Math 146B.001 (Ordinary and Partial Differential Equations)
Homework 02 Solutions

Problem 1 (4.2.1)**.** Express $1 + i$ in the form $R(\cos(\theta) + i\sin(\theta) = Re^{i\theta}$.

Solution. Recall that
$$x + iy = Re^{i\theta},$$
where
$$R^2 = |x + iy|^2 = x^2 + y^2 \qquad \text{and} \qquad \tan(\theta) = \frac{y}{x}.$$
With $x = 1$ and $y = 1$, this gives
$$R^2 = 1^2 + 1^2 = 2 \implies R = \sqrt{2},$$
and
$$\tan(\theta) = \frac{1}{1} = 1 \implies \theta \in \{\arctan(1) + k\pi \mid k \in \mathbb{Z}\}.$$
The arctan function will always give a value between $-\pi/2$ and $\pi/2$, which corresponds to an angle in the first or fourth quadrant, hence it is necessary to choose a value of k which gives a point in the correct quadrant. In this case, $1 + i$ is in the first quadrant, hence we may take $k = 0$. Thus
$$\theta = \arctan(1) + 0\pi = \frac{\pi}{4}.$$
Therefore
$$1 + i = \sqrt{2}e^{i\pi/4} = \sqrt{2}\left(\cos\left(\tfrac{\pi}{4}\right) + i\sin\left(\tfrac{\pi}{4}\right)\right).$$
Note that this polar representation is not unique: we chose $k = 0$ as the given point is in the first quadrant, and $\arctan(1) + 0\pi$ is an appropriate angle in the first quadrant. However, if k is *any* even integer, then $\arctan(1) + k\pi$ will be coterminal with (i.e., it will be the same angle as) $\pi/4$. Thus other answers are possible. □

Problem 2 (4.2.5)**.** Express $\sqrt{3} - i$ in the form $R(\cos(\theta) + i\sin(\theta) = Re^{i\theta}$.

Solution. The process is the same as in the previous problem, with $x = \sqrt{3}$ and $y = -1$. Thus
$$R^2 = \sqrt{3}^2 + (-1)^2 = 4 \implies R = 2$$
(note that the solution $R^2 = 4$ has two solutions, but R represents the modulus (absolute value, magnitude) of a complex number, hence $R > 0$). The argument θ satisfies
$$\tan(\theta) = -\frac{1}{\sqrt{3}} \implies \theta \in \{\arctan(-1/\sqrt{3}) + k\pi \mid k \in \mathbb{Z}\}.$$
As the point $\sqrt{3} - i$ is in quadrant IV, we may take $k = 0$. Therefore, since $\arctan(-1/\sqrt{3}) = -\pi/6$, we have
$$\sqrt{3} - i = 2e^{-i\pi/6} = 2\left(\cos\left(-\tfrac{\pi}{6}\right) + i\sin\left(-\tfrac{\pi}{6}\right)\right).$$
Again, this representation is not unique, as we may choose $k = 2m\pi$ for any $m \in \mathbb{Z}$. □

Problem 3 (4.2.7)**.** Determine the roots $1^{1/3}$, i.e. the third roots of 1, thought of as a complex number (NB: these are called *third roots of unity*).

Solution. If a complex number z is written in the polar form

$$Re^{i\theta},$$

then the n-th roots of R, denoted by $R^{1/n}$, are given by

$$R^{1/n} = \left\{ \sqrt[n]{R}e^{i(\theta+2k\pi)/n} \;\middle|\; k = 0, 1, \ldots, n-1 \right\},$$

where $\sqrt[n]{R}$ denotes the *principle n-th root* of R, which is the unique nonnegative real number ω with the property that $\omega^n = R$. Since $z = 1 = 1 \cdot e^{i \cdot 0}$, it follows that

$$1^{1/3} = \left\{ 1 \cdot e^{i(0+2k\pi)/3} \;\middle|\; k = 0, 1, 2 \right\} = \left\{ 1, e^{i2\pi/3}, e^{i4\pi/3} \right\}.$$

Using the identity $Re^{i\theta} = R(\cos(\theta) + i\sin(\theta))$, these may be written in Cartesian form as

$$1^{1/3} = \left\{ 1, \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right), \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) \right\} = \left\{ 1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2} \right\}.$$

□

Problem 4 (4.2.8)**.** Determine the roots $(1-i)^{1/2}$, i.e. the square roots of $1-i$.

