Math 146B.001 (Ordinary and Partial Differential Equations)
Homework 04 Solutions

Important Power Series

It is worth recalling a few important power series expansions (including those discussed above):

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n,$$
$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \qquad \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}.$$

These identities will be used without further comment in the sequel. These identies are not tremendously difficult to prove—each is a Taylor series expansion of the given function around the point $x_0 = 0$—but the derivations are a little tedious, and not an essential part of this course. At this point in your mathematical career, you should probably *recognize* the series presented above (as well as a few others), but you need not have them *memorized*—you should be able to easily look them up if you need them.

Problem 1 (5.2.1)**.** Solve the differential equation

$$y'' - y = 0$$

around the point $x_0 = 0$ using power series techniques. Find a recurrence relation for the coefficients in the power series, give the first four terms and, if possible, find the general term.

Solution. We seek solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

where $\{a_n\}$ is a sequence of real numbers. Assuming that a solution of this form exists,

$$\begin{aligned}
0 &= y'' - y \\
&= \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} - \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=-2}^{\infty} a_{n+2}(n+2)(n+1) x^n - \sum_{n=0}^{\infty} a_n x^n && \text{(reindex the first series)} \\
&= \sum_{n=0}^{\infty} \left(a_{n+2}(n+2)(n+1) - a_n\right) x^n.
\end{aligned}$$

Note that in the last line of this computation, the terms corresponding to $n = -2$ and $n = -1$ drop out, as they include the factors $n + 2 = 0$ and $n + 1 = 0$, respectively. This series is zero if and only if the coefficient of each term is zero, which implies that

$$a_{n+2}(n+2)(n+1) + a_n = 0 \implies a_{n+2} = -\frac{1}{(n+1)(n+2)}a_n. \tag{1}$$

This recurrence relation gives two solutions: one which corresponds to choosing a value for a_0 (and taking $a_1 = 0$), and a second which corresponds to choosing a value of a_1 (and taking $a_0 = 0$). If a_0 is fixed, then the first four terms of the recurrence relation are given by

$$\begin{aligned}
a_2 &= \frac{1}{1 \cdot 2}a_0 = \frac{1}{2!}a_0, && (n = 0)\\
a_4 &= \frac{1}{3 \cdot 4}a_2 = \frac{1}{4!}a_0, && (n = 2)\\
a_6 &= \frac{1}{5 \cdot 6}a_4 = \frac{1}{6!}a_0, && (n = 4)
\end{aligned}$$

where the parenthetical indicates the value of n in the recurrence relation (1). It can be shown via an induction argument that if n is even, then

$$a_n = \frac{1}{n!}a_0$$

Therefore one solution to the original differential equation is given by

$$y_1(x) = \sum_{n \text{ even}} \frac{1}{n!}a_0 x^n = a_0 \sum_{m=0}^{\infty} \frac{1}{(2m)!}x^{2m},$$

where a_0 is an arbitrarily chosen constant. With a_1 fixed, the first four terms of the recurrence relation are given by

$$\begin{aligned}
a_3 &= \frac{1}{2 \cdot 3}a_1 = \frac{1}{3!}a_1, && (n = 1)\\
a_5 &= \frac{1}{4 \cdot 5}a_3 = \frac{1}{5!}a_1, && (n = 3)\\
a_7 &= \frac{1}{6 \cdot 7}a_5 = \frac{1}{7!}a_1. && (n = 5)
\end{aligned}$$

Again, via an induction argument, the general term (with n odd) is given by

$$a_n = \frac{1}{n!}a_1.$$

Hence a second solution to the original differential equation is given by

$$y_2(x) = \sum_{n \text{ odd}} \frac{1}{n!}a_1 x^n = a_1 \sum_{m=0}^{\infty} \frac{1}{(2m+1)!}x^{2m+1}$$

While this isn't specifically asked for in the statement of the question, it might be work spending some time to determine whether or not these series solutions have "nice" closed forms. In particular, using the techniques from Section 4.3 of the course text, we might expect solutions

$$\tilde{y}_1(x) = \mathrm{e}^x \qquad \text{and} \qquad \tilde{y}_2(x) = \mathrm{e}^{-x}.$$

Do our series solutions give the same solution space?

