Chap 5. Nonlinear Planar System III: Poincaré – Bendixon Thm

For a planar linear system $\vec{x}' = A\vec{x}$, we know that the past and the future of all trajectories, namely they are infinity, equilibria, or stay on a closed curve (cycle).

For a planar nonlinear system, we've seen that life may become more complicated & interesting. Other than those similar to linear systems, past & future of trajectories in a nonlinear planar system may also be a limit cycle (period solutions attract / ~~or~~ repel nearby solutions), homoclinic orbits, or heteroclinic orbits.

Question: For a planar nonlinear system, are there any other type of past & future that trajectories may have?

We will address this question by introducing the Poincaré – Bendixon theorem. Roughly speaking, the answer is no. In other words, lifes in a planar nonlinear system are still ~~p~~ simple. In particular, there are no chaotic behaviors in a continuous time planar system.

How can we possibly address this question for a general planar system $\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$ without even knowing what are f & g?

How can we talk about past & future of trajectories without knowing explicitly ~~th~~ a system?

One of the principles in studying math or doing math research: simplify a complicated math problem into simpler ones until you face the essences of the difficulties.

Given a dynamical system, if we don't know what can do at the beginning, we will try to reduce the big system into smaller subsystem. This is exactly why we were looking for first integrals & their level sets. ~~But~~ Because level sets are invariant and are in some sense subsystems in the whole system. In general, we have the following definition:

Def 5.1: Consider a dynamical system $\phi: \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ on $\mathbb{R}^n$. 1) A set $\Lambda_+ \subseteq \mathbb{R}^n$ is said to be a positively invariant set of ϕ if for any $\vec{x}\in\Lambda_+$, it holds that

$$\{\phi(t,\vec{x}),\ t\geq 0\} \subseteq \Lambda_+,$$

i.e. the entire future of $\vec{x}\in\Lambda_+$ remains in Λ_+.

2) A set $\Lambda_- \subseteq \mathbb{R}^n$ is said to be a negatively invariant set if for any $\vec{x}\in\Lambda_-$, it holds that

$$\{\phi(t,\vec{x}),\ t\leq 0\} \subseteq \Lambda_-,$$

i.e. the entire past of $\vec{x}\in\Lambda_-$ remains in Λ_-

3) A set $\Lambda \subseteq \mathbb{R}^n$ is said to be invariant if it's both positively & negatively invariant.

Note if Λ is invariant, then the follow map

$$\phi : \mathbb{R} \times \Lambda \to \Lambda$$

is well-defined. Thus the restriction of ϕ on Λ is like a sub system of ϕ on $\mathbb{R}^n$.

As discussed before, the reasons we consider invariant set is that it may reduce the complexity of understanding the system, like what we did in a Hamiltonian syseem.

Moreover, invariant sets may have long term means for nearby trajectories as they themselves are invariant. In fact, it's clear that equilibria & cycles are invariant.

Next seep is to look for invariant sets that may have long term meaning for trajectories nearby, e.g. sink, source, saddle, or ~~limit~~ limit cycles. Because for a given initial state, we don't know in general where it will go. How to determine their future & past & relate them to some invariant sets.

Idea: follow a trajectory?

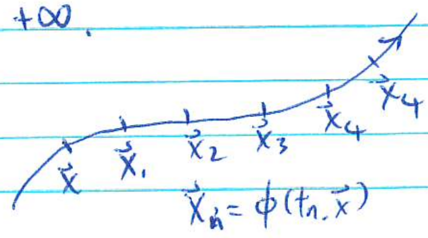
Concretely, we have the following definition:

Def 5.2: For a given DS $\phi(t,\vec{x})$ on $\mathbb{R}^n$, & a point $\vec{x} \in \mathbb{R}^n$, we say $\vec{y} \in \mathbb{R}^n$ is a ω-limit pt of $\vec{x}$ if there is a sequence of time $\{t_n\}_{n\geq 1}$ goes to infinity, i.e. $\lim_{n\to\infty} t_n = \infty$ s.t.

$$\lim_{n\to\infty} \phi(t_n, \vec{x}) = \vec{y}.$$

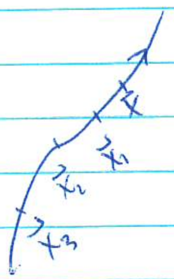
i.e. $\vec{y}$ is a limit of a sequence of points on the trajectory of $\vec{x}$, moreover the time associated with this sequence tends to $+\infty$.

