Worksheet 2 = Homework 4 — Due Monday, May 25th.

A philosophical note to students: We should not try to solve problems or memorize methods. Rather we should seek to simplify problems until their solutions reveal themselves to us. We try to solve first order PDEs by reducing them to first order ODEs. Worksheet 1 covers how this can be done. This worksheet should help us to understand how to solve second order PDEs by reducing them to first order PDEs that we already know how to solve. In this worksheet, we will explore this thinking and see exactly how it allows us to solve the wave equation, the diffusion equation, and other similar equations. Have fun!

Problem 1. Factor the following polynomials:

(1) $t^2 - x^2$,

(2) $t^2 + tx - 2x^2$, and

(3) $t^2 + 6tx + 9x^2$.

Problem 2. Write the following second order differential operators as products of first order operators:

(1) $\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$,

(2) $\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial t \partial x} - 2\frac{\partial^2}{\partial x^2}$, and

(3) $\frac{\partial^2}{\partial t^2} + 6\frac{\partial^2}{\partial t \partial x} + 9\frac{\partial^2}{\partial x^2}$.

Problem 3. Solve for the functions a, b, and c, where

(1) $\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) a = 0$,

(2) $\left(\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x}\right) b = 0$, and

(3) $\left(\frac{\partial}{\partial t} + 3\frac{\partial}{\partial x}\right) c = 0$.

Problem 4. Solve for the functions u, v, and w, where

(1) $\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) u = a$,

(2) $\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) v = b$, and

(3) $\left(\frac{\partial}{\partial t} + 3\frac{\partial}{\partial x}\right) w = c$,

where a, b, and c are the functions that you calculated in Problem 3.

Problem 5. Use Problem 4 to solve the following second order differential equations.

(1) $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$,

(2) $\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial t \partial x} - 2\frac{\partial^2 v}{\partial x^2} = 0$, and

(3) $\frac{\partial^2 w}{\partial t^2} + 6\frac{\partial^2 w}{\partial t \partial x} + 9\frac{\partial^2 w}{\partial x^2} = 0$.

Problem 6. In each of the problems above, suppose that f is any one of the three functions u, v, or w above. Given that

$$f(0, x) = \phi(x) \quad \text{and} \quad f_t(0, x) = \psi(x),$$

solve for f in terms of ϕ and ψ. You may wish to study how Strauss does this in his derivation of the solution to the wave equation.