§10.2. Fourier Series

Problem 1. (Symmetric functions)

(1) Assume f is odd, show that $f(x)\cos(\frac{n\pi x}{L})$ is odd, and $f(x)\sin(\frac{n\pi x}{L})$ is even.

(2). Assume f is even, show that $f(x)\cos(\frac{n\pi x}{L})$ is even. and $f(x)\sin(\frac{n\pi x}{L})$ is odd.

Solution. Recall: g is odd if $g(-x)=-g(x)$.

g is even if $g(-x)=g(x)$.

Check (1).

$$f(-x)\cos(-\frac{n\pi x}{L}) = f(-x)\cos(\frac{n\pi x}{L}) = -f(x)\cos(\frac{n\pi x}{L}).$$

$$f(-x)\sin(-\frac{n\pi x}{L}) = -f(-x)\sin(\frac{n\pi x}{L}) = (-1)^2 f(x)\sin(\frac{n\pi x}{L}) = f(x)\sin(\frac{n\pi x}{L}).$$

(2)

$$f(-x)\cos(-\frac{n\pi x}{L}) = f(-x)\cos(\frac{n\pi x}{L}) = f(x)\cos(\frac{n\pi x}{L}).$$

$$f(-x)\sin(-\frac{n\pi x}{L}) = -f(-x)\sin(\frac{n\pi x}{L}) = -f(x)\sin(\frac{n\pi x}{L}).$$

Problem 2. Show that $f(x) = |x|$, $-\pi < x < \pi$.

is even.

Check: $f(-x) = |-x| = |x| = f(x)$ on $-\pi < x < \pi$.

Problem 3. (Fourier expansion).

Expand $f(x) = |x|$, $-\pi < x < \pi$ as Fourier Series.

Solution. By previous two problems, $f(x)$ is even.

$L = \pi$. $f(x)\cos(nx)$ is even and $f(x)\sin(nx)$ is odd.

So $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = 0$, $n = 1, 2, \cdots$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} |x|\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx.$$

$$= \frac{2}{\pi}\cdot\frac{1}{n^2}\int_0^{n\pi} u\cos(u)\,du. \qquad \left\{\begin{array}{l} u = nx \\ du = n\,dx. \end{array}\right. \quad \left\{\begin{array}{l} \pi \rightsquigarrow n\pi \\ 0 \rightsquigarrow 0. \end{array}\right.$$

$$= \frac{2}{\pi n^2}\cdot\left[u\sin(u)\Big|_0^{n\pi} - \int_0^{n\pi}\sin(u)\,du\right] \quad \text{(integration by parts)}.$$

$$= \frac{2}{\pi n^2}\left[-\int_0^{n\pi}\sin(u)\,du\right]$$

$$= \frac{2}{\pi n^2}\cdot\cos(u)\Big|_0^{n\pi} = \frac{2}{\pi n^2}\left[\cos(n\pi) - 1\right] = \frac{2\left[(-1)^n - 1\right]}{n^2\pi}, \quad n = 1, 2, \cdots$$

So $$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2\left[(-1)^n - 1\right]}{n^2\pi}\cos(nx).$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\cdot\left(\cos(x) + \frac{1}{9}\cos(3x) + \frac{1}{25}\cos(5x) + \cdots\right).$$

Remark: If f is even, then its Fourier series expansion only consists of cosine terms. (including $\cos(\frac{0\cdot\pi x}{L}) = 1$).

If f is odd, then its Fourier series expansion only consists of sine terms. (example in lecture).

Problem 4. (Fourier Series).

Let. $f(x) = |x|$ on $-\pi < x < 0$ and $0 < x < \pi$.

then $$f'(x) = \begin{cases} -1, & -\pi < x < 0. \\ 1, & 0 < x < \pi. \end{cases}$$ (example in lecture).

Verify that the differentiation can be taken term by term on the Fourier expansion.

Solution. Previous problem $f(x) \sim \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{n^2\pi}\cos(nx)$.

So $$\left(\frac{2[(-1)^n - 1]}{n^2\pi}\cos(nx)\right)' = \frac{2\left[1 - (-1)^n\right]}{n\pi}\sin(nx). \quad n = 1, 2, \cdots$$

Example in lecture: $f'(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1 - (-1)^n\right)\sin(nx)$.