146C — 2

RJ Acuña

(862079740)

I.3

5 Derive the equation of one-dimensional diffusion in a medium that is moving along the x axis to the right at constant speed V.

slu.

The mass function of the dye is given by the integral of the concentration between the start a up to some $x > a$,

$$M(x,t) = \int_a^x u(y,t)\,\mathrm{d}y \implies \frac{\partial M}{\partial t} = \int_a^x u_t(y,t)\,\mathrm{d}y$$

Fick's law says that the mass in this section of pipe cannot change except by flowing in or out of its ends. So,

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \text{flow in} - \text{flow out}$$

Since the rod is moving with constant speed V to the right, we have,

flow in $= ku_x(x,t) - Vu(x,t)$ and flow out $= ku_x(a,t) - Vu(a,t)$

$$\implies \int_a^x u_t(y,t)dy = ku_x(x,t) - Vu(x,t) - (ku_x(a,t) - Vu(a,t))$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x u_t(y,t)dy = \frac{\mathrm{d}}{\mathrm{d}x}\left(ku_x(x,t) - Vu(x,t) - (ku_x(a,t) - Vu(a,t))\right)$$

$$\overset{\text{FTC 1}}{\implies} u_t(x,t) = ku_{xx}(x,t) - Vu_x(x,t) - (k\cdot 0 - V\cdot 0)$$

$$\implies u_t = ku_x - Vu_{xx} \quad \blacksquare$$

6 Consider heat flow in a long circular cylinder where the temperature depends only on t and on the distance r to the axis of the cylinder. Here $r = \sqrt{x^2+y^2}$ is the cylindrical coordinate. From the three-dimensional heat equation derive the equation $u_t = k(u_{rr} + u_r/r)$.

slu.

The heat equation is 1.3 equation (10)

$$c\rho\frac{\partial u}{\partial t} = \nabla\cdot(\kappa\nabla u) \iff u_t = \frac{\kappa}{c\rho}\nabla^2 u$$

We have a cylindrical coordinate system,

$$\begin{cases} x = \cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \iff \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \operatorname{atan}(\frac{y}{x}) \\ z = z \end{cases}$$

$$\mathcal{J} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note, $|\mathcal{J}| = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r$, then

$$\mathcal{J}^{-1} = \frac{1}{r}\begin{pmatrix} \begin{vmatrix} r\cos\theta & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & \sin\theta \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} \sin\theta & 0 \\ r\cos\theta & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & -r\sin\theta \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & \cos\theta \\ 0 & -r\sin\theta \end{vmatrix} \\ \begin{vmatrix} -r\sin\theta & r\cos\theta \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} \sin\theta & \cos\theta \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} \end{pmatrix}$$
$$= \frac{1}{r}\begin{pmatrix} r\cos\theta & -\sin\theta & 0 \\ r\sin\theta & \cos\theta & 0 \\ 0 & 0 & r \end{pmatrix} = \begin{pmatrix} \cos\theta & -\frac{1}{r}\sin\theta & 0 \\ \sin\theta & \frac{1}{r}\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then we can see that,

$$\mathcal{J}^{-1} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} & \frac{\partial z}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} & \frac{\partial z}{\partial y} \\ \frac{\partial r}{\partial z} & \frac{\partial \theta}{\partial z} & \frac{\partial z}{\partial z} \end{pmatrix}$$

And from the chain rule we can conclude,

$$\frac{\partial}{\partial z} = 0\frac{\partial}{\partial r} + 0\frac{\partial}{\partial \theta} + 1\frac{\partial}{\partial z} = \frac{\partial}{\partial z} \text{ lol!}$$

Now from the calculations in page 157, we can conclude after squaring the preceding operator that,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

Since u depends only on t we have,

$$\frac{\partial^2 u}{\partial \theta^2} = 0 \text{ and } \frac{\partial^2}{\partial u^2}[z] = 0 \implies \nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}$$

Let $k = \frac{\kappa}{c\rho}$ and the result follows,

$$u_t = k(u_{rr} + u_r/r) \quad \blacksquare$$

7 Solve Exercise 6 in a ball except that the temperature depends only on the spherical coordinate $\sqrt{x^2+y^2+z^2}$. Derive the equation $ut = k(u_{rr} + 2u_r/r)$.

slu.

