MATH 148 HOMEWORK 7

1. Review and make sure you understand the following notions and theorems. When understanding abstract notions, try to have concrete examples in mind. Below we consider planar dynamical systems, i.e. $n = 2$:

- positively and negatively invariant sets, invariant set
- ω-limit point and set, α-limit point and set, and their dynamical meaning
- ω or α-limit sets are invariant
- local section, flow box, times of first arrival and return to a local section
- monotone sequences along a trajectory or along a line segament
- Proposition 5.1: consider a sequence that is both on a local section and on a trajectory. If it's monotone on the trajectory, then it's monotone on the local section.
- Proposition 5.2: if $\vec{y}$ is a ω-limit point of some point $\vec{x}$, then the trajectory of $\vec{y}$ meet any local section at most at one point.
- **Poincaré-Bendixon theorem**: for any $\vec{x}$, if $\omega(\vec{x})$ ($\alpha(\vec{x})$, resp.) is bounded, nonempty, and contains no equlibirium point, then $\omega(\vec{x})$ ($\alpha(\vec{x})$, resp.) is a cycle. Moreover, either $x \in \omega(\vec{x})$ and the trajectory of $\vec{x}$ is the cycle $\omega(\vec{x})$, or $x \notin \omega(\vec{x})$ and $\omega(\vec{x})$ is a limit cycle that attracts (repels, resp.) some nearby trajectories.

2. Suppose the trajectory of $\vec{x}$, $\phi(t, \vec{x})$, is a cycle i.e. there is a $T > 0$ such that $\phi(t + T, \vec{x}) = \phi(t, \vec{x})$ for all $t \in \mathbb{R}$. Let γ denotes this cycle. Show that $\omega(\vec{x}) = \alpha(\vec{x}) = \gamma$.

3. Recall in HW 6, we consider the Hamiltonian system $\begin{cases} x' = -y \\ y' = x^3 - x \end{cases}$. We know:

(1) Its Hamiltonian function may be chosen as: $H = \frac{1}{2}y^2 + \frac{1}{4}(x^2 - 1)^2$.
(2) For each $c \geq 0$, $\{(x, y) \in \mathbb{R}^2 : H(x, y) \leq c\}$ is a bounded, invariant set of the dynamical system.
(3) There are three equilibria $(0, 0), (\pm 1, 0)$. Moreover, $(0, 0) \in H^{-1}(\frac{1}{4})$ is a homoclinic point with two homoclinic orbits and $H^{-1}(0) = \{(\pm 1, 0)\}$.

Using the information above and following the steps below to show that for each (x, y) with $H(x, y) \neq 0$ or $\frac{1}{4}$, its trajectory is a cycle.

- step 1: for each c, $H^{-1}(c)$ is a bounded, invariant set.
- step 2: fix any $\vec{x} = (x, y)$ with $H(x, y) = c$. By continuity of H, show that $\omega(\vec{x})$ is a subset of $H^{-1}(c)$. Thus, $\omega(\vec{x})$ is bounded and nonempty. *Here you may directly use the fact that $\omega(x) \neq \varnothing$ which can be obtained via Bolzano-Weirstrass theorem.*
- step 3: if $c \neq 0, 1/4$, then by Poincaré-Bendixon, $\omega(\vec{x})$ must be a cycle.
- step 4: if $\vec{x} \notin \omega(\vec{x})$, then $\omega(\vec{x})$ is a limit cycle. We will show that Hamiltonian systems may not have limit cycles (or it's actually a consequence of Liouville's theorem that we mentioned in the end of Chapter 4). Thus $\vec{x} \in \omega(\vec{x})$ and the cycle $\omega(\vec{x})$ is the trajectory of $\vec{x}$.

I strongly recommend that you think about the following questions. I will answer these quesitons for you at some point in class.

(1) Based on the steps of problem 3 on page 1, can you draw the following conclusion: for a planar Hamiltonian system, if all the level sets of the Hamiltonian function are bounded, then trajectories may only be one of the following types: equilibrium points, homoclinic or heteroclinic type of trajectories (i.e. curves consists of equilibria and orbits connecting them), or cycles?

(2) Based on the conclusion in question (1), can you construct other hamiltonian systems with such types of trajectories as described in question (1)?

(3) In this conclusion of question (1), can we replace "planar Hamiltonian system" by "planar system with a non-trival first integeral"? Here a non-triviality function means a function $f : \mathbb{R}^2 \to \mathbb{R}$ that doesn't stay constant on any open ball, i.e. $B_r(\vec{x}) = \{\vec{y} : |\vec{x} - \vec{y}| < r\}$ where $|\vec{x} - \vec{y}|$ is the distance between the two points $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^2$.