**MATH 148—HOMEWORK 5**

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**1** Consider a planar nonlinear system

$$\begin{cases} x' = y(y-2) \\ y' = x \end{cases}$$

(1) For each equilibrium point, discuss if one can apply the Linearization Theorem to determine the local dynamics near it. If yes, sketch such local phase portraits. If not, explain why it cannot be applied.

(2) Determine if the system is a Hamiltonian system. If yes, find its Hamiltonian function.

(3) If it's a Hamiltonian system, sketch the phase portraits via plotting the level curves of the Hamiltonian function.

*Hint: Compare this problem with example 2 of Chapter 4 that we did in class. You may need to switch the roles of x and y here.*

Slu .

(1)—

$$(*) \begin{cases} 0 = y(y-2) = y^2 - 2y \\ 0 = x \end{cases} \quad \implies (y = 0 \text{ or } y = 2) \text{ and } x = 0$$

So, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ are the equilibria of $(*)$.

Let $F(x,y) = \begin{pmatrix} y^2 - 2y \\ x \end{pmatrix}$

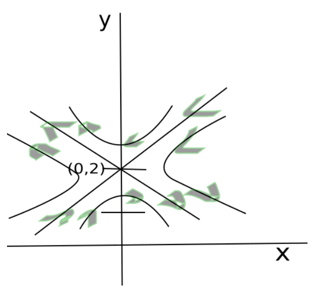
$$\implies DF(x,y) = \begin{pmatrix} 0 & 2y-2 \\ 1 & 0 \end{pmatrix}$$

For $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$: $DF(0,0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \implies \text{Tr}(DF(0,0)) = 0$ and $|DF(0,0)| = 0$.

So, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a center. We cannot apply the Linearization Theorem.

For $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$: $DF(0,2) = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \implies \text{Tr}(DF(0,0)) = 0$ and $|DF(0,0)| = -2$.

So, $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ is a saddle. So we can apply the Linearization Theorem.



(2)—

It is!

$H(x,y) = -\int y^2 - 2y \ dy + \int x \ dx = -\frac{y^3}{3} + y^2 + \frac{x^2}{2}$

Since, $x' = -\frac{\partial H}{\partial y} = \frac{3y^2}{3} - 2y + 0 = y(y-2)$ and $y' = \frac{\partial H}{\partial x} = \frac{2x}{2} = x$

(3)—

$H^{-1}(C) = \{x, y \in \mathbb{R}| \ -\frac{y^3}{3} + y^2 + \frac{x^2}{2} = C\}$

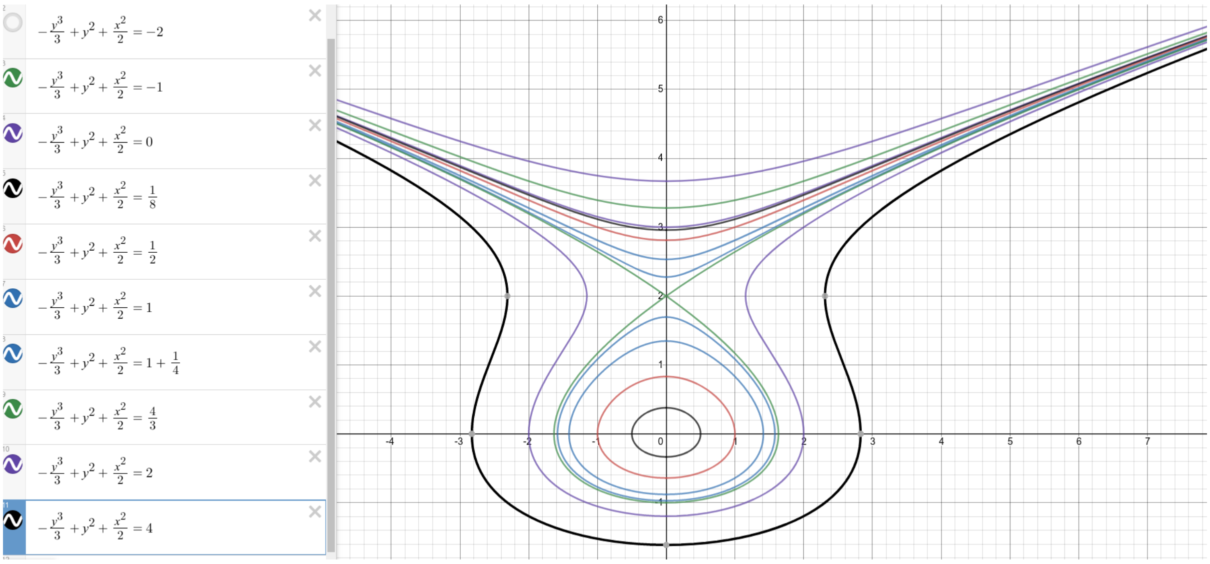
The constant can absorb the fractions. Equivalently:

$H^{-1}(C) = \{x, y \in \mathbb{R}| \ -2y^3 + 6y^2 + 3x^2 = C\}$ $(**)$

We can express $x$ as a function of $y$:

$H^{-1}(C) = \{y \in \mathbb{R}| \ f(y) = \pm\sqrt{\frac{C+2y^3-6y^2}{3}}\}$

So now we need to graph for different values of $C$, but reflect with respect to the $x$ axis. I did that in my notebook, but I have a computer so... I used expression $(**)$ in desmos.com to generate the following phase-portrait.




Again in practice, we need to sketch the graph of $f(y)$ for different values of $C$.

It's not defined when $\frac{C+2y^3-6y^2}{3} = 0$ so that's why we get disconnected chunks.

**2** Let $\vec{x_0}$ be an equilibrium point of a planar dynamical system $\Phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by a nice ODE system $\vec{x}' = F(\vec{x})$. Recall that we say $\vec{x_0}$ is a sink if $\vec{0}$ is a sink of the linearized system $\vec{u}' = DF(\vec{x_0})\vec{u}$, where $\vec{u} = \vec{x} - \vec{x_0}$ and $DF(\vec{x_0})$ is the derivative or Jacobian matrix of $F$ at $\vec{x_0}$. Similarly, we can define a source or a saddle for a nonlinear system.

Let $\Psi$ be a dynamical system defined by $\vec{x}' = G(\vec{x})$. Suppose $\Phi$ is topologically conjugate to $\Psi$ via homeomorphism $h$.

Suppose no eigenvalue of $DF(\vec{x_0})$ or $DG(h(\vec{x_0}))$ has zero real part. In other words, the linearized systems at both $\vec{x_0}$ and $h(\vec{x_0})$ are hyperbolic. Show that $h(\vec{x_0})$ is a saddle if and only if $\vec{x_0}$ is a saddle. Show that the same holds true if one replaces a saddle by a source or a sink. In other words, topological conjugacy preserves the types of hyperbolic equilibria for nonlinear systems.

*Hint: Previously, we did it for planar linear systems. Try to apply here the Linearization Theorem and the fact that topological conjugacy is an equivalence condition. Then you may reduce the nonlinear cases to linear cases.*

Slu .

Denote the dynamical systems corresponding to the Jacobians DF($\vec{x_0}$) and DG($h(\vec{x_0})$), respectively by $\Phi^l$ and $\Psi^l$. Since, there exist a homomorphism $h$ for the conjugate pairs $\Psi$ and $\Psi$, it follows that $h$ is a homomorphism of $\Phi^l$ and $\Psi^l$ in a neighborhood around DF($\vec{x_0}$) and DG($h(\vec{x_0})$) respectively.

($\Longleftarrow$) Ass. $\vec{x_0}$ is a saddle of $\vec{x}' = F(\vec{x})$

$W^s(\vec{0}) := \{\vec{x} \in \mathbb{R}^2 | \vec{x} \neq 0$ and $\lim_{t\to\infty} \Phi^l(t, \vec{x}) = \vec{0}\}$ and

$W^u(\vec{0}) := \{\vec{x} \in \mathbb{R}^2 | \vec{x} \neq 0$ and $\lim_{t\to-\infty} \Phi^l(t, \vec{x}) = \vec{0}\}$

$\vec{0}$ is a saddle $\implies$ both $W^s$ and $W^u$are not empty.

