Math 146B.010 (Ordinary and Partial Differential Equations)
Homework 02 Solutions

Variation of Parameters

Suppose that

$$L[y] = y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1} y' + p_n y = g. \tag{1}$$

Some would say that the use of ellipses $(\cdots)$ is slightly non-rigorous. The notation and following arguments can be made rigorous with big sigma notation, but this is unnecessary, and (in my opinion) obscures some of what is going on. Here and below, I will use ellipses.

The essential idea behind the method of variation of parameters is to assume that the solution to a differential equation of the form

$$y^{(n)} + \cdots + y = g$$

is of the form

$$y = u_1 y_1 + u_2 y_2 + \cdots + u_n y_n,$$

where $\{y_j\}_{j=1}^n$ is a fundamental set of solutions to the differential equation, and $\{u_j\}_{j=1}^n$ is a set of unknown coefficient (or *parameter*) functions, which act on the same independent variable as the y_j. By making a series of assumptions about the relation between the parameter functions and the fundamental solutions, it is possible to set up a linear system of equations which can be solved for the parameter functions. In particular, observe that the product rule gives

$$y' = (u_1 y_1' + u_2 y_2' + \cdots + u_n y_n') + (u_1' y_1 + u_2' y_2 + \cdots + u_n' + y_n).$$

As it is preferable not to introduce more unknown variables, make the simplifying assumption that

$$u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n = 0,$$

so that

$$y' = u_1 y_1' + u_2 y_2' + \cdots + u_n y_n'.$$

Applying a similar process to y', note that

$$y'' = (u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n') + (u_1 y_1'' + u_2 y_2'' + \cdots + u_n y_n'').$$

Again, assume that the terms involving derivatives of the u_j sum to zero:

$$u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n' = 0.$$

Continue in this manner (taking derivatives and assuming that the terms involving derivatives of the parameter functions vanish) in order to obtain the equations

$$u_1' y_1^{(m)} + u_2' y_2^{(m)} + \cdots + u_n' y_n^{(m)} = 0, \tag{2}$$

and

$$y^{(m)} = u_1 y_1^{(m)} + u_2 y_2^{(m)} + \cdots + u_n y_n^{(m)},$$

where $m = 0, 1, \ldots, n-1$. Substituting this final set of results into the original differential equation at (1) gives

$$\begin{aligned}
g &= L[y] \\
&= y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1} y' + p_n y \\
&= u_1 y_1^{(n)} + u_2 y_2^{(n)} + \cdots + u_n y_n^{(n)} + u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} \\
&\qquad + p_1 \left(u_1 y_1^{(n-1)} + u_2 y_2^{(n-1)} + \cdots + u_n y_n^{(n-1)} \right) \\
&\qquad + \cdots + p_{n-1} \left(u_1 y_1' + u_2 y_2' + \cdots + u_n y_n' \right) \\
&\qquad + p_n \left(u_1 y_1 + u_2 y_2 + \cdots + u_n y_n \right) \\
&= u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' \\
&\qquad + u_1 \left(y_1^{(n)} + p_1 y_1^{(n-1)} + \cdots + p_{n-1} y_1' + p_n y_1 \right) \\
&\qquad + u_2 \left(y_2^{(n)} + p_1 y_2^{(n-1)} + \cdots + p_{n-1} y_2' + p_n y_2 \right) \\
&\qquad + \cdots + u_n \left(y_n^{(n)} + p_1 y_n^{(n-1)} + \cdots + p_{n-1} y_n' + p_n y_n \right) \\
&= u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' + u_1 L[y_1] + u_2 L[y_2] + \cdots + u_n L[y_n] \\
&= u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)},
\end{aligned}$$

where the final equality follows from the hypothesis that each of the y_j is a solution to the homogeneous equation $L[y] = 0$. In short,

$$u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = g. \tag{3}$$

Combining the $n-1$ equations obtained at (2) with the equation at (3) gives the system

$$\begin{cases}
u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n = 0, \\
u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n' = 0, \\
\vdots \\
u_1' y_1^{(n-2)} + u_2' y_2^{(n-2)} + \cdots + u_n' y_n^{(n-2)} = 0, \\
u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = g,
\end{cases}$$

which is a system of n equations in n unknowns (specifically, the u_j' are unknown). Equivalently, this system may be written as

$$W(y_1, y_2, \ldots, y_n) \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g \end{pmatrix}, \tag{4}$$

where $W(y_1, y_2, \dots, y_n)$ is the Wronskian matrix. Because the y_j constitute a fundamental set of solutions to the homogeneous equation, the Wronskian matrix is non-singular, and so this system has a unique solution; that is, the u_j' may be found. Once these functions are determined, integrate to obtain a solution to the differential equation $L[y]$.

While this looks like a lot of work, the application can often be a straight-forward computation. The matrix $W(y_1, y_2, \dots, y_n)$ might be easily simplified via row reduction, or it may be possible to apply Cramer's rule.

