MATH 146B.001 (ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS)
HOMEWORK 06 SOLUTIONS

The Method of Frobenius

Define the differential operator L by

$$L[y] = (x - x_0)^2 y'' + (x - x_0)[(x - x_0)p(x)]y' + [(x - x_0)^2 q(x)]y,$$

where p and q are "nice" functions (e.g. functions which are analytic). This operator is similar to the operator in the Euler equation, but involves non-constant coefficient functions in terms of p and q, rather than constants. Near ordinary points, it is reasonable to seek power series solutions using the usual techniques, as described in previous sections.

Near regular singular points, things can be little more complicated. It is still reasonable to seek power series solutions, but by making the observation that if x is very close to a regular singular point x_0, then both $xp(x)$ and $x^2q(x)$ will be very nearly constant, and so solutions will behave like solutions to the Euler equation. Thus solutions of the slightly modified form

$$y(x) = (x - x_0)^\lambda \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{1}$$

are sought, were $a_0 \neq 0$ and λ is an unknown constant which will be determined through computation.

Let p_j and q_j denote the j-th terms in the Taylor series expansions of $xp(x)$ and $x^2q(x)$ near x_0, respectively. Note that, under the hypothesis that x_0 is a regular singular point,

$$\lim_{x \to x_0} (x - x_0)p(x - x_0) = p_0 \qquad \text{and} \qquad \lim_{x \to x_0} (x - x_0)^2 q(x - x_0) = q_0.$$

Note that the existence of these two limits is equivalent to the conclusion that x_0 is a regular singular point. Assuming that such a solution exists, the differential equation $L[y] = 0$ expands to

$$0 = a_0 F(\lambda)(x - x_0)^\lambda + \sum_{n=1}^{\infty} \left[a_n F(\lambda + n) + \sum_{k=0}^{n-1} a_k [(\lambda + n)p_{n-k} + q_{n-k}\right] (x - x_0)^{\lambda + n}, \tag{2}$$

where F is the *indicial polynomial*

$$F(\lambda) = \lambda(\lambda - 1) + p_0 \lambda + q_0.$$

A complete derivation of this expansion involves taking the product of series, and is outlined in the course text on pages 288–9. As $a_0 \neq 0$, it follows that a modified power series solution of the form given in (1) exists only if $F(\lambda) = 0$. Thus each root of the indicial polynomial corresponds to a power series solution to the original differential equation. These roots are called the Frobenius indices or the *exponents at the singularity*.

Once these exponents have been determined, the remaining terms in the modified power series expansion can be found via the techniques developed in previous sections: the original assumption was that solutions of the form

$$y(x) = |x - x_0|^\lambda \sum_{n=0}^{\infty} a_n (x - x_0)^n = |x - x_0|^\lambda \left[1 + \sum_{n=0}^{\infty} a_n(\lambda) x^n \right] \tag{3}$$

exist, where it has been further assumed that $a_0 = 1$ and the notation has been modified to match the observation that the n-th coefficient of the power series expansion depends on λ (i.e. write $a_n = a_n(\lambda)$). Note that if F has repeated roots or if the difference between the two roots is an integer, then one solution will be of the form given above, and a second solution can be found via a modification described in Theorem 5.6.1 in the text.

Problem 1 (5.6.1)**.** Find all of the the regular singular points of

$$xy'' + 2xy' + 6e^x y = 0.$$

Determine the indicial equation and the exponents at the singularity for each regular singular point.

Solution. This equation has a single singular point at $x_0 = 0$. Multiplying through by x, this differential equation is equivalent to

$$x^2 y'' + 2x^2 + 6xe^x y = 0.$$

In the language and notation of the introductory discussion,

$$xp(x) = 2x \qquad \text{and} \qquad x^2 q(x) = 6xe^x.$$

Hence

$$p_0 = \lim_{x\to 0} 2x = 0 \qquad \text{and} \qquad q_0 = \lim_{x\to 0} 6xe^x = 0.$$

As both of these limits exist, the singular point at $x_0 = 0$ is a regular singular point. The indicial polynomial is then given by

$$F(\lambda) = \lambda(\lambda - 1),$$

which has roots $\lambda_{1,2} = 0, 1$. □

Problem 2 (5.6.3)**.** Find all of the the regular singular points of

$$x(x-1)y'' + 6x^2 y' + 3y = 0.$$

Determine the indicial equation and the exponents at the singularity for each regular singular point.

Solution. This equation has singular points at $x_0 = 0$ and $x_1 = 1$. Near $x_0 = 0$, this equation can be rewritten as

$$x^2y'' + \frac{6x^3}{x-1} + \frac{3x}{x-1}y = 0.$$

Thus

$$xp(x) = \frac{6x^2}{x-1} \xrightarrow{x\to 0} 0 = p_0 \qquad \text{and} \qquad x^2q(x) = \frac{3x}{x-1} \xrightarrow{x\to 0} 0 = q_0.$$

As these limits exist, the singular point $x_0 = 0$ is a regular singular point. Near this singular point, the indicial polynomial is given by

$$F(\lambda) = \lambda(\lambda - 1),$$

which has roots $\lambda_{1,2} = 0, 1$.

