Math 146B.010 (Ordinary and Partial Differential Equations)
Homework 01 Solutions

Problem 1 (4.1.7)**.** Determine whether

$$f_1(t) = 2t - 3, \qquad f_2(t) = t^2 + 1, \qquad \text{and} \qquad f_3(t) = 2t^2 - 1$$

are linearly independent. If they are linearly dependent, find a linear relation among them.

Solution. Recall that a collection of functions is linearly independent if and only if the determinant of the Wronskian matrix is nonzero. Observe that

$$\begin{aligned}
W(f_1, f_2, f_3)(t) &= \begin{vmatrix} f_1(t) & f_2(t) & f_3(t) \\ f_1'(t) & f_2'(t) & f_3'(t) \\ f_1''(t) & f_2''(t) & f_3''(t) \end{vmatrix} \\
&= \begin{vmatrix} 2t-3 & t^2+1 & 2t^2-1 \\ 2 & 2t & 4t \\ 0 & 0 & 4 \end{vmatrix} \\
&= (2t-3)\begin{vmatrix} 2t & 4t \\ 0 & 4 \end{vmatrix} - 2\begin{vmatrix} t^2+1 & 2t^2-1 \\ 0 & 4 \end{vmatrix} \\
&\qquad \text{(expand along the first column)} \\
&= (2t-3)(8t) - 2(4t^2+4) \\
&= 16t^2 - 24t - 8t^2 + 8 \\
&= 8t^2 - 24t + 8,
\end{aligned}$$

which is not the zero function. As the determinant of the Wronskian matrix is not zero, the functions are linearly independent. □

Problem 2 (4.1.11)**.** Verify that that the functions defined by

$$f_1(t) = 1, \qquad f_2(t) = \cos(t), \qquad \text{and} \qquad f_3(t) = \sin(t)$$

are solutions of the differential equation

$$y''' + y' = 0.$$

Determine the Wronskian determinant, $W(f_1, f_2, f_3)(t)$.

Solution. Verification is a matter of computation:

$$\begin{aligned}
&f_1'''(t) + f_1'(t) = 0 + 0 = 0, && \checkmark \\
&f_2'''(t) + f_2'(t) = \sin(t) - \sin(t) = 0, && \checkmark \\
&f_3'''(t) + f_3'(t) = -\cos(t) + \cos(t) = 0. && \checkmark
\end{aligned}$$

The Wronskian is given by

$$\begin{aligned} W(f_1, f_2, f_3)(t) &= \begin{vmatrix} 1 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \\ 0 & -\cos(t) & -\sin(t) \end{vmatrix} && = \begin{vmatrix} -\sin(t) & \cos(t) \\ -\cos(t) & -\sin(t) \end{vmatrix} \\ & && \text{(expand along the first column)} \\ &= \sin(t)^2 - (-\cos(t)^2) \\ &= 1. \end{aligned}$$

Therefore $W(f_1, f_2, f_3)(t) = 1$, which indicates that the three solutions are linearly independent and form a fundamental set of solutions. □

Problem 3 (4.2.11)**.** Find the general solution to the differential equation

$$y''' - y'' - y' + y = 0.$$

Solution. Begin by making the *ansatz* that solutions are of the form $y(t) = e^{\lambda t}$. Then

$$\begin{aligned} y''' - y'' - y' + y &= \lambda^3 e^{\lambda t} - \lambda^2 e^{\lambda t} - \lambda e^{\lambda t} + e^{\lambda t} \\ &= (\lambda^3 - \lambda^2 - \lambda + 1)e^{\lambda t} \\ &= \left[\lambda^2(\lambda - 1) - (\lambda - 1)\right] e^{\lambda t} \\ &= (\lambda^2 - 1)(\lambda - 1)e^{\lambda t} \\ &= (\lambda - 1)^2(\lambda + 1)e^{\lambda t} \\ &= 0. \end{aligned}$$

Hence $\lambda_1 = -1$ (with multiplicity 1), and $\lambda_2 = 1$ (with multiplicity 2). Two solutions corresponding to these two values of λ are given by

$$y_1(t) = e^{-t}, \qquad \text{and} \qquad y_2(t) = e^t.$$

Because $\lambda_2 = 1$ is a repeated root of the characteristic polynomial, a third solution corresponding to this root is given by

$$y_3(t) = te^t.$$

See page 232 of Boyce and DiPrima for a more detailed discussion, and Problem 4.2.41 for a proof of this fact. It can be verified that y_1, y_2, and y_3 are linearly independent, hence they form a fundamental set of solutions of the differential equation. It therefore follows from Theorem 4.1.2 that every solution of the differential equation is a linear combination of these solutions, i.e.

