146C — 2

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Let's solve the problem $a, b \in \mathbb{R}$,

$$\begin{cases} u_{tt} + au_{tx} + bu_{xx} = 0 \\ u(0,x) = \phi(x) \text{ and } u_t(0,x) = \psi(x) \end{cases}$$

$t^2+atx+bx^2 = 0 \implies t = \frac{-ax\pm\sqrt{(ax)^2-4bx^2}}{2} = \frac{-a\pm\sqrt{a^2-4b}}{2}x \implies t^2+atx+bx^2 = \left(t + \frac{a+\sqrt{a^2-4b}}{2}x\right)\left(t + \frac{a-\sqrt{a^2-4b}}{2}x\right)$
From the factorization we get two cases,

In the case that $a^2 - 4b = 0$, we can factor the operator and see that the solution for $u$ in the following system, corresponds to the solution to the original equation,

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{a}{2}\frac{\partial}{\partial x}\right) d = 0 \\ \left(\frac{\partial}{\partial t} + \frac{a}{2}\frac{\partial}{\partial x}\right) u = d \end{cases}$$

Then, $d(x,t) = f(x - \frac{a}{2}t)$, solves the first equation in the system.

Let $y = x - \frac{a}{2}t$, then $u_t = \tau_t u_\tau - \frac{a}{2}u_y$, and $u_x = \tau_x u_\tau + u_y$.So,

$$u_t + \frac{a}{2}u_x = (\tau_t + \frac{a}{2}\tau_x)u_\tau = f(y)$$

Let $\tau = t$, so $\tau_t + \frac{a}{2}\tau_x = 1$,

$$u_\tau = f(y) \implies u(\tau, y) = (y) + g(y)$$

So,

$$u(t,x) = tf(x - \frac{a}{2}t) + g(x - \frac{a}{2}t)$$

$$u_(t,x) = f(x - \frac{a}{2}t) - t\frac{a}{2}f'(x - \frac{a}{2}t) - \frac{a}{2}g'(x - \frac{a}{2}t)$$

Now,

$$u(0,x) = g(x) = \phi(x) \text{ and } u_t(0,x) = f(x) - \frac{a}{2}g'(x) = \psi(x)$$

So,

$$f(x) = \psi(x) + \frac{a}{2}\phi'(x) \text{ and } g(x) = \phi(x)$$

Therefore,

$$u(t,x) = t\left(\psi(x - \frac{a}{2}t) + \frac{a}{2}\phi'(x - \frac{a}{2}t)\right) + \phi(x - \frac{a}{2}t).$$

In that case you can check that if $a = 0$, then $b = 0$. So,

$$u(t,x) = t\left(\psi(x)\right) + \phi(x),$$

which is the same result that we would've gotten by integrating $u_{tt} = 0$.

In the case that $a^2 - 4b \neq 0$, we can factor the operator, and see that the solution for $u$ in the following system, correspond to the solution to the original equation.

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{a+\sqrt{a^2-4b}}{2}\frac{\partial}{\partial x}\right) d = 0 \\ \left(\frac{\partial}{\partial t} + \frac{a-\sqrt{a^2-4b}}{2}\frac{\partial}{\partial x}\right) u = d \end{cases}$$

So,

$$u(t,x) = f\left(x - \frac{a+\sqrt{a^2-4b}}{2}t\right) + g\left(x - \frac{a-\sqrt{a^2-4b}}{2}t\right)$$

$$u(0,x) = f(x) + g(x) = \phi(x) \implies \phi'(x) = f'(x) + g'(x)$$

$$u_t(t,x) = -\frac{a+\sqrt{a^2-4b}}{2} f'\left(x - \frac{a+\sqrt{a^2-4b}}{2}t\right) - \frac{a-\sqrt{a^2-4b}}{2} g'\left(x - \frac{a-\sqrt{a^2-4b}}{2}t\right)$$

$$u_t(0,x) = -\frac{a+\sqrt{a^2-4b}}{2} f'(x) - \frac{a-\sqrt{a^2-4b}}{2} g'(x) = \psi(x) \implies \psi(x) = -\frac{a+\sqrt{a^2-4b}}{2} f'(x) - \frac{a-\sqrt{a^2-4b}}{2} g'(x)$$