Near $x_1 = 1$, the original differential equation can be rewritten as

$$(x-1)^2y'' + 6x(x-1)y' + \frac{3(x-1)}{x}y = 0.$$

Thus

$$(x-1)p(x) = 6x \xrightarrow{x\to 1} 6 = p_0 \qquad \text{and} \qquad (x-1)^2q(x) = \frac{3(x-1)}{x} \xrightarrow{x\to 1} 0 = q_0.$$

As these limits exist, the singular point $x_1 = 1$ is a regular singular point. Near this singular point, the indicial polynomial is given by

$$F(\lambda) = \lambda(\lambda - 1) + 6\lambda = \lambda(\lambda + 5),$$

which has roots $\lambda_{1,2} = 0, -5$. ◻

Problem 3 (5.6.5)**.** Find all of the the regular singular points of

$$x^2y'' + 3\sin(x)y' - 2y = 0.$$

Determine the indicial equation and the exponents at the singularity for each regular singular point.

Solution. This equation has a singular point at $x_0 = 0$. Near this singular point

$$xp(x) = \frac{3\sin(x)}{x} \xrightarrow{x\to 0} 3 = p_0 \qquad \text{and} \qquad x^2q(x) = -2 \xrightarrow{x\to 0} -2 = q_0.$$

As these limits exist, the singular point $x_0 = 0$ is a regular singular point. Near this singular point, the indicial polynomial is given by

$$F(\lambda) = \lambda(\lambda - 1) + 3\lambda - 2 = \lambda^2 + 2\lambda - 2.$$

This does not obviously factor in a "nice" way, hence the quadratic formula gives the exponents at the singularity as

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} = \frac{-2 \pm \sqrt{12}}{2} = -1 \pm \sqrt{3}.$$

□

Problem 4 (5.6.13)**.** Show that $x_0 = 0$ is a regular singular point of the differential equation

$$xy'' + y' - y = 0.$$

Find the exponents at the singular point $x_0 = 0$, then determine the first three nonzero terms in each of two linearly lindependent solutions power series solutions near $x_0 = 0$.

Solution. This equation can be rewritten as

$$x^2y'' + xy' - xy = 0.$$

In the context of the discussion on generalized Euler equations,

$$xp(x) = 1 \xrightarrow{x\to 0} 1 = p_0 \qquad \text{and} \qquad x^2q(x) = -x \xrightarrow{x\to 0} 0 = q_0.$$

As these limits exist, the singular point $x_0 = 0$ is a regular singular point. The indicial polynomial is given by

$$F(\lambda) = \lambda(\lambda - 1) + \lambda = \lambda^2,$$

which has the repeated root $\lambda_{1,2} = 0$. Thus two linearly independent solutions are of the form

$$y_1(x) = 1 + \sum_{n=1}^{\infty} a_n(0)x^n \qquad \text{and} \qquad y_2(x) = y_1(x)\log|x| + \sum_{n=1}^{\infty} b_n(0)x^n.$$

As y_1 solves the original differential equation,

$$\begin{aligned}
0 &= xy_1''(x) + y_1'(x) - y(x) \\
&= x\sum_{n=1}^{\infty} a_n(0)n(n-1)x^{n-2} + \sum_{n=1}^{\infty} a_n(0)nx^{n-1} - 1 - \sum_{n=1}^{\infty} a_n(0)x^n \\
&= -1 + \sum_{n=1}^{\infty} a_n(0)n(n-1)x^{n-1} + \sum_{n=1}^{\infty} a_n(0)nx^{n-1} - \sum_{n=2}^{\infty} a_{n-1}(0)x^{n-1} \\
&= -1 + a_1(0) + \sum_{n=2}^{\infty} \left[a_n(0)(n(n-1) + n) + a_{n-1}(0)\right] x^{n-1}.
\end{aligned}$$

As the constant terms must add to zero, $a_1(0) = 1$. The remaining coefficients must also be zero, and so

$$a_n(0)(n(n-1) + n) - a_{n-1}(0) = 0 \implies a_n(0) = \frac{a_{n-1}(0)}{n^2}.$$

The first several terms are given by

$$a_0(0) = 1, \qquad a_1(0) = 1, \qquad a_2(0) = \frac{1}{4}, \qquad a_3(0) = \frac{1}{4 \cdot 9}, \qquad \ldots .$$

It is, perhaps, worth noting that the general term is

$$a_n(0) = \frac{1}{(n!)^2},$$

though it is not immediately clear how this might be useful. In any event, the first solution is therefore

$$y_1(x) = 1 + x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \frac{1}{900}x^4 + \cdots .$$

