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RJ Acuña
(862079740)

1.1

2 Which of the following operators are linear?

Let u, v be real valued functions, $c \in \mathbb{R}$.

(a) $\mathcal{L}u = u_x + xu_y$

$$\begin{aligned}
\mathcal{L}[c(u+v)] &= [c(u+v)]_x + x[c(u+v)]_y \\
&= c(u+v)_x + xc(u+v)_y \\
&= c(u_x+v_x) + xc(u_y+v_y) \\
&= c[(u_x+v_x) + x(u_y+v_y)] \\
&= c[u_x+v_x+xu_y+xv_y] \\
&= c[u_x+xu_y+v_x+xv_y] \\
&= c[\mathcal{L}u + \mathcal{L}v]
\end{aligned}$$

(b) $\mathcal{L}u = u_x + uu_y$

$$\begin{aligned}
\mathcal{L}[cu] &= [cu]_x + [cu][cu]_y \\
&= cu_x + cucu_y \\
&= cu_x + c^2uu_y \\
&\neq cu_x + cuu_y \\
&= c(u_x + uu_y) \\
&= c\mathcal{L}u
\end{aligned}$$

(c) $\mathcal{L}u = u_x + u_y^2$

$$\begin{aligned}
\mathcal{L}[cu] &= [cu]_x + [cu]_y^2 \\
&= cu_x + (cu_y)^2 \\
&= cu_x + c^2u_y^2 \\
&\neq cu_x + cu_y^2 \\
&= c(u_x + u_y^2) \\
&= c\mathcal{L}u
\end{aligned}$$

(d) $\mathcal{L}u = u_x + u_y + 1$

$$\begin{aligned}
\mathcal{L}[cu] &= [cu]_x + [cu]_y + 1 \\
&= cu_x + cu_y + 1 \\
&= cu_x + cu_y + 1 \\
&\neq cu_x + cu_y + c \\
&= c(u_x + u_y + 1) \\
&= c\mathcal{L}u
\end{aligned}$$

(e) $\mathcal{L}u = \sqrt{1+x^2}(cos(y))u_x + u_{yxy} - [\text{arctan}(x/y)]u$

$$\begin{aligned}\mathcal{L}[c(u+v)] &= \sqrt{1+x^2}(cos(y))[c(u+v)]_x + [c(u+v)]_{yxy} - [\text{arctan}(x/y)][c(u+v)] \\ &= c[\sqrt{1+x^2}(cos(y))(u+v)_x + (u+v)_{yxy} - [\text{arctan}(x/y)](u+v)] \\ &= c[\sqrt{1+x^2}(cos(y))(u_x+v_x) + (u_{yxy}+v_{yxy}) - [\text{arctan}(x/y)](u+v)] \\ &= c[\sqrt{1+x^2}(cos(y))u_x + \sqrt{1+x^2}(cos(y))v_x + u_{yxy} + v_{yxy} - [\text{arctan}(x/y)]u - [\text{arctan}(x/y)]v)] \\ &= c[\sqrt{1+x^2}(cos(y))u_x + u_{yxy} - [\text{arctan}(x/y)]u + \sqrt{1+x^2}(cos(y))v_x + v_{yxy} - [\text{arctan}(x/y)]v)] \\ &= c[\mathcal{L}u + \mathcal{L}v]\end{aligned}$$

slu. by the previous observations just (a) and (e) are linear $\Diamond$

3 For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous; provide reasons.

(a) $u_t - u_{xx} + 1 = 0$

$$u_t - u_{xx} + 1 = 0 \iff u_t - u_{xx} = -1$$

So, second order linear inhomogeneous.

(b) $u_t - u_{xx} + xu = 0$

Second order linear homogeneous.

(c) $u_t - u_{xxt} + uu_x = 0$

Third order nonlinear.

(d) $u_{tt} - u_{xx} + x^2 = 0$

$$u_{tt} - u_{xx} + x^2 = 0 \iff u_{tt} - u_{xx} = -x^2$$

So, second order linear inhomogeneous.

(e) $iu_t - u_{xx} + u/x = 0$

Second order linear homogeneous.

(f) $u_x(1+u_x^2)^{-1/2} + u_y(1+u_y^2)^{-1/2} = 0$

First order nonlinear.

(g)$u_x + e^y u_y = 0$

First order linear.