Problem 1 (3.6.3)**.** Us the method of variation of parameters to find a particular solution of the differential equation

$$y'' + 2y' + y = 3\mathrm{e}^{-t}$$

Then check the answer using the method of undetermined coefficients.

Solution. Using techniques which have been developed in detail earlier in the course, a fundamental set of solutions to the homogeneous equation

$$y'' + 2y' + y = 0$$

is given by

$$\left\{ y_1(t) = \mathrm{e}^{-t}, y_2(t) = t\mathrm{e}^{-t} \right\}.$$

With $g(t) = 3\mathrm{e}^{-t}$, this may be substituted into (4) in order to obtain

$$\begin{pmatrix} \mathrm{e}^{-t} & t\mathrm{e}^{-t} \\ -\mathrm{e}^{-t} & (1-t)\mathrm{e}^{-t} \end{pmatrix} \begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}.$$

This system can be solved by performing row reductions on the augmented matrix:

$$\begin{aligned}
\left(\begin{array}{cc|c} \mathrm{e}^{-t} & t\mathrm{e}^{-t} & 0 \\ -\mathrm{e}^{-t} & (1-t)\mathrm{e}^{-t} & 3\mathrm{e}^{-t} \end{array}\right) &\rightsquigarrow \left(\begin{array}{cc|c} \mathrm{e}^{t} & t\mathrm{e}^{t} & 0 \\ 0 & \mathrm{e}^{t} & 3\mathrm{e}^{-t} \end{array}\right) && \begin{cases} \\ R_2 \leftarrow R_2 - R_1 \end{cases} \\
&\rightsquigarrow \left(\begin{array}{cc|c} 1 & t & 0 \\ 0 & 1 & 3 \end{array}\right) && \begin{cases} \\ R_2 \leftarrow \mathrm{e}^{t} R_2 \end{cases} \\
&\rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & -3t \\ 0 & 1 & 3 \end{array}\right). && \left\{ R_1 \leftarrow R_1 - tR_2 \right.
\end{aligned}$$

Therefore

$$u_1'(t) = -3t \qquad \text{and} \qquad u_2'(t) = 3.$$

As per the discussion preceding this problem, a solution to the original differential equation

$$y'' + 2y' + y = 3\mathrm{e}^{-t}$$

is given by

$$\begin{aligned} y(t) &= y_1(t)u_1(t) + y_2(t)u_2(t) \\ &= \mathrm{e}^{-t} \int -3t \, \mathrm{d}t + t\mathrm{e}^{-t} \int 3 \, \mathrm{d}t \\ &= \left(-\frac{3}{2}t^2 + C_1\right) \mathrm{e}^{-t} + (3t + C_2)\, t\mathrm{e}^{-t} \\ &= C_1 \mathrm{e}^{-t} + C_2 t\mathrm{e}^{-t} + \frac{3}{2}t^2\mathrm{e}^{-t}, \end{aligned}$$

where C_1 and C_2 are arbitrary constants of integration.

Using the method of undetermined coefficients as a verification, we guess that a particular solution should be of the form

$$y_p = At^2\mathrm{e}^{-t}.$$

Substituting this into the original differential equation gives

$$3\mathrm{e}^{-t} = y_p''(t) + 2y_p'(t) + y_p(t) = 2A\mathrm{e}^{-t}.$$

Hence $A = \frac{3}{2}$, which is consistent with the result obtained above. □

Problem 2 (3.6.5)**.** Find a general solution to the differential equation

$$y'' + y = \tan(t),$$

where $t \in (0, \pi/2)$.

Solution. A fundamental set of solutions for the homogeneous differential equation

$$y'' + y = 0$$

is given by

$$\{y_1(t) = \cos(t), y_2(t) = \sin(t)\}.$$

Substituting this into (4) gives

$$\begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \tan(t) \end{pmatrix}.$$

This system can be solved via Cramer's rule. The function u_1' is given by

$$u_1'(t) = \frac{\begin{vmatrix} 0 & \sin(t) \\ \tan(t) & \cos(t) \end{vmatrix}}{\begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix}} = \frac{-\tan(t)\sin(t)}{1} = -\tan(t)\sin(t).$$

The function u_2' is given by

$$u_2'(t) = \frac{\begin{vmatrix} \cos(t) & 0 \\ -\sin(t) & \tan(t) \end{vmatrix}}{\begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix}} = \frac{\tan(t)\cos(t)}{1} = \sin(t).$$

Integrate to obtain

$$\begin{aligned} u_1(t) &= -\int \tan(t)\sin(t)\,\mathrm{d}t = \sin(t) - \log(\sec(t) - \tan(t)) + C_1, \\ u_2(t) &= \int \sin(t)\,\mathrm{d}t = -\cos(t) + C_2. \end{aligned}$$

This implies that

$$\begin{aligned} y(t) &= \cos(t)(\sin(t) - \log(\sec(t) - \tan(t)) + C_1) + \sin(t)(-\cos(t) + C_2) \\ &= \cos(t)\sin(t) - \cos(t)\log(\sec(t) - \tan(t)) + C_1\cos(t) - \cos(t)\sin(t) + C_2\sin(t) \\ &= C_1\cos(t) + C_2\sin(t) - \cos(t)\log(\sec(t) - \tan(t)), \end{aligned}$$

where C_1 and C_2 are arbitrary constants of integration. □

Problem 3 (3.6.7)**.** Find a general solution to the differential equation

$$y'' + 4y' + 4y = t^{-2}\mathrm{e}^{2t},$$

where $t > 0$.