Substituting the second solution into the original differential equation gives

$$\begin{aligned}
&xy_2''(x) + y_2'(x) - y_2(x) \\
&= x\left(\log|x|y_1''(x) + \frac{2}{x}y_1'(x) - \frac{1}{x^2}y_1(x)\right) + \left(\log|x|y_1'(x) + \frac{1}{x}y_1(x)\right) - \log|x|y_1(x) \\
&\qquad\qquad + x\sum_{n=1}^{\infty} b_n(0)n(n-1)x^{n-2} + \sum_{n=1}^{\infty} b_n(0)nx^{n-1} - \sum_{n=1}^{\infty} b_n(0)x^n \\
&= \log|x| \overbrace{\big(xy_1''(x) + y_1'(x) - y_1(x)\big)}^{=0} + 2y_1'(x) \\
&\qquad\qquad + \sum_{n=1}^{\infty} b_n(0)n(n-1)x^{n-1} + \sum_{n=1}^{\infty} b_n(0)nx^{n-1} - \sum_{n=2} b_{n-1}(0)x^{n-1} \\
&= 2\sum_{n=0}^{\infty} a_n(0)nx^{n-1} + \sum_{n=1}^{\infty} b_n(0)n(n-1)x^{n-1} + \sum_{n=1}^{\infty} b_n(0)nx^{n-1} - \sum_{n=2} b_{n-1}(0)x^{n-1} \\
&= 2a_1(0) + b_1(0) + \sum_{n=2}^{\infty}\big[2a_n(0) + b_n(0)n(n-1) + b_n(0)n - b_{n-1}(0)\big]x^{n-1} \\
&= \big(2a_1(0) + b_1(0)\big) + \sum_{n=2}\big[b_n(0)n^2 - b_{n-1}(0) + 2na_n(0)\big]x^{n-1}
\end{aligned}$$

As all of the coefficients must be zero, the constants, in particular, must sum to zero. Thus

$$b_1(0) = -2a_1(0) = -2.$$

For the remaining coefficients, observe that

$$b_n(0)n^2 - b_{n-1}(0) + 2na_n(0) = 0 \implies b_n(0) = \frac{b_{n-1}(0) - 2na_n(0)}{n^2}.$$

Hence

$$b_1(0) = -2, \qquad b_2(0) = -\frac{3}{4}, \qquad b_3(0) = -\frac{11}{108}, \qquad b_4(0) = -\frac{25}{3456}, \qquad \ldots .$$

There does not seem to be any “nice” closed form for this sequence. Therefore, with y_1 as determined above, a second solution to the original differential equation is given by

$$y_2(x) = \log|x| y_1(x) - 2x - \frac{3}{4}x^2 - \frac{11}{108}x^3 - \frac{25}{3456}x^4 + \cdots.$$

Note that this solution is linearly independent from the previous solution. □

Problem 5 (5.6.14)**.** Show that $x_0 = 0$ is a regular singular point of the differential equation

$$xy'' + 2xy' + 6e^x y = 0.$$

Find the exponents at the singular point $x_0 = 0$, then determine the first three nonzero terms in each of two linearly lindependent solutions power series solutions near $x_0 = 0$.

Solution. The first part of this problem was completed above, in Problem 1. In that problem, it was determined the differential equation has a regular singularity at $x_0 = 0$, with the exponents at the singularity given by $\lambda_{1,2} = 1, 0$. As the difference between these two indices is an integer, Theorem 5.6.1 in the text suggests that there are solutions of the form

$$y_1(x) = |x| \left[1 + \sum_{n=1}^{\infty} a_n x^n\right] \qquad \text{and} \qquad y_2(x) = b y_1(x) \log|x| + \left[1 + \sum_{n=1}^{\infty} b_n x^n\right]$$

The first solution corresponds to the Frobenius index $\lambda_1 = 1$ with the choice $a_0 = 1$, while the second solution corresponds to the Frobenius index $\lambda_2 = 0$ with the choice $b_0 = 1$, and b is a constant to be determined. Per equation (8) on page 289 of the text, the a_j are given by the recurrence relation

$$F(1+n)a_n + \sum_{k=0}^{n-1} a_k \left[(1+k)p_{n-k} + q_{n-k}\right] = 0, \qquad (n \geq 1) \tag{4}$$

where F is the Frobenius polynomial, and p_k and q_k are the k-th terms in the Taylor series expansions of $xp(x) = 2x$ and $x^2 q(x) = 6xe^x$, respectively. The power series expansion of $xp(x)$ about zero is given by

$$xp(x) = 0 + 2x + 0x^2 + 0x^3 + \cdots \implies p_k = \begin{cases} 2 & \text{if } k = 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Via the Taylor series expansion of the exponential function, a power series expansion of $x^2 q(x)$ about zero is given by

$$x^2 q(x) = 6xe^x = 6x \sum_{k=0}^{\infty} \frac{1}{k!} x^k = \sum_{k=0}^{\infty} \frac{6}{k!} x^{k+1} = \sum_{k=1}^{\infty} \frac{6}{(k-1)!} x^k.$$

Thus for any $k \geq 1$,

$$q_k = \frac{6}{(k-1)!}$$

As the Frobenius polynomial is given by $F(\lambda) = \lambda(\lambda - 1)$, the recurrence relation (4) may be rewritten as

$$a_n = -\frac{1}{(n+1)n} \sum_{k=0}^{n-1} a_k \left[(1+k)p_{n-k} + q_{n-k}\right].$$

