146C — 5

RJ Acuña

(862079740)

Diffusion Equation (DE),

$$u_t = ku_{xx}\,.$$

Wave Equation (WE),

$$u_{tt} = ku_{xx}\,.$$

1 Suppose that u is the solution to (DE) on the closed interval $[0, l]$. Suppose that ϕ is a continuous function on $[0, l]$ and that

$$u(0, x) = \phi(x) \text{ and } u(t, 0) = 0 = u(t, l)$$

Use the weak maximal property to argue that if u and v are two solutions to the above equation with these initial conditions and boundary conditions, then for all (t, x) in $[0, T] \times [0, l]$,

$$u(t, x) = v(t, x).$$

pf.

u, and v solve (DE). Then,

$$u_t = ku_{xx} \text{ and } v_t = kv_{xx} \implies (u - v)_t = k(u - v)_{xx}.$$

$$u(0, x) = \phi(x) = v(0, x) \implies (u - v)(0, x) = 0.$$

$$u(t, 0) = v(t, 0) = 0 = v(t, l) = u(t, l) \implies (u - v)(t, 0) = 0 = (u - v)(t, l).$$

Therefore, $(u - v)(t, x) \equiv 0$, for (t, x) on the boundary of the semi-infinite strip $[0, \infty) \times [0, l]$. By the weak maximum principle,the function $u - v$ has it's maximum on the boundary. Since, solutions of (DE) are concentrations, or temperatures, there are no negative solutions if one uses an absolute scale, i.e. Kelvin. Therefore, everywhere on that domain $(u - v)(t, x) \equiv 0$, so

$$u(t, x) = v(t, x) \quad \blacksquare$$

2 Suppose that u is the solution to (DE) on $\mathbb{R}^2$. Suppose that ϕ is a positive continuous function on $\mathbb{R}$ that is integrable on all of $\mathbb{R}$ (meaning the integral over $(-\infty, \infty)$ is finite) and that

$$u(0, x) = \phi(x) \text{ and } \lim_{x \to -\infty} u(t, x) = 0 = \lim_{x \to \infty} u(t, x)$$

Using the previous problem, argue that if u and v are two solutions to the above equation with this initial condition, then for all (t, x) in $[0, \infty) \times (-\infty, \infty)$,

$$u(t, x) = v(t, x).$$

pf.

The boundary of the region $[0, \infty) \times (-\infty, \infty)$, is $\{0\} \times (-\infty, \infty)$, there

$$u(0, x) = \phi(x) = v(0, x) \implies (u - v)(0, x) = 0.$$

By the weak maximum property $(u - v)(t, x) \equiv 0$ as $u - v$ is non-negative, for the same reasons as above. So,

$$u(t, x) = v(t, x) \quad \blacksquare$$

3 Solve Strauss 2.2 # 1. Use this exercise to prove the following. Suppose that u is the solution to (WE) on $\mathbb{R}^2$. Suppose that ϕ is continuously differentiable and that ψ is continuous on $\mathbb{R}$ and that

$$u(0,x) = \phi(x), (2) u_t(0,x) = \psi(x).$$

Prove that if u and v are two solutions to the above equation with this initial condition, then for all (t,x) in $(-\infty,\infty)\times(-\infty,\infty)$, $u(t,x) = v(t,x)$.

slu of Strauss 2.2 # 1 See the first problem in my HW # 3 ■

pf.

$$u_{tt} = k u_{xx} \text{ and } v_{tt} = k v_{xx} \implies (u-v)_{tt} = k(u-v)_{xx}$$
$$u(0,x) = \phi(x), u_t(0,x) = \psi(x), v(0,x) = \phi(x), \text{ and } v_t(0,x) = \psi(x)$$
$$\implies (u-v)(0,x) \equiv 0 \text{ and } (u-v)_t(0,x) \equiv 0$$

By Strauss 2.2 # 1,

$$(u-v)(t,x) \equiv 0 \implies u(x,t) = v(x,t) \quad ■$$

4 Suppose that u is the solution to (DE) on the closed interval $[0,l]$ and suppose that v is the solution to (WE) on the closed interval $[0,l]$. Suppose further that $v(t,0) = v(t,l) = 0$ and $u(t,0) = u(t,l) = 0$. Given functions T and X, find all solutions to these boundary value problems, where

$$u(t,x) = T(t)X(x) \text{ and } v(t,x) = T(t)X(x).$$

Note that you are only being asked to find a single solution for each problem.

slu.

$$u_t = T'X = kTX'' = ku_{xx} \text{ and } v_{tt} = T''X = kTX'' = v_{xx}$$
$$\implies \frac{T'}{T} = k\frac{X''}{X} = -\lambda \text{ and } \frac{T''}{T} = k\frac{X''}{X} = -\lambda \text{ both are constant}$$
$$\implies X'' + \frac{\lambda}{k}X = 0 \text{ and } (T' + \lambda T = 0 \text{ or } T'' + \lambda T = 0 \text{ respectively.})$$