$$y(t) = k_1 e^{-t} + k_2 e^t + k_3 te^t,$$

where k_1, k_2, and k_3 are arbitrary constants. □

Problem 4 (4.2.13)**.** Find the general solution to the differential equation

$$2y''' - 4y'' - 2y' + 4y = 0.$$

Solution. Using the same technique as in Problem 3, begin by looking for solutions of the form $y(t) = e^{\lambda t}$, where λ is a root of the characteristic polynomial

$$\begin{aligned} 2\lambda^3 - 4\lambda^2 - 2\lambda + 4 &= 2\lambda^2(\lambda - 2) - 2(\lambda - 2) \\ &= 2(\lambda^2 - 1)(\lambda - 2) \\ &= 2(\lambda - 1)(\lambda + 1)(\lambda - 2). \end{aligned}$$

The roots are $\lambda_1 = 1$, $\lambda_2 = -1$, and $\lambda_3 = 2$, corresponding to the linearly independent solutions

$$y_1(t) = e^t, \qquad y_2(t) = e^{-t}, \qquad \text{and} \qquad y_3(t) = e^{2t}.$$

Thus the general solution of the differential equation is given by

$$y(t) = k_1 e^t + k_2 e^{-t} + k_3 e^{2t},$$

where k_1, k_2, and k_3 are arbitrary constants. □

Problem 5 (4.2.17)**.** Find the general solution to the differential equation

$$y^{(6)} - 3y^{(4)} + 3y'' - y = 0.$$

Solution. Using the same technique as in Problem 3, begin by looking for solutions of the form $y(t) = e^{\lambda t}$, where λ is a root of the characteristic polynomial

$$\lambda^6 - 3\lambda^4 + 3\lambda^2 - 1 = (\lambda^2)^3 - 3(\lambda^2)^2 + 3(\lambda^2) - 1 = (\lambda^2 - 1)^3,$$

where the last equality comes from recognizing the binomial expansion of $(a-b)^3$. The roots of the characteristic polynomial are given by $\lambda_1 = 1$ and $\lambda_2 = -1$, each having multiplicity 3. Thus six linearly independent solutions which form a fundamental set of solutions are given by

$$\begin{aligned} &y_1(t) = e^t, && y_2(t) = te^t, && y_3(t) = t^2 e^t, \\ &y_4(t) = e^{-t}, && y_5(t) = te^{-t}, && y_6(t) = t^2 e^{-t}. \end{aligned}$$

Thus the general solution of the differential equation is given by

$$y(t) = k_1 e^t + k_2 te^t + k_3 t^2 e^t + k_4 e^{-t} + k_5 te^{-t} + k_6 t^2 e^{-t},$$

where $\{k_j\}_{j=1}^6$ are arbitrary constants. □

Problem 6 (4.2.29)**.** Find the solution to the initial value problem

$$y''' + y' = 0; \qquad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 2.$$

Solution. Begin by making the *ansatz* that solutions are of the form $y(t) = \mathrm{e}^{\lambda t}$. Then

$$y'''(t) + y'(t) = \lambda^3 \mathrm{e}^{\lambda t} + \lambda \mathrm{e}^{\lambda t} = \lambda(\lambda^2 + 1)\mathrm{e}^{\lambda t} = 0.$$

This last equation has three solutions: $\lambda_1 = i$, $\lambda_2 = -i$, and $\lambda_3 = 0$. The two solutions corresponding to λ_1 and λ_2 are given by

$$\tilde{y}_1(t) = \mathrm{e}^{it}, \qquad \text{and} \qquad \tilde{y}_2(t) = \mathrm{e}^{-it}.$$

Note that any linear combination of solutions is also a solution. Hence, in order to avoid complex solutions, it is convenient to define two alternative solutions

$$y_1(t) = \frac{1}{2}(\tilde{y}_1(t) + \tilde{y}_2(t)) = \frac{1}{2}(\mathrm{e}^{it} + \mathrm{e}^{-it}) = \cos(t),$$

and

$$y_2(t) = \frac{1}{2}(\tilde{y}_1(t) - \tilde{y}_2(t)) = \frac{1}{2}(\mathrm{e}^{it} - \mathrm{e}^{-it}) = \sin(t).$$

Finally, the solution corresponding λ_3 is given by $y_3(t) = \mathrm{e}^{0t} = 1$. Hence the differential equation has a fundamental set of solutions given by

$$\{y_1(t) = \cos(t), y_2(t) = \sin(t), y_3(t) = 1\}.$$

Note that this is a set of three solutions to a third order differential equation, hence it forms a fundamental set of solutions if and only if the solutions are linearly independent. Checking that these solutions are indeed linearly independent is left as an exercise (see Problem 2).

