

## Problem 1. (Symmetric functions)

- (1) Assume  $f$  is odd, show that  $f(x) \cos(\frac{n\pi x}{L})$  is odd,  
and  $f(x) \sin(\frac{n\pi x}{L})$  is even.
- (2) Assume  $f$  is even, show that  $f(x) \cos(\frac{n\pi x}{L})$  is even.  
and  $f(x) \sin(\frac{n\pi x}{L})$  is odd.

Solution. Recall:  $g$  is odd if  $g(-x) = -g(x)$ .  
 $g$  is even if  $g(-x) = g(x)$ .

Check (1). 
$$\begin{aligned} f(-x) \cos(-\frac{n\pi x}{L}) &= f(-x) \cos(\frac{n\pi x}{L}) \\ &= -f(x) \cos(\frac{n\pi x}{L}). \end{aligned}$$

$$\begin{aligned} f(-x) \sin(-\frac{n\pi x}{L}) &= -f(-x) \sin(\frac{n\pi x}{L}) \\ &= (-1)^2 f(x) \sin(\frac{n\pi x}{L}) = f(x) \sin(\frac{n\pi x}{L}). \end{aligned}$$

(2) 
$$\begin{aligned} f(-x) \cos(-\frac{n\pi x}{L}) &= f(-x) \cos(\frac{n\pi x}{L}) \\ &= f(x) \cos(\frac{n\pi x}{L}). \end{aligned}$$

$$\begin{aligned} f(-x) \sin(-\frac{n\pi x}{L}) &= -f(-x) \sin(\frac{n\pi x}{L}) \\ &= -f(x) \sin(\frac{n\pi x}{L}). \end{aligned}$$

Problem 2. Show that  $f(x) = |x|$ ,  $-\pi < x < \pi$  is even.

$$\text{Check: } f(-x) = |-x| = |x| = f(x) \quad \text{on } -\pi < x < \pi.$$

Problem 3. (Fourier expansion).

Expand  $f(x) = |x|$ ,  $-\pi < x < \pi$  as Fourier series.

Solution. By previous two problems,  $f(x)$  is even.

$L = \pi$ .  $f(x) \cos(nx)$  is even and  $f(x) \sin(nx)$  is odd.

$$\text{So } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = 0, \quad n=1, 2, \dots$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} |x| dx = \frac{2}{\pi} \int_0^{\pi} x dx = \pi.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} |x| \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx.$$

$$= \frac{2}{\pi} \cdot \frac{1}{n^2} \int_0^{n\pi} u \cos(u) du. \quad \begin{cases} u = nx \\ du = n dx. \end{cases} \quad \begin{cases} \pi \rightarrow n\pi \\ 0 \rightarrow 0. \end{cases}$$

$$= \frac{2}{\pi n^2} \cdot \left[ u \sin(u) \Big|_0^{n\pi} - \int_0^{n\pi} \sin(u) du \right] \quad (\text{integration by parts})$$

$$= \frac{2}{\pi n^2} \left[ - \int_0^{n\pi} \sin(u) du \right]$$

$$= \frac{2}{\pi n^2} \cdot \cos(u) \Big|_0^{n\pi} = \frac{2}{\pi n^2} [\cos(n\pi) - 1] = \frac{2[(-1)^n - 1]}{n^2 \pi}, \quad n=1, 2, \dots$$

$$\begin{aligned}
 \text{So } f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \\
 &= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{n^2 \pi} \cos(nx) \\
 &= \frac{\pi}{2} - \frac{4}{\pi} \cdot \left( \cos(x) + \frac{1}{9} \cos(3x) + \frac{1}{25} \cos(5x) + \dots \right)
 \end{aligned}$$

Remark: If  $f$  is even, then its Fourier series expansion only consists of cosine terms. (including  $\cos(0 \cdot \frac{\pi x}{L}) = 1$ ).

If  $f$  is odd, then its Fourier series expansion only consists of sine terms. (example in lecture).

Problem 4. (Fourier Series).

Let  $f(x) = |x|$  on  $-\pi < x < 0$  and  $0 < x < \pi$ .

then  $f'(x) = \begin{cases} -1, & -\pi < x < 0. \\ 1, & 0 < x < \pi. \end{cases}$  (example in lecture).

Verify that the differentiation can be taken term by term on the Fourier expansion.

Solution. Previous problem  $f(x) \sim \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{n^2 \pi} \cos(nx)$ .

$$\text{So } \left( \frac{2[(-1)^n - 1]}{n^2 \pi} \cos(nx) \right)' = \frac{2[1 - (-1)^n]}{n \pi} \sin(nx) \quad n=1, 2, \dots$$

Example in lecture:  $f'(x) \sim \sum_{n=1}^{\infty} \frac{2}{n \pi} (1 - (-1)^n) \sin(nx)$ .