The first ODE in X is common to both equations, we know from a previous course that with the boundary conditions as stated, the only possible eigenvalues that yield non-trivial solutions are positive, since $\lambda/k > 0$, we have

$$X(x) = A\cos\left(\sqrt{\frac{\lambda}{k}}x\right) + B\sin\left(\sqrt{\frac{\lambda}{k}}x\right)$$
$$X(0) = 0 = A\cos(0) + B\sin(0) = A$$
$$X(l) = 0 = B\sin\left(\sqrt{\frac{\lambda}{k}}l\right)$$

If $B = 0$, we have the trivial solution, but we don't want trivial solutions so,

$$\sin\left(\frac{\sqrt{\lambda}l}{\sqrt{k}}\right) = 0 \implies \frac{\sqrt{\lambda}l}{\sqrt{k}} = n\pi \implies \lambda = k\left(\frac{n\pi}{l}\right)^2$$
$$\implies X_n(x) = \sin\left(\frac{n\pi x}{l}\right) \text{ as the } \sqrt{k} \text{ cancels.}$$
$$\implies T' + k\left(\frac{n\pi}{l}\right)^2 T = 0 \implies T_n(t) = C_n \exp\left(-k\left(\frac{n\pi}{l}\right)^2 t\right)$$
$$\implies T'' + k\left(\frac{n\pi}{l}\right)^2 T = 0 \implies T_n(t) = A_n\cos\left(\frac{\sqrt{k}n\pi t}{l}\right) + B_n\sin\left(\frac{\sqrt{k}n\pi t}{l}\right).$$
$$\implies u(t,x) = \sum_{n=1}^{\infty} C_n \exp\left(-k\left(\frac{n\pi}{l}\right)^2 t\right)\sin\left(\frac{n\pi x}{l}\right),$$
$$\text{and } v(t,x) = \sum_{n=1}^{\infty}\left(A_n\cos\left(\sqrt{k}\left(\frac{n\pi}{l}\right)t\right) + B_n\sin\left(\sqrt{k}\left(\frac{n\pi}{l}\right)t\right)\right)\sin\left(\frac{n\pi x}{l}\right) \quad \blacklozenge$$

5 Suppose that u is the solution to (DE) on the closed interval $[0, l]$ and

$$u(t, 0) = u(t, l) = 0.$$

Find all solutions to this boundary value problem, where

(1) $u(0, x) = 2\sin\left(\frac{\pi x}{l}\right)$,

(2) $u(0, x) = 5\sin\left(\frac{3\pi x}{l}\right) + 7\sin\left(\frac{4\pi x}{l}\right)$.

slu.

By 4,

$$u(0, x) = \sum_{n=1}^{\infty} C_n \exp(0) \sin\left(\frac{n\pi x}{l}\right) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)$$

for (1) $C_1 = 2$ and $C_n = 0 \quad \forall n : n \neq 2$, then

$$\implies u(t, x) = 2\exp\left(-k\left(\frac{\pi}{l}\right)^2 t\right)\sin\left(\frac{\pi x}{l}\right)$$

for (2) $C_3 = 5, C_4 = 7$ and $C_n = 0 \quad \forall n : n \neq 2$ and $n \neq 7$, then

$$\implies u(t, x) = 5\exp\left(-k\left(\frac{3\pi}{l}\right)^2 t\right)\sin\left(\frac{3\pi x}{l}\right) + 7\exp\left(-k\left(\frac{4\pi}{l}\right)^2 t\right)\sin\left(\frac{4\pi x}{l}\right) \quad \Diamond$$

6 Suppose that v is the solution to (WE) on the closed interval $[0, l]$, with l positive and

$$v(t, 0) = u(t, l) = 0.$$

Find all solutions to this boundary value problem, where

(1) $v(0, x) = 2\sin\left(\frac{\pi x}{l}\right)$,

(2) $v(0, x) = 5\sin\left(\frac{3\pi x}{l}\right) + 7\sin\left(\frac{4\pi x}{l}\right)$.

(3) Do you get unique solutions to the above problems? If not, what additional information will give you unique solutions? Supply such information and find the solutions to these problems with the additional information.

slu.

To simplify the form we let $\sqrt{k} = c$, the wave speed,

$$v(t, x) = \sum_{n=1}^{\infty}\left(A_n \cos\left(c\left(\frac{n\pi}{l}\right)t\right) + B_n \sin\left(c\left(\frac{n\pi}{l}\right)t\right)\right)\sin\left(\frac{n\pi x}{l}\right).$$

Then,

$$\text{and } v(0, x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right).$$

for (1) $A_1 = 2$ and $A_n = 0 \quad \forall n : n \neq 2$, then

$$\implies u(t, x) = 2\cos\left(c\left(\frac{\pi}{l}\right)t\right)\sin\left(\frac{\pi x}{l}\right) + \sum_{n=1}^{\infty} B_n \sin\left(c\left(\frac{n\pi}{l}\right)t\right)\sin\left(\frac{n\pi x}{l}\right).$$

for (2) $C_3 = 5, C_4 = 7$ and $C_n = 0 \quad \forall n : n \neq 2$ and $n \neq 7$, then

$$\implies u(t, x) = 5\cos\left(c\left(\frac{3\pi}{l}\right)t\right)\sin\left(\frac{3\pi x}{l}\right) + 7\cos\left(c\left(\frac{4\pi}{l}\right)t\right)\sin\left(\frac{4\pi x}{l}\right) + \sum_{n=1}^{\infty} B_n \sin\left(c\left(\frac{n\pi}{l}\right)t\right)\sin\left(\frac{n\pi x}{l}\right)$$

for (3), no it's not enough information, we need to characterize the B_ns by specifying $v_t(0, x) = \psi(x)$. If we let $v_t(0, x) = 0$, then all of the B_ns above are 0, and we get unique solutions $\quad \Diamond$