It follows from Theorem 4.1.2 that every solution to the differential equation can be written as a linear combination of solutions from this fundamental set. Thus every solution is of the form

$$y(t) = k_1 \cos(t) + k_2 \sin(t) + k_3,$$

where k_1, k_2, and k_3 are constants. To solve the initial value problem, it is necessary to determine the values of these constants. Observe that

$$\begin{aligned} y(0) = 0 \qquad &\Longrightarrow \qquad k_1 \cos(0) + k_2 \sin(0) + k_3 = k_1 + k_3 = 0, \\ y'(0) = 1 \qquad &\Longrightarrow \qquad -k_1 \sin(0) + k_2 \cos(0) = k_2 = 1, \\ y''(0) = 2 \qquad &\Longrightarrow \qquad -k_1 \cos(0) - k_2 \sin(0) = -k_1 = 2. \end{aligned}$$

It can be immediately seen that $k_1 = -2$ and that $k_2 = 1$. From this, it follows that

$$k_1 + k_3 = 0 \implies k_3 = -k_1 = 2.$$

Therefore the solution to the initial value problem is

$$y(t) = -2\cos(t) + \sin(t) + 2.$$

□

Problem 7 (4.2.31)**.** Find the solution to the initial value problem

$$y^{(4)} - 4y''' + 4y'' = 0; \qquad y(1) = -1, \quad y'(1) = 2, \quad y''(1) = 0, \quad y'''(1) = 0.$$

Solution. Using the techniques from Problem 6, solutions are of the form $y(t) = \mathrm{e}^{\lambda t}$, where λ is a root of the characteristic polynomial

$$\lambda^4 - 4\lambda^3 + 4\lambda^2 = \lambda^2(\lambda^2 - 4\lambda + 4) = \lambda^2(\lambda - 2)^2.$$

The roots are given by $\lambda_{1,2} = 0$ and $\lambda_{3,4} = 2$. Note that both roots have multiplicity two. Thus a fundamental set of solutions is given by

$$\{y_1(t) = 1, y_2(t) = t, y_3(t) = \mathrm{e}^{2t}, y_4(t) = t\mathrm{e}^{2t}\}.$$

Every solution is then of the form

$$y(t) = k_1 + k_2 t + k_3 \mathrm{e}^{2t} + k_4 t\mathrm{e}^{2t},$$

where the k_j are arbitrary constants. To solve the initial value problem, it is necessary to determine the values of these constants. Observe that

$$\begin{aligned} y(1) = -1 &= k_1 + k_2 + k_3\mathrm{e}^2 + k_4\mathrm{e}^2, \\ y'(1) = 2 &= k_2 + 2k_3\mathrm{e}^2 + 3k_4\mathrm{e}^2, \\ y''(1) = 0 &= 4k_3\mathrm{e}^2 + 8k_4\mathrm{e}^2, \\ y'''(1) = 0 &= 8k_3\mathrm{e}^2 + 20k_4\mathrm{e}^2. \end{aligned}$$

Equivalently,

$$\begin{pmatrix} 1 & 1 & \mathrm{e}^2 & \mathrm{e}^2 \\ 0 & 1 & 2\mathrm{e}^2 & 3\mathrm{e}^2 \\ 0 & 0 & 4\mathrm{e}^2 & 8\mathrm{e}^2 \\ 0 & 0 & 8\mathrm{e}^2 & 20\mathrm{e}^2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix},$$

which is a problem that can be approached using techniques from linear algebra. Specifically, a solution can be found by manipulating an augmented matrix via row reduction:

$$\begin{aligned} \left(\begin{array}{cccc|c} 1 & 1 & \mathrm{e}^2 & \mathrm{e}^2 & -1 \\ 0 & 1 & 2\mathrm{e}^2 & 3\mathrm{e}^2 & 2 \\ 0 & 0 & 4\mathrm{e}^2 & 8\mathrm{e}^2 & 0 \\ 0 & 0 & 8\mathrm{e}^2 & 20\mathrm{e}^2 & 0 \end{array}\right) &\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 1 & \mathrm{e}^2 & \mathrm{e}^2 & -1 \\ 0 & 1 & 2\mathrm{e}^2 & 3\mathrm{e}^2 & 2 \\ 0 & 0 & 4 & 8 & 0 \\ 0 & 0 & 8 & 20 & 0 \end{array}\right) \\ &\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \\ &\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right). \end{aligned}$$

Therefore $k_1 = -3$, $k_2 = 2$, and $k_3 = k_4 = 0$. Therefore the solution to the initial value problem is $y(t) = -3 + 2t$. □

Problem 8 (4.2.35)**.** Find the solution to the initial value problem

$$6y''' + 5y'' + y' = 0; \qquad y(0) = -2, \quad y'(0) = 2, \quad y''(0) = 0.$$

Solution. Using the techniques from Problem 6, solutions are of the form $y(t) = \mathrm{e}^{\lambda t}$, where λ is a root of the characteristic polynomial

$$6\lambda^3 + 5\lambda^2 + \lambda = 0 = \lambda(6\lambda^2 + 5\lambda + 1) = \lambda(2\lambda + 1)(3\lambda + 1).$$

The roots are given by $\lambda_1 = 0$, $\lambda_2 = -\frac{1}{2}$, and $\lambda_3 = -\frac{1}{3}$. Thus the general solution to the differential equation is given by

$$y(t) = k_1 + k_2\mathrm{e}^{-t/2} + k_3\mathrm{e}^{-t/3}.$$

To solve the initial value problem, it is necessary to determine the values of these constants. Observe that

$$\begin{aligned} y(0) &= -2 = k_1 + k_2 + k_3, \\ y'(0) &= 2 = -\frac{1}{2}k_2 - \frac{1}{3}k_3, \\ y''(0) &= 0 = \frac{1}{4}k_2 + \frac{1}{9}k_3. \end{aligned}$$

The last equation implies that $k_2 = -\frac{4}{9}k_3$. Substituting this into the second equation gives

$$2 = -\frac{1}{2}\left(-\frac{4}{9}k_3\right) - \frac{1}{3}k_3 = -\frac{1}{9}k_3 \implies k_3 = -18.$$

Hence $k_2 = 8$. Finally, substituting these into the first equation gives

$$-2 = k_1 + 8 - 18 = k_1 - 10 \implies k_1 = 8.$$

Therefore the solution to the initial value problem is given by

$$y(t) = 8 + 8\mathrm{e}^{-t/2} - 18\mathrm{e}^{-t/3}.$$

□

Problem 9 (4.3.1)**.** Determine the general solution to the differential equation

$$y''' - y'' - y' + y = 2\mathrm{e}^{-t} + 3.$$

Solution. As per the result in Problem 3, a general solution to the homogeneous differential equation

$$y''' - y'' - y' + y = 0$$

is given by

$$y_h(t) = k_1\mathrm{e}^{-t} + k_2\mathrm{e}^{t} + k_3t\mathrm{e}^{t}.$$

To solve the inhomogeneous solution via the method of undetermined coefficients, we make a "guess" as to the form of a particular solution. An initial guess might be something of the form

$$y_p(t) = Ae^{-t} + B,$$

where A and B are unknown constants (the undetermined coefficients). However, Ae^{-t} is linearly dependent on one of the solutions to the homogeneous equation. To obtain a linearly independent "guess", multiply by t to get

$$y_p(t) = Ate^{-t} + B.$$

As y_p solves the original differential equation,

$$\begin{aligned}
2e^{-t} + 3 &= y_p'''(t) - y_p''(t) - y_p'(t) + y_p(t) \\
&= (3Ae^{-t} - Ate^{-t}) - (-2Ae^{-t} + Ate^{-t}) - (Ae^{-t} - Ate^{-t}) + (Ate^{-t} + B) \\
&= 4Ae^{-t} + B.
\end{aligned}$$

As e^{-t} and 1 (a constant) are linearly independent, it follows that

$$2e^{-t} + 3 = 4Ae^{-t} + B \iff 2 = 4A \text{ and } 3 = B \iff A = \frac{1}{2} \text{ and } B = 3.$$