As the original differential equation is linear, linear combinations of solutions are also solutions. Hence another solution to the original differential equation is given by

$$\frac{1}{a_0}y_1(x) + \frac{1}{a_1}y_2(x) = \sum_{n \text{ even}} \frac{1}{n!}x^n + \sum_{n \text{ odd}} \frac{1}{n!}x^n = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = \mathrm{e}^x = \tilde{y}_1(x).$$

Similarly,

$$\begin{aligned} \frac{1}{a_0}y_1(x) - \frac{1}{a_1}y_2(x) &= \sum_{n \text{ even}} \frac{1}{n!}x^n - \sum_{n \text{ odd}} \frac{1}{n!}x^n \\ &= \sum_{n \text{ even}} (-1)^n \frac{1}{n!}x^n + \sum_{n \text{ odd}} (-1)^n \frac{1}{n!}x^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(-x)^n \\ &= \mathrm{e}^{-x} = \tilde{y}_2(x). \end{aligned}$$

Thus the solutions obtained via power series techniques give us exactly the same solution space as the solutions obtained via eigenvalue / eigenfunction arguments. Finally, observe that (with $a_0 = a_1 = 1$)

$$y_1(x) + y_2(x) = \mathrm{e}^x \qquad \text{and} \qquad y_1(x) - y_2(x) = \mathrm{e}^{-x}.$$

Adding these two equations together renders

$$2y_1(x) = \mathrm{e}^x + \mathrm{e}^{-x} \implies y_1(x) = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} = \cosh(x),$$

and subtracting the second equation from the first renders

$$2y_2(x) = \mathrm{e}^x - \mathrm{e}^{-x} \implies y_2(x) = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} = \sinh(x).$$

Therefore

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n} \qquad \text{and} \qquad \sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n+1}.$$

These two power series are also useful to have in your quiver for future work. That is, if you encounter one of these power series, you should recognize that it is one you have seen before, and you should have a handy place to go look it up (e.g. WolframAlpha, Wikipedia, etc.). □

Problem 2 (5.2.2)**.** Solve the differential equation

$$y'' - xy' - y = 0$$

around the point $x_0 = 0$ using power series techniques. Find a recurrence relation for the coefficients in the power series, give the first four terms and, if possible, find the general term.

Solution. Again, a series solution of the form $y(x) = \sum a_n x^n$, when substituted into the differential equation, gives

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - x\sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0} a_n x^n \\
&= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} \left(a_{n+2}(n+2)(n+1) - (n+1)a_n\right) x^n.
\end{aligned}$$

For this to hold, it must be the case that

$$a_{n+2}(n+2)(n+1) - (n+1)a_n = 0 \implies a_{n+2} = \frac{1}{n+2}a_n.$$

Taking a_0 arbitrary and $a_1 = 0$, this implies that

$$a_2 = \frac{1}{2}a_0, \qquad a_4 = \frac{1}{2\cdot 4}a_0, \qquad a_6 = \frac{1}{2\cdot 4\cdot 6}a_0.$$

The general form of this sequence can be obtained by noticing that each of the $n/2$ terms in the denominator contains a factor of 2. Factoring this out gives

$$a_n = \frac{1}{2\cdot 4\cdot\cdots\cdot(n-2)\cdot n}a_0 = \frac{1}{2^{n/2}(1\cdot 2\cdot\cdots\cdot\frac{n}{2})}a_0 = \frac{1}{2^{n/2}(\frac{n}{2})!}a_0.$$

This gives a solution to the original differential equation of the form

$$y_1(x) = \sum_{n\text{ even}} \frac{1}{2^{n/2}(\frac{n}{2})!}a_0 x^n = a_0\sum_{m=0}^{\infty}\frac{1}{2^m m!}x^{2m} = a_0\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{x^2}{2}\right)^m = a_0 e^{x^2/2}.$$

It can be quickly verified (either by hand computation or with the aid of a computer algebra system) that this is, indeed a solution to the original differential equation.

Now, with $a_0 = 0$ and a_1 an arbitrary constant,

$$a_3 = \frac{1}{3}a_1, \qquad a_5 = \frac{1}{3\cdot 5}a_1, \qquad a_7 = \frac{1}{3\cdot 5\cdot 7}a_1.$$

In general, when n is odd, the coefficient a_n is given by

$$a_n = \frac{1}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (n-2) \cdot n} a_1.$$

The fraction can be manipulated into something a bit more tractable by multiplying upstairs and downstairs by the "missing" even terms:

$$\frac{1}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot n} a_1 = \frac{2 \cdot 4 \cdot \cdots \cdot (n-1)}{\big(2 \cdot 4 \cdot \cdots \cdot (n-1)\big)\big(1 \cdot 3 \cdot 5 \cdot \cdots \cdot n\big)} a_1 = \frac{2^{(n-1)/2}(\frac{n-1}{2})!}{n!} a_1.$$

Therefore a second solution to the original differential equation is given by

$$y_2(x) = \sum_{n \text{ odd}} \frac{2^{(n-1)/2}(\frac{n}{2})!}{n!} a_1 x^n = a_1 \sum_{m=0}^{\infty} \frac{2^m m!}{(2m+1)!} x^n.$$

This particular power series does not have a "nice" closed form in terms of elementary functions. If we use a computer algebra system to find a closed form, we might get something like

$$\sum_{m=0}^{\infty} \frac{2^m m!}{(2m+1)!} x^m) = \frac{\sqrt{\frac{\pi}{2}} e^{x/2} \operatorname{erf}\left(\frac{\sqrt{x}}{\sqrt{2}}\right)}{\sqrt{x}},$$

where the erf function (the "error function"), which comes from integrating a Gaussian curve (specifically, the standard normal distribution from probability), and which has no closed form in terms of elementary functions. On the bright side, the power series can be used to find numerical approximations of arbitrary precision. □

Problem 3 (5.2.3)**.** Solve the differential equation

$$y'' - xy' - y = 0 = 0$$

around the point $x_0 = 1$ using power series techniques. Find a recurrence relation for the coefficients in the power series, give the first four terms and, if possible, find the general term.