Solution. A fundamental set of solutions for the homogeneous differential equation

$$y'' + 4y' + 4y = 0$$

is given by

$$\{y_1(t) = \mathrm{e}^{-2t}, y_2(t) = t\mathrm{e}^{-2t}\}.$$

Substituting this into (4) gives

$$\begin{pmatrix} \mathrm{e}^{-2t} & t\mathrm{e}^{-2t} \\ -2\mathrm{e}^{-2t} & (1-2t)\mathrm{e}^{-2t} \end{pmatrix} \begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ t^{-2}\mathrm{e}^{-2t} \end{pmatrix}.$$

This system can be solved by performing row reductions on the augmented matrix:

$$\begin{aligned} &\left(\begin{array}{cc|c} \mathrm{e}^{-2t} & t\mathrm{e}^{-2t} & 0 \\ -2\mathrm{e}^{-t} & (1-2t)\mathrm{e}^{-2t} & t^{-2}\mathrm{e}^{-2t} \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} \mathrm{e}^{-2t} & t\mathrm{e}^{-2t} & 0 \\ 0 & \mathrm{e}^{-2t} & t^{-2}\mathrm{e}^{-2t} \end{array}\right) \\ &\qquad \rightsquigarrow \left(\begin{array}{cc|c} 1 & t & 0 \\ 0 & 1 & t^{-2} \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & -t^{-1} \\ 0 & 1 & t^{-2} \end{array}\right). \end{aligned}$$

Hence
$$u_1'(t) = -\frac{1}{t} \qquad \text{and} \qquad u_2'(t) = \frac{1}{t^2},$$
from which it follows that
$$u_1(t) = -\int \frac{1}{t}\,\mathrm{d}t = -\log(t) + C_1, \qquad \text{and} \qquad u_2(t) = \int \frac{1}{t^2}\,\mathrm{d}t = -\frac{1}{t} + C_2,$$
where C_1 and C_2 are arbitrary constants of integration. Therefore
$$\begin{aligned} y(t) &= u_1(t)y_1(t) + u_2(t)y_2(t) \\ &= (-\log(t) + C_1)\,\mathrm{e}^{-2t} + \left(-\frac{1}{t} + C_2\right) t\mathrm{e}^{-2t} \\ &= -\mathrm{e}^{-2t}\log(t) + C_1\mathrm{e}^{-2t} - \mathrm{e}^{-2t} + C_2 t\mathrm{e}^{-2t} \\ &= (C_1 - 1)\mathrm{e}^{-2t} + C_2 t\mathrm{e}^{-2t} - \mathrm{e}^{-2t}\log(t) \\ &= k_1\mathrm{e}^{-2t} + k_2 t\mathrm{e}^{-2t} - \mathrm{e}^{-2t}\log(t), \end{aligned}$$
where $k_1 = C_1 - 1$ and $k_2 = C_2$ are arbitrary contants. □

The Ratio Test

For the next two problems, recall the *ratio test* from elementary calculus: let $\{a_n\}$ be a sequence of real numbers and define
$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \qquad \text{and} \qquad S := \sum_{n=0}^{\infty} a_n.$$
Then

- if $L < 1$, then the series S converges absolutely,
- if $L > 1$, then the series S diverges, and
- if $L = 1$, then the ratio test is inconclusive.

In the context of the current section of our text, we consider power series of the form
$$\sum_{n=0}^{\infty} a_n(x - x_0)^n$$
Applying the ratio test, the series will converge whenever
$$1 > \lim_{n\to\infty} \left|\frac{a_{n+1}(x - x_0)^n}{a_n}(x - x_0)\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x - x_0|.$$
Hence the power series converges whenever
$$|x - x_0| \le \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \frac{1}{L} =: \rho.$$

In other words, the power series converges whenever

$$x \in (x_0 - \rho, x_0 + \rho).$$

This interval, the *interval of convergence* has diameter (or length) 2ρ, and so it has radius ρ. Thus ρ is the *radius of convergence*.

It might seem somewhat odd to talk about the radius of an interval, but in higher dimensions, power series can be said to converge on a disk (in two dimensions), or a ball (in higher dimensions). The radius of a disk is a fairly natural concept, and the notion of radius extends in a straightforward manner to higher dimensional balls. By analogy, the radius of a one-dimensional ball (i.e. an interval) also "makes sense."

