146C — 4

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1

Factor the following polynomials:

(1) $t^2 - x^2 = (t-x)(t+x)$

(2) $t^2 + tx - 2x^2 = (t-x)(t+2x)$

(3) $t^2 + 6tx + 9x^2 = (t+3x)^2$

2

Write the following second order differential operators as products of first order operators:

(1) $\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = (\frac{\partial}{\partial t} - \frac{\partial}{\partial x})(\frac{\partial}{\partial t} + \frac{\partial}{\partial x})$

(2) $\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x \partial t} - 2\frac{\partial^2}{\partial x^2} = (\frac{\partial}{\partial t} - \frac{\partial}{\partial x})(\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x})$

(3) $\frac{\partial^2}{\partial t^2} + 6\frac{\partial^2}{\partial x \partial t} + 9\frac{\partial^2}{\partial x^2} = (\frac{\partial}{\partial t} + 3\frac{\partial}{\partial x})^2$

3

Solve for the functions a, b, and c

(1) $(\frac{\partial}{\partial t} - \frac{\partial}{\partial x})a = 0$ slu. $a(t,x) = \alpha(t+x)$ $\lozenge$

(2) $(\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x})b = 0$ slu. $b(t,x) = \beta(2t-x)$ $\lozenge$

(3) $(\frac{\partial}{\partial t} + 3\frac{\partial}{\partial x})c = 0$slu. $c(t,x) = \gamma(3t-x)$ $\lozenge$

4

Solve for the functions u, v, and w

(1) $(\frac{\partial}{\partial t} + \frac{\partial}{\partial x})u = a$

slu.

Let $\tau = t - x$, and $y = t + x$,

Then $u_t = u_\tau + u_y$ and $u_x = -u_\tau + u_y$, so

$$u_t + u_x = 2u_y = \alpha(y) \implies u = f(y) + g(\tau)$$

where $f'(y) = \frac{1}{2}\alpha(y)$.

Therefore,

$$u(t, x) = f(t + x) + g(t - x) \quad \blacklozenge$$

(2) $(\frac{\partial}{\partial t} - \frac{\partial}{\partial x})v = b$

slu.

Let $\tau = t + x$, and $y = 2t - x$,

Then $v_t = v_\tau + 2v_y$ and $v_x = v_\tau - v_y$, so

$$v_t - v_x = v_y = \beta(y) \implies v = f(y) + g(\tau)$$

where $f'(y) = \beta(y)$.

Therefore,

$$v(t, x) = f(t + x) + g(2t - x) \quad \lozenge$$

(3) $(\frac{\partial}{\partial t} + 3\frac{\partial}{\partial x})w = c$

Let $y = 3t - x$,

Then $w_t = \tau_t w_\tau + 3w_y$ and $w_x = \tau_x w_\tau - w_y$, so

$$w_t + 3w_x = (\tau_t + 3\tau_x)w_\tau = \gamma(y)$$

Let $\tau = t$, then $w_\tau = \gamma(y) \implies w = \tau\gamma(y) + g(y)$

Plug-in τ and y and verify,

$$w_t = \gamma(3t - x) + 3t\gamma'(3t - x) + 3g'(3t - x)$$
$$w_x = -t\gamma'(3t - x) - g'(3t - x)$$
$$\implies w_t + 3w_x = \gamma(3t - x)$$

So,

$$w(t, x) = t\gamma(3t - x) + g(3t - x) \quad \blacklozenge$$

5

Use Problem 4 to solve the following second order differential equations

(1) $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$

slu.

Factor the operator $(\frac{\partial}{\partial t} - \frac{\partial}{\partial x})(\frac{\partial}{\partial t} + \frac{\partial}{\partial x})u = 0$.

Let $a = (\frac{\partial}{\partial t} + \frac{\partial}{\partial x})u$, to reduce (1) to $\begin{cases} (\frac{\partial}{\partial t} - \frac{\partial}{\partial x})a = 0 \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial x})u = a \end{cases}$

By 4.(1), the solution is,

$$u(t,x) = f(t+x) + g(t-x) \quad \Diamond$$

(2) $\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial x \partial t} - 2\frac{\partial^2 v}{\partial x^2} = 0$

slu.

Factor the operator $(\frac{\partial}{\partial t} - \frac{\partial}{\partial x})(\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x})v = 0$

Let $b = (\frac{\partial}{\partial t} - \frac{\partial}{\partial x})v$, to reduce (1) to $\begin{cases} (\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x})b = 0 \\ (\frac{\partial}{\partial t} - \frac{\partial}{\partial x})v = b \end{cases}$

By 4.(2), the solution is,

$$u(t,x) = f(t+x) + g(2t-x) \quad \Diamond$$

(3) $\frac{\partial^2 w}{\partial t^2} + 6\frac{\partial^2 w}{\partial x \partial t} + 9\frac{\partial^2 w}{\partial x^2} = 0$

slu.

