\lambda x} + k_2 e^{-\lambda x}.$$

The boundary conditions imply that

$$0 = k_1 + k_2 \qquad \text{and} \qquad 0 = k_1 e^{\lambda \pi} - k_2 e^{-\lambda \pi}.$$

The first equation implies that $k_1 = -k_2$, which, when substituted into the second equation, gives

$$2k_1 \left(e^{\lambda \pi} + e^{-\lambda \pi} \right) = 0.$$

This is possible if and only if $k_1 = 0$, as the exponential term never vanishes. As this corresponds to the trivial solution, no negative value of λ can be a eigenvalue of this problem.

(iii) $\lambda > 0$. If $\lambda < 0$, then $y'' + \lambda y = 0$ has a general solution of the form

$$y(x) = k_1 \cos(\sqrt{-\lambda} x) + k_2 \sin(\sqrt{-\lambda} x). \tag{1}$$

The boundary conditions imply that

$$0 = k_1 \cos(0) + k_2 \sin(0) = k_1 \qquad \text{and} \qquad 0 = k_2 \cos(\sqrt{-\lambda} \pi).$$

If $k_2 = 0$, then the solution is the trivial one, so assume that $k_2 \neq 0$. Then it must be the case that

$$\cos(\sqrt{-\lambda} \pi) = 0,$$

which implies that $\sqrt{-\lambda}$ is an odd multiple of $1/2$. That is,

$$\sqrt{-\lambda} \in \left\{ \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \dots \right\} = \left\{ \frac{2n+1}{2} \,\middle|\, n \in \mathbb{Z} \right\}.$$

Isolating λ, this implies that

$$\lambda \in \left\{ \frac{(2n+1)^2}{4} \,\middle|\, n \in \mathbb{Z} \right\} = \left\{ \frac{(2n+1)^2}{4} \,\middle|\, n \in \mathbb{N}_0 \right\},$$

where $\mathbb{N}_0$ denotes the set of natural numbers (including zero). With $k_1 = 0$, any value of k_2 will give rise to an eigenfunction in (1). For the sake of convenience, choose $k_2 = -1$.

The eigenpairs (λ, y_λ) (e.g. the collection of eigenvalues λ with their corresponding eigenfunctions y_λ) of the given boundary value problem are, therefore,

$$\left\{ \left(\lambda_n = \frac{(2n+1)^2}{4}, y_n(x) = \sin\left(\frac{2n+1}{2} x \right) \right) \,\middle|\, n \in \mathbb{N}_0 \right\}.$$

□

Problem 6 (10.1.20)**.** Find the eigenvalues and eigenfunctions of the boundary value problem

$$x^2y'' - xy' + \lambda y = 0; \qquad y(1) = 0, \quad y(L) = 0, \quad L > 1.$$

Assume that all eigenvalues are real.

Solution. Assume that a solution of the form $y(x) = x^\mu$ exists. Then the differential equation becomes

$$\mu(\mu-1)x^\mu - \mu x^\mu + \lambda x^\mu = (\mu(\mu-1) - \mu + \lambda)x^\mu = 0.$$

This is possible only if

$$\mu^2 - 2\mu + \lambda = 0 \implies \mu_{1,2} = 1 \pm \sqrt{1-\lambda}.$$

The corresponding solutions to the differential equation will be different, depending on whether the roots $\mu_{1,2}$ are distinct and whether or not they are real. This leads to three cases:

(i) $\lambda = 1$. In this case, the general solution to the original differential equation is of the form

$$y(x) = k_1x + k_2x\log(x).$$

The boundary values imply that

$$0 = y(1) = k_1 \qquad \text{and} \qquad 0 = k_2L\log(L).$$

As it has been assumed that $L > 0$, the second identity holds only when $k_2 = 0$. This implies that the only solution in this case is the trivial solution, which further implies that $\lambda = 1$ is not an eigenvalue of the original boundary value problem.

