Diffusion Equation

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The following equation looks deceptively simple.

$$u_t = ku_{xx} \tag{1}$$

It is not! As we shall see.

First, we do the standard trick of passing to operators,

$$u_t = ku_{xx} \iff u_t - ku_{xx} = 0 \iff (\partial_t - k\partial_x^2)u = 0$$

So we obtain the following equation,

$$(\partial_t - k\partial_x^2)u = 0 \tag{2}$$

Now, make the reasonable assumption that we can factor the operator,

$$(\partial_t - k\partial_x^2)u = 0 \implies (\partial_t^{\frac{1}{2}} + k\partial_x)(\partial_t^{\frac{1}{2}} - k\partial_x)u = 0$$

We need to make sense of $\partial_t^{\frac{1}{2}}$, for that we turn to Fractional Equations and Models by Sandev and Tomovski page 31 formula (2.7). With suitable modifications we can see that

$$\begin{aligned}
\partial_t^{\frac{1}{2}} w(t,x) &= \frac{1}{\Gamma(1-\frac{1}{2})} \int_0^t (t-\tau)^{1-\frac{1}{2}-1} w_\tau(\tau,x)\,\mathrm{d}\tau \\
&= \frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t-\tau)^{-\frac{1}{2}} w_\tau(\tau,x)\,\mathrm{d}\tau \\
&= \frac{1}{\sqrt{\pi}} \int_0^t \frac{w_\tau(\tau,x)}{\sqrt{t-\tau}}\,\mathrm{d}\tau
\end{aligned}$$

Do a change of variables,

$$\begin{aligned}
z = \sqrt{t-\tau} &\implies z^2 = t-\tau \implies \tau = t - z^2 \\
&\implies \mathrm{d}z = -\frac{1}{2\sqrt{t-\tau}}\,\mathrm{d}\tau = -\frac{1}{2z}\,\mathrm{d}\tau \\
&\implies \mathrm{d}\tau = -2z\,\mathrm{d}z \\
&\implies v(\tau,x) = v(\tau(t,z),x) = v(t-z^2,x) \\
&\implies \frac{\partial v}{\partial z} = \frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial x}{\partial z} = \frac{\partial v}{\partial \tau}\cdot(-2z) + \frac{\partial v}{\partial x}\cdot 0 = -2z\frac{\partial v}{\partial \tau} \\
&\implies \frac{\partial v}{\partial \tau} = -\frac{1}{2z}\frac{\partial v}{\partial z}
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{1}{\sqrt{\pi}} \int_0^t \frac{v_\tau(\tau,x)}{\sqrt{t-\tau}}\,\mathrm{d}\tau &= \frac{1}{\sqrt{\pi}} \int_{\sqrt{t-0}}^{\sqrt{t-t}} \frac{-\frac{1}{2z}v_z(t-z^2,x)}{z}(-2z)\,\mathrm{d}z \\
&= -\frac{1}{\sqrt{\pi}} \int_0^{\sqrt{t}} \frac{v_z(t-z^2,x)}{z}\,\mathrm{d}z \\
&= -\frac{1}{\sqrt{\pi}} \int_0^{\sqrt{t}} v_z(t-z^2,x)\,\mathrm{d}\log(z)
\end{aligned}$$