Problem 4 (5.1.5)**.** Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n^2}.$$

Solution. Per the above discussion, the radius of convergence of a series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is given by

$$\rho = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

However, the given series is not of this form, so it is necessary to first massage the general term a little. This can be done by factoring out a 2 in the numerator:

$$\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(2(x+\frac{1}{2}))^n}{n^2} = \sum_{n=1}^{\infty} \frac{2^n}{n^2} \left(x - \left(-\frac{1}{2} \right) \right)^n.$$

Thus the given power series is of the required form with

$$a_n = \frac{2^n}{n^2} \qquad \text{and} \qquad x_0 = -\frac{1}{2}.$$

Therefore the radius of convergence is

$$\rho = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{2^n/n^2}{2^{n+1}(n+1)^2} \right| = \frac{1}{2} \lim_{n\to\infty} \left| \frac{n^2}{(n+1)^2} \right| = \frac{1}{2},$$

which completes the problem. □

Problem 5 (5.1.7)**.** Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (x+2)^n}{3^n}.$$

Solution. Unlike the previous problem, this series is of the required form, with

$$a_n = (-1)^n \frac{n^2}{3^n} \qquad \text{and} \qquad x_0 = -2.$$

As discussed above,

$$\rho = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^n n^2/3^n}{(-1)^{n+1}(n+1)^2 3^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{3n^2}{(n+1)^2} \right| = 3.$$

Therefore the radius of convergence is 3. □

Problem 6 (5.1.15)**.** Determine a Taylor series expansion of the function f, defined by

$$f(x) = \frac{1}{1-x}$$

about the point $x_0 = 0$. Then determine the radius of convergence of this series.

Solution. A naïve approach to the problem is to simply expand the function out via Taylor's theorem:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n. \tag{5}$$

To do this, it is necessary to determine $f^{(n)}(0)$ for arbitrary n. Observe that

$$\begin{aligned}
f'(x) &= \frac{1}{(1-x)^2} & \implies && f'(0) &= 1, \\
f''(x) &= \frac{2}{(1-x)^3} & \implies && f''(0) &= 2, \\
f'''(x) &= \frac{2\cdot 3}{(1-x)^4} & \implies && f'''(0) &= 6, \\
f^{(4)}(x) &= \frac{2\cdot 3\cdot 4}{(1-x)^5} & \implies && f^{(4)}(0) &= 24,
\end{aligned}$$

and so on. By inspection, it appears that

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \implies f^{(n)}(0) = n!. \tag{6}$$

This can be verified by induction. It certainly holds for $n = 0$. For induction, suppose that (6) holds for some $k > 0$. Then, applying the induction hypothesis,

$$\begin{aligned}
f^{(k+1)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x} f^{(k)}(x) \\
&= \frac{\mathrm{d}}{\mathrm{d}x} \left(k!(1-x)^{-(k+1)} \right) \\
&= k! \left(-(k+1)(1-x)^{-(k+2)}(-1) \right) \\
&= k!(k+1)(1-x)^{-(k+2)} \\
&= \frac{(k+1)!}{(1-x)^{k+2}},
\end{aligned}$$

which is the desired result—that is, the identity at (6) holds for all $n \geq 0$. Substitute this into (5) in order to obtain

$$\frac{1}{1-x} = f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{n!}{n!} x^n = \sum_{n=0}^{\infty} x^n,$$

which gives the desired power series. To determine the radius of convergence, apply the result from above to get

$$\rho = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{1}{1} \right| = 1.$$

It is worth noting that a part of this argument is to prove that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for all $|x| < 1$. This identity is incredibly useful, and pops up over and over again in mathematics. As such, I think that it is worth seeing an alternative approach, which works from the other direction. Define

$$S := \sum_{n=0}^{\infty} x^n.$$

By the ratio test, this series will converge whenever $|x| < 1$. Assuming that this holds,

$$xS = x \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1}.$$

This can be reindexed to get

$$xS = \sum_{n=1}^{\infty} x^n.$$

This series also converges, and so

$$S - xS = \sum_{n=0}^{\infty} x^n - \sum_{n=1}^{\infty} x^n = \left(1 + \sum_{n=0}^{\infty} x^n\right) - \sum_{n=1}^{\infty} x^n = 1.$$

Solve this for S to obtain

$$\sum_{n=0}^{\infty} x^n = S = \frac{1}{1-x}.$$

This result remains valid whenever $|x| < 1$. □

Problem 7 (5.1.18)**.** Given that

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

compute y' and y'' and write out the first four terms of each series, as well as the coefficient of x^n in the general term. Show that if $y'' = y$, then the coefficients of a_0 and a_1 are arbitrary, and determine a_2 and a_3 in terms of a_0 and a_1. Show that

$$a_{n+1} = \frac{a_n}{(n+1)(n+2)}$$

for all $n \in \mathbb{N}$.