(ii) $\lambda < 1$. In this case, the general solution to the original equation is of the form

$$y(x) = k_1\mathrm{e}^{(1+\sqrt{1-\lambda})x} + k_2\mathrm{e}^{(1-\sqrt{1-\lambda})x}.$$

The first boundary value implies that

$$0 = y(1) = k_1\mathrm{e}^{1+\sqrt{1-\lambda}} + k_2\mathrm{e}^{1-\sqrt{1-\lambda}} \implies k_1 = -k_2\mathrm{e}^{-2\sqrt{1-\lambda}},$$

while the second boundary value implies that

$$0 = y(L) = k_1\mathrm{e}^{(1+\sqrt{1-\lambda})L} + k_2\mathrm{e}^{(1-\sqrt{1-\lambda})L} \implies k_1 = -k_2\mathrm{e}^{-2L\sqrt{1-\lambda}}.$$

These two conditions can be satisfied if $L = 1$, but $L > 1$ by hypothesis. The only other solution is $k_1 = k_2 = 0$, which gives rise to the trivial solution. Thus the original boundary value problem has no eigenvalues smaller than 1.

(iii) $\lambda > 1$. In this case, the general solution ot the original differential equation is of the form

$$y(x) = k_1x\cos\left(\sqrt{\lambda-1}\log(x)\right) + k_2x\sin\left(\sqrt{\lambda-1}\log(x)\right). \qquad (2)$$

Note that, via the assumption that $\lambda > 1$, the radicand $\lambda - 1$ is positive, which implies that $\sqrt{\lambda - 1}$ is real. The first boundary value implies that

$$0 = y(1) = k_1,$$

which, when combined with the second boundary value, implies that

$$0 = y(L) = k_2 L \sin\left(\sqrt{\lambda - 1}\log(L)\right).$$

Nontrivial solutions to the original differential equation therefore exist if

$$\begin{aligned}\sin\left(\sqrt{\lambda-1}\log(L)\right) = 0 &\implies \sqrt{\lambda-1}\log(L) = n\pi \qquad (n \in \mathbb{Z})\\ &\implies \lambda - 1 = \left(\frac{n\pi}{\log(L)}\right)^2\\ &\implies \lambda = 1 + \left(\frac{n\pi}{\log(L)}\right)^2.\end{aligned}$$

As $\lambda > 1$, it must be the case that $n \neq 0$. Moreover, negative values of n are redundant (both n and $-n$ give the same eigenvalue). Therefore for each $n \in \mathbb{N}$ (not including zero), the original problem has an eigenvalue of the above form.

In summary, the original boundary problem has eigenvalues

$$\lambda_n = 1 + \left(\frac{n\pi}{\log(L)}\right)^2,$$

where $n \in \mathbb{N}$. Substituting this into the general solution, with $k_1 = 0$ and $k_2 = 1$ (this value is an arbitrary choice), the corresponding eigenfunctions are given by

$$y_n(x) = x\sin\left(\frac{n\pi}{\log(L)}\log(x)\right).$$

Note that the logarithm is the natural logarithm, though, in this case, it doesn't matter (think about the change of base formula for logarithms). □

Definition 1. A function $f : \mathbb{R} \to \mathbb{R}$ (or $\to \mathbb{C}$) is said to be *periodic with period T*, or *T-periodic*, if

$$f(x+T) = f(x)$$

for all $x \in \mathbb{R}$. A function is said to be *periodic* if it is T-periodic for some $T > 0$. The smallest value of $T > 0$ such f is T-periodic (if such a value exists) is called the *fundamental period* of f. In notation, the fundamental period of f is

$$(\text{fundamental period}) = \min\{T > 0 \mid f(x+T) = f(x) \forall x \in \mathbb{R}\}.$$

Note that there exist periodic functions which lack a fundamental period (e.g. any constant function).

Problem 7 (10.2.1)**.** Is the function defined by the formula

$$f(x) = \sin(5x)$$

periodic? If so, what is its fundamental period?