By assumption, $a_0 = 1$. Thus the next several terms are given by

$$\begin{aligned}
a_1 &= -\frac{1}{2}\left[a_0(1p_1 + q_1)\right] \\
&= -\frac{1}{2}\left[1(1 \cdot 2 + 6)\right] \\
&= -4, && (n = 1) \\
a_2 &= -\frac{1}{6}\left[a_0(1p_2 + q_2) + a_1(2p_1 + q_1)\right] \\
&= -\frac{1}{6}\left[1(1 \cdot 0 + 6) - 4(2 \cdot 2 + 6)\right] \\
&= \frac{17}{3}, && (n = 2) \\
a_3 &= -\frac{1}{12}\left[a_0(1p_3 + q_3) + a_1(2p_2 + q_2) + a_2(3p_1 q_1)\right] \\
&= -\frac{1}{12}\left[1(1 \cdot 0 + 3) - 4(2 \cdot 0 + 6) + \frac{17}{3}(3 \cdot 2 + 6)\right] \\
&= -\frac{47}{12}, && (n = 3)
\end{aligned}$$

and so on. Therefore

$$y_1(x) = x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots = x - 4x^2 + \frac{17}{3}x^3 - \frac{47}{12}x^4 + \cdots.$$

Finding the first several terms of a power series expansion of the second solution is slightly more involved. Let L denote teh differential operator

$$L[y] := xy'' + 2xy' + 6e^x y.$$

Then

$$\begin{aligned}
0 &= L[y_2(x)] \\
&= bL[y_1(x)\log(x)] + L\left[1 + \sum_{n=1}^{\infty} b_n x^n\right] \\
&= x\left(by_1''(x)\log|x| + \frac{2b}{x}y_1'(x) - \frac{b}{x^2}y_1(x)\right) + 2x\left(by_1'(x)\log|x| + \frac{b}{x}y_1(x)\right) \\
&\qquad + xq(x)\left(by_1(x)\log|x|\right) + L\left[1 + \sum_{n=1}^{\infty} b_n x^n\right] \\
&= bL[y_1(x)]\log|x| + 2by_1'(x) - \frac{b}{x}y_1(x) + 2by_1(x) + L\left[1 + \sum_{n=1}^{\infty} b_n x^n\right].
\end{aligned}$$

As y_1 is a solution to the differential equation, it follows that $L[y_1] = 0$. Therefore

$$L\left[1 + \sum_{n=1}^{\infty} b_n x^n\right] = -2by_1'(x) - 2by_1(x) + \frac{b}{x}y_1(x)$$

The right-hand side expands to

$$\begin{aligned}
-2by_1'(x) - 2by_1(x) + \frac{b}{x}y_1(x) &= -2b\left[1 - 8x + 17x^2 - \frac{47}{3}x^3 + \cdots\right] \\
&\quad - 2b\left[x - 4x^2 + \frac{17}{3}x^3 - \frac{47}{12}x^4 + \cdots\right] \\
&\quad + \frac{b}{x}\left[x - 4x^2 + \frac{17}{3}x^3 - \frac{47}{12}x^4 + \cdots\right] \\
&= b\left[-2 + 16x - 34x^2 + \frac{94}{3}x^3 + \cdots\right] \\
&\quad + b\left[-2x + 8x^2 - \frac{34}{3}x^3 + \frac{47}{6}x^4 + \cdots\right] \\
&\quad + b\left[1 - 4x + \frac{17}{3}x^2 - \frac{47}{12}x^3 + \cdots\right] \\
&= -b + 10bx - \frac{61}{3}bx^2 + \frac{193}{12}bx^3 + \cdots . \qquad (5)
\end{aligned}$$

The left-hand side is a little more complicated to expand. By definition of L,

$$\begin{aligned}
&L\left[1 + \sum_{n=1}^{\infty} b_n x^n\right] \\
&\quad = x\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[1 + \sum_{n=1}^{\infty} b_n x^n\right] + 2x\frac{\mathrm{d}}{\mathrm{d}x}\left[1 + \sum_{n=1}^{\infty} b_n x^n\right] + 6\mathrm{e}^x\left[1 + \sum_{n=1}^{\infty} b_n x^n\right] \\
&\quad = \sum_{n=1}^{\infty} b_n n(n-1)x^{n-1} + \sum_{n=1}^{\infty} 2b_n n x^n + \mathrm{e}^x\left[6 + \sum_{n=1}^{\infty} 6b_n x^n\right] \\
&\quad = \left[2b_2x + 6b_3x^2 + 12b_4x^3 + \cdots\right] + \left[2b_1x + 4b_2x^2 + 6b_3x^3 + 8b_4x^4 + \cdots\right] \\
&\qquad + \underbrace{\left[1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots\right]}_{=\mathrm{e}^x}\left[6 + 6b_1x + 6b_2x^2 + 6b_3x^3 + \cdots\right] \\
&\quad = \left[2b_2x + 6b_3x^2 + 12b_4x^3 + \cdots\right] + \left[2b_1x + 4b_2x^2 + 6b_3x^3 + \cdots\right] \\
&\qquad + \left[6 + (6b_1 + 6)x + (6b_2 + 6b_1 + 3)x^2 + (6b_3 + 6b_2 + 3b_1 + 1)x^3 + \cdots\right] \\
&\quad = 6 + (2b_2 + 8b_1 + 6)\,x + (6b_3 + 10b_2 + 6b_1 + 3)\,x^2 \\
&\qquad + (12b_4 + 12b_3 + 6b_2 + 3b_1 + 1)\,x^3 + \cdots \qquad (6)
\end{aligned}$$

