§5.4. Euler Equations and Regular Singular Points.

Problem 1. (distinct real roots).

Find the general solution of the Euler equation

$$x^2y'' - 4xy' + 4y = 0. \qquad (x>0).$$

Solution. The indicial equation is $r(r-1) - 4r + 4 = 0$.

$\Rightarrow$ $r^2 - 5r + 4 = 0$, $(r-1)(r-4) = 0$.

So $r_1 = 4$, $r_2 = 1$.

Then $y_1 = x^4$ and $y_2 = x$.

The general solution is $y = c_1x^4 + c_2x$.

Problem 2. (repeated real roots).

Find the general solution of the Euler equation:

$$x^2y'' - 5xy' + 9y = 0. \qquad (x>0).$$

Solution. The indicial equation is $r(r-1) - 5r + 9 = 0$.

$\Rightarrow$ $r^2 - 6r + 9 = 0$, $(r-3)^2 = 0$.

So $r_1 = r_2 = 3$.

Then $y_1 = x^3$ and $y_2 = x^3 \ln(x)$.

The general solution is $y = c_1x^3 + c_2x^3\ln(x)$.

Problem 3. (complex roots).

Find the general solution of the Euler equation:

$$2x^2y'' - 4xy' + 6y = 0. \qquad (x>0)$$

Solution. The indicial equation is $2r(r-1) - 4r + 6 = 0$.

$$\Rightarrow \quad r^2 - 3r + 3 = 0.$$

$$r_{1,2} = \frac{3 \pm \sqrt{3^2 - 4\cdot 3}}{2} = \frac{3}{2} \pm \frac{\sqrt{3}}{2} i.$$

i.e. $\lambda = \frac{3}{2}$ and $\mu = \frac{\sqrt{3}}{2}$.

Then $y_1 = x^{\frac{3}{2}} \cos\left(\frac{\sqrt{3}}{2} \ln(x)\right)$, $y_2 = x^{\frac{3}{2}} \sin\left(\frac{\sqrt{3}}{2} \ln(x)\right)$.

The general solution is $y = c_1 x^{\frac{3}{2}} \cos\left(\frac{\sqrt{3}}{2}\ln(x)\right) + c_2 x^{\frac{3}{2}} \sin\left(\frac{\sqrt{3}}{2}\ln(x)\right)$.

Problem 4. (singular points).

Find all singular points and determine which one is regular.

$$(x+2)^2(x-1)\, y'' + 3(x-1)\, y' - 2(x+2)\, y = 0.$$

Solution. $P(x) = (x+2)^2(x-1)$, $Q(x) = 3(x-1)$, $R(x) = -2(x+2)$.

polynomials. Only check $P(x) = 0$. $\Rightarrow$ $x = 1, -2$.

So $x_0 = 1$, $x_0 = -2$ are singular points.

At $x_0 = 1$:

$$(x-x_0)\frac{Q(x)}{P(x)} = (x-1)\frac{3(x-1)}{(x+2)^2(x-1)} = \frac{3(x-1)}{(x+2)^2} \quad \text{analytic at } x_0 = 1$$

$$(x-x_0)^2 \frac{R(x)}{P(x)} = (x-1)^2 \cdot \frac{-2(x+2)}{(x+2)^2(x-1)} = \frac{-2(x-1)}{(x+2)} \quad \text{analytic at } x_0 = 1.$$

So $x_0 = 1$ is a regular singular point.

At $x_0 = -2$:

$$(x-x_0)\frac{Q(x)}{P(x)} = (x+2)\cdot\frac{3(x-1)}{(x+2)^2(x-1)} = \frac{3}{x+2}. \quad \text{not analytic at } x_0 = -2.$$

So $x_0 = -2$ is irregular.