Non-homogeneous ODE (2^{nd} order).

Problem 1. (Undetermined Coefficients).

Find a particular solution of $y''+4y'+4y=3e^{-2t}$.

First look at $y''+4y'+4y=0$. Its characteristic equation is $r^2+4r+4=0$. $\Rightarrow$ $r=-2$ (repeated).

So $y_1=e^{-2t}$ and $y_2=te^{-2t}$.

The right side $g(t)=3e^{-2t}$ solves $y''+4y'+4y=0$.
(involve one of the fundamental solutions).

So $Y(t)=At^se^{-2t}$ for some integer $s \geq 0$.

We must take $s=2$. $\Rightarrow$ $Y(t)=At^2e^{-2t}$.

Then $Y'(t)=2Ate^{-2t}-2At^2e^{-2t}$ and $Y''(t)=2Ae^{-2t}-8Ate^{-2t}+4At^2e^{-2t}$.

Plug in Y : $Y''+4Y'+4Y=2Ae^{-2t}$. $=$ right hand side $g(t)$ $=3e^{-2t}$.

$\Rightarrow$ $2A=3$. i.e. $A=\frac{3}{2}$.

Hence, $Y(t)=\frac{3}{2}t^2e^{-2t}$ is a particular solution. ※.

Problem 2. (Variation of Parameters).

Find a particular solution of $y''+4y=3\csc(t)$.

First look at $y''+4y=0$. Its characteristic equation is $r^2+4=0$. $\Rightarrow$ $r = \pm 2i$ ($\lambda=0$ and $\mu=2$).

So $y_1 = \cos(2t)$ and $y_2 = \sin(2t)$.

Right hand side $g(t) = 3\csc(t)$, assume $Y = u_1 y_1 + u_2 y_2$.

$$W(y_1, y_2) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \det\begin{pmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{pmatrix} = 2.$$

$$u_1 = \int \frac{-g(t)\, y_2(t)}{W(y_1,y_2)(t)}\,dt = -\int \frac{3\csc(t)\cdot\sin(2t)}{2}\,dt$$

$$= -\frac{3}{2}\int \frac{1}{\sin(t)}\cdot 2\cdot\sin(t)\cos(t)\,dt = -3\sin(t) + C_1.$$

$$u_2 = \int \frac{g(t)\, y_1(t)}{W(y_1,y_2)(t)}\,dt = \int \frac{3\csc(t)\cos(2t)}{2}\,dt$$

$$= \frac{3}{2}\int \csc(t)\cdot(1-2\sin^2(t))\,dt = \frac{3}{2}\int \csc(t)\,dt - 3\int \sin(t)\,dt$$

$$= \frac{3}{2}\ln|\csc(t)-\cot(t)| + 3\cos(t) + C_2.$$

$$Y = -3\sin(t)\cos(2t) + \frac{3}{2}\ln|\csc(t)-\cot(t)|\sin(2t) + 3\cos(t)\sin(2t)$$
$$+ C_1\cos(2t) + C_2\sin(2t).$$

Problem 3. (Power Series).

Assume $\sum_{n=2}^{\infty} a_n\cdot n\cdot(n-1)x^{n-2} = -\sum_{n=0}^{\infty} a_n x^n$, determine a_n.

Shift index: $\sum_{n=2}^{\infty} a_n\, n(n-1)x^{n-2} = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n$

So $\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n = \sum_{n=0}^{\infty}(-a_n)x^n$. $\Rightarrow$ $a_{n+2}(n+2)(n+1) = -a_n$.

Hence $(n+2)!\, a_{n+2} = (-1)\cdot n!\, a_n$. for $n=0,1,2,\cdots$

Let $n=2m$, $(2m+2)!\,(-1)^{m+1} a_{2m+2} = (-1)^m (2m)!\, a_{2m} = \cdots = a_0$.

Let $n=2m+1$, $(2m+3)!\,(-1)^{m+1} a_{2m+3} = (-1)^m (2m+1)!\, a_{2m+1} = \cdots = a_1$.

So $a_n = \begin{cases} (-1)^m a_0/(2m)! & \text{if } n=2m \\ (-1)^m a_1/(2m+1)! & \text{if } n=2m+1 \end{cases}$ $m=0,1,2,\cdots$.