7 Suppose that u is the solution to (DE) on the closed interval $[0, l]$ and

$$u_x(t, 0) = u_x(t, l) = 0.$$

Find all solutions to this boundary value problem, where

(1) $u(0, x) = 5\cos\left(\frac{\pi x}{l}\right)$,

(2) $u(0, x) = 4 + 2\cos\left(\frac{3\pi x}{l}\right) + 7\sin\left(\frac{4\pi x}{l}\right)$.

slu.

Since we have the condition,

$$u_x(t, 0) = u_x(t, l) = 0.$$

It follows that,

$$X'(0)T(t) = X'(L)T(t) = 0 \implies X'(0) = X'(l) = 0.$$

Consider the case $\lambda = 0$, then

$$X'' = 0 \implies X(x) = d_1 x + d_2.$$

$$X''(x) = d_1 \text{ and } X''(0) = 0 \implies d_1 = 0.$$

However, d_2 is not determined by the boundary conditions, so $\lambda = 0$ is an eigenvalue, with eigenfunction $X_0(x) = C_0$.

We know that there are no negative eigenvalues from a previous class.

Furthermore, the solution for the positive eigenvalues is the same, with the difference that the eigenfunctions are $\cos(\frac{n\pi x}{l})$.Therefore,

$$u(t, x) = C_0 + \sum_{n=1}^{\infty} C_n \exp\left(-k\left(\frac{n\pi}{l}\right)^2 t\right)\cos\left(\frac{n\pi x}{l}\right)$$

for (1) $C_1 = 5$, and $\forall n : n \neq 1 C_n = 0 \implies C_1 = \frac{5l}{\pi}$. So,

$$u(t, x) = 5\exp\left(-k\left(\frac{\pi}{l}\right)^2 t\right)\cos\left(\frac{n\pi x}{l}\right).$$

for (2) $C_0 = 4$, $C_3 = 2$, and $C_4 = 7$, $\forall n : n \neq 3$ and $n \neq 4, C_n = 0$. So,

$$u(t, x) = 4 + 2\exp\left(-k\left(\frac{3\pi}{l}\right)^2 t\right)\cos\left(\frac{3\pi x}{l}\right) + 7\exp\left(-k\left(\frac{4\pi}{l}\right)^2 t\right)\cos\left(\frac{3\pi x}{l}\right) \quad \Diamond$$

8 Suppose that u is the solution to (WE) on the closed interval $[0, l]$ and

$$v_x(t, 0) = v_x(t, l) = 0.$$

Find all solutions to this boundary value problem, where

(1) $v(0, x) = 5\cos\left(\frac{\pi x}{l}\right)$,

(2) $v(0, x) = 4 + 2\cos\left(\frac{3\pi x}{l}\right) + 7\sin\left(\frac{4\pi x}{l}\right)$.

(3) Do you get unique solutions to the above problems? If not, what additional information will give you unique solutions? Supply such information and find the solutions to these problems with the additional information.

slu. The same way as number 7,

$$v(t, x) = A_0 + \sum_{n=0}^{\infty}\left(A_n \cos\left(c\left(\frac{n\pi}{l}\right)t\right) + B_n \sin\left(c\left(\frac{n\pi}{l}\right)t\right)\right)\cos\left(\frac{n\pi x}{l}\right)$$

for (1) $C_1 = 5$, and $\forall n : n \neq 1 C_n = 0 \implies C_1 = \frac{5l}{\pi}$. So,

$$v(t, x) = 5\cos\left(c\left(\frac{\pi}{l}\right)t\right)\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=0}^{\infty} B_n \sin\left(c\left(\frac{n\pi}{l}\right)t\right)\cos\left(\frac{n\pi x}{l}\right).$$

for (2) $C_0 = 4$, $C_3 = 2$, and $C_4 = 7$, $\forall n : n \neq 3$ and $n \neq 4, C_n = 0$. So,

$$v(t, x) = 4+2\cos\left(c\left(\frac{3\pi}{l}\right)t\right)\cos\left(\frac{3\pi x}{l}\right)+7\cos\left(c\left(\frac{4\pi}{l}\right)t\right)\cos\left(\frac{4\pi x}{l}\right)+\sum_{n=0}^{\infty} B_n \sin\left(c\left(\frac{n\pi}{l}\right)t\right)\cos\left(\frac{n\pi x}{l}\right).$$

for (3), no it's not enough information, we need to characterize the B_ns by specifying $v_t(0, x) = \psi(x)$. If we let $v_t(0, x) = 0$, then all of the B_ns above are 0, and we get unique solutions $\Diamond$