Solution. The computation is similar to that in the previous problem. Note that

$$1 - i = \sqrt{2}e^{-i\pi/4}.$$

Therefore

$$(1-i)^{1/2} = \left\{ \sqrt[4]{2}e^{i(-\pi/4+2k\pi)/2} \;\middle|\; k = 0, 1 \right\} = \left\{ \sqrt[4]{2}e^{-i\pi/8}, \sqrt[4]{2}e^{i7\pi/8} \right\}.$$

It is possible to express these values exactly in Cartesian form using the "half-angle formulæ" for sine and cosine, but this is rather tedious and unenlightening in the current context. Decimal approximations can be found using a calculator, but this is similarly unenlightening, and has therefore been omitted. □

Problem 5 (4.2.11)**.** Find the general solution to the differential equation

$$y''' - y'' - y' + y = 0.$$

Solution. Begin by making the *ansatz* that solutions are of the form $y(t) = e^{\lambda t}$. Then

$$\begin{aligned} y''' - y'' - y' + y &= \lambda^3 e^{\lambda t} - \lambda^2 e^{\lambda t} - \lambda e^{\lambda t} + e^{\lambda t} \\ &= (\lambda^3 - \lambda^2 - \lambda + 1)e^{\lambda t} \end{aligned}$$

$$
\begin{aligned}
&= \left[\lambda^2(\lambda-1) - (\lambda-1)\right] e^{\lambda t} \\
&= (\lambda^2-1)(\lambda-1)e^{\lambda t} \\
&= (\lambda-1)^2(\lambda+1)e^{\lambda t} \\
&= 0.
\end{aligned}
$$

Hence $\lambda_1 = -1$ (with multiplicity 1), and $\lambda_2 = 1$ (with multiplicity 2). Two solutions corresponding to these two values of λ are given by

$$y_1(t) = e^{-t}, \qquad \text{and} \qquad y_2(t) = e^{t}.$$

Because $\lambda_2 = 1$ is a repeated root of the characteristic polynomial, a third solution corresponding to this root is given by

$$y_3(t) = te^{t}.$$

See page 232 of Boyce and DiPrima for a more detailed discussion, and Problem 4.2.41 for a proof of this fact. It can be verified that y_1, y_2, and y_3 are linearly independent, hence they form a fundamental set of solutions of the differential equation. It therefore follows from Theorem 4.1.2 that every solution of the differential equation is a linear combination of these solutions, i.e.

$$y(t) = k_1e^{-t} + k_2e^{t} + k_3te^{t},$$

where k_1, k_2, and k_3 are arbitrary constants. □

Problem 6 (4.2.13)**.** Find the general solution to the differential equation

$$2y''' - 4y'' - 2y' + 4y = 0.$$

Solution. Using the same technique as in Problem 5, begin by looking for solutions of the form $y(t) = e^{\lambda t}$, where λ is a root of the characteristic polynomial

$$
\begin{aligned}
2\lambda^3 - 4\lambda^2 - 2\lambda + 4 &= 2\lambda^2(\lambda-2) - 2(\lambda-2) \\
&= 2(\lambda^2-1)(\lambda-2) \\
&= 2(\lambda-1)(\lambda+1)(\lambda-2).
\end{aligned}
$$

The roots are $\lambda_1 = 1$, $\lambda_2 = -1$, and $\lambda_3 = 2$, corresponding to the linearly independent solutions

$$y_1(t) = e^{t}, \qquad y_2(t) = e^{-t}, \qquad \text{and} \qquad y_3(t) = e^{2t}.$$