(h) $u_t + u_{xxxx} + \sqrt{1+u} = 0$

Fourth order nonlinear.

4 Show that the difference of two solutions of an inhomogeneous linear equation $\mathcal{L}u = g$ with the same g is a solution of the homogeneous equation $\mathcal{L}u = 0$.

pf.

Let u, and v solve the equation $\mathcal{L}u = g$.

$$\mathcal{L}u = g \text{ and } \mathcal{L}v = g \implies \mathcal{L}(u-v) = \mathcal{L}u - \mathcal{L}v = g - g = 0$$

So, $w = u - v$ is a solution to $\mathcal{L}w = 0$ $\square$

9 Show that the functions $(c_1 + c_2 \sin^2 x + c_3 \cos^2 x)$ form a vector space. Find a basis of it. What is its dimension?

slu.

$$\sin^2 + \cos^2 = 1$$

So, Span$\{1, \sin^2 x\}$ is the 2-dimensional vector space with basis $\{1, \sin^2 x\}$ that contains all the functions of the form $(c_1 + c_2 \sin^2 x + c_3 \cos^2 x)$, where $c_1, c_2, c_3 \in \mathbb{R}$ $\Diamond$

11 Verify that $u(x, y) = f(x)g(y)$ is a solution of the PDE $uu_{xy} = u_x u_y$ for all pairs of (differentiable) functions f and g of one variable.

slu.

$$u(x,y) = f(x)g(y) \implies u_x = f'(x)g(y) \text{ and } u_y = f(x)g'(y) \text{ and } u_{xy} = f'(x)g'(y)$$
$$\text{So, } uu_{xy} = f(x)g(y)f'(x)g'(y) = f'(x)g(y)f(x)g'(y) = u_x u_y$$

So, $uu_{xy} = u_x u_y$ if $u(x, y) = f(x)g(y)$, and f and g are differentiable functions of x, and y respectively $\Diamond$

1.2

1 Solve the first-order equation $2u_t + 3u_x = 0$ with the auxiliary condition $u = \sin x$ when $t = 0$.

slu.

The solution is $u(x, t) = f(3x - 2t)$.

We know that $u(x, 0) = \sin x \implies u(x, 0)f(3x - 2 \cdot 0) = f(3x) = \sin x$.

Let $g(x) = \frac{x}{3}$. Thus,

$$f(3x) = \sin(x) \implies f = \sin \circ g \ .$$

So,

$$u(x,t) = \sin\left(\frac{3x - 2t}{3}\right) \quad \Diamond$$

2 Solve the equation $3u_y + u_{xy} = 0$. (Hint: Let $v = u_y$.)

slu.

Let $v = u_y$. Since, $u_{xy} = u_{yx} = (u_y)_x$. It follows that,

$$3v + v_x = 0 \implies v_x = -3v \implies v = C(y)e^{-3x} \implies u_y = C(y)e^{-3x} \implies u = \left(\int C(y)dy\right)e^{-3x} + f(x)$$

$$3u_y = 3C(y)e^{-3x} \text{ and } \left(u_x = -3\left(\int C(y)dy\right)e^{-3x} + f'(x) \implies u_{xy} = -3C(y)e^{-3x}\right) \implies 3u_y + u_{xy} = 0$$

So, $u(x, y) = \left(\int C(y)dy\right)e^{-3x} + f(x)$ $\Diamond$

3 Solve the equation $(1 + x^2)u_x + u_y = 0$. Sketch some of the characteristic curves.

$$\begin{aligned}
&(1 + x^2)u_x + u_y = \langle 1 + x^2, 1\rangle \nabla u = 0 \\
&\text{So, } \frac{dy}{dx} = \frac{1}{1 + x^2} \implies y = \int \frac{1}{1+x^2}dx = \operatorname{atan}(x) + C \\
&\qquad\qquad\qquad\quad \implies y - \operatorname{atan}(x) = C \\
&\qquad\qquad\qquad\quad \implies u(x,y) = f(y - \operatorname{atan}(x)) \quad \Diamond
\end{aligned}$$

5 Solve the equation $xu_x + yu_y = 0$.

slu.

$$\begin{aligned} xu_x + yu_y &= \langle x, y\rangle \nabla u = 0 \\ \text{So, } \frac{dy}{dx} = \frac{y}{x} &\implies \int \frac{1}{y} dy = \int \frac{1}{x} dx \\ &\implies \ln y = \ln x + C \\ &\implies y = e^{(\ln x + C)} = Cx \\ &\implies C = \frac{y}{x} \\ &\implies u(x,y) = f\left(\frac{y}{x}\right) \quad \Diamond \end{aligned}$$