We have the following linear system of equations

$$\begin{cases} f'(x) + g'(x) = \phi'(x) \\ -\frac{a+\sqrt{a^2-4b}}{2} f'(x) - \frac{a-\sqrt{a^2-4b}}{2} g'(x) = \psi(x) \end{cases} \implies \begin{pmatrix} 1 & 1 \\ -\frac{a+\sqrt{a^2-4b}}{2} & -\frac{a-\sqrt{a^2-4b}}{2} \end{pmatrix} \begin{pmatrix} f'(x) \\ g'(x) \end{pmatrix} = \begin{pmatrix} \phi'(x) \\ \psi(x) \end{pmatrix}$$

$$\begin{vmatrix} 1 & 1 \\ -\frac{a+\sqrt{a^2-4b}}{2} & -\frac{a-\sqrt{a^2-4b}}{2} \end{vmatrix} = -\frac{a-\sqrt{a^2-4b}}{2} + \frac{a+\sqrt{a^2-4b}}{2} = \sqrt{a^2-4b}$$

$$\begin{pmatrix} 1 & 1 \\ -\frac{a+\sqrt{a^2-4b}}{2} & -\frac{a-\sqrt{a^2-4b}}{2} \end{pmatrix}^{-1} = \frac{1}{\sqrt{a^2-4b}} \begin{pmatrix} -\frac{a-\sqrt{a^2-4b}}{2} & -1 \\ \frac{a+\sqrt{a^2-4b}}{2} & 1 \end{pmatrix} = \begin{pmatrix} -\frac{a}{2\sqrt{a^2-4b}} + \frac{1}{2} & -\frac{1}{\sqrt{a^2-4b}} \\ \frac{a}{2\sqrt{a^2-4b}} + \frac{1}{2} & \frac{1}{\sqrt{a^2-4b}} \end{pmatrix}$$

$$\implies \begin{pmatrix} f'(x) \\ g'(x) \end{pmatrix} = \begin{pmatrix} -\frac{a}{2\sqrt{a^2-4b}} + \frac{1}{2} & -\frac{1}{\sqrt{a^2-4b}} \\ \frac{a}{2\sqrt{a^2-4b}} + \frac{1}{2} & \frac{1}{\sqrt{a^2-4b}} \end{pmatrix} \begin{pmatrix} \phi'(x) \\ \psi(x) \end{pmatrix}$$

So,

$$f'(x) = -\left(\frac{a}{2\sqrt{a^2-4b}} - \frac{1}{2}\right)\phi'(x) - \frac{1}{\sqrt{a^2-4b}}\psi(x)$$

$$f(x) = -\left(\frac{a}{2\sqrt{a^2-4b}} - \frac{1}{2}\right)\phi(x) - \frac{1}{\sqrt{a^2-4b}}\int_0^x \psi(s)\,\mathrm{d}s$$

$$g'(x) = \left(\frac{a}{2\sqrt{a^2-4b}} + \frac{1}{2}\right)\phi'(x) + \frac{1}{\sqrt{a^2-4b}}\psi(x)$$

$$g(x) = \left(\frac{a}{2\sqrt{a^2-4b}} + \frac{1}{2}\right)\phi(x) + \frac{1}{\sqrt{a^2-4b}}\int_0^x \psi(s)\,\mathrm{d}s$$

Then,

$$\begin{aligned} u(t,x) = &-\left(\frac{a}{2\sqrt{a^2-4b}} - \frac{1}{2}\right)\phi\left(x - \frac{a+\sqrt{a^2-4b}}{2}t\right) - \frac{1}{\sqrt{a^2-4b}}\int_0^{x-\frac{a+\sqrt{a^2-4b}}{2}t} \psi(s)\,\mathrm{d}s \\ &+ \left(\frac{a}{2\sqrt{a^2-4b}} + \frac{1}{2}\right)\phi\left(x - \frac{a-\sqrt{a^2-4b}}{2}t\right) + \frac{1}{\sqrt{a^2-4b}}\int_0^{x-\frac{a-\sqrt{a^2-4b}}{2}t} \psi(s)\,\mathrm{d}s \end{aligned}$$

Therefore,

$$\begin{aligned} u(t,x) = &\frac{1}{2}\left(\phi\left(x - \frac{a+\sqrt{a^2-4b}}{2}t\right) + \phi\left(x - \frac{a-\sqrt{a^2-4b}}{2}t\right)\right) \\ &+ \frac{a}{2\sqrt{a^2-4b}}\left(\phi\left(x - \frac{a-\sqrt{a^2-4b}}{2}t\right) - \phi\left(x - \frac{a+\sqrt{a^2-4b}}{2}t\right)\right) \\ &+ \frac{1}{\sqrt{a^2-4b}}\int_{x-\frac{a+\sqrt{a^2-4b}}{2}t}^{x-\frac{a-\sqrt{a^2-4b}}{2}t} \psi(s)\,\mathrm{d}s \end{aligned}$$

In the case $a = 0$ and $b < 0$, putting $b = c^2$, you can check it's exactly the same solution found in Strauss p.36 equation (8).