The expressions at (5) and (6) are equal. From this, it follows that the coefficients of each power of x must be equal, as well. Therefore

$$
\begin{aligned}
-b &= 6, && \text{(constant terms)}\\
10b &= 6 + 8b_1 + 2b_2, && \text{(coefficient of } x)\\
-\frac{61}{3}b &= 3 + 3b_1 + 10b_2 + 6b_3, && \text{(coefficient of } x^2)\\
\frac{193}{12}b &= 1 + 3b_1 + 6b_2 + 12b_3 + 12b_4, && \text{(coefficient of } x^3)
\end{aligned}
$$

and so on. This is an infinite system of linear equations, which can be solved using the usual techniques (e.g. solve the n-th equation, then substitute that result into the $(n+1)$-st equation). However, the system is underdetermined, as the n-th equation depends on $n+1$ variables. Thus a choice needs to be made—choosing $b_1 = 0$ will be a convenient choice moving forward. Then

$$
\begin{aligned}
b &= -6,\\
b_2 &= \frac{1}{2}(10b - 6) = \frac{1}{2}(-60 - 6) = -33,\\
b_3 &= \frac{1}{6}\left(-\frac{61}{3}b - 10b_2 - 3\right) = \frac{1}{6}\left(\frac{61}{3}\cdot 6 + 10\cdot 33 - 3\right) = \frac{449}{6},\\
b_4 &= \frac{1}{12}\left(\frac{193}{12}b - 12b_3 - 6b_2 - 3b_1 - 1\right) = \cdots = -\frac{1595}{24},
\end{aligned}
$$

and so on. Thus the second solution to the original differential equation is given by

$$
y_2(x) = -6\log|x|y_1(x) + 1 - 33x^2 + \frac{449}{6}x^3 - \frac{1595}{24}x^4 + \cdots .
$$

□

Problem 6 (5.7.4)**.** Show that the differential equation

$$
x^2y'' + 4xy' + (2+x)y = 0 \tag{7}
$$

has a regular singular point at $x = 0$, and determine two linearly independent solutions for $x > 0$.

Solution. Once again employing the method of Frobenius, first note that both $xp(x) = 4$ and $x^2q(x) = 2 + x$ are both analytic, hence the differential equation has a regular singularity at $x = 0$. Observe that

$$
\lim_{x\to 0} xp(x) = \lim_{x\to 0} 4 = 4 = p_0 \qquad \text{and} \qquad \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} 2 + x = 2 = q_0.
$$

It then follows that the Frobenius polynomial is given by

$$
F(\lambda) = \lambda(\lambda - 1) + p_0\lambda + q_0 = \lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1).
$$

Therefore the Frobenius indices (i.e. the exponents at the singularity) are $\lambda_{1,2} = -1, -2$. Per the discussion of the Bessel problem in Section 5.7 of the text, let

$$y = \varphi(\lambda, x) = \sum_{n=0}^{\infty} a_n(\lambda) x^{\lambda+n}$$

denote a power series solution to (7) which depends on the Frobenius index λ, and define

$$L[y](x) = x^2 y''(x) + 4xy'(x) + (2+x)y(x).$$

But then

$$\begin{aligned}
0 &= L[\varphi](\lambda, x) \\
&= \sum_{n=0}^{\infty} a_n(\lambda)(\lambda+n)(\lambda+n-1)x^{\lambda+n} + \sum_{n=0}^{\infty} 4a_n(\lambda)(\lambda+n)x^{\lambda+n} \\
&\qquad + \sum_{n=0}^{\infty} 2a_n(\lambda)x^{\lambda+n} + \sum_{n=0}^{\infty} a_n(\lambda)x^{\lambda+n+1} \\
&= \sum_{n=0}^{\infty} \big[(\lambda+n)(\lambda+n-1) + 4(\lambda+n) + 2\big] a_n(\lambda)x^{\lambda+n} + \sum_{n=1}^{\infty} a_{n-1}(\lambda)x^{\lambda+n} \\
&= \sum_{n=1}^{\infty} \big[(\lambda+n+1)(\lambda+n+2)a_n(\lambda) + a_{n-1}(\lambda)\big] x^{\lambda+n}.
\end{aligned}$$

This gives rise to the recurrence relation

$$a_n(\lambda) = -\frac{a_{n-1}(\lambda)}{(\lambda+n+1)(\lambda+n+2)}.$$

With $\lambda = \lambda_1 = -1$, this becomes

$$a_n(-1) = -\frac{a_{n-1}(-1)}{n(n+1)}, \qquad (n \geq 1).$$

Via an induction argument,

$$a_n(-1) = \frac{(-1)^n}{n!(n+1)!} a_0,$$

where $a_0 := a_0(-1)$ is some arbitrary constant. Taking $a_0 = 1$, this implies that there is a solution of the form

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} x^{n-1}.$$

Per the discussion on pages 292–3 of the course text, a second solution is given by

$$y_2(x) = a y_1(x) \log(x) + \frac{1}{x^2} + x^{\lambda_2} \sum_{n=1}^{\infty} c_n(\lambda_2) x^n,$$

where

$$a = \lim_{\lambda \to \lambda_2} (\lambda - \lambda_2) a_{\lambda_1 - \lambda_2}(\lambda) \qquad \text{and} \qquad c_n(\lambda_2) = \frac{\mathrm{d}}{\mathrm{d}\lambda} \left[(\lambda - \lambda_2) a_n(\lambda) \right|_{\lambda = \lambda_2}.$$