6 Solve the equation $\sqrt{1-x^2}u_x + u_y = 0$ with the condition $u(0, y) = y$.

slu.

$$\begin{aligned} \sqrt{1-x^2}u_x + u_y &= \langle \sqrt{1-x^2}, 1\rangle \nabla u = 0 \\ \text{So, } \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} &\implies y = \int \frac{1}{\sqrt{1-x^2}} dx = \text{asin}(x) + C \\ &\implies C = y - \text{asin}(x) \\ &\implies u(x,y) = f(y - \text{asin}(x)) \\ u(0,y) = y &\implies u(0,y) = f(y - \text{asin}(0)) = f(y-0) = f(y) = y \\ &\implies f \text{ is the identity map.} \\ &\implies u(x,y) = y - \text{asin}(x) \quad \Diamond \end{aligned}$$

10 Solve $u_x + u_y + u = e^{x+2y}$ with $u(x,0) = 0$.

slu.

$$u_x + u_y = \langle 1, 1 \rangle \nabla u$$

Suggests the change of coordinates $x' = x + y$ and $y' = x - y$ will yield results. Thus,

$$x = \frac{x' + y'}{2} \text{ and } y = \frac{x' - y'}{2}$$

So, $u(x,y) = u(x', y')$:

$$\begin{aligned}
u_x &= u_{x'} x'_x + u_{y'} y'_x \\
&= u_{x'} + u_{y'} \\
u_y &= u_{x'} x'_y + u_{y'} y'_y \\
&= u_{x'} - u_{y'} \\
\text{So, } u_x + u_y &= 2u_{x'} \\
\text{and } x + 2y &= \frac{x' + y'}{2} + \frac{2(x' - y')}{2} \\
&= \frac{3x' - y'}{2} \\
\text{So, } e^{x+2y} &= e^{\frac{3x'-y'}{2}} \\
\text{So,} u_x + u_y + u = e^{x+2y} &\implies 2u_{x'} + u = e^{\frac{3x'-y'}{2}} \\
&\iff u_{x'} + \frac{1}{2}u = \frac{1}{2}e^{\frac{3x'-y'}{2}}
\end{aligned}$$

Is a first order ode in the variable x', if we regard y' as fixed. The method of integrating factors gives,

$$\begin{aligned}
v(x') = \int \frac{1}{2} dx' = \frac{x'}{2} \text{ and } u(x', y') &= e^{-v(x')} \int e^{v(x')} \frac{1}{2} e^{\frac{3x'-y'}{2}} dx' \\
&= \frac{1}{2} e^{-\frac{x'}{2}} \int e^{\frac{x'}{2}} e^{\frac{3x'}{2}} e^{\frac{-y'}{2}} dx' \\
&= \frac{1}{2} e^{-\frac{x'}{2}} e^{\frac{-y'}{2}} \int e^{\frac{x'+3x'}{2}} dx' \\
&= \frac{1}{2} e^{-\frac{x'}{2}} e^{\frac{-y'}{2}} \int e^{2x'} dx' \\
&= \frac{1}{2} e^{-\frac{x'}{2}} e^{\frac{-y'}{2}} \frac{e^{2x'} + f(y')}{2} \\
&= \frac{e^{2x'} + f(y')}{4e^{\frac{x'+y'}{2}}} \\
\implies u(x,y) &= \frac{e^{2(x+y)} + f(x-y)}{4e^x} \\
&= \frac{e^{x+2y}}{4} + \frac{f(x-y)}{4e^x} \\
u(x,0) = 0 \implies u(x,0) &= \frac{e^x}{4} + \frac{f(x)}{4e^x} = 0 \\
\implies -\frac{e^x}{4} &= \frac{f(x)}{4e^x} \\
\implies f(z) &= -e^{2z}
\end{aligned}$$

Put $z = x - y$. Finally, $u(x,y) = \frac{e^{x+2y}}{4} - \frac{e^{2(x-y)}}{4e^x} = \frac{e^{x+2y}}{4} - \frac{e^{x-2y}}{4} = \frac{e^{x+2y} - e^{x-2y}}{4}$ ♦