Worksheet 3 = Homework 5 — Due Friday, June 5th.

A philosophical note to students: Uniqueness theorems are important at least in part because they give us freedom in making intuitive leaps in solving problems. We may expect that a given approach should solve a problem, and a given approach may indeed solve a problem, but without a uniqueness theorem we cannot expect that this solution is important or physically relevant. Uniqueness theorems in conjunction with our principles of degree reduction and dimensional reduction can together be powerful tools in solving problems, tools we explore in this worksheet in the setting of the wave and diffusion equations. Once again, have fun!

For the problems below, let T , ℓ, c, and k be positive real numbers. The diffusion equation, (DE), and the wave equation, (WE), are respectively given below by

$$\text{(DE)} \qquad u_t = ku_{xx},$$

$$\text{(WE)} \qquad \text{and} \quad u_{tt} = ku_{xx}.$$

Problem 1. Suppose that u is the solution to (DE) on the closed interval $[0, \ell]$. Suppose that ϕ is a continuous function on $[0, \ell]$ and that

$$(1)\ u(0, x) = \phi(x) \quad \text{and} \quad (2)\ u(t, 0) = u(t, \ell) = 0.$$

Use the weak maximal property to argue that if u and v are two solutions to the above equation with these initial conditions and boundary conditions, then for all (t, x) in $[0, T] \times [0, \ell]$,

$$u(t, x) = v(t, x).$$

Problem 2. Suppose that u is the solution to (DE) on $\mathbb{R}$. Suppose that ϕ is a positive continuous function on $\mathbb{R}$ that is integrable on all of $\mathbb{R}$ (meaning the integral over $(-\infty, \infty)$ is finite) and that

$$(1)\ u(0, x) = \phi(x) \quad \text{and} \quad (2)\ u(t, 0) = u(t, \ell) = 0.$$

Using the previous problem, argue that if u and v are two solutions to the above equation with this initial condition, then for all (t, x) in $[0, \infty) \times (-\infty, \infty)$,

$$u(t, x) = v(t, x).$$

Problem 3. Solve Strauss 2.2 # 1. Use this exercise to prove the following. Suppose that u is the solution to (WE) on $\mathbb{R}$. Suppose that ϕ is continuously differentiable and that ψ is continuous on $\mathbb{R}$ and that

$$(1)\ u(0, x) = \phi(x), \quad (2)\ u_t(0, x) = \psi(x).$$

Prove that if u and v are two solutions to the above equation with this initial condition, then for all (t, x) in $(-\infty, \infty) \times (-\infty, \infty)$,

$$u(t, x) = v(t, x).$$

Problem 4. Suppose that u is the solution to (DE) on the closed interval $[0, \ell]$ and suppose that v is the solution to (WE) on the closed interval $[0, \ell]$. Suppose further that

$$v(t, 0) = v(t, \ell) = 0 \quad \text{and} \quad u(t, 0) = u(t, \ell) = 0.$$

Given functions T and X, find all solutions to these boundary value problems, where

$$u(t, x) = T(t)X(x) \quad \text{and} \quad v(t, x) = T(t)X(x).$$

Note that you are only being asked to find a single solution for each problem.

Problem 5. Suppose that u is the solution to (DE) on the closed interval $[0, \ell]$ and

$$u(t, 0) = u(t, \ell) = 0.$$

Find all solutions to this boundary value problem, where

(1) $u(0, x) = 2\sin\left(\frac{\pi x}{\ell}\right)$,

(2) $u(0, x) = 5\sin\left(\frac{3\pi x}{\ell}\right) + 7\sin\left(\frac{4\pi x}{\ell}\right)$.

Problem 6. Suppose that u is the solution to (WE) on the closed interval $[0, \ell]$, with ℓ positive and

$$u(t, 0) = u(t, \ell) = 0.$$

Find all solutions to this boundary value problem, where

(1) $u(0, x) = 2\sin\left(\frac{\pi x}{\ell}\right)$,

(2) $u(0, x) = 5\sin\left(\frac{3\pi x}{\ell}\right) + 7\sin\left(\frac{4\pi x}{\ell}\right)$.

(3) Do you get unique solutions to the above problems? If not, what additional information will give you unique solutions? Supply such information and find the solutions to these problems with the additional information.

Problem 7. Suppose that u is the solution to (DE) on the closed interval $[0, \ell]$, with

$$u_x(t, 0) = u_x(t, \ell) = 0.$$

Find all solutions to this boundary value problem, where

(1) $u(0, x) = 5\cos\left(\frac{\pi x}{\ell}\right)$,

(2) $u(0, x) = 4 + 2\cos\left(\frac{3\pi x}{\ell}\right) + 7\cos\left(\frac{4\pi x}{\ell}\right)$.

Problem 8. Suppose that u is the solution to (WE) on the closed interval $[0, \ell]$, with

$$u_x(t, 0) = u_x(t, \ell) = 0.$$

Find all solutions to this boundary value problem, where

(1) $u(0, x) = 5\cos\left(\frac{\pi x}{\ell}\right)$,

(2) $u(0, x) = 4 + 2\cos\left(\frac{3\pi x}{\ell}\right) + 7\cos\left(\frac{4\pi x}{\ell}\right)$.

(3) Do you get unique solutions to the above problems? If not, what additional information will give you unique solutions? Supply such information and find the solutions to these problems with the additional information.