With $\lambda_1 = -1$ and $\lambda_2 = -2$, it follows that

$$a = \lim_{\lambda \to -2} (\lambda + 2) a_1(\lambda) = -\lim_{\lambda \to -2} (\lambda + 2) \frac{a_0(\lambda)}{(\lambda + 2)(\lambda + 3)} = -a_0(-2).$$

Again, $a_0(-2)$ is an arbitrary nonzero constant, so taking it to be 1,

$$a = -1.$$

In order to evaluate $c_n(-2)$, it will be useful to have a general formula for $a_n(\lambda)$. Via an induction argument, it can be shown that

$$a_n(\lambda) = \frac{(-1)^n}{(\lambda + 2)(\lambda + 3)^2 (\lambda + 4)^2 \cdots (\lambda + n + 1)^2 (\lambda + n + 2)},$$

under the usual assumption that $a_0(\lambda) = 1$. Making use of the Pochhammer symbol to simplify notation, this can be rewritten as

$$a_n(\lambda) = \frac{(-1)^n}{(\lambda + 2)(\lambda + n + 2) \left[(\!(\lambda + 3)\!)_n \right]^2}, \tag{8}$$

where the Pochhammer symbol $(\!(x)\!)_k$ denotes

$$(\!(x)\!)_k = \prod_{j=0}^{k-1} (x + j) = x(x + 1)(x + 2) \cdots (x + k - 1).$$

Observe that $(\!(1)\!)_n = (n - 1)!$. Note that everything which follows can be done *without* the Pochhammer symbol, but the notation drastically reduces the amount of notation required. As the goal is to differentiate $a_n(\lambda)$ with respect to λ in order to find the coefficients $c_n(-2)$, it might be helpful to recall the following facts: if f is a positive differentiable function, then the chain rule implies that

$$\frac{\mathrm{d}}{\mathrm{d}x} \log(f(x)) = \frac{f'(x)}{f(x)}.$$

In the case that $f(x)$ is a product of powers of linear terms, i.e.

$$f(x) = \prod_{j=1}^{k} (x - a_j)^{b_j},$$

then the derivative may be computed explicitly as

$$\frac{\mathrm{d}}{\mathrm{d}x} \log(f(x)) = \frac{\mathrm{d}}{\mathrm{d}x} \log\left(\prod_{j=1}^{k} (x - a_j)^{b_j} \right) = \frac{\mathrm{d}}{\mathrm{d}x} \sum_{j=1}^{k} b_j \log(x - a_j) = \sum_{j=1}^{k} \frac{b_j}{x + j}.$$

Of particular interest, this implies that, with $f(x) = (\!(x)\!)_k$,

$$\frac{f'(x)}{f(x)} = \sum_{j=0}^{k-1} \frac{1}{x+j}$$

The value of $c_n(-2)$ depends on the derivative of $(\lambda+2)a_n(\lambda)$. This derivative can be obtained by applying the above results:

$$\begin{aligned}
\frac{\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[(\lambda+2)a_n(\lambda)\right]}{(\lambda+2)a_n(\lambda)} &= \left[-\frac{1}{\lambda+n+2} - 2\sum_{j=0}^{n-1}\frac{1}{\lambda+3+j}\right] \\
\implies \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[(\lambda+2)a_n(\lambda)\right] &= (\lambda+2)a_n(\lambda)\left[-\frac{1}{\lambda+n+2} - 2\sum_{j=0}^{n-1}\frac{1}{\lambda+3+j}\right] = (*).
\end{aligned}$$

Taking a limit as $\lambda \to -2$ on the left-hand side gives $c_n(-2)$. Thus it remains only to simplify the right-hand side of the equation and take that limit. The right-hand side can be expanded by recalling the formula for $a_n(\lambda)$, which is given at (8):

$$\begin{aligned}
c_n(-2) &= \lim_{\lambda\to-2}(*) \\
&= \lim_{\lambda\to-2}\left[(\lambda+2)a_n(\lambda)\left(-\frac{1}{\lambda+n+2} - 2\sum_{j=0}^{n-1}\frac{(-1)^n}{\lambda+3+j}\right)\right] \\
&= \lim_{\lambda\to-2}\left[\frac{(-1)^n}{(\lambda+n+2)\left[(\!(\lambda+3)\!)_n\right]^2}\left(-\frac{1}{\lambda+n+2} - 2\sum_{j=0}^{n-1}\frac{1}{\lambda+3+j}\right)\right] \\
&= -\frac{(-1)^n}{n\left[(\!(1)\!)_n\right]^2}\left(\frac{1}{n} + 2\sum_{j=0}^{n-1}\frac{1}{j+1}\right) \\
&= -\frac{(-1)^n}{n\left[(n-1)!\right]^2}\left(\sum_{j=0}^{n}\frac{1}{j+1} + \sum_{j=0}^{n-1}\frac{1}{j+1}\right) \\
&= -\frac{(-1)^n}{n!(n-1)!}(H_n + H_{n-1}),
\end{aligned}$$

where H_n is the n-th harmonic number. Therefore (finally!) a second solution to the original differential equation is given by

$$\begin{aligned}
y_2(x) &= a\log(x)y_1(x) + \frac{1}{x^2} + x^{-2}\sum_{n=1}^{\infty} c_n(-2)x^n \\
&= -\log(x)y_1(x) + \frac{1}{x^2} - \sum_{n=1}^{\infty}\frac{H_n + H_{n-1}}{n!(n-1)!}x^{n-2}.
\end{aligned}$$