Since, $\Psi^l$ is conjugate to $\Phi^l$ via homomorphism $h$. It follows that $h \circ \Phi^l(t, \vec{x}) = \Psi^l(t, h(\vec{x}))$. It follows, that:

$W^s(h(\vec{0})) := \{\vec{x} \in \mathbb{R}^2 | \vec{x} \neq 0$ and $\lim_{t\to\infty} h \circ \Phi^l(t, \vec{x}) = \Psi^l(t, h(\vec{x})) = h(\vec{0})\}$ and

$W^u(h(\vec{0})) := \{\vec{x} \in \mathbb{R}^2 | \vec{x} \neq 0$ and $\lim_{t\to-\infty} h \circ \Phi^l(t, \vec{x}) = \Psi^l(t, h(\vec{x})) = h(\vec{0})\}$

are both non-empty, as $h$ is continuous and bijective.

So, $h(\vec{x_0})$ a saddle of $\vec{x}' = G(\vec{x})$

($\Longrightarrow$) Ass. $h(\vec{x_0})$ is a saddle of $\vec{x}' = G(\vec{x})$

Since, $\Psi^l$ is conjugate to $\Phi^l$ via homomorphism $h$. There exists a continuous inverse $h^{-1}$ of $h$. It follows that $\Psi^l$ is conjugate to $\Phi^l$ via $h^{-1}$. So, $h^{-1} \circ \Psi^l(t, h(\vec{x})) = \Phi^l(t, h^{-1} \circ h(\vec{x})) = \Phi^l(t, \vec{x})$ .

Since $h(\vec{0})$ a saddle

$W^s(h(\vec{0})) := \{\vec{x} \in \mathbb{R}^2 | \vec{x} \neq 0$ and $\lim_{t\to\infty} \Psi^l(t, h(\vec{x})) = h(\vec{0})\}$ and

$W^u(h(\vec{0})) := \{\vec{x} \in \mathbb{R}^2 | \vec{x} \neq 0$ and $\lim_{t\to-\infty} \Psi^l(t, h(\vec{x})) = h(\vec{0})\}$

are both non-empty.

Via the conjugacy $h^{-1}$ we obtain the following sets:

$W^s(\vec{0}) = W^s(h^{-1} \circ h(\vec{0})) = \{\vec{x} \in \mathbb{R}^2 | \vec{x} \neq 0$ and $\lim_{t\to\infty} h^{-1} \circ \Psi^l(t, h(\vec{x})) = \Phi^l(t, \vec{x}) = \vec{0}\}$ and

$W^u(\vec{0}) = W^s(h^{-1} \circ h(\vec{0})) = \{\vec{x} \in \mathbb{R}^2 | \vec{x} \neq 0$ and $\lim_{t\to-\infty} h^{-1} \circ \Psi^l(t, h(\vec{x})) = \Phi^l(t, \vec{x}) = \vec{0}\}$

Since $h^{-1}$ is a continuous bijection, these sets are not empty.

So, $\vec{x_0}$ is a saddle of $\vec{x}' = F(\vec{x})$

Similarly, it is clear that the homomorphism preserves the type of the systems, because the type changes depending only on whether the initial sets $W^s$ and $W^u$ are initially empty. The argument breaks only in the case when both $W^s$ and $W^u$ are empty, as the center type doesn't give any information for the Linearization Theorem. ■

3. For a planar linear system $\vec{x}' = A\vec{x}$ with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, determine for what choices of $a, b, c, d \in \mathbb{R}$, it's a Hamiltonian system. Assume in addition that $|A| \neq 0$. Then without using the Liouville's Theorem, explain why $\vec{0}$ cannot be a sink or source for such choices of $a, b, c, d$. Or equivalently, why $\vec{0}$ can only be a center or a saddle point in such system.

Slu .

$$(***) \begin{cases} x' = ax + by \\ y' = cx + dy \end{cases}$$

$(***)$ is Hamiltonian $\iff \frac{\partial ax+by}{\partial x} = -\frac{\partial cx+dy}{\partial y} \quad [1]$

$\implies a = -d$

So, $a, b$, and $c$ are free variables. However, $d = -a$ as [1] is an iff statement.

$\implies A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \implies \text{Tr}(A) = 0$

$|A| \neq 0 \implies |A| < 0$ or $0 < |A|$

Case I: $|A| < 0$ and $\text{Tr}(A) = 0 \implies \vec{0}$ is a saddle of $(***)$

Case I: $0 < |A|$ and $\text{Tr}(A) = 0 \implies \vec{0}$ is a center of $(***)$

■