Thus the general solution of the differential equation is given by

$$y(t) = k_1e^{t} + k_2e^{-t} + k_3e^{2t},$$

where k_1, k_2, and k_3 are arbitrary constants. □

Problem 7 (4.2.15)**.** Find the general solution to the differential equation

$$y^{(6)} + y = 0.$$

Solution. Using the same techniques as in Problem 5, begin by looking for solutions of the form $y(t) = e^{\lambda t}$, where λ is a root of the characteristic polynomial

$$\lambda^6 + 1 = (\lambda^2 + 1)(\lambda^4 - \lambda^2 + 1). \qquad \text{(sum/difference of cubes)}$$

The first factor has roots $\lambda_{1,2} = \pm i$. Using the quadratic formula, the second factor has roots which solve the equation

$$\lambda^2 = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

Hence the remaining roots of the characteristic polynomial are given by

$$\lambda_{3,5} = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{1/2} \qquad \text{and} \qquad \lambda_{4,6} = \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^{1/2}$$

(NB: the numbering here may not make sense, but it will be useful to label the roots in the manner farther down). In polar form

$$\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\pi/3},$$

from which it follows that

$$\lambda_{3,5} = \left(e^{i\pi/3}\right)^{1/2} = \left\{e^{i\pi/6}, e^{i2\pi/3}\right\} = \left\{\frac{\sqrt{3}}{2} + i\frac{1}{2}, -\frac{\sqrt{3}}{2} + i\frac{1}{2}\right\}.$$

It is possible to find $\lambda_{5,6}$ by a similar computation, but in this case it is sufficient to recall that the roots of a polynomial with real coefficients always come in conjugate pairs, thus

$$\lambda_{4,6} = \overline{\lambda_{3,5}} = \left\{\frac{\sqrt{3}}{2} - i\frac{1}{2}, -\frac{\sqrt{3}}{2} - i\frac{1}{2}\right\}.$$

We therefore obtain a linearly independent set of solutions of the form

$$\left\{\tilde{y}_j(t) = e^{\lambda_j t} \;\middle|\; j = 1, 2, \ldots, 6\right\}.$$

However, as the λ_j come in conjugate pairs, it may be more useful to express the fundamental set of solutions in terms of sines and cosines. Since $\lambda_1 = \overline{\lambda_2} = i$, the functions

$$y_1(t) = \frac{\tilde{y}_1(t) + \tilde{y}_2(t)}{2} = \cos(t) \qquad \text{and} \qquad y_2(t) = \frac{\tilde{y}_1(t) - \tilde{y}_2(t))}{2i} = \sin(t)$$

are solutions to the original differential equation. By a similar computation,

$$y_3(t) = \frac{\tilde{y}_3(t) + \tilde{y}_4(t)}{2} = \frac{e^{(\frac{\sqrt{3}}{2} + i\frac{1}{2})t} + e^{(\frac{\sqrt{3}}{2} - i\frac{1}{2})t}}{2} = e^{\sqrt{3}t/2}\frac{e^{it/2} + e^{-it/2}}{2} = e^{\sqrt{3}t/2}\cos\left(\frac{t}{2}\right),$$

and

$$y_4(t) = \frac{\tilde{y}_3(t) - \tilde{y}_4(t)}{2} = \mathrm{e}^{\sqrt{3}t/2} \sin\left(\frac{t}{2}\right).$$

Finally, by a similar sequence of computations involving $\tilde{y}_5$ and $\tilde{y}_6$, the remaining two fundamental solutions are

$$y_5(t) = \mathrm{e}^{-\sqrt{3}t/2} \cos\left(-\frac{t}{2}\right), \qquad \text{and} \qquad y_6(t) = \mathrm{e}^{-\sqrt{3}t/2} \sin\left(-\frac{t}{2}\right).$$