Moreover, we define the ω-limit set, denoted $\omega(\vec{x})$, to be the set of all ω-limit pts of $\vec{x}$, i.e.



$$\omega(\vec{x}) = \{\vec{y} \in \mathbb{R}^n : \lim_{n\to\infty} \phi(t_n, \vec{x}) = \vec{y} \text{ for some time sequence } t_n \text{ s.t. } \lim_{n\to\infty} t_n = +\infty\}.$$

Similarly, we say $\vec{z} \in \mathbb{R}^n$ is a α-limit pt of $\vec{x}$ if $\exists \{t_n\}$ with $\lim_{n\to\infty} t_n = -\infty$ &



$$\lim_{n\to\infty} \phi(t_n, \vec{x}) = \vec{z}$$

& $\alpha(\vec{x}) = \{\vec{z} \in \mathbb{R}^n : \lim_{n\to\infty} \phi(t_n, \vec{x}) = \vec{z}$ for some t_n s.t. $\lim_{n\to\infty} t_n = -\infty\}$.

Basically, $\omega(\vec{x})$ is the future of $\vec{x}$ & $\alpha(\vec{x})$ is the past of $\vec{x}$.

Roughly speaking, a set S of $\mathbb{R}^n$ is closed, if any convegent sequence of S converges to some pt in S. i.e. when you take limit in S, you may only not get outside of S

Next, we are going to determine what are $\omega(\vec{x})$ & $\alpha(\vec{x})$ look like? If $n \geq 3$, then it's basically hopeless as chaotic system may occur. But for $n=2$, we do can determine them. First, we note

Theorem 5.1.: For any $\vec{x} \in \mathbb{R}^n$ & DS ϕ on $\mathbb{R}^n$, $\omega(\vec{x})$ & $\alpha(\vec{x})$ are closed invariant sets.

Proof: We shall ignore the closedness as it involves topology and/or analysis. However for invariance, we can do it. Let $\vec{y} \in \omega(\vec{x})$, we want to show for any $s \in \mathbb{R}$, $\phi(s,\vec{y}) \in \omega(\vec{x})$.

$$\vec{y} \in \omega(\vec{x}) \Rightarrow \exists \{t_n\}_{n\geq 1} \text{ with } \lim_{n\to\infty} t_n = +\infty \text{ s.t. }$$

$$\lim_{n\to\infty} \phi(t_n, \vec{x}) = \vec{y}$$

time s map $\phi(s,\cdot)$ is continuous, thus $\phi(\cancel{\lim}, \lim_{n\to\infty} \phi(t_n,\vec{x}))$

$$= \lim_{n\to\infty} \phi(s, \phi(t_n,\vec{x})) = \phi(s,\vec{y}).$$

But $\phi(s,\phi(t_n,\vec{x})) = \phi(t_n+s, \vec{x})$, i.e.

$$\lim_{n\to\infty} \phi(t_n+s, \vec{x}) = \phi(s,\vec{y}). \text{ clearly } t_n+s \to \infty \text{ as } n\to\infty.$$

$\Rightarrow \phi(s,\vec{y}) \in \omega(\vec{x})$.

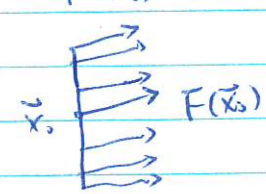
Similarly, one may show $\alpha(\vec{x})$ is invariant.

Fact: if Λ is a closed & invariant set, then $\forall \vec{x} \in \Lambda$, $\omega(\vec{x})$ & $\alpha(\vec{x}) \subset \Lambda$. Roughly speaking, Λ is invariant implies that the entire trajectory of $\vec{x}$ is in Λ for any $\vec{x} \in \Lambda$. Since Λ is closed, any limit pts of sequence of pts on the trajectory of $\vec{x}$ are still in Λ. Note $\omega(\vec{x})$ & $\alpha(\vec{x})$ are then closed & invariant as well. Thus $\omega(\vec{x})$ & $\alpha(\vec{x})$ are kind of "minimal" invariant & closed sets. They are referred to as limit sets.

How to determine them for $n=2$? We start with the following definitions. From now on, $n=2$.

Def 5.3 (local section & flow box).

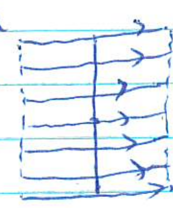
Let ϕ be defined by $\vec{x}' = F(\vec{x})$. Consider $\vec{x}_0 \in \mathbb{R}^2$ where $F(\vec{x}_0) \neq 0$. Then we may pick a line segament I passing through $\vec{x}_0$ s.t. the angle between $F(\vec{x}_0)$ & I is not 0 or π. Moreover, by continuity & suitable choice of I, we may assume for all $\vec{x} \in I$, $F(\vec{x}) \neq 0$ & $\angle(F(\vec{x}), I) \neq 0$



Such a I is called a local section around $\vec{x}_0$.