Chapter 6.1 formula (6) gives us ∇^2 in spherical coordinates,

$$\nabla^2 u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}[u_{\theta\theta} + (\cot\theta)u_\theta + \frac{1}{\sin^2\theta}u_{\phi\phi}]$$

Since u only depends on t and r we have,

$$u_\theta = 0 \text{ and } u_\phi = 0 \implies \nabla^2 u = u_{rr} + \frac{2}{r}u_r$$

Let $k = \frac{\kappa}{c\rho}$ and the result follows,

$$u_t = k(u_{rr} + 2u_r/r) \quad \blacksquare$$

10 If $\mathrm{f}(\mathrm{x})$ is continuous and $|\mathrm{f}(\mathrm{x})| \leq \frac{1}{|\mathrm{x}|^3+1}$ for all x show that

$$\iiint_{\text{all space}} \nabla \cdot \mathrm{f}\, \mathrm{d}s = 0$$

(*Hint:* Take D to be a large ball, apply the divergence theorem, and let its radius tend to infinity)

slu. Let $B_R = \{\mathrm{x} | |x| < R\}$ and bdy $B_R = S_R = \{\mathrm{x} \in \mathbb{R}^3 | \quad |x| = R\}$

$$\begin{aligned}
\left| \iiint_{B_R} \nabla \cdot \mathrm{f}\, \mathrm{d}s \right| &\overset{\text{div. thm.}}{=} \left| \iint_{S_R} \mathrm{f} \cdot \mathrm{n}\, \mathrm{d}s \right| \\
&\leq \iint_{S_R} |\mathrm{f} \cdot \mathrm{n}| ds \leq \iint_{S_R} |\mathrm{f}||\mathrm{n}|\, \mathrm{d}s \leq \iint_{S_R} |\mathrm{f}| 1\, \mathrm{d}s \\
&\leq \iint_{S_R} |\mathrm{f}|\, \mathrm{d}s \leq \iint_{S_R} \left| \frac{1}{|\mathrm{x}|^3+1} \right| \mathrm{d}s = \iint_{S_R} \frac{1}{R^3+1}\, \mathrm{d}s \\
&= \frac{1}{R^3+1} \iint_{S_R} \mathrm{d}s = \frac{4\pi R^2}{R^3+1}
\end{aligned}$$

$$\iiint_{\text{all space}} \nabla \cdot \mathrm{f}\, \mathrm{d}s = \lim_{R\to\infty} \frac{4\pi R^2}{R^3+1} = \lim_{R\to\infty} \frac{4\pi}{R+\frac{1}{R^2}} = 0 \quad \blacksquare$$

11 If curl $\mathrm{v} = 0$ in all of three-dimensional space, show that there exists a scalar function $\phi(x,y,z)$ such that $v =$ grad ϕ.

slu. Suppose curl $\mathrm{v} = 0$,

$$\begin{aligned}
\nabla \times \mathrm{v} &= \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \\
&= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, -\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z} \right), \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \\
&= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) = 0
\end{aligned}$$

$$\Longrightarrow \begin{cases} \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} = 0 \\ \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} = 0 \\ \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0 \end{cases} \quad \Longrightarrow \begin{cases} \frac{\partial v_3}{\partial y} = \frac{\partial v_2}{\partial z} \\ \frac{\partial v_1}{\partial z} = \frac{\partial v_3}{\partial x} \\ \frac{\partial v_2}{\partial x} = \frac{\partial v_1}{\partial y} \end{cases}$$

