9 Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0, u(x,0) = x^2, u_t(x,0) = e^x$. (*Hint:* Factor the operator as we did for the wave equation.)

slu.

$$u_{xx} - 3u_{xt} - 4u_{tt} = 0 \iff \left(\frac{\partial^2}{\partial x^2} - 3\frac{\partial^2}{\partial x \partial t} - 4\frac{\partial^2}{\partial t^2}\right)u = 0 \iff \left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)u = 0$$

Let $v = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)u$,

$$\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right)v = 0 \implies v_x - 4v_t = 0 \implies -v_x + 4v_t = 0 \implies v(x,t) = f(4x+t)$$

$$v = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)u \implies u_x + u_t = f(4x+t)$$

$$u_x(x,0) = 2x \text{ and } u_t(x,0) = e^x \implies 2x + e^x = f(4x)$$

Let $g(s) = \frac{1}{4}s$ and $h(s) = 2s + e^s$

$$f = h \circ g \implies f(4x+t) = h(x + \frac{1}{4}t) = 2x + \frac{1}{2}t + e^{x+\frac{1}{4}t} \implies f(4x+0) = 2x + 0 + e^x + 0$$

So, f works, and we have,

$$u_x + u_t = 2x + \frac{1}{2}t + e^{x+\frac{1}{4}t} \tag{1}$$

Now the left hand side factors as,

$$u_x + u_t = \langle 1, 1\rangle \nabla u$$

This suggests the following change of variables, $y = x + t$ and $z = x - t$ which gives,

$$u_x = u_y + u_z \text{ and } u_t = u_y - u_z$$

$$x = \frac{y+z}{2} \text{ and } t = \frac{y-z}{2}$$

Plugging into (1)

$$u_y + u_z + u_y - u_z = 2u_y = 2\frac{y+z}{2} + \frac{1}{2}\frac{y-z}{2} + e^{\frac{y+z}{2} + \frac{1}{4}\frac{y-z}{2}}$$

$$\implies u_y = \frac{y+z}{2} + \frac{1}{4}\frac{y-z}{2} + \frac{1}{2}e^{\frac{y+z}{2} + \frac{1}{4}\frac{y-z}{2}}$$

$$\implies u_y = \frac{5}{8}y + \frac{3}{8}z + \frac{1}{2}e^{\frac{5}{8}y + \frac{3}{8}z}$$

$$\implies u_y = \frac{5}{8}y + \frac{3}{8}z + \frac{1}{2}e^{\frac{3}{8}z}e^{\frac{5}{8}y}$$

Regarding z as fixed we can integrate with respect to y,

$$\begin{aligned} u(y,z) &= \int \frac{5}{8}y + \frac{3}{8}z + \frac{1}{2}e^{\frac{3}{8}z}e^{\frac{5}{8}y}\,\mathrm{d}y \\ &= \frac{5}{16}y^2 + \frac{3}{8}yz + \frac{8}{10}e^{\frac{3}{8}z}e^{\frac{5}{8}y} + k(z) \end{aligned}$$

$$u(y,z) = \frac{5}{16}y^2 + \frac{3}{8}yz + \frac{4}{5}e^{\frac{3}{8}z}e^{\frac{5}{8}y} + k(z) \tag{2}$$

Then plugging x and t back into (2)

$$u(x,t) = \frac{5}{16}(x+t)^2 + \frac{3}{8}(x+t)(x-t) + \frac{4}{5}e^{\frac{3}{8}(x-t)}e^{\frac{5}{8}(x+t)} + k(x-t)$$

Now simplifying gives

$$\begin{aligned}
u(x,t) &= \frac{5}{16}(x+t)^2 + \frac{3}{8}(x+t)(x-t) + \frac{4}{5}e^{\frac{3}{8}(x-t)}e^{\frac{5}{8}(x+t)} + k(x-t) \\
&= \frac{5}{16}(x^2+2xt+t^2) + \frac{3}{8}(x^2-t^2) + \frac{4}{5}e^{\frac{3}{8}(x-t)+\frac{5}{8}(x+t)} + k(x-t) \\
&= \frac{11}{16}x^2 + \frac{5}{8}xt - \frac{1}{16}t^2 + \frac{4}{5}e^{x+\frac{1}{4}t} + k(x-t)
\end{aligned}$$

$$u(x,t) = \frac{11}{16}x^2 + \frac{5}{8}xt - \frac{1}{16}t^2 + \frac{4}{5}e^{x+\frac{1}{4}t} + k(x-t) \tag{3}$$

Now we need k such that $u(x,0) = x^2$, so we plug 0 into (3)

$$\begin{aligned}
u(x,0) &= \frac{11}{16}x^2 + \frac{5}{8}x0 - \frac{1}{16}0^2 + \frac{4}{5}e^{x+\frac{1}{4}0} + k(x-0) \\
&= \frac{11}{16}x^2 + \frac{4}{5}e^x + k(x) = x^2 \\
\implies k(x) &= x^2 - \frac{11}{16}x^2 - \frac{4}{5}e^x \\
\implies k(x) &= \frac{5}{16}x^2 - \frac{4}{5}e^x \\
\implies k(x-t) &= \frac{5}{16}(x-t)^2 - \frac{4}{5}e^{x-t}
\end{aligned}$$

Now we can plug $k(x-t)$ back into (3)

$$\begin{aligned}
u(x,t) &= \frac{11}{16}x^2 + \frac{5}{8}xt - \frac{1}{16}t^2 + \frac{4}{5}e^{x+\frac{1}{4}t} + \frac{5}{16}(x-t)^2 - \frac{4}{5}e^{x-t} \\
&= \frac{11}{16}x^2 + \frac{5}{8}xt - \frac{1}{16}t^2 + \frac{4}{5}e^{x+\frac{1}{4}t} + \frac{5}{16}(x^2-2xt+t^2) - \frac{4}{5}e^{x-t} \\
&= \left(\frac{11}{16} + \frac{5}{16}\right)x^2 + \frac{5}{8}xt - \frac{5}{8}xt + \left(\frac{5}{16} - \frac{1}{16}\right)t^2 + \frac{4}{5}e^{x+\frac{1}{4}t} - \frac{4}{5}e^{x-t} \\
&= x^2 + \frac{1}{4}t^2 + \frac{4}{5}e^{x+\frac{1}{4}t} - \frac{4}{5}e^{x-t}
\end{aligned}$$

Thus,

$$u(x,t) = x^2 + \frac{1}{4}t^2 + \frac{4}{5}e^{x-t}\left(e^{\frac{5}{4}t} - 1\right) \tag{4}$$

We get $u(x,0) = x^2$ and $u_t(x,0) = e^x$, and,

$u_{xx} = \frac{4}{5}\left(e^{\left(\frac{5}{4}t\right)} - 1\right)e^{(-t+x)} + 2$ and $u_{xt} = -\frac{4}{5}\left(e^{\left(\frac{5}{4}t\right)} - 1\right)e^{(-t+x)} + e^{\left(\frac{1}{4}t+x\right)}$ and $u_{tt} = \frac{4}{5}\left(e^{\left(\frac{5}{4}t\right)} - 1\right)e^{(-t+x)} - \frac{3}{4}e^{\left(\frac{1}{4}t+x\right)} + \frac{1}{2}$

$$u_{xx} - 3u_{xt} - 4u_{tt} = 0$$