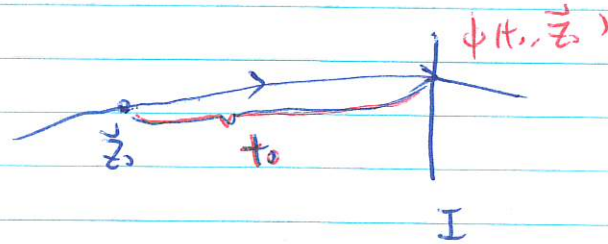
Then for a small number $\delta > 0$, we define

$V_\delta = \{\phi(t, \vec{x}) : \vec{x} \in I \;\&\; |t| < \delta\}$, called a flow box

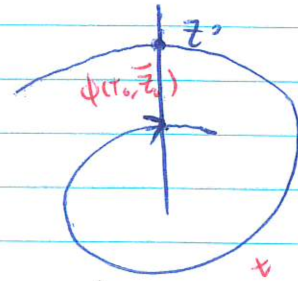


Time of first arrival: Let I be a local section & $\vec{z} \in \mathbb{R}^n$. Suppose $t_0 > 0$ is the smallest positive time s.t. $\phi(t_0, \vec{z}) \in I$, then ~~new~~ call t_0 the time of first arrival of z to I.

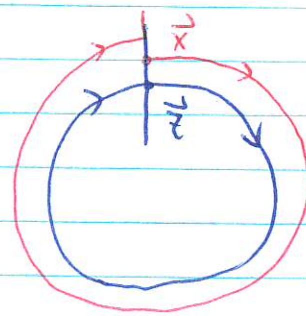
If $z_0 \in I$, then we call such a t_0 the time of first return.



Note in general there might not no first arrival time or first return time.



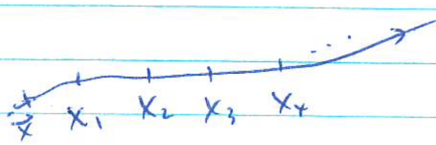
However if γ is a cycle & $\vec{z} \in \gamma$. I is a local section at $\vec{z}$. Then the first return time of $\vec{z}$ is nothing then the period of γ. Moreover for $\vec{x} \neq \vec{z}$ on I, we may also find their first return time



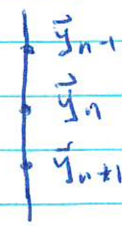
Definition 5.4: (Montone sequence)

① Consider a system $\phi: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. Let $\{\vec{x}_n\}_{n \geq 1}$ be a sequence on the trajectory of $\vec{x}$. We say $\{\vec{x}_n\}_{n \geq 1}$ is monotone along the trajectory if $\exists \{t_n\}_{n \geq 1}$ s.t.

- $0 \leq t_1 < t_2 < t_3 < \cdots < t_n < t_{n+1} < \cdots$
- $x_n = \phi(t_n, \vec{x})$, $n \geq 1$.



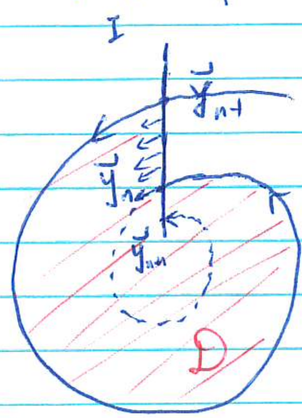
② Let $\{\vec{y}_n\}_{n\geq 1}$ be a sequence on a line segament, e.g. local seg section. We say $\{\vec{y}_n\}$ is monotone if $\forall n \geq 2$, $\vec{y}_n$ is between $\vec{y}_{n-1}$ & $\vec{y}_{n+1}$ in the natural order



Proposition 4.1: Let I be a local section for a planar system ϕ. Let $\vec{y}_0, \dots, \vec{y}_n$ be a sequence on I that lies on the same trajectory. If this sequence is monotone along the trajectory, then they are monotone along I.

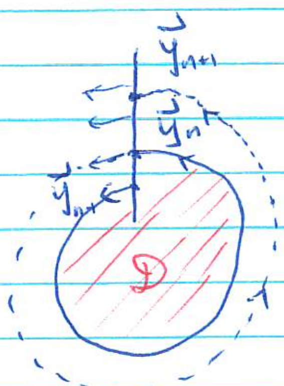
Idea of proof:

either



This D is positively invariant. In particular, one you are in, you may not get out. Thus next time $\vec{y}_n$ comes back to I, it arrives below it.