Solution. Solutions of the form

$$\sum_{n=0}^{\infty} a_n (x-1)^n$$

are sought. To simplify notation a little, write $t = x - 1$, and observe that $x = t + 1$.

Then, substituting this into the original differential equation gives

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} a_n n(n-1)(x-1)^{n-2} - x\sum_{n=0}^{\infty} a_n n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n \\
&= \sum_{n=0}^{\infty} a_n n(n-1)t^{n-2} - (t+1)\sum_{n=0}^{\infty} a_n n t^{n-1} - \sum_{n=0}^{\infty} a_n t^n \\
&= \sum_{n=0}^{\infty} a_n n(n-1)t^{n-2} - \sum_{n=0}^{\infty} a_n n t^{n} - \sum_{n=0}^{\infty} a_n n t^{n-1} - \sum_{n=0}^{\infty} a_n t^n \\
&= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)t^{n} - \sum_{n=0}^{\infty} a_{n+1}(n+1) t^{n} - \sum_{n=0}^{\infty} a_n (n+1) t^n \\
&= \sum_{n=0}^{\infty} \big(a_{n+2}(n+2)(n+1) - a_{n+1}(n+1) - a_n(n+1)\big)t^n.
\end{aligned}$$

This is possible if and only if each coefficient is zero—that is, for any $n \geq 0$,

$$a_{n+2}(n+2)(n+1) - a_{n+1}(n+1) - a_n(n+1) = 0 \implies a_{n+2} = \frac{a_{n+1} + a_n}{n+2}.$$

There is a solution corresponding to $a_1 = 0$ and an arbitrary choice of a_0, and a second solution corresponding to $a_0 = 0$ and an arbitrary choice of a_1. If these two solutions are linearly independent, then they form a fundamental set of solutions. With $a_1 = 0$, the first several terms of the sequence $\{a_n\}$ are given by

$$a_0 = a_0, \quad a_1 = 0, \quad a_2 = \frac{1}{2}a_0, \quad a_3 = \frac{1}{6}a_0, \quad a_4 = \frac{1}{6}a_0, \quad a_5 = \frac{1}{15}a_0, \quad \ldots .$$

There is not an obvious pattern here, thus we remain content with giving the solution as

$$y_1(x) = a_0\left(1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \frac{1}{15}(x-1)^5 + \cdots\right).$$

With $a_0 = 0$, the first several terms of the sequence $\{a_n\}$ are given by

$$a_0 = 0, \quad a_1 = a_1, \quad a_2 = \frac{1}{2}a_1, \quad a_3 = \frac{1}{2}a_1, \quad a_4 = \frac{1}{4}a_1, \quad a_5 = \frac{3}{20}a_1, \quad \ldots .$$

Again, there is no obvious pattern, so we must be content with the solution

$$y_2(x) = a_1\left((x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \frac{3}{20}(x-1)^5 + \cdots\right).$$

□

Problem 4 (5.2.4)**.** Let $k \in \mathbb{R}$ be a fixed constant. Solve the differential equation

$$y'' + k^2x^2y = 0$$

around the point $x_0 = 0$ using power series techniques. Find a recurrence relation for the coefficients in the power series, give the first four terms and, if possible, find the general term.

Solution. A power series solution satisfies

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + k^2 x^2 \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} k^2 a_n x^{n+2} \\
&= \sum_{n=-2}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=2}^{\infty} k^2 a_{n-2} x^n \\
&= 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} \big(a_{n+2}(n+2)(n+1) + k^2 a_{n-2}\big)x^n.
\end{aligned}$$

This occurs if and only if

$$a_2 = a_3 = 0 \qquad \text{and} \qquad a_{n+4} = -\frac{k^2}{(n+3)(n+4)} a_n \text{ for all } n \geq 4.$$

Note that the recurrence relation follows from the fact that

$$a_{n+2}(n+1)(n+2)k^2 a_{n-2} = 0 \implies a_{n+2} = -\frac{k^2}{(n+1)(n+2)} a_{n-2},$$

and the transformation $n \mapsto n+2$. Since a_0, it follows that

$$a_6 = -\frac{k^2}{30} a_2 = 0, \qquad a_8 = -\frac{k^2}{90} a_4 = 0, \qquad \ldots.$$

Indeed, $a_n = 0$ for any integer which is two more than a multiple of 4. In other words, if $n = 2 \pmod 4$, then $a_n = 0$. This can be proved by induction, though it is straightforward to do so, hence the proof is omitted. By similar arguments, $a_n = 0$ whenever $n = 3 \pmod 4$ (that is, $n = 4m + 3$ for some integer m). Further solutions may be found by taking a_0 and a_1 to be arbitrary.