Solution. As stated without proof on page 250 of the course text, if the series defining y converges absolutely (i.e. if x is such that $|x - x_0| < \rho$, where ρ is the radius of convergence of the series), then derivatives of y may be found by differentiating the series term-by-term, i.e.

$$y^{(m)}(x) = \sum_{n=0}^{\infty} a_n \frac{\mathrm{d}^m}{\mathrm{d}x^m} x^n.$$

Therefore

$$y'(x) = \sum_{n=0}^{\infty} a_n n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

and

$$y''(x) = \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} = 2a_2 + (2\cdot 3)a_3 x + (3\cdot 4)a_4 x^2 + (4\cdot 5)a_5 x^3 + \cdots .$$

If $y'' = y$, then

$$\begin{aligned}
0 &= y'' - y \\
&= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} \\
&= \sum_{n=0}^{\infty} a_n x^n - \sum_{n+2=0} a_{n+2}(n+2)(n+2-1)x^{n+2-2} && \text{(reindex the second sum)} \\
&= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=-2} a_{n+2}(n+2)(n+1)x^n \\
&= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0} a_{n+2}(n+2)(n+1)x^n \\
& && \text{(since } a_n(n+2)(n+1)x^n = 0 \text{ when } n = -2, -2) \\
&= \sum_{n=0}^{\infty} \underbrace{(a_n - (n+2)(n+1)a_{n+2})}_{=:A_n} x^n.
\end{aligned}$$

As the monomials $\{x^n\}_{n=0}^{\infty}$ form a linearly independent set, the series

$$\sum_{n=0}^{\infty} A_n x^n = 0$$

if and only if $A_n = 0$ for all n. Note that this is not precisely the definition of linear independence as it has been stated in class (which asserts that a finite set of functions is linearly independent if and only if a linear combination of those functions summing to zero implies that each coefficient is zero), but it follows from this definition without too much difficulty. But

$$A_n = a_n - (n+2)(n+1)a_{n+2} = 0 \implies a_{n+2} = \frac{1}{(n+1)(n+2)} a_n,$$

which is the claimed result. □

Problem 8 (5.1.28)**.** Determine the a_n so that the equation

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

is satisfied. Determine the function represented by the power series $\sum_{n=0}^{\infty} a_n x^n$.

Solution. Note that

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2 \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} \left((n+1) a_{n+1} + 2 a_n \right) x^n. \end{aligned}$$

Hence

$$(n+1)a_{n+1} + 2a_n = 0 \implies a_{n+1} = -\frac{2}{n+1} a_n \tag{7}$$

for all $n \geq 0$. Fixing an arbitrary a_0, the first several terms of the sequence $\{a_n\}$ are given by

$$\begin{aligned} a_1 &= a_{0+1} = -\frac{2}{0+1} a_0 = -\frac{2^1}{1} a_0 \\ a_2 &= a_{1+1} = -\frac{2}{1+1} a_1 = \frac{2^2}{1 \cdot 2} a_0 \\ a_3 &= a_{2+1} = -\frac{2}{2+1} a_2 = -\frac{2^3}{1 \cdot 2 \cdot 3} a_0 \\ a_4 &= a_{3+1} = -\frac{2}{3+1} a_3 = \frac{2^4}{1 \cdot 2 \cdot 3 \cdot 4} a_0 \\ a_5 &= a_{4+1} = -\frac{2}{4+1} a_4 = -\frac{2^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} a_0, \end{aligned}$$

and so on. It appears that the general term might be of the form

$$a_n = (-1)^n \frac{2^n}{n!} a_0. \tag{8}$$

This certainly holds for $n = 0$, so assume for induction that (8) holds for some $k > 0$. Then, by (7),

$$a_{k+1} = -\frac{2}{k+1} a_k = -\frac{2}{k+1} \left((-1)^k \frac{2^k}{k!} a_0 \right) = -(1)^{k+1} \frac{2^{k+1}}{(k+1)!} a_0,$$

which is the claimed result. Therefore the general term is given by

$$a_n = (-1)^n \frac{2^n}{n!} a_0.$$

To determine a closed form for this series, recall that the Taylor series expansion of the exponential function around $x_0 = 0$ is given by

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

But then

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{n!} a_0 x^n = a_0 \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} = a_0 \mathrm{e}^{-2x}.$$

As the radius of convergence of this series is infinite (this can be quickly verified by the studious reader), this series defines the function

$$f : \mathbb{R} \to \mathbb{R} : x \mapsto a_0 \mathrm{e}^{-2x},$$

where a_0 is a free variable. □

Important Power Series

Before continuing with the next section, it is worth recalling a few important power series expansions (including those discussed above):

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad \mathrm{e}^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n,$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}, \qquad \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}.$$

These identities will be used without further comment in the sequel. These identies are not tremendously difficult to prove—each is a Taylor series expansion of the given function around the point $x_0 = 0$—but the derivations are a little tedious, and not an essential part of this course. At this point in your mathematical career, you should probably *recognize* the series presented above (as well as a few others), but you need not have them *memorized*—you should be able to easily look them up if you need them.