If $a^2 - 4b < 0$ we don't get a real solution. However, any linear combination solves the problem. Since the operator is linear, it's real and imaginary parts must solve the problem and any combinations of them as well,

$$\mathfrak{R}[u(t,x)] = \frac{u(t,x) + \overline{u(t,x)}}{2} \text{ and } \mathfrak{I}[u(t,x)] = \frac{u(t,x) - \overline{u(t,x)}}{2i}$$

Let $u(t,x) = p(t,x) + iq(t,x)$, then $u_t(t,x) = p_t(t,x) + q_t(t,x)$

If there exists a linear combination of the real and imaginary parts of $u$ that solves the IVP, that's our solution.

let $\sqrt{a^2 - 4b} = di$,

$$\begin{aligned} u(t,x) = &\frac{1}{2}\left(\phi\left(x - \frac{a+di}{2}t\right) + \phi\left(x - \frac{a-di}{2}t\right)\right) \\ &+ \frac{a}{2di}\left(\phi\left(x - \frac{a-di}{2}t\right) - \phi\left(x - \frac{a+di}{2}t\right)\right) \\ &+ \frac{1}{di}\int_{x - \frac{a+di}{2}t}^{x - \frac{a-di}{2}t} \psi(s)\,\mathrm{d}s \end{aligned}$$

Verify $u(0,x) = \phi(x)$,

$$u(0,x) = \frac{1}{2}\left(\phi(x) + \phi(x)\right) + \frac{a}{2di}\left(\phi(x) - \phi(x)\right) + \frac{1}{di}\int_x^x \psi(s)\,\mathrm{d}s = \phi(x)$$

Compute $u_t$,

$$\begin{aligned} u_t(t,x) = &-\left(\frac{a+di}{4}\right)\phi'\left(x - \frac{a+di}{2}t\right) - \left(\frac{a-di}{4}\right)\phi'\left(x - \frac{a-di}{2}t\right) \\ &+ \left(\frac{a}{2di}\frac{a+di}{2}\right)\phi'\left(x - \frac{a+di}{2}t\right) - \left(\frac{a}{2di}\frac{a-di}{2}\right)\phi'\left(x - \frac{a-di}{2}t\right) \\ &- \left(\frac{1}{di}\frac{a-di}{2}\right)\psi\left(x - \frac{a-di}{2}t\right) + \left(\frac{1}{di}\frac{a+di}{2}\right)\psi\left(x - \frac{a+di}{2}t\right) \\ = &-\left(\frac{a}{4} + \frac{d}{4}i\right)\phi'\left(x - \frac{a+di}{2}t\right) - \left(\frac{a}{4} - \frac{d}{4}i\right)\phi'\left(x - \frac{a-di}{2}t\right) \\ &+ \left(\frac{a}{4} - \frac{a^2}{4d}i\right)\phi'\left(x - \frac{a+di}{2}t\right) + \left(\frac{a}{4} + \frac{a^2}{4d}i\right)\phi'\left(x - \frac{a-di}{2}t\right) \\ &+ \left(\frac{1}{2} + \frac{a}{2d}i\right)\psi\left(x - \frac{a-di}{2}t\right) + \left(\frac{1}{2} - \frac{a}{2d}i\right)\psi\left(x - \frac{a+di}{2}t\right) \end{aligned}$$

Verify $u_t(0,x) = \psi(x)$

$$\begin{aligned} u_t(0,x) = &-\left(\frac{a}{4} + \frac{d}{4}i\right)\phi'(x) - \left(\frac{a}{4} - \frac{d}{4}i\right)\phi'(x) \\ &+ \left(\frac{a}{4} - \frac{a^2}{4d}i\right)\phi'(x) + \left(\frac{a}{4} + \frac{a^2}{4d}i\right)\phi'(x) \\ &+ \left(\frac{1}{2} + \frac{a}{2d}i\right)\psi(x) + \left(\frac{1}{2} - \frac{a}{2d}i\right)\psi(x) = \psi(x) \end{aligned}$$

Verify $w(t,x) = \mathfrak{R}[u(t,x)] + \mathfrak{I}[u(t,x)]$ solves the IVP,

$$w(0,x) = p(0,x) + q(0,x) = \phi(x) \text{ and } w_t(t,x) = p_t(x,t) + q_t(x,t)$$

We've shown,

$$u_t(0,x) = \psi(x) \in \mathbb{R} \implies q_t(0,x) = 0 \text{ and } w_t(0,x) = p_t(0,x) = \psi(x)$$

So, for $a^2 - 4b < 0$ the solution is,

$$w(t,x) = \mathfrak{R}[u(t,x)] + \mathfrak{I}[u(t,x)]$$