Solution. Yes, this function is periodic with fundamental period $T = 2\pi/5$. Observe that if $x \in \mathbb{R}$, then

$$\begin{aligned} f\left(x + \tfrac{2\pi}{5}\right) &= \sin\left(5\left(x + \tfrac{2\pi}{5}\right)\right) \\ &= \sin(5x + 2\pi) \\ &= \cos(5x)\sin(2\pi) + \sin(5x)\cos(2\pi) \qquad \text{(angle addition formula)} \\ &= \sin(5x) \\ &= f(x). \end{aligned}$$

Thus f is $(2\pi/5)$-periodic. To see that this is the fundamental period, suppose for contradiction that there is some $0 < T < 2\pi/5$ such that f is T-periodic. Then for any $x \in \mathbb{R}$,

$$\begin{aligned} \sin(5x) &= f(x) \\ &= f(x+T) \\ &= \sin(5(x+T)) \\ &= \sin(5x + 5T) \\ &= \cos(5x)\sin(5T) + \sin(5x)\cos(5T). \end{aligned} \tag{3}$$

As this must be true for any $x \in \mathbb{R}$, it must be true for $x = 0$. Hence

$$0 = \sin(5 \cdot 0) = \cos(5 \cdot 0)\sin(5T) + \sin(5 \cdot 0)\cos(5T) = \sin(5T).$$

But $5T \in (0, 2\pi)$, and the equation $\sin(5T) = 0$ has only one solution in that interval, namely $T = \pi/5$. On the other hand, with $x = \pi/10$, the identity at (3) becomes

$$0 = \sin\left(5 \cdot \frac{\pi}{10}\right) = \cos\left(5 \cdot \frac{\pi}{10}\right)\sin(5T) + \sin\left(5 \cdot \frac{\pi}{10}\right)\cos(5T) = \sin\left(\frac{\pi}{2}\right)\cos(5T).$$

But $\cos(5T) = \cos(\pi) = -1$, which implies that

$$0 = \sin\left(\frac{\pi}{2}\right)\cos(5T) = -1.$$

This is a contradiction, which implies that the fundamental period of f cannot be less than $2\pi/5$. □

Problem 8 (10.2.2)**.** Is the function defined by the formula

$$f(x) = \cos(2\pi x)$$

periodic? If so, what is its fundamental period?

Solution. Yes, this function is periodic with fundamental period 1. The argument is nearly identical to that given in the previous exercise. Alternatively, this can be proved via the result given in the Remark following Problem 12. □

Problem 9 (10.2.3)**.** Is the function defined by the formula

$$f(x) = \sinh(2x)$$

periodic? If so, what is its fundamental period?

Solution. No, this function is not periodic. For contradiction, suppose that there is some $T > 0$ such that $f(x + T) = f(x)$ for all $x \in \mathbb{R}$. Then, by definition of the hyperbolic sine function,

$$0 = \sinh(0) = f(0) = f(T) = \sinh(2T) = \frac{1}{2}\left(\mathrm{e}^{2T} - \mathrm{e}^{-2T}\right) = \frac{1}{2}\mathrm{e}^{-2T}\left(\mathrm{e}^{4T} - 1\right).$$

This is possible if and only if

$$\mathrm{e}^{4T} - 1 = 0 \iff 4T = 0 \iff T = 0.$$

This contradicts the hypothesis that $T > 0$, therefore f cannot be periodic. □

Problem 10 (10.2.4)**.** Let $L > 0$. Is the function defined by the formula

$$f(x) = \sin\left(\frac{\pi x}{L}\right)$$

periodic? If so, what is its fundamental period?

Solution. Yes, this function is periodic with fundamental period $2L$. This can be justified via an argument identical to that given in Problem **??**, changing every instance of 5 to π/L. □

Problem 11 (10.2.5)**.** Is the function defined by the formula

$$f(x) = \tan(\pi x)$$

periodic? If so, what is its fundamental period?

Solution. Yes, this function is periodic with fundamental period 1. This conclusion may be justified via the argument presented in the Remark following Problem 12, though there is a small technicality regarding the fact that tan is not defined on all of $\mathbb{R}$.

The solution is to broaden the definition of periodicity to allow functions which are not defined everywhere: let $f : D \to \mathbb{R}$ (where $D \subseteq \mathbb{R}$) and $T > 0$. Then f is T-periodic if

$$f(x) = f(x + T)$$

for all $x \in D$ (i.e. for all x such that f is defined). This implicitly requires that $f(x+T)$ also be defined.