Problem 9 (5.2.1)**.** Solve the differential equation

$$y'' - y = 0$$

around the point $x_0 = 0$ using power series techniques. Find a recurrence relation for the coefficients in the power series, give the first four terms and, if possible, find the general term.

Solution. We seek solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

where $\{a_n\}$ is a sequence of real numbers. Assuming that a solution of this form exists,

$$\begin{aligned}
0 &= y'' - y \\
&= \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=-2}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n \qquad \text{(reindex the first series)} \\
&= \sum_{n=0}^{\infty} \left(a_{n+2}(n+2)(n+1) - a_n\right) x^n.
\end{aligned}$$

Note that in the last line of this computation, the terms corresponding to $n = -2$ and $n = -1$ drop out, as they include the factors $n+2 = 0$ and $n+1 = 0$, respectively. This series is zero if and only if the coefficient of each term is zero, which implies that

$$a_{n+2}(n+2)(n+1) + a_n = 0 \implies a_{n+2} = -\frac{1}{(n+1)(n+2)} a_n. \tag{9}$$

This recurrence relation gives two solutions: one which corresponds to choosing a value for a_0 (and taking $a_1 = 0$), and a second which corresponds to choosing a value of a_1 (and taking $a_0 = 0$). If a_0 is fixed, then the first four terms of the recurrence relation are given by

$$\begin{aligned}
a_2 &= \frac{1}{1\cdot 2} a_0 = \frac{1}{2!} a_0, && (n = 0) \\
a_4 &= \frac{1}{3\cdot 4} a_2 = \frac{1}{4!} a_0, && (n = 2) \\
a_6 &= \frac{1}{5\cdot 6} a_4 = \frac{1}{6!} a_0, && (n = 4)
\end{aligned}$$

where the parenthetical indicates the value of n in the recurrence relation (9). It can be shown via an induction argument that if n is even, then

$$a_n = \frac{1}{n!} a_0$$

Therefore one solution to the original differential equation is given by

$$y_1(x) = \sum_{n \text{ even}} \frac{1}{n!} a_0 x^n = a_0 \sum_{m=0}^{\infty} \frac{1}{(2m)!} x^{2m},$$

where a_0 is an arbitrarily chosen constant. With a_1 fixed, the first four terms of the recurrence relation are given by

$$\begin{aligned}
a_3 &= \frac{1}{2 \cdot 3} a_1 = \frac{1}{3!} a_1, && (n = 1)\\
a_5 &= \frac{1}{4 \cdot 5} a_3 = \frac{1}{5!} a_1, && (n = 3)\\
a_7 &= \frac{1}{6 \cdot 7} a_5 = \frac{1}{7!} a_1. && (n = 5)
\end{aligned}$$

Again, via an induction argument, the general term (with n odd) is given by

$$a_n = \frac{1}{n!} a_1.$$

Hence a second solution to the original differential equation is given by

$$y_2(x) = \sum_{n \text{ odd}} \frac{1}{n!} a_1 x^n = a_1 \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} x^{2m+1}$$

While this isn't specifically asked for in the statement of the question, it might be work spending some time to determine whether or not these series solutions have "nice" closed forms. In particular, using the techniques from Section 4.3 of the course text, we might expect solutions

$$\tilde{y}_1(x) = \mathrm{e}^x \qquad \text{and} \qquad \tilde{y}_2(x) = \mathrm{e}^{-x}.$$

Do our series solutions give the same solution space?

As the original differential equation is linear, linear combinations of solutions are also solutions. Hence another solution to the original differential equation is given by

$$\frac{1}{a_0} y_1(x) + \frac{1}{a_1} y_2(x) = \sum_{n \text{ even}} \frac{1}{n!} x^n + \sum_{n \text{ odd}} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \mathrm{e}^x = \tilde{y}_1(x).$$

Similarly,

$$\begin{aligned}
\frac{1}{a_0} y_1(x) - \frac{1}{a_1} y_2(x) &= \sum_{n \text{ even}} \frac{1}{n!} x^n - \sum_{n \text{ odd}} \frac{1}{n!} x^n\\
&= \sum_{n \text{ even}} (-1)^n \frac{1}{n!} x^n + \sum_{n \text{ odd}} (-1)^n \frac{1}{n!} x^n\\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (-x)^n\\
&= \mathrm{e}^{-x} = \tilde{y}_2(x).
\end{aligned}$$