Hence the particular solution is given by

$$y_p(t) = \frac{1}{2}te^{-t} + 3.$$

Therefore a general solution to the original differential equation is given by

$$y(t) = y_h(t) + y_p(t) = k_1e^{-t} + k_2e^{t} + k_3te^{t} + \frac{1}{2}te^{-t} + 3,$$

where k_1, k_2, and k_3 are arbitrary constants. □

Problem 10 (4.3.3)**.** Determine the general solution to the differential equation

$$y''' + y'' + y' + y = e^{-t} + 4t.$$

Solution. Begin by solving the homogeneous equation

$$y''' + y'' + y' + y = 0$$

in order to obtain a homogeneous solution. Using the techniques used above, roots of the characteristic polynomial

$$\lambda^3 + \lambda^2 + \lambda + 1$$

are $\lambda_{1,2} = \pm i$ and $\lambda_3 = -1$. As in Problem 6, the imaginary roots give rise to real solutions in terms of sines and cosines. A fundamental set of solutions is given by

$$\{y_1(t) = \cos(t), y_2(t) = \sin(t), y_3(t) = e^{-t}\},$$

and so a general solution to the homogeneous problem is

$$y_h(t) = k_1 \cos(t) + k_2 \sin(t) + k_3 \mathrm{e}^{-t},$$

where k_1, k_2, and k_3 are arbitrary constants.

Now, to find a particular solution to inhomogenous problem, we "guess" that such a solution will be of the form

$$y_p(t) = At\mathrm{e}^{-t} + Bt + C,$$

where A, B, and C are unknown constants. Substituting this into the original differential equation, we obtain

$$\mathrm{e}^{-t} + 4t = y_p'''(t) + y_p''(t) + y_p'(t) + y_p(t) = 2A\mathrm{e}^{-t} + Bt + (B + C).$$

Again taking advantage of the linear independence of e^{-t} and the remaining terms, we conclude that

$$1 = 2A \implies A = \frac{1}{2}, \qquad 4t = Bt + (B + C) \implies B = 4 \text{ and } C = -4.$$

Therefore a general solution to the original differential equation is given by

$$y(t) = y_h(t) + y_p(t) = k_1 \cos(t) + k_2 \sin(t) + k_3 \mathrm{e}^{-t} + \frac{1}{2}\mathrm{e}^{-t} + 4t - 4,$$

where k_1, k_2, and k_3 are arbitrary constants. □

Problem 11 (4.3.13)**.** Determine a suitable form for a particular solution $y_p(t)$ if the method of undetermined coefficients is to be used to solve

$$y''' - 2y'' + y' = t^3 + 2\mathrm{e}^t.$$

Solution. The characteristic polynomial is given by

$$\lambda^3 - 2\lambda^2 + \lambda,$$

which has roots $\lambda_1 = 0$, $\lambda_{2,3} = 1$. Thus a homogeneous solution to the differential equation is given by

$$y_h(t) = k_1 + k_2\mathrm{e}^t + k_3 t\mathrm{e}^t.$$

A reasonable guess of a particular solution is then

$$y_p(t) = At^2\mathrm{e}^t + (B_3t^3 + B_2t^2 + B_1t + B_0),$$

where A and the B_j are unknown constants. Note that this guess consists of two basic terms: an exponential term which is related to, but linearly independent from, the term $2\mathrm{e}^t$ on the right hand side, and an arbitrary cubic polynomial to account for the term t^3. □

Problem 12 (4.3.15)**.** Determine a suitable form for a particular solution $y_p(t)$ if the method of undetermined coefficients is to be used to solve

$$y^{(4)} - 2y'' + y = \mathrm{e}^t + \sin(t).$$

Solution. The characteristic polynomial is given by

$$\lambda^4 - 2\lambda + 1$$

which has roots $\lambda_{1,2} = -1$, $\lambda_{3,4} = 1$. Thus a homogeneous solution to the differential equation is given by

$$y_h(t) = k_1\mathrm{e}^{-t} + k_2 t\mathrm{e}^{-t} + k_3\mathrm{e}^t + k_4 t\mathrm{e}^t.$$

A reasonable guess of a particular solution is then

$$y_p(t) = At^2\mathrm{e}^t + (B_1\cos(t) + B_2\sin(t)),$$

where A, B_1, and B_2 are unknown constants. □