Problem 3. (12 points)

Suppose that u is a twice differentiable function.

(3.1)

Find all solutions to the wave equation

$$u_{tt} = c^2 u_{xx}. \tag{2}$$

(3.2)

Find all solutions to (2) with

$$u(0,x) = \frac{1}{1+x^2}.$$

(3.3)

Let L be the differential operator

$$L = \frac{\partial^2}{\partial t^2} - 2\frac{\partial^2}{\partial t \partial x} - 3\frac{\partial^2}{\partial x^2}.$$

Use the factorisation

$$L = \left(\frac{\partial}{\partial t} - 3\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)$$

to solve the equation

$$Lu = 0. \tag{3}$$

(3.4)

Find all solutions to (3) with the initial conditions

$$u(0,x) = \phi(x) \quad \text{and} \quad u_t(0,x) = \psi(x).$$

(3.5)

Find coordinate functions s and y so that in these coordinates, L factors as

$$L = \left(\frac{\partial}{\partial s} - c\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial s} + c\frac{\partial}{\partial y}\right),$$

so that u solves

$$Lu = 0 \quad \text{if and only if} \quad u_{ss} = c^2 u_{yy}.$$

Using these coordinates, solve (3.3) directly from your solution to (3.1) and write your solution u as function of the (t,x) coordinates. Does this solution agree with you solution to (3.3)?