146B — Fourier Series

Let $f : [-L, L] \to \mathbb{R}$.

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(\frac{n\pi x}{L}) + b_n sin(\frac{n\pi x}{L})]$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})dx \ \ , n \in \mathbb{N}_0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})dx, n \in \mathbb{N}$$

$1^\circ f$ is odd: $a_n = 0; 2^\circ f$ is even: $b_n = 0$

Convergence

f is said to be piecewise continuous on $[a, b]$, if the interval is divided into a finite number of subintervals, $(x_0, x_1), (x_1, x_2), \cdots, (x_{n-1}, x_n)$, $a = x_0$, $b = x_n$. So that,

$1^\circ f$ is continuous on(x_{i-1}, x_i), for $i = 1, 2, \ldots, n$

$2^\circ f$ has finite (one-sided) limit at the endpoints x_i, for $i = 1, 2, \ldots, n$

Ass. f and its derivative f' are piecewise continuous on $[-L, L]$. f has a period of $2L$. Then f has a Fourier expansion

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})]$$

The Fourier series converges to $f(x)$ where f is continuous, and converges to,

$$\frac{f(x+) + f(x-)}{2}$$

at points where f is discontinuous. The value of f at the discontinuities need not be the average of the left and right hand limits.

The heat equation

$$\begin{cases} u_t = u_{xx}, 0 < x < L \text{ and } t > 0 \\ u(0, t) = u(L, t) = 0 \\ u(x, 0) = f(x) \end{cases}$$

Separation of variables, for the eigenvalues

$$\lambda_n = (\frac{n\pi}{L})^2,$$

with corresponding eigenfunctions

$$X_n(x) = \sin(\frac{n\pi}{L}x), \text{ and } T_n(x) = e^{-(\frac{n\pi}{L})^2 t}$$

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And $u_n(x, t) = X_n(x)T_n(x) = sin(\frac{n\pi}{L}x)e^{-(\frac{n\pi}{L})^2 t}$ for $n \in \mathbb{N}$.

Then the general solution is

$$u(x, t) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi}{L}x)e^{-(\frac{n\pi}{L})^2 t}$$

We can see that the last condition gives us that,

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi}{L}x)e^{-(\frac{n\pi}{L})^2 0} = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi}{L}x) = f(x)$$

We need to extend $f : [0, L] \to \mathbb{R}$ $[-L, L]$ to a function $F : [-L, L] \to \mathbb{R}$ to get a Fourier expansion of F.

And we need the extension to be an odd function to get a Fourier expansion involving only $\sin(\frac{n\pi}{L}x)$.

Then define $F : [-L, L] \to \mathbb{R}; F(x) = \begin{cases} f(x) & \text{, for} \quad 0 < x < L \\ -f(-x), & \text{for} \ -L < x < 0 \end{cases}$

$$b_n = \frac{1}{L}\int_{-L}^{L} F(x)\sin(\frac{n\pi}{L}x)dx = \frac{2}{L}\int_{0}^{L} F(x)\sin(\frac{n\pi}{L}x)dx \ , \quad \text{since } F(x)\sin(\frac{n\pi}{L}x) \text{ is even.}$$

Notice $F = f$ on $[0, L]$ and c_n corresponds to b_n so,

$$c_n = \frac{2}{L}\int_{0}^{L} f(x)\sin(\frac{n\pi}{L}x)dx.$$

Now to consider general classes of solutions of partial differential equations, we need to see if the first two conditions determine a series of $\cos(\frac{n\pi}{L}x)$. So, we need to consider the even extension F of f,

Define $F : [-L, L] \to \mathbb{R}; F(x) = \begin{cases} f(x) & \text{, for} \quad 0 < x < L \\ f(-x) & \text{, for} \ -L < x < 0 \end{cases}$

Similarly, $a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi}{L}x)dx$. This is called the cosine expansion, and the c_n are called the sine expansion. Also called the half-range expansions for $f : [0, L] \to \mathbb{R}$, corresponding to sine and cosine series.