Once this technicality is settled, it can be observed that tan is π-periodic, from which it follows that f is 1-periodic. $\square$

Problem 12 (10.2.6)**.** Is the function defined by the formula

$$f(x) = x^2$$

periodic? If so, what is its fundamental period?

Solution. No, this function is not periodic. For contradiction, suppose that it is T-periodic for some $T > 0$. Then

$$0 = f(0) = f(T) = T^2.$$

But $T > 0$ implies that $T^2 > 0$, which is a contradiction. $\square$

Remark. Problems 7, 8, 10, and 11 suggest a more general theorem: let $a, b \in \mathbb{R}$ with $a > 0$. If a function f is periodic with fundamental period T, then the function g defined by

$$g(x) = f(ax + b)$$

is periodic with fundamental period T/a. To prove this, first observe that if $x \in \mathbb{R}$, then

$$g\left(x + \frac{T}{a}\right) = f\left(a\left(x + \frac{T}{a}\right) + b\right) = f((ax+b)+T) = f(ax+b) = g(x).$$

Therefore g is periodic with period T/a, as claimed. To see that this is the fundamental period of g, suppose for contradiction that there is some $T' > 0$ with $T' < T/a$ such that g is T'-periodic. Then for any $x \in \mathbb{R}$,

$$f(x) = g\left(\frac{1}{a}(x-b)\right) = g\left(\frac{1}{a}(x-b) + T'\right) = f(x + aT').$$

Thus f must be aT'-periodic. But $aT' < T$, which contradicts the hypothesis that the fundamental period of f is T. Therefore the fundamental period of g is T/a, as claimed.

Note that the hypothesis that $a > 0$ is not necessary. If $a < 0$, then g (as defined above) is periodic with fundamental period $T/|a|$. The argument is identical to that given above, with some extra care taken to keep track of signs.

Problem 13 (10.2.13)**.** Let $L > 0$ and define the function $f : [-L, L) \to \mathbb{R}$ by

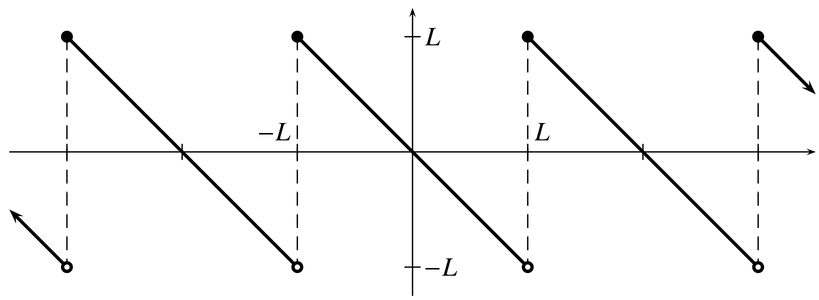
$$f(x) = -x.$$

Extend this function to $\mathbb{R}$ by periodicity. That is, define f on the remainder of $\mathbb{R}$ so that

$$f(x + 2L) = f(x).$$

Sketch a graph of f, and determine the Fourier series expansion of f.

Solution. The graph of the function is given by



The corresponding Fourier series is of the form

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right),$$

where the coefficients are given by

$$a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{m\pi x}{L}\right)\, \mathrm{d}x \qquad \text{and} \qquad b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{m\pi x}{L}\right)\, \mathrm{d}x.$$

First evaluating the a_m, observe that if $m = 0$, then $\cos(m\pi x/L) = 1$. Thus

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, \mathrm{d}x = -\frac{1}{L} \int_{-L}^{L} x\, \mathrm{d}x = -2L \left(\frac{1^2}{2} - \frac{(-1)^2}{2} \right) = 0.$$