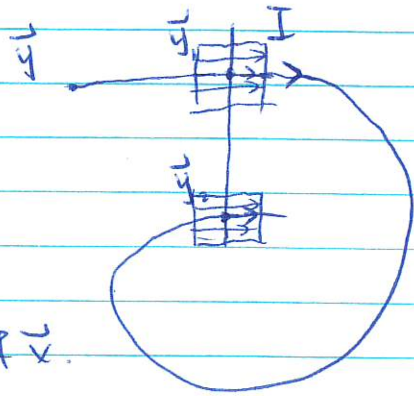
or



The compliement of D is positively invariant. In particular, once you are outside of it, you may not get in again. Thus, next time $\vec{y}_n$ comes back to I, it arrives above it

Prop 4.2: For a planar system ϕ, let $\omega(\vec{x})$ be some ω limit set for some $\vec{x}$ & $\vec{y} \in \omega(\vec{x})$. Then $\phi(t, \vec{y})$ meet any local section at most ~~once~~ at one pt.

Idea of proof:



Suppose $\phi(t, \vec{y})$ meets I at $\vec{y}_1$, $\vec{y}_2$ & $\vec{y}_2 \neq \vec{y}_1$. ~~Since~~

Both $\vec{y}_1$ & $\vec{y}_2$ $\in \phi(t, \vec{y}) \subset \omega(\vec{x})$, i.e. they are both ω-limit pt of $\vec{x}$.

Thus the trajectory of $\vec{x}$, $\phi(t, \vec{x})$, comes arbitrarily close to $\vec{y}_1$ & $\vec{y}_2$ as $t \to \infty$. In particular, one may obtain a monotone sequence on $\phi(t, \vec{x})$ that jump ~~forms~~ from near $\vec{y}_1$ to near $\vec{y}_2$ ~~& back~~, which in particular is not monotone on I. Contradicts Prop 4.1.

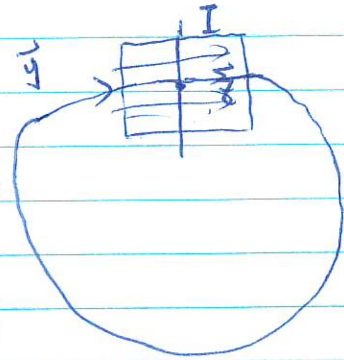
Theorem 5.2 (Poincaré-Bendixon Theorem).

Suppose Ω is a bounded, non-empty limit set of a planar system that contains no equilibria. Then it's a ~~limit~~ cycle.

Idea of proof: Without loss of generality, we assume $\Omega = \omega(\vec{x})$ for some $\vec{x} \in \mathbb{R}^2$.

① First, we show Ω contains a cycle. Let $\vec{y} \in \omega(\vec{x})$. Then $\phi(t, \vec{y}) \subset \omega(\vec{x})$, which is bounded, closed & invariant. By Bolzano-Weirstrass (any bounded

sequence has a convergent subsequence), $\omega(\vec{y}) \neq \phi$.
Pick $\vec{z} \in \omega(\vec{y})$ & construct a local section I & flow box V_δ around $\vec{z}$. Since there are sequence on $\phi(t,\vec{y})$ tends to $\vec{z}$, $\phi(t,\vec{y})$ must visit I infinitely many times. However, since $\vec{y} \in \omega(\vec{x})$, $\phi(t,\vec{y})$ may only meets I at one pt. Thus $\phi(t,\vec{y})$ come back to I infinitely many times & each time it combes back to the same pt, i.e. $\vec{z}$.
Thus $\phi(t,\vec{x})$ is a cycle, say γ.



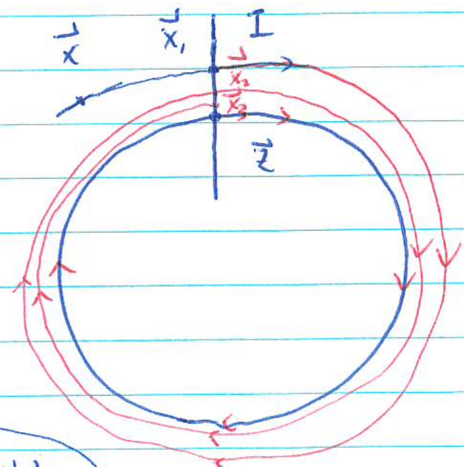
(iii) $\omega(\vec{x}) = \gamma$. If $\vec{x} \in \gamma$, then $\omega(\vec{x}) = \gamma$ & we are done. Otherwise $\vec{x} \notin \gamma$.
$\vec{z} \in \gamma$ is in $\omega(\vec{x})$, thus $\phi(t,\vec{x})$ visit V_δ, hence I, infinitely many times. Then we may arrange its visits at I as follows:

$\vec{x}_1$: first arrival

$\vec{x}_2$: first return of $\vec{x}_1$

⋮

$\vec{x}_n$: first return of $\vec{x}_{n-1}$



Thus, in this case, γ is a limit cycle $\Leftarrow$ $\omega(\vec{x}) = \gamma$

Thus, we will get a monotone sequence $\{\vec{x}_n\}$ on I. The fact $\vec{z} \in \omega(\vec{x})$ forces $\lim_{n\to\infty} \vec{x}_n = \vec{z}$, which in turn force the piece of orbit between $\vec{x}_{n-1}$ & $\vec{x}_n$ tends to γ, i.e. $\phi(t,\vec{x})$ winds around & approach γ as $t \to \infty$ □

Poincaré - Bendixon Theorem implies:
Any bounded, nonempty $\omega(\vec{x})$ or $\alpha(\vec{x})$ is ~~either~~ an equilibrium pt, a cycle, or a closed curve consists of equilibria & orbits connecting them, i.e. orbits like homoclinic or heteroclinic orbits. This answers the main question of this chaper.

But Poincaré - Bendixon Thm tells us more.

Corollary 1: Let γ be as in Thm 5.2. Moreover, suppose $\gamma = \omega(\vec{x})$ for some $\vec{x} \notin \gamma$, i.e. γ is a limit cycle. Then $\exists\, r>0$, s.t. $\forall\, y \in B(\vec{x}, r)$, $\omega(\vec{y}) = \gamma$. In the language of topology, the points whose trajectories tend to a limit cycle form an open set.

Corollary 2: A bounded, closed, invariant set contains either a cycle or an equilibrium pt.

Corollary 3: Let γ be a cycle & $\mathcal{U}$ be the open region enclosed by γ. Then $\mathcal{U}$ contains either a ~~limit~~ cycle or an equilibrium pt.

Corollary 4: Let γ & $\mathcal{U}$ be as in corollary 3, then $\mathcal{U}$ contains an equilibrium pt.

Corollary 5: Let ϕ be a planar system that has a first integral which is non-constant on any open ball. Then ϕ may not have a sink, source, or a limit cycle.

Proof: If $\vec{x}_0$ is a sink or source of ϕ, ~~then~~ say it's a sink, then $W^s(\vec{x}_0)$ contains an open ball. Let f be the first integral. Note $\forall \vec{y} \in W^s(\vec{x}_0)$,

$$\lim_{t\to\infty} \phi(t, \vec{y}) = \vec{x}_0$$

$$\Rightarrow \quad f\left(\lim_{t\to\infty} \phi(t, \vec{y})\right) = \vec{x}_0$$

$$\Rightarrow \quad \lim_{t\to\infty} f\circ\phi(t, \vec{y}) = f(\vec{x}_0)$$

However, f is a first integral $\Rightarrow$ f stay constant on any trajectory $\Rightarrow$ $f\circ\phi(t,\vec{y}) = f(\vec{x}_0)$, $\forall \vec{y} \in W^s(\vec{x}_0)$
$\Rightarrow$ $f(\vec{y}) = f(\vec{x}_0)$ $\forall \vec{y} \in W^s(\vec{x}_0)$ violates the fact that f is nonconstant on any open ball.

Similarly, one may prove it if $\vec{x}_0$ is a source.

If γ is a limit cycle of ϕ, then by corollary 1, $\mathcal{B}_\gamma = \{\vec{x} : \omega(\vec{x}) = \gamma,\ \vec{x} \notin \gamma\}$ ~~is~~ contains open balls. Again, $\forall \vec{x} \in \mathcal{B}_\gamma$,

$$\lim_{t\to\infty} \phi(t, \vec{x}) = \gamma \quad \Rightarrow \quad \lim_{t\to\infty} f\circ\phi(t,\vec{x}) = f(\gamma) = c$$

$\Rightarrow$ $f(\vec{x}) = c$, $\forall \vec{x} \in \mathcal{B}_\gamma$ which contains open balls, a contradiction.

In particular, Hamiltonian system may not have source, sink, or limit cycles which matches with Liouville's Theorem.