With $a_1 = 0$, the first several terms of the sequence $\{a_n\}$ (for $n \geq 2$) are given by

$$a_0 = a_0, \quad a_4 = -\frac{k^2}{3 \cdot 4} a_0, \quad a_8 = \frac{k^4}{3 \cdot 4 \cdot 7 \cdot 8} a_0, \quad a_{12} = -\frac{k^6}{3 \cdot 4 \cdot 7 \cdot 8 \cdot 11 \cdot 12} a_0,$$

and so on. By an induction argument (and by making the substitution $n = 4m$)

$$a_{4m} = (-1)^m \frac{k^{2m}}{\prod_{j=0}^{m-1}(4j+3)(4j+4)} a_0,$$

where the "large Π" product notion is like the large Σ notation for a sum:

$$\prod_{j=0}^{m-1}(4j+3)(4j+4) = 3 \cdot 4 \cdot 7 \cdot 8 \cdot \cdots \cdot (4m-1) \cdot 4m.$$

To keep the notation consistent, define the "empty product" to be 1. Hence when $m = 0$, the product is given by

$$\prod_{j=0}^{-1}(4j+3)(4j+4) = 1.$$

This gives rise to the solution

$$\begin{aligned} y_1(x) = a_0 \sum_{m=0}^{\infty} a_{4m}x^{4m} &= a_0 \sum_{m=0}^{\infty} (-1)^m \frac{k^{2m}}{\prod_{j=0}^{m-1}(4j+3)(4j+4)} x^{4m} \\ &= a_0 \sum_{m=0}^{\infty} \frac{(-k^2x^4)^m}{\prod_{j=0}^{m-1}(4j+3)(4j+4)}. \end{aligned}$$

Similarly, with $a_0 = 0$ and a_1 an arbitrary constant,

$$a_1 = a_1, \quad a_5 = -\frac{k^2}{4 \cdot 5} a_1, \quad a_9 = \frac{k^4}{4 \cdot 5 \cdot 8 \cdot 9} a_1, \quad a_{13} = -\frac{k^6}{4 \cdot 5 \cdot 8 \cdot 9 \cdot 12 \cdot 13} a_1,$$

and so on. Making the substitution $n = 4m + 1$,

$$a_{4m+1} = (-1)^m \frac{k^{2m}}{\prod_{j=0}^{m-1}(4m+4)(4m+5)} a_1.$$

This gives the second solution

$$y_2(x) = a_1 x \sum_{m=0}^{\infty} \frac{(-k^2x^4)^m}{\prod_{j=0}^{m-1}(4m+4)(4m+5)}.$$

Neither of these solutions has an obvious closed form. □

Problem 5 (5.2.13)**.** Solve the differential equation

$$2y'' + xy' + 3y = 0$$

around the point $x_0 = 0$ using power series techniques. Find a recurrence relation for the coefficients in the power series, give the first four terms and, if possible, find the general term.

Solution. Substituting a power series solution into this differential equation gives

$$\begin{aligned} 0 &= 2 \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + x \sum_{n=0}^{\infty} a_n n x^{n-1} + 3 \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} 2a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} 3a_n x^n \\ &= \sum_{n=0}^{\infty} \big(2a_{n+2}(n+1)(n+2) + a_n(n+3)\big)x^n. \end{aligned}$$

This is possible if and only if

$$a_{n+2} = -\frac{n+3}{2(n+1)(n+2)} a_n$$

for all $n \geq 2$, where a_0 and a_1 may be chosen arbitrarily. There are two solutions, the first several terms of each are given by

$$\begin{aligned} y_1(x) &= a_0 \left(1 - \frac{3}{4}x^2 + \frac{5}{32}x^4 - \frac{7}{384}x^6 + \frac{3}{2048}x^8 + \cdots \right) \\ y_2(x) &= a_1 \left(x - \frac{1}{3}x^3 + \frac{1}{20}x^5 - \frac{1}{210}x^7 + \frac{1}{3024}x^9 + \cdots \right). \end{aligned}$$

There is no clear pattern in either series solution, and no obvious closed form for the coefficients. It is, however, worth noting that WolframAlpha does come up with a closed form in terms of the Gamma function, which is a continuous extension of the factorial function to the complex plane. □