Let $\phi(x,y,z) = \int_0^z v_3(x,y,z)\, \mathrm{d}z$

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= \int_0^z \frac{\partial v_3}{\partial x}\, \mathrm{d}z = \int_0^z \frac{\partial v_1}{\partial z}\, \mathrm{d}z = v_1 \\
\frac{\partial \phi}{\partial y} &= \int_0^z \frac{\partial v_3}{\partial y}\, \mathrm{d}z = \int_0^z \frac{\partial v_2}{\partial z}\, \mathrm{d}z = v_2 \\
\frac{\partial \phi}{\partial z} &= \int_0^z \frac{\partial v_3}{\partial z}\, \mathrm{d}z = v_3 \\
&\Longrightarrow \mathrm{v} = \nabla \phi \quad \blacksquare
\end{aligned}$$

I.4

(Hint: Use equations (4) and (5) on Page 23. For (a), note that the curl of a gradient is zero.)

1 By trial and error, find a solution of the diffusion equation $u_t = u_{xx}$ with the initial condition $u(x,0) = x^2$.

slu.

Assume $u(x,t) = f(x) + g(t)$,

$$u(x,0) = f(x) + g(0) = x^2 \implies f(x) = x^2 \text{ and } g(0) = 0$$
$$u_t = f_t + g_t = f_{xx} + g_{xx} = u_{xx}$$
$$\implies 0 + g_t = 2 + 0 \implies \frac{\mathrm{d}g}{\mathrm{d}t} = 2 \implies g(t) = 2t + C$$
$$g(0) = 0 \implies C = 0$$

So,

$$u(x,t) = x^2 + 2t \quad \blacklozenge$$

7 In linearized gas dynamics (sound), verify the following.

(a) If curl $\mathrm{v} = 0$ at $t = 0$, then curl $\mathrm{v} = 0$ at all later times.

(b) Each component of v and ρ satifies the wave equation.

slu.

Let $\mathrm{x} \in \mathbb{R}^3$ we know $\nabla \times \mathrm{v}(0,\mathrm{x}) = 0$, we want to show $\forall t > 0,\ \nabla \times \mathrm{v}(\mathrm{t},\mathrm{x}) = 0$. That is that the curl doesn't change. So it's enough to show,

$$\frac{\mathrm{d}}{\mathrm{d}t} \nabla \times \mathrm{v} = 0$$

Now sound satisfies the following system of equations in page 23,

$$\begin{cases} \frac{\partial \mathrm{v}}{\partial t} + \frac{c_0^2}{\rho_0} \nabla \rho = 0 \\ \frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \mathrm{v} = 0 \end{cases}$$

Since,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \nabla \times \mathrm{v} &= \nabla \times \frac{\mathrm{d}\mathrm{v}}{\mathrm{d}t} \\ &= \nabla \times -\frac{c_0^2}{\rho_0} \nabla \rho \\ &= -\frac{c_0^2}{\rho_0} \nabla \times \nabla \rho \\ &= -\frac{c_0^2}{\rho_0} 0 = 0 \\ &\implies \nabla \times \mathrm{v} \equiv C \in \mathbb{R} \\ \nabla \times \mathrm{v}(0,\mathrm{x}) = 0 &\implies C = 0 \end{aligned}$$

So curl v is 0 for all $t > 0$. That completes part (a).

For part (b), differentiating with respect to t the system of eqns. in page 23, should yield the result:

$$
\begin{aligned}
\begin{cases} \frac{\partial \mathrm{v}}{\partial t} + \frac{c_0^2}{\rho_0}\nabla\rho = 0 \\ \frac{\partial \rho}{\partial t} + \rho_0 \nabla\cdot \mathrm{v} = 0 \end{cases}
&\implies \begin{cases} \frac{\partial^2 \mathrm{v}}{\partial t^2} + \frac{\partial}{\partial t}\frac{c_0^2}{\rho_0}\nabla\rho = 0 \\ \frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial t}\rho_0 \nabla\cdot \mathrm{v} = 0 \end{cases} \\
&\implies \begin{cases} \frac{\partial^2 \mathrm{v}}{\partial t^2} + \frac{c_0^2}{\rho_0}\nabla\frac{\partial \rho}{\partial t} = 0 \\ \frac{\partial^2 \rho}{\partial t^2} + \rho_0 \nabla\cdot \frac{\partial \mathrm{v}}{\partial t} = 0 \end{cases} \\
&\implies \begin{cases} \frac{\partial^2 \mathrm{v}}{\partial t^2} + \frac{c_0^2}{\rho_0}\nabla\left(-\rho_0 \nabla\cdot \mathrm{v}\right) = 0 \\ \frac{\partial^2 \rho}{\partial t^2} + \rho_0 \nabla\cdot -\frac{c_0^2}{\rho_0}\nabla\rho = 0 \end{cases} \\
&\implies \begin{cases} \frac{\partial^2 \mathrm{v}}{\partial t^2} - \frac{c_0^2\cancel{\rho_0}}{\cancel{\rho_0}}\nabla\cdot\nabla \mathrm{v} = 0 \\ \frac{\partial^2 \rho}{\partial t^2} - \frac{\cancel{\rho_0}c_0^2}{\cancel{\rho_0}}\nabla\cdot\nabla\rho = 0 \end{cases} \\
&\implies \begin{cases} \frac{\partial^2 \mathrm{v}}{\partial t^2} = c_0^2\nabla^2 \mathrm{v} \\ \frac{\partial^2 \rho}{\partial t^2} = c_0^2\nabla^2\rho \end{cases}
\end{aligned}
$$

2.1

For 3: Assume that the string has an initial square displacement and that waves propagating in opposite directions are produced. Do not assume an initial velocity as you might considering it is a hammer strike. I'd like you to try problems 5 and 6 from this section, but won't require them for homework.

1 Solve $u_{tt} = c^2 u_{xx}$, $u(x,0) = e^x$, $u_t(x,0) = \sin x$.

slu.

$$
\begin{aligned}
u_{tt} = c^2 u_{xx} &\implies u(x,t) = \frac{1}{2}\left(e^{x+ct} + e^{x-ct}\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin x dx \\
&\implies u(x,t) = \frac{1}{2}\left(e^{x+ct} + e^{x-ct}\right) + \frac{1}{2c}\left(\cos(x+ct) - \cos(x-ct)\right) \\
&\implies u(x,t) = e^x\left(\frac{e^{ct}+e^{-ct}}{2}\right) + \frac{1}{2c}\left(\cos(x+ct) - \cos(x-ct)\right) \\
&\implies u(x,t) = e^x\cosh ct + \frac{-2}{2c}\sin\left(\frac{x+ct+x-ct}{2}\right)\sin\left(\frac{x+ct-(x-ct)}{2}\right) \\
&\implies u(x,t) = e^x\cosh ct - \frac{1}{c}\sin x\sin ct \quad \blacklozenge
\end{aligned}
$$

2 Solve $u_{tt} = c^2 u_{xx}, u(x,0) = \log(1+x^2), u_t(x,0) = 4+x$.

slu.

$$
\begin{aligned}
u_{tt} = c^2 u_{xx} &\implies u(x,t) = \frac{1}{2}\left(\log(1+(x+ct)^2) + \log(1+(x-ct)^2)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} 4 + x\,\mathrm{d}x \\
&\implies u(x,t) = \frac{1}{2}\log((1+(x+ct)^2)(1+(x-ct)^2)) + 4t + xt \\
&\implies u(x,t) = \log\left(\sqrt{x^4 + 2(1-c^2t^2)x^2 + (1+c^2t^2)^2}\right) + 4t + xt \quad \blacklozenge
\end{aligned}
$$

3 The midpoint of a piano string of tension T, density ρ, and length l is hit by a hammer whose head diameter is $2a$. A flea is sitting at a distance $l/4$ from one end. (Assume that $a < l/4$; otherwise, poor flea!) How long does it take for the disturbance to reach the flea?

slu.