For completeness, note that it can be verified that the set

$$\{y_1, y_2, \ldots, y_6\}$$

is a linearly independent set of six solutions to the original sixth order differential equation. Therefore every solution to this equation is of the form

$$\begin{aligned} y(t) &= k_1 y_1(t) + k_2 y_2(t) + k_3 y_3(t) + k_4 y_4(t) + k_5 y_5(t) + k_6 y_6(t) \\ &= k_1 \cos(t) + k_2 \sin(t) + k_3 \mathrm{e}^{\sqrt{3}t/2} \cos\left(\frac{t}{2}\right) + k_4 \mathrm{e}^{\sqrt{3}t/2} \sin\left(\frac{t}{2}\right) \\ &\quad + k_5 \mathrm{e}^{-\sqrt{3}t/2} \cos\left(\frac{t}{2}\right) + k_6 \mathrm{e}^{-\sqrt{3}t/2} \sin\left(\frac{t}{2}\right). \end{aligned}$$

Modulo any typos (which are very likely to exist—"extra credit" points to anyone that finds one), this gives a general solution to the original differential equation. □

Problem 8 (4.2.38)**.** Consider the equation $y^{(4)} - y = 0$.

(i) In Problem 4.1.20, it is shown that if

$$y^{(n)} + p_1(t) y^{(n-1)} + \cdots + p_{n-1}(t) y = 0,$$

then the Wronskian of any set of fundamental solutions is given by

$$W(y_1, y_2, \ldots, y_n)(t) = C \exp\left(-\int p_1(t)\,\mathrm{d}t\right),$$

where C is a constant. This result is Abel's Theorem. Use Abel's theorem to determine the Wronskian of a fundamental set of solutions to the above differential equation.

(ii) Note that $\{\mathrm{e}^t, \mathrm{e}^{-t}, \cos(t), \sin(t)\}$ is a fundamental set of solutions to this differential equation. Determine the Wronskian of this set.

(iii) Note that $\{\cosh(t), \sinh(t), \cos(t), \sin(t)\}$ is a fundamental set of solutions to this differential equation. Determine the Wronskian of this set.

Solution (*i*). In the differential equation $y^{(4)} - y = 0$, the coefficient function p_1 is the zero function; that is, $p_1(t) = 0$. Hence Abel's theorem implies that if $\{y_1, y_2, y_3, y_4\}$ is any fundamental set of solutions to this differential equation, then

$$W(y_1, y_2, y_3, y_4)(t) = C \exp\left(-\int p_1(t)\,\mathrm{d}t\right) = C \exp\left(-\int 0\,\mathrm{d}t\right) = C\mathrm{e}^0 = C.$$

Thus Abel's theorem implies that the Wronskian is a constant. □

Solution (*ii*). By direct computation, using the fact that adding a multiple of one row to another does not change the determinant,

$$
\begin{aligned}
&W(y_1, y_2, y_3, y_4)(t) \\
&\quad = \begin{vmatrix} \mathrm{e}^t & \mathrm{e}^{-t} & \cos(t) & \sin(t) \\ \mathrm{e}^t & -\mathrm{e}^{-t} & -\sin(t) & \cos(t) \\ \mathrm{e}^t & \mathrm{e}^{-t} & -\cos(t) & -\sin(t) \\ \mathrm{e}^t & -\mathrm{e}^{-t} & \sin(t) & -\cos(t) \end{vmatrix} \\
&\quad = \begin{vmatrix} \mathrm{e}^t & \mathrm{e}^{-t} & \cos(t) & \sin(t) \\ 0 & -2\mathrm{e}^{-t} & -\sin(t) - \cos(t) & \cos(t) - \sin(t) \\ 0 & 0 & -2\cos(t) & -2\sin(t) \\ 0 & -2\mathrm{e}^{-t} & \sin(t) - \cos(t) & -\cos(t) - \sin(t) \end{vmatrix} \\
&\qquad\qquad \text{(subtract row 1 from every other row)} \\
&\quad = \mathrm{e}^t \begin{vmatrix} -2\mathrm{e}^{-t} & -\sin(t) - \cos(t) & \cos(t) - \sin(t) \\ 0 & -2\cos(t) & -2\sin(t) \\ -2\mathrm{e}^{-t} & \sin(t) - \cos(t) & -\cos(t) - \sin(t) \end{vmatrix} \\
&\qquad\qquad \text{(expand along the first column)} \\
&\quad = \mathrm{e}^t \begin{vmatrix} -2\mathrm{e}^{-t} & -\sin(t) - \cos(t) & \cos(t) - \sin(t) \\ 0 & -2\cos(t) & -2\sin(t) \\ 0 & 2\sin(t) & -2\cos(t) \end{vmatrix} \qquad \text{(subtract row 1 from row 3)} \\
&\quad = \mathrm{e}^t \left(-2\mathrm{e}^{-t}\right) \begin{vmatrix} -2\cos(t) & -2\sin(t) \\ 2\sin(t) & -2\cos(t) \end{vmatrix} \qquad \text{(expand along the first column)} \\
&\quad = -2\left(4\cos(t)^2 + 4\sin(t)^2\right) \\
&\quad = -8,
\end{aligned}
$$

