MATH 148 HOMEWORK 6

1. Consider the pendulum model that we discussed in class. It's also on page 71 of the lecture notes 4. Again, for simplicity, we assume $m = l = g = 1$.

Suppose in addition we consider a damping force of this model as follows. The magnitude of the damping force is proptional to angular velocity while its direction is opposite to the one of the angular velocity. For simplicity, we may assume the damping force to be $-\theta'$.

Then the net external torque become $\tau = -\sin\theta - \theta'$. Moment of inertia and angular acceleration remain the same as before. Then do the following problem:

(1) Construct the second order ODE that describes damped pendulum.
(2) Convert the second order ODE into a nonlinear planar system in terms of (θ, v) where $v = \theta'$ is the angular velocity.
(3) Find all the equilibia of the dynamical system from part (2) and use linearization theorem to study the local dynamical behavior near them.
(4) Describe the motion of the damped pendulum by using the local dynamical behavior you found in part (3).

Part (5) of the problem 2 below might be difficult. It won't be graded. But I strongly recommend that you try your best to think about it.

2. For the Hamiltonian system we did in class:

$$\begin{cases} x' = -y \\ y' = x^3 - x. \end{cases}$$

(1) Show that it's a Hamiltonian system with a Hamiltonian function
$$H = \frac{1}{2}y^2 + \frac{1}{4}(x^2 - 1)^2.$$
(2) Show that for each $c \geq 0$, $\{(x, y) \in \mathbb{R}^2 : H(x, y) \leq c\}$ is a bounded invariant set of the dynamical system (in fact, it's also closed).
(3) Find all the equilibria of this system. Show that $H^{-1}(\frac{1}{4})$ is made up of one equilibium point and two homoclinic orbits attached to it.
(4) Sketch the invariant set $\{(x, y) \in \mathbb{R}^2 : H(x, y) \leq c\}$ on the xy-plane. You may divide the cases into $c = 0$, $0 < c < \frac{1}{4}$, $c = \frac{1}{4}$, and $c > \frac{1}{4}$.
(5) Using all the information above, Poincaré-Bendixon theorem and its corollaires to show that all other trajectories (than those you found in part (3)) are cycles.