For the remaining a_m,

$$\begin{aligned}
a_m &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{m\pi x}{L}\right)\, \mathrm{d}x \\
&= -\frac{1}{L} \int_{-L}^{L} x \cos\left(\frac{m\pi x}{L}\right)\, \mathrm{d}x \\
&= -\frac{1}{L} \left(\left[\frac{Lx}{m\pi} \sin\left(\frac{m\pi x}{L}\right) \right]_{x=-L}^{L} - \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\, \mathrm{d}x \right) && \text{(integration by parts)} \\
&= \frac{1}{L} \left(\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\, \mathrm{d}x \right) \\
&= -\frac{1}{L} \left[\frac{L}{m\pi} \cos\left(\frac{m\pi x}{L}\right) \right]_{x=-L}^{L} \\
&= -\frac{1}{m\pi} \left(\cos(m\pi) - \cos(-m\pi)\right) \\
&= -\frac{1}{m\pi} \left(\cos(m\pi) - \cos(m\pi)\right) && \text{(cosine is even)} \\
&= 0.
\end{aligned}$$

This is, perhaps, the expected result: the function f is an odd function, thus we expect the "even" terms of the Fourier series (i.e. those terms involving cosines) to vanish. Finally, the b_m are given by

$$
\begin{aligned}
b_m &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x \\
&= -\frac{1}{L}\int_{-L}^{L} x\sin\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x \\
&= -\frac{1}{L}\left(-\frac{Lx}{m\pi}\cos\left(\frac{m\pi x}{L}\right)\Big|_{x=-L}^{L} + \int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x\right) && \text{(integration by parts)} \\
&= -\frac{1}{L}\left(-\frac{L^2}{m\pi}\left(\cos(m\pi)-\cos(-m\pi)\right) + \int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\,\mathrm{d}x\right) \\
&= -\frac{1}{L}\left(-\frac{(-1)^m 2L^2}{m\pi} + \left[\frac{L}{m\pi}\sin\left(\frac{m\pi x}{L}\right)\right]_{x=-L}^{L}\right) && (\cos(m\pi) = (-1)^m) \\
&= -\frac{1}{L}\left(-\frac{(-1)^m 2L^2}{m\pi} + \frac{L}{m\pi}\left(\sin(m\pi)-\sin(-m\pi)\right)\right) \\
&= \frac{(-1)^m 2L}{m\pi}
\end{aligned}
$$

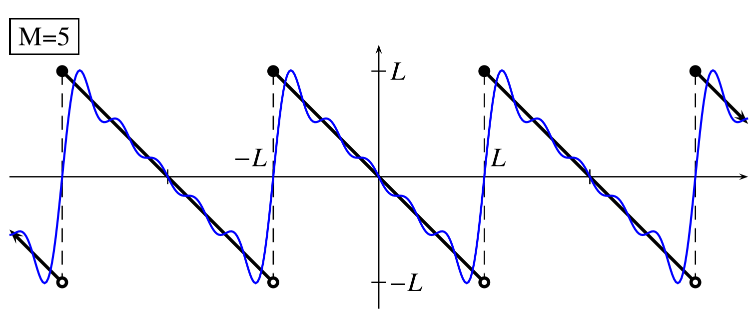
Therefore the Fourier series expansion of f is given by

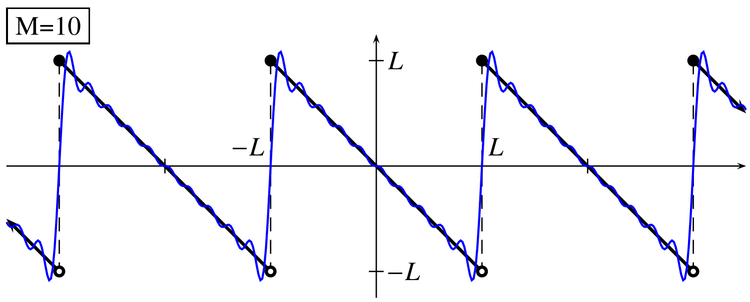
$$
f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m\cos\left(\frac{m\pi x}{L}\right) + b_m\sin\left(\frac{m\pi x}{L}\right)\right) = \frac{2L}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{m}\sin\left(\frac{m\pi x}{L}\right).
$$