which is a constant, consistent with the result in Part i. □

Solution (*iii*). Again, the result is a direct computation. Recall that

$$\frac{\mathrm{d}}{\mathrm{d}t}\cosh(t) = \sinh(t) \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\sinh(t) = \cosh(t),$$

and that the hyperbolic cosine and sine functions satisfy the relation

$$\cosh(t)^2 - \sinh(t) = 1$$

for all t. Hence

$$
\begin{aligned}
&W(\cosh(t), \sinh(t), \cos(t), \sin(t)) \\
&\quad = \begin{vmatrix} \cosh(t) & \sinh(t) & \cos(t) & \sin(t) \\ \sinh(t) & \cosh(t) & -\sin(t) & \cos(t) \\ \cosh(t) & \sinh(t) & -\cos(t) & -\sin(t) \\ \sinh(t) & \cosh(t) & \sin(t) & -\cos(t) \end{vmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{vmatrix} \cosh(t) & \sinh(t) & \cos(t) & \sin(t) \\ \sinh(t) & \cosh(t) & -\sin(t) & \cos(t) \\ 0 & 0 & -2\cos(t) & -2\sin(t) \\ 0 & 0 & 2\sin(t) & -2\cos(t) \end{vmatrix} \\
&\qquad\qquad\qquad\qquad\text{(subtract: } R_3 \leftarrow R_3 - R_1 \text{ and } R_4 \leftarrow R_4 - R_2\text{)} \\
&= \cosh(t) \begin{vmatrix} \cosh(t) & -\sin(t) & \cos(t) \\ 0 & -2\cos(t) & -2\sin(t) \\ 0 & 2\sin(t) & -2\cos(t) \end{vmatrix} \\
&\qquad\qquad - \sinh(t) \begin{vmatrix} \sinh(t) & \cos(t) & \sin(t) \\ 0 & -2\cos(t) & -2\sin(t) \\ 0 & 2\sin(t) & -2\cos(t) \end{vmatrix} \\
&\qquad\qquad\qquad\qquad\text{(expand along the first column)} \\
&= \cosh(t)^2 \begin{vmatrix} -2\cos(t) & -2\sin(t) \\ 2\sin(t) & -2\cos(t) \end{vmatrix} - \sinh(t)^2 \begin{vmatrix} -2\cos(t) & -2\sin(t) \\ 2\sin(t) & -2\cos(t) \end{vmatrix} \\
&\qquad\qquad\qquad\qquad\text{(expand along the first column of each matrix)} \\
&= \cosh(t)^2 \left(4\cos(t)^2 + 4\sin(t)^2\right) - \sinh(t)^2 \left(4\cos(t)^2 + 2\sin(t)^2\right) \\
&= 4\left(\cosh(t)^2 - \sinh(t)^2\right) \\
&= 4,
\end{aligned}
$$

which is again a constant. □