Thus the solutions obtained via power series techniques give us exactly the same solution space as the solutions obtained via eigenvalue / eigenfunction arguments. Finally, observe that (with $a_0 = a_1 = 1$)

$$y_1(x) + y_2(x) = e^x \qquad \text{and} \qquad y_1(x) - y_2(x) = e^{-x}.$$

Adding these two equations together renders

$$2y_1(x) = e^x + e^{-x} \implies y_1(x) = \frac{e^x + e^{-x}}{2} = \cosh(x),$$

and subtracting the second equation from the first renders

$$2y_2(x) = e^x - e^{-x} \implies y_2(x) = \frac{e^x - e^{-x}}{2} = \sinh(x).$$

Therefore

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} \qquad \text{and} \qquad \sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}.$$

These two power series are also useful to have in your quiver for future work. That is, if you encounter one of these power series, you should recognize that it is one you have seen before, and you should have a handy place to go look it up (e.g. WolframAlpha, Wikipedia, etc.). □

Problem 10 (5.2.2)**.** Solve the differential equation

$$y'' - xy' - y = 0$$

around the point $x_0 = 0$ using power series techniques. Find a recurrence relation for the coefficients in the power series, give the first four terms and, if possible, find the general term.

Solution. Again, a series solution of the form $y(x) = \sum a_n x^n$, when substituted into the differential equation, gives

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - x \sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0} a_n x^n \\ &= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} \left(a_{n+2}(n+2)(n+1) - (n+1)a_n\right) x^n. \end{aligned}$$

For this to hold, it must be the case that

$$a_{n+2}(n+2)(n+1) - (n+1)a_n = 0 \implies a_{n+2} = \frac{1}{n+2} a_n.$$

Taking a_0 arbitrary and $a_1 = 0$, this implies that

$$a_2 = \frac{1}{2}a_0, \qquad a_4 = \frac{1}{2\cdot 4}a_0, \qquad a_6 = \frac{1}{2\cdot 4\cdot 6}a_0.$$

The general form of this sequence can be obtained by noticing that each of the $n/2$ terms in the denominator contains a factor of 2. Factoring this out gives

$$a_n = \frac{1}{2\cdot 4\cdot\dots\cdot(n-2)\cdot n}a_0 = \frac{1}{2^{n/2}(1\cdot 2\cdot\dots\cdot\frac{n}{2})}a_0 = \frac{1}{2^{n/2}(\frac{n}{2})!}a_0.$$

This gives a solution to the original differential equation of the form

$$y_1(x) = \sum_{n \text{ even}} \frac{1}{2^{n/2}(\frac{n}{2})!}a_0x^n = a_0\sum_{m=0}^{\infty}\frac{1}{2^m m!}x^{2m} = a_0\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{x^2}{2}\right)^m = a_0\mathrm{e}^{x^2/2}.$$

It can be quickly verified (either by hand computation or with the aid of a computer algebra system) that this is, indeed a solution to the original differential equation.

Now, with $a_0 = 0$ and a_1 an arbitrary constant,

$$a_3 = \frac{1}{3}a_1, \qquad a_5 = \frac{1}{3\cdot 5}a_1, \qquad a_7 = \frac{1}{3\cdot 5\cdot 7}a_1.$$

In general, when n is odd, the coefficient a_n is given by

$$a_n = \frac{1}{1\cdot 3\cdot 5\cdot\dots\cdot(n-2)\cdot n}a_1.$$

The fraction can be manipulated into something a bit more tractable by multiplying upstairs and downstairs by the "missing" even terms:

$$\frac{1}{1\cdot 3\cdot 5\cdot\dots\cdot n}a_1 = \frac{2\cdot 4\cdot\dots\cdot(n-1)}{\big(2\cdot 4\cdot\dots\cdot(n-1)\big)\big(1\cdot 3\cdot 5\cdot\dots\cdot n\big)}a_1 = \frac{2^{(n-1)/2}(\frac{n-1}{2})!}{n!}a_1.$$

Therefore a second solution to the original differential equation is given by

$$y_2(x) = \sum_{n \text{ odd}} \frac{2^{(n-1)/2}(\frac{n}{2})!}{n!}a_1x^n = a_1\sum_{m=0}^{\infty}\frac{2^m m!}{(2m+1)!}x^n.$$

This particular power series does not have a "nice" closed form in terms of elementary functions. If we use a computer algebra system to find a closed form, we might get something like

$$\sum_{m=0}^{\infty}\frac{2^m m!}{(2m+1)!}x^m) = \frac{\sqrt{\frac{\pi}{2}}\mathrm{e}^{x/2}\,\mathrm{erf}\Big(\frac{\sqrt{x}}{\sqrt{2}}\Big)}{\sqrt{x}},$$

where the erf function (the "error function"), which comes from integrating a Gaussian curve (specifically, the standard normal distribution from probability), and which has no closed form in terms of elementary functions. On the bright side, the power series can be used to find numerical approximations of arbitrary precision. □

Problem 11 (5.2.10)**.** Solve the differential equation

$$(4 - x^2)y'' + 2y = 0$$

around the point $x_0 = 0$ using power series techniques. Find a recurrence relation for the coefficients in the power series, give the first four terms and, if possible, find the general term.