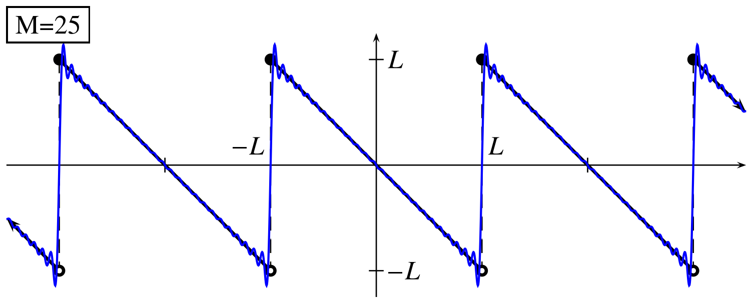
It might be edifying to plot an approximation of this series. For any integer $M > 0$, the function f can be approximated by the first M terms of the series. That is,

$$
f(x) \approx \frac{2L}{\pi}\left(-\sin(\pi x) + \frac{1}{2}\sin(2\pi x) - \frac{1}{3}\sin(3\pi x) + \cdots + \frac{(-1)^M}{M}\sin(M\pi x)\right).
$$

Plots of several approximations are shown below, for various values of M. It is worth noting that the plots appear to converge, though there is something interesting happening near the discontinuities. For the interested reader, the term "Gibbs phenomenon" is worth Googling.







□

Problem 14 (10.2.15)**.** Define the function $f : [-\pi, \pi) \to \mathbb{R}$ by

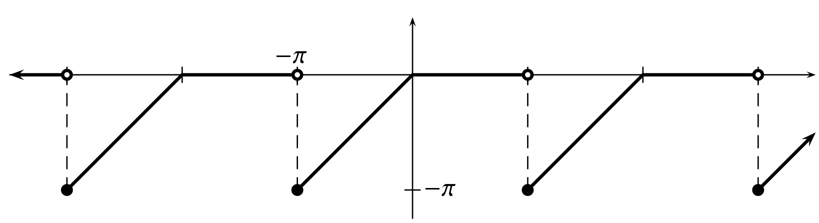
$$f(x) = \begin{cases} x & \text{if } -\pi \le x < 0, \text{ and} \\ 0 & \text{if } 0 \le x < \pi. \end{cases}$$

Extend this function to $\mathbb{R}$ by periodicity. That is, define f on the remainder of $\mathbb{R}$ so that

$$f(x + 2\pi) = f(x).$$

Sketch a graph of f, and determine the Fourier series expansion of f.

Solution. The graph of the function is given by



The "zeroth" Fourier coefficient is given by

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi} \int_{-\pi}^{0} x\,\mathrm{d}x = \frac{1}{\pi} \left(\frac{x^2}{2} \right]_{x=-\pi}^{0} = -\frac{\pi}{2}.$$

The remaining "even" coefficients are

$$
\begin{aligned}
a_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\!\left(\frac{m\pi x}{\pi}\right)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{0} x\cos(mx)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left(\left[\frac{x}{m}\sin(mx)\right]_{x=-\pi}^{0} - \frac{1}{m}\int_{-\pi}^{0}\sin(mx)\,\mathrm{d}x\right) \\
&= -\frac{1}{m\pi}\int_{-\pi}^{0}\sin(mx)\,\mathrm{d}x \\
&= -\frac{1}{m\pi}\left[-\frac{1}{m}\cos(mx)\right]_{x=-\pi}^{0} \\
&= \frac{1}{m^2\pi}\left(\cos(0)-\cos(m\pi)\right) \\
&= \frac{1}{m^2\pi}\left(1-(-1)^m\right) \\
&= \frac{1}{m^2\pi}\left\{\begin{matrix} 2 & \text{if } m \text{ is odd, and} \\ 0 & \text{if } m \text{ is even} \end{matrix}\right\}.
\end{aligned}
$$