As this solution matches the solution in the book, there is even a really good chance that it is correct! $\square$

Problem 7 (5.7.5)**.** Find two linearly independent solutions of the Bessel equation of order $\frac{3}{2}$:

$$x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right)y = 0. \tag{9}$$

Solution. The method in this problem is similar to that used in the previous problems. Assume that $x > 0$ and suppose that a solution of the form $\varphi(x, \lambda)$ exists and has the form

$$\varphi(x, \lambda) = \sum_{n=0}^{\infty} a_n(\lambda)x^{\lambda+n}.$$

Then, substituting this into (9), it follows that

$$\begin{aligned}
0 &= x^2\varphi''(x,\lambda) + x\varphi'(x,\lambda) + \left(x^2 - \frac{9}{4}\right)\varphi(x,\lambda) \\
&= \sum_{n=0}^{\infty} a_n(\lambda)(\lambda+n)(\lambda+n-1)x^{\lambda+n} + \sum_{n=0}^{\infty} a_n(\lambda)(\lambda+n)x^{\lambda+n} \\
&\qquad + \sum_{n=0}^{\infty}\left[\frac{9}{4}a_n(\lambda)x^{\lambda+n} - a_n(\lambda)x^{\lambda+n+2}\right] \\
&= \sum_{n=0}^{\infty}\left((\lambda+n)(\lambda+n-1) + (\lambda+n) - \frac{9}{4}\right)a_n(\lambda)x^{\lambda+n} + \sum_{n=2}^{\infty} a_{n-2}(\lambda)x^{\lambda+n} \\
&= \sum_{n=0}^{\infty}\left((\lambda+n)^2 - \frac{9}{4}\right)a_n(\lambda)x^{\lambda+n} + \sum_{n=2}^{\infty} a_{n-2}(\lambda)x^{\lambda+n} \\
&= \left(\lambda^2 - \frac{9}{4}\right)a_0(\lambda)x^{\lambda} + \left((\lambda+1)^2 - \frac{9}{4}\right)a_1(\lambda)x^{\lambda+1} \\
&\qquad + \sum_{n=2}^{\infty}\left[\left((\lambda+n)^2 - \frac{9}{4}\right)a_n(\lambda) + a_{n-2}(\lambda)\right]x^{\lambda+n}.
\end{aligned} \tag{10}$$

In order for this equation to hold, the coefficient of $x^{\lambda+n}$ must be zero for each $n \in \mathbb{N}$. The series coefficients give rise to the recurrence relation

$$a_n(\lambda) = -\frac{a_{n-2}(\lambda)}{(\lambda+n)^2 - \frac{9}{4}}, \qquad n \geq 2. \tag{11}$$

Because $a_n(\lambda)$ depends on $a_{n-2}(\lambda)$ for each n, the coefficients $a_n(\lambda)$ with even index ultimately depend on a choice of $a_0(\lambda)$, while the coefficients with odd index ultimately depend on $a_1(\lambda)$. This implies that judicious choices of $a_0(\lambda)$ and $a_1(\lambda)$ should give useful results.

To obtain a first solution, suppose that $a_0(\lambda) = 1$ and that $a_1(\lambda) = 0$ (indeed, if $a_0(\lambda) \neq 0$, then it must be the case that $a_1(\lambda) = 0$). This second assumption implies that

$$a_{2k+1}(\lambda) = 0, \qquad \text{for all } k \in \mathbb{N}.$$

On the other hand,

$$\left(\lambda^2 - \frac{9}{4}\right) a_0(\lambda) = \left(\lambda + \frac{3}{2}\right)\left(\lambda - \frac{3}{2}\right) = 0 \implies \lambda = \pm\frac{3}{2}.$$

Either choice of λ will eventually render a solution so, making an arbitrary choice, let $\lambda = 3/2$. With this choice, the recurrence relation (11) becomes

$$a_n(\tfrac{3}{2}) = -\frac{a_{n-2}(\frac{3}{2})}{n(n+3)}$$

The first several terms are given by

$$a_0(\tfrac{3}{2}) = 1, \quad a_2(\tfrac{3}{2}) = -\frac{1}{2\cdot 5}, \quad a_4(\tfrac{3}{2}) = \frac{1}{2\cdot 5\cdot 4\cdot 7}, \quad a_6(\tfrac{3}{2}) = \frac{1}{2\cdot 5\cdot 4\cdot 7\cdot 6\cdot 9},$$

and so on. Via an induction argument, it can be shown that the general term is given by

$$a_{2k}(\tfrac{3}{2}) = (-1)^k \frac{1}{(2\cdot 4\cdot 6\cdot\cdots\cdot 2k)(5\cdot 7\cdot 9\cdot\cdots\cdot(2k+3))}.$$