The hammer hits the midpoint of the string at $l/2$, since the hammer head is $2a$, at $t = 0$, the hammer extends from $l/2 - a$ to $l/2 + a$. This generates two waves going at wave speed $c = \sqrt{T/\rho}$.

The wave speed is constant thus $\Delta x = \sqrt{T/\rho}t$.

The distance between the right side of the string that hits the hammer to the point $3l/4$ where the flea is sitting is,

$$3l/4 - (l/2 + a) = l/4 - a = \Delta x$$

$$\implies l/4 - a = \sqrt{T/\rho}t \implies t = \sqrt{\rho/T}(l/4 - a) \quad \Diamond$$

9 Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0, u(x,0) = x^2, u_t(x,0) = e^x$. (*Hint:* Factor the operator as we did for the wave equation.)

slu.

$$u_{xx} - 3u_{xt} - 4u_{tt} = 0 \iff \left(\frac{\partial^2}{\partial x^2} - 3\frac{\partial^2}{\partial x \partial t} - 4\frac{\partial^2}{\partial t^2}\right)u = 0 \iff \left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)u = 0$$

Let $v = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)u$,

$$\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right)v = 0 \implies v_x - 4v_t = 0 \implies -v_x + 4v_t = 0 \implies v(x,t) = f(4x + t)$$

$$v = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)u \implies u_x + u_t = f(4x + t)$$

$$u_x(x,0) = 2x \text{ and } u_t(x,0) = e^x \implies 2x + e^x = f(4x)$$

Let $g(s) = \frac{1}{4}s$ and $h(s) = 2s + e^s$

$$f = h \circ g \implies f(4x + t) = h(x + \frac{1}{4}t) = 2x + \frac{1}{2}t + e^{x + \frac{1}{4}t} \implies f(4x + 0) = 2x + 0 + e^x + 0$$

So, f works, and we have,

$$u_x + u_t = 2x + \frac{1}{2}t + e^{x + \frac{1}{4}t} \tag{1}$$

Now the left hand side factors as,

$$u_x + u_t = \langle 1, 1\rangle \nabla u$$

This suggests the following change of variables, $y = x + t$ and $z = x - t$ which gives,

$$u_x = u_y + u_z \text{ and } u_t = u_y - u_z$$

$$x = \frac{y+z}{2} \text{ and } t = \frac{y-z}{2}$$

Plugging into (1)

$$u_y + u_z + u_y - u_z = 2u_y = 2\frac{y+z}{2} + \frac{1}{2}\frac{y-z}{2} + e^{\frac{y+z}{2} + \frac{1}{4}\frac{y-z}{2}}$$

$$\implies u_y = \frac{y+z}{2} + \frac{1}{4}\frac{y-z}{2} + \frac{1}{2}e^{\frac{y+z}{2} + \frac{1}{4}\frac{y-z}{2}}$$

$$\implies u_y = \frac{5}{8}y + \frac{3}{8}z + \frac{1}{2}e^{\frac{5}{8}y+\frac{3}{8}z}$$
$$\implies u_y = \frac{5}{8}y + \frac{3}{8}z + \frac{1}{2}e^{\frac{3}{8}z}e^{\frac{5}{8}y}$$

Regarding z as fixed we can integrate with respect to y,

$$\begin{aligned} u(y,z) &= \int \frac{5}{8}y + \frac{3}{8}z + \frac{1}{2}e^{\frac{3}{8}z}e^{\frac{5}{8}y}\,\mathrm{d}y \\ &= \frac{5}{16}y^2 + \frac{3}{8}yz + \frac{8}{10}e^{\frac{3}{8}z}e^{\frac{5}{8}y} + k(z) \end{aligned}$$

$$u(y,z) = \frac{5}{16}y^2 + \frac{3}{8}yz + \frac{4}{5}e^{\frac{3}{8}z}e^{\frac{5}{8}y} + k(z) \tag{2}$$

Then plugging x and t back into (2)

$$u(x,t) = \frac{5}{16}(x+t)^2 + \frac{3}{8}(x+t)(x-t) + \frac{4}{5}e^{\frac{3}{8}(x-t)}e^{\frac{5}{8}(x+t)} + k(x-t)$$