Finally, the "odd" coefficients are given by

$$
\begin{aligned}
b_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\!\left(\frac{m\pi x}{\pi}\right)\,\mathrm{d}x \\
&= -\frac{1}{\pi}\int_{-\pi}^{0} x\sin(mx)\,\mathrm{d}x \\
&= \frac{1}{\pi}\left(-\left[\frac{x}{m}\cos(mx)\right]_{x=-\pi}^{0} + \frac{1}{m}\int_{-\pi}^{0}\cos(mx)\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left(-\frac{\pi}{m}\cos(m\pi) + \frac{1}{m}\left[-\frac{1}{m}\sin(mx)\right]_{x=-\pi}^{0}\right) \\
&= -\frac{1}{m}\cos(m\pi) \\
&= \frac{(-1)^{m+1}}{m}.
\end{aligned}
$$

The Fourier series is therefore given by

$$
f(x) = -\frac{\pi}{4} + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m+1)^2}\cos((2m+1)x) + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\sin(mx).
$$

□

Problem 15 (10.3.2)**.** Let $L > 0$ and define the function $f : [-\pi, \pi) \to \mathbb{R}$ by

$$
f(x) = \begin{cases} 0 & \text{if } -\pi \le x < 0, \text{ and} \\ x & \text{if } 0 \le x < \pi. \end{cases}
$$

Extend this function to $\mathbb{R}$ by periodicity. That is, define f on the remainder of $\mathbb{R}$ so that

$$f(x + 2\pi) = f(x).$$

Sketch a graph of f, and determine the Fourier series expansion of f.

Solution. The zeroth Fourier coefficient is given by

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \int_0^{\pi} x\,\mathrm{d}x = \frac{\pi^2}{2}.$$

The remaining "even" coefficients are

$$\begin{aligned} a_m &= \int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x \\ &= \int_0^{\pi} x\cos(mx)\,\mathrm{d}x \\ &= \frac{1}{n^2}\big((-1)^m - 1\big) \\ &= \begin{cases} \dfrac{2}{m^2} & \text{if } m \text{ is odd, and} \\ 0 & \text{if } m \text{ is even.} \end{cases} \end{aligned}$$

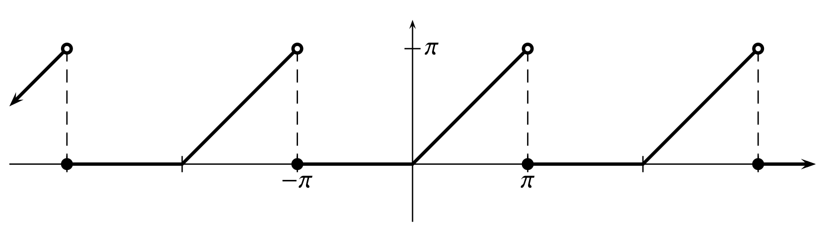
Much of the computation here has been elided as an almost identical computation has already been done (modulo a couple of signs, perhaps) in Problem **??**, above. Similarly, the "odd" coefficients are given by

$$b_m = \frac{(-1)^{m+1}\pi}{m}.$$

Therefore the Fourier series corresponding to f is given by

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{1}{(2m-1)}\cos((2m-1)x) - \sum_{m=1}^{\infty} \frac{(-1)^m}{m}\sin(mx).$$

The graph of this function is given by



Another approach to this problem is to note that the function in this problem may be obtained from the function in Problem 14 via elementary transformations. Specifically, reflect the function in the previous problem across both axes to obtain the function in this problem. More explicitly, define

$$f(x) = \begin{cases} 0 & \text{if } -\pi \le x < 0, \text{ and} \\ x & \text{if } 0 \le x < \pi, \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} x & \text{if } -\pi \le x < 0, \text{ and} \\ 0 & \text{if } 0 \le x < \pi. \end{cases}$$

Hence g is the function from Problem 14. Observe that

$$f(x) = -g(-x);$$

the mapping $x \mapsto -x$ reflects the graph of the function across the x-axis, while the mapping $g(x) \mapsto -g(x)$ reflects the graph across the y-axis. But, from the previous problem, we have

$$g(x) = -\frac{\pi}{4} + \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{1}{(2m+1)^2}\cos((2m+1)x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}\sin(mx).$$