This formula is sufficient to answer the given question, but can be massaged into something a little easier to work with. First, observe that

$$2\cdot 4\cdot 6\cdot\cdots\cdot 2k = \prod_{j=1}^{k} 2k = 2^k \prod_{j=1}^{k} k = 2^k k!.$$

The other factor in the denominator gives

$$5\cdot 7\cdot 9\cdot\cdots\cdot(2k+3) = \prod_{j=1}^{k}(2k+3) = \prod_{j=1}^{k} 2\left(k + \frac{3}{2}\right) = 2^k \prod_{j=0}^{k-1}\left(\frac{1}{2} + k\right) = 2^k (\!(\tfrac{1}{2})\!)_k,$$

where, again $(\!(\frac{1}{2})\!)_k$ is the Pochhammer symbol. Substituting these values back into the general formula for $a_n(\frac{3}{2})$ renders

$$a_{2k}(\tfrac{3}{2}) = \frac{(-1)^k}{2^{2k} k! (\!(\frac{1}{2})\!)_k},$$

where $k \in \mathbb{N}_0$ (the set of natural numbers, including zero). Therefore one solution to the original differential equation (9) is given by

$$\varphi(x, \tfrac{3}{2}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k! (\!(\frac{1}{2})\!)_k} x^{2k+3/2}.$$

By Theorem 5.6.1, there is a second solution of the form

$$\varphi(x, -\tfrac{3}{2}) = a\log(x) y_1(x) + \sum_{n=0}^{\infty} c_n(-\tfrac{3}{2}) x^{n-3/2},$$

where

$$a = \lim_{\lambda \to -3/2} \left(\lambda + \frac{3}{2}\right) a_3(\lambda) \qquad \text{and} \qquad c_n(-\tfrac{3}{2}) = \frac{\mathrm{d}}{\mathrm{d}\lambda} \left[\left(\lambda + \frac{3}{2}\right) a_n(\lambda) \right|_{\lambda=-3/2}.$$

As noted above, $a_0(\lambda) \neq 0$ implies that $a_1(\lambda)$ must be zero. From this, it follows that $a = 0$, and so there is no logarithmic term. At this point, it may be possible to use the formula for $c_n(-3/2)$ in terms of the derivative of $a_n(\lambda)$. However, this computation is likely to be quite difficult. Instead, observe that since there is no logarithmic term, the solution is of the same form as (10), with $\lambda = -3/2$. The recurrence relation at (11) holds, and so

$$c_n(\tfrac{3}{2}) = a_n(\tfrac{3}{2}) = -\frac{a_{n-2}(-\frac{3}{2})}{(-\frac{3}{2} + n)^2 - \frac{9}{4}} = -\frac{a_{n-2}(-\frac{3}{2})}{n(n-3)}.$$

Via analysis similar to that presented above, the general term is

$$a_{2k}(-\tfrac{3}{2}) = \frac{(-1)^k}{2^{2k} k! (\!(-\frac{1}{2})\!)_k}.$$

Therefore a second solution to the differential equation (9) is given by

$$\varphi(x, -\tfrac{3}{2}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k! (\!(-\frac{1}{2})\!)_k} x^{2k-3/2}.$$

Summarizing the above results, the differential equation (9) has two solutions, given by

$$y_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k! (\!(\frac{1}{2})\!)_k} x^{2k+3/2} \qquad \text{and} \qquad y_2(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k! (\!(-\frac{1}{2})\!)_k} x^{2k-3/2}.$$

While it is not immediately obvious, it can be shown that these two solutions are, in fact linearly independent. This can be verified by observing that the exponents in the series expansion of y_1 are (abusing notation a bit) of the form

$$\frac{4k+3}{2} \equiv \frac{1}{2} \left[3 \pmod 4\right]$$

for some integer k, while the exponents in the series expansion of y_2 are of the form

$$\frac{4k-3}{2} \equiv \frac{1}{2} \left[1 \pmod 4\right].$$

Since $1 \not\equiv 3 \pmod 4$ and monomials of the form x^j and x^k are linearly independent for all $j \neq k$, it follows that $y_1(x)$ and $y_2(x)$ are linearly independent. Another approach is to compute the Wronskian, which can be simplified by use of big-oh notation. Near zero,

$$y_1(x) = x^{3/2} + O(x^{7/2}) \qquad \text{and} \qquad y_2(x) = x^{-3/2} + O(x^{1/2}).$$

Therefore

$$\begin{aligned} W(y_1, y_2)(x) &= \begin{vmatrix} x^{3/2} + O(x^{7/2}) & x^{-3/2} + O(x^{1/2}) \\ \frac{3}{2}x^{1/2} + O(x^{5/2}) & -\frac{3}{2}x^{-5/2} + O(x^{-1/2}) \end{vmatrix} \\ &= \left(-\frac{3}{2}x^{-1} + O(x)\right) - \left(\frac{3}{2}x^{-1} + O(x)\right) = -3x^{-1} + O(x). \end{aligned}$$

This is not zero, and so the solutions are linearly independent. □