Now simplifying gives

$$\begin{aligned} u(x,t) &= \frac{5}{16}(x+t)^2 + \frac{3}{8}(x+t)(x-t) + \frac{4}{5}e^{\frac{3}{8}(x-t)}e^{\frac{5}{8}(x+t)} + k(x-t) \\ &= \frac{5}{16}(x^2+2xt+t^2) + \frac{3}{8}(x^2-t^2) + \frac{4}{5}e^{\frac{3}{8}(x-t)+\frac{5}{8}(x+t)} + k(x-t) \\ &= \frac{11}{16}x^2 + \frac{5}{8}xt - \frac{1}{16}t^2 + \frac{4}{5}e^{x+\frac{1}{4}t} + k(x-t) \end{aligned}$$

$$u(x,t) = \frac{11}{16}x^2 + \frac{5}{8}xt - \frac{1}{16}t^2 + \frac{4}{5}e^{x+\frac{1}{4}t} + k(x-t) \tag{3}$$

Now we need k such that $u(x,0) = x^2$, so we plug 0 into (3)

$$\begin{aligned} u(x,0) &= \frac{11}{16}x^2 + \frac{5}{8}x0 - \frac{1}{16}0^2 + \frac{4}{5}e^{x+\frac{1}{4}0} + k(x-0) \\ &= \frac{11}{16}x^2 + \frac{4}{5}e^x + k(x) = x^2 \\ \implies k(x) &= x^2 - \frac{11}{16}x^2 - \frac{4}{5}e^x \\ \implies k(x) &= \frac{5}{16}x^2 - \frac{4}{5}e^x \\ \implies k(x-t) &= \frac{5}{16}(x-t)^2 - \frac{4}{5}e^{x-t} \end{aligned}$$

Now we can plug $k(x-t)$ back into (3)

$$\begin{aligned} u(x,t) &= \frac{11}{16}x^2 + \frac{5}{8}xt - \frac{1}{16}t^2 + \frac{4}{5}e^{x+\frac{1}{4}t} + \frac{5}{16}(x-t)^2 - \frac{4}{5}e^{x-t} \\ &= \frac{11}{16}x^2 + \frac{5}{8}xt - \frac{1}{16}t^2 + \frac{4}{5}e^{x+\frac{1}{4}t} + \frac{5}{16}(x^2-2xt+t^2) - \frac{4}{5}e^{x-t} \\ &= \left(\frac{11}{16} + \frac{5}{16}\right)x^2 + \frac{5}{8}xt - \frac{5}{8}xt + \left(\frac{5}{16} - \frac{1}{16}\right)t^2 + \frac{4}{5}e^{x+\frac{1}{4}t} - \frac{4}{5}e^{x-t} \\ &= x^2 + \frac{1}{4}t^2 + \frac{4}{5}e^{x+\frac{1}{4}t} - \frac{4}{5}e^{x-t} \end{aligned}$$

Thus,

$$u(x,t) = x^2 + \frac{1}{4}t^2 + \frac{4}{5}e^{x-t}\left(e^{\frac{5}{4}t} - 1\right) \tag{4}$$

We get $u(x,0) = x^2$ and $u_t(x,0) = e^x$, and,

$u_{xx} = \frac{4}{5}\left(e^{\left(\frac{5}{4}t\right)} - 1\right)e^{(-t+x)} + 2$ and $u_{xt} = -\frac{4}{5}\left(e^{\left(\frac{5}{4}t\right)} - 1\right)e^{(-t+x)} + e^{\left(\frac{1}{4}t+x\right)}$ and $u_{tt} = \frac{4}{5}\left(e^{\left(\frac{5}{4}t\right)} - 1\right)e^{(-t+x)} - \frac{3}{4}e^{\left(\frac{1}{4}t+x\right)} + \frac{1}{2}$

$$u_{xx} - 3u_{xt} - 4u_{tt} = 0$$