Solution. Using the techniques employed above, but eliding much of the discussion:

$$\begin{aligned}
0 &= (4 - x^2)y'' + 2y \\
&= (4 - x^2)\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + 2\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} 4a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} a_n n(n-1)x^n + \sum_{n=0}^{\infty} 2a_n x^n \\
&= \sum_{n=0}^{\infty} \Big(4a_{n+2}(n+2)(n+1) - a_n(n^2 - n - 2)\Big) x^n \\
&= \sum_{n=0}^{\infty} (4a_{n+2}(n+2)(n+1) - a_n(n-2)(n+1))\, x^n.
\end{aligned}$$

Then

$$4a_{n+2}(n+1)(n+2) - a_n(n-2)(n+1) = 0 \implies a_{n+2} = \frac{n-2}{4(n+2)}a_n.$$

With a_0 arbitrary, a_2 is given by

$$a_2 = -\frac{1}{4}a_0.$$

But then $a_n = 0$ for all larger even values of n. Hence one solution is given by

$$y_1(x) = a_0\left(1 - \frac{1}{4}x^2\right).$$

With a_1 arbitrary, the first several odd coefficients are given by

$$a_3 = -\frac{1}{4 \cdot 3}a_1, \quad a_5 = -\frac{1}{4^2 \cdot 3 \cdot 5}a_1, \quad a_7 = -\frac{1}{4^3 \cdot 5 \cdot 7}a_1, \quad a_9 = -\frac{1}{4^4 \cdot 7 \cdot 9}a_1.$$

For odd n, the general term is given by

$$a_n = -\frac{1}{4^{(n-1)/2}(n-2)n}a_1.$$

Therefore a second solution to the original differential equation is given by

$$y_2(x) = a_1 - \sum_{n \text{ odd}} \frac{1}{4^{(n-1)/2}(n-2)n}a_1 x^n = a_1\left(1 - \sum_{m=1}^{\infty} \frac{1}{4^m(2m-1)(2m+1)}x^{2m+1}\right).$$

This series does not have a "nice" closed form expression in terms of a geometric series, or in terms of exponential, logarithmic, trigonometric, or hyperbolic trigonometric functions, which are the kinds of series which you are expected to recognize. However, this series can be simplified to a function in terms of the inverse hyperbolic tangent function. Demonstrating this rigorously requires techniques with which you are likely unfamiliar at this time (such as summation by parts). □

Problem 12 (5.3.6)**.** Determine a lower bound for the radius of convergence of series solutions of the differential equation

$$(x^2 - 2x - 3)y'' + xy' + 4y = 0 \tag{10}$$

about the points $x_0 = 4$, $x_0 = -4$, and $x_0 = 0$.

Solution. It is stated (though not rigorously proved) on page 251 of the course text that, under certain hypotheses, the radius of convergence of a power series solution of the differential equation

$$Py'' + Qy' + Ry = 0$$

can be bounded in terms of the coefficient function P. Specifically, if P is a polynomial, then the differential equation has a power series solution centered at x_0 with a radius of convergence which is at least the minimum distance from x_0 to any root of P. That is

$$\rho \geq \min\{|x_0 - \omega| \mid P(\omega) = 0\}. \tag{11}$$

The essential idea behind this bound is that on any open set where P is nonzero, the differential equation can be written in normal form as

$$Py'' + Qy' + Ry = 0 \iff y'' + \frac{Q}{P}y' + \frac{R}{P}y = 0.$$

Heuristically, we expect solutions to "blow up" near singular points, i.e. those points where P vanishes. Away from these singularities, power series solutions will converge, which means that the radius of convergence is at least the distance from the center of the power series expansion to the nearest zero of P. On the other hand, zeros of Q and R might cancel with zeros of P, giving a larger radius of convergence. In either event, the estimate at (11) holds.

In the given problem,

$$P(x) = x^2 - 2x - 3 = (x-3)(x+1),$$

which has roots at $x = 3$ and $x = -1$. To simplify notation, let ρ_x denote the radius of convergence of a power series solution of (10). As above,

$$\begin{aligned}\rho_x &\geq \min\{|x_0 - \omega| \mid \omega^2 - 2\omega - 3 = 0\} \\ &= \min\big\{|x_0 - \omega| \mid \omega \in \{-1, 3\}\big\} \\ &= \min\{|x_0 + 1|, |x_0 - 3|\}.\end{aligned}$$

Therefore

$$\begin{aligned}\rho_4 &= \min\{|4+1|, |4-3|\} = \min\{5, 1\} = 1,\\ \rho_{-4} &= \min\{|-4+1|, |-4-3|\} = \min\{3, 7\} = 3,\\ \rho_0 &= \min\{|0+1|, |0-3|\} = \min\{1, 3\} = 1.\end{aligned}$$

In this case, we might expect these bounds to be sharp (i.e. as good as possible), as there is no obvious cancelation of zeros in with the remaining coefficient functions. □