Factor the operator

$$(\frac{\partial}{\partial t} + 3\frac{\partial}{\partial x})^2 w = 0$$

Let $c = (\frac{\partial}{\partial t} + 3\frac{\partial}{\partial x})w$, to reduce (1) to $\begin{cases} (\frac{\partial}{\partial t} + 3\frac{\partial}{\partial x})c = 0 \\ (\frac{\partial}{\partial t} - \frac{\partial}{\partial x})w = c \end{cases}$

By 4.(3), the solution is,

$$w(t,x) = t\gamma(3t-x) + g(3t-x) \quad \Diamond$$

6

6. In each of the problems above, suppose that f is any one of the three functions u, v, or w above. Given that

$$f(0,x) = \phi(x) \text{ and } f_t(0,x) = \psi(x).$$

,solve for f in terms of ϕ and ψ. You may wish to study how Strauss does this in his derivation of the solution to the wave equation.

slu.

(1) The equation 5.(1) is the wave equation with $c = 1$, thus the solution with initial conditions is,

$$u(x,t) = \frac{1}{2}[\phi(x+t) + \phi(x-t)] + \int_{x-t}^{x+t} \psi(s)\,\mathrm{d}s$$

The solution is due to Jean-Baptiste le Rond d'Alembert in 1746.

(2) The equation in 5.(2) has solution

$$u(t,x) = f(t+x) + g(2t-x)$$

Multiplying the argument of g by -1 doesn't change the form of the solution as g is arbitrary, so,

$$u(t,x) = f(x+t) + g(x-2t)$$

Differentiating with respect to t, we get

$$u_t(t,x) = f'(x+t) - 2g'(x-2t)$$

Plugging in 0, we get

$$u(0,x) = f(x) + g(x) = \phi(x) \text{ and } u_t(0,x) = f'(x) - 2g'(x) = \psi(x)$$

Then

$$\phi'(x) = f'(x) + g'(x) \text{ and } \psi(x) = f'(x) - 2g'(x)$$

Solving the system for $f'(x)$, and $g'(x)$ we get

$$f'(x) = \frac{1}{3}(2\phi'(x) + \psi(x)) \text{ and } g'(x) = \frac{1}{3}(\phi'(x) - \psi(x))$$

So,

$$f(x) = \frac{1}{3}\left(2\phi(x) - \int_0^x \psi(s)\,\mathrm{d}s\right) \text{ and } g(x) = \frac{1}{3}\left(\phi(x) - \int_0^x \psi(s)\,\mathrm{d}s\right)$$

Plugging f and g into the solution,

$$v(t,x) = \frac{1}{3}\left(2\phi(x+t) + \int_0^{x+t} \psi(s)\,\mathrm{d}s\right) + \frac{1}{3}\left(\phi(x-2t) - \int_0^{x-2t} \psi(s)\,\mathrm{d}s\right)$$

Rearranging,

$$v(t,x) = \frac{1}{3}(2\phi(x+t) + \phi(x-2t)) + \frac{1}{3}\int_{x-2t}^{x+t} \psi(s)\,\mathrm{d}s$$

(3) The equation in 4.(3), has solution in the form,

$$w(t,x) = tf(x-3t) + g(x-3t)$$

$$w_t(t,x) = f(x-3t) - 3tf'(x-3t) - 3g'(x-3t)$$

$$w(0,x) = g(x) = \phi(x) \text{ and } w_t(0,x) = f(x) - 3g'(x) = \psi(x)$$

So,

$$\phi(x) = g(x) \text{ and } \psi(x) = f(x) - 3\phi'(x)$$

Solving for f, and g,

$$f(x) = \psi(x) + 3\phi'(x) \text{ and } g(x) = \phi(x)$$

$$w(t,x) = t\left(\psi(x-3t) + 3\phi'(x-3t)\right) + \phi(x-3t)$$

Since the solution is a bit different I want to verify,

$$w_t(t,x) = \psi(x-3t) + 3\phi'(x-3t) - 3t\psi'(x-3t) - 9t\phi''(x-3t) - 3\phi'(x-3t)$$

$$w_t(t,x) = \psi(x-3t) - 3t\psi'(x-3t) - 9t\phi''(x-3t) \implies w_t(0,x) = \psi(x) \text{ and } w(0,x) = \phi(x)$$

$$w_{tt}(t,x) = -3\psi'(x-3t) - 3\psi'(x-3t) - 9\phi''(x-3t) + 9t\psi''(x-3t) + 27t\phi'''(x-3t)$$

$$w_{tt}(t,x) = -6\psi'(x-3t) - 9\phi''(x-3t) + 9t\psi''(x-3t) + 27t\phi'''(x-3t)$$

$$w_{tx}(t,x) = \psi'(x-3t) - 3t\psi''(x-3t) - 9t\phi'''(x-3t)$$

$$6w_{tx}(t,x) = 6\psi'(x-3t) - 18t\psi''(x-3t) - 54t\phi'''(x-3t)$$

$$w_x(t,x) = t\left(\psi'(x-3t) + 3\phi''(x-3t)\right) + \phi'(x-3t)$$

$$w_{xx}(t,x) = t\left(\psi''(x-3t) + 3\phi'''(x-3t)\right) + \phi''(x-3t) = t\psi''(x-3t) + 3t\phi'''(x-3t) + \phi''(x-3t)$$

$$9w_{xx}(t,x) = 9t\psi''(x-3t) + 27t\phi'''(x-3t) + 9\phi''(x-3t)$$

So,

$$w_{tt} + 6w_{tx} + 9w_{xx} = 0$$

So the solutions (1),(2), and (3) solve the equations in consideration for the corresponding initial conditions ⧫