Therefore

$$\begin{aligned} f(x) &= -g(-x) \\ &= -\left[-\frac{\pi}{4} + \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{1}{(2m+1)^2}\cos(-(2m+1)x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}\sin(-mx)\right] \\ &= \frac{\pi}{4} - \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{1}{(2m+1)^2}\cos(-(2m+1)x) - \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}\sin(-mx) \\ &= \frac{\pi}{4} - \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{1}{(2m+1)^2}\cos((2m+1)x) - \sum_{m=1}^{\infty} \frac{(-1)^{m}}{m}\sin(mx), \end{aligned}$$

which is precisely the series given above. $\square$

Problem 16 (10.3.4)**.** Define the function $f : [-1, 1) \to \mathbb{R}$ by

$$f(x) = 1 - x^2$$

Extend this function to $\mathbb{R}$ by periodicity. That is, define f on the remainder of $\mathbb{R}$ so that

$$f(x+2) = f(x).$$

Sketch a graph of f, and determine the Fourier series expansion of f.

Solution. Observe that this function is even: for any $x \in [-1, 1]$,

$$f(-x) = 1 - (-x)^2 = 1 - x^2 = f(x).$$

The periodic extension of the function to the entire real line preserves this parity, hence f is even. This implies that the "odd" Fourier coefficients all vanish; that is, $b_m = 0$ for all m. The zeroth Fourier coefficient is given by

$$a_0 = \int_{-1}^{1} 1 - x^2 \,\mathrm{d}x = \left[x - \frac{1}{3}x^3\right]_{x=-1}^{1} = \left[1 - \frac{1}{3}\right] - \left[-1 + \frac{1}{3}\right] = \frac{4}{3}.$$

The remaining "even" terms are given by

$$
\begin{aligned}
a_m &= \int_{-1}^{1} (1 - x^2) \cos(m\pi x)\, \mathrm{d}x \\
&= 2\int_0^1 \cos(m\pi x)\, \mathrm{d}x - 2\int_0^1 x^2 \cos(m\pi x)\, \mathrm{d}x \qquad (4) \\
&= \left[\frac{2}{m\pi}\sin(m\pi x)\right]_0^1 - 2\left(\left[\frac{x^2}{m\pi}\sin(m\pi x)\right]_{x=0}^1 - \frac{2}{m\pi}\int_0^1 x\sin(m\pi x)\, \mathrm{d}x\right) \\
&= 0 - 2\left(0 - \frac{2}{m\pi}\left(\left[-\frac{x}{m\pi}\cos(m\pi x)\right]_{x=0}^1 + \frac{1}{m\pi}\int_0^1 \cos(m\pi x)\, \mathrm{d}x\right)\right) \\
&= \left[-\frac{4x}{(m\pi)^2}\cos(m\pi x)\right]_{x=0}^1 + \frac{4}{(m\pi)^2}\int_0^1 \cos(m\pi x)\, \mathrm{d}x \\
&= -\frac{4}{(m\pi)^2}(-1)^m + 0 \\
&= (-1)^{m+1}\frac{4}{(m\pi^2)}.
\end{aligned}
$$

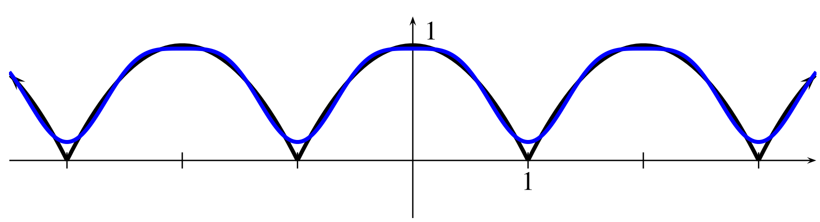
At (4), we are taking advantage of a property of even functions. Specifically, if g is an even function, then

$$\int_{-a}^{a} g(x)\, \mathrm{d}x = 2\int_0^a g(x)\, \mathrm{d}x.$$

The Fourier series is given by

$$f(x) = \frac{2}{3} + \frac{4}{\pi^2}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^2}\cos(m\pi x).$$

A graph of the function (in black), as well as a Fourier series approximation with (just!) three terms (in blue), is given by



$\square$