Chap 4. Nonlinear Planar Systems II: Hamiltonian Systems

Definition 4.1: A planar system is said to be a Hamiltonian system if it's in the form

$$\begin{cases} x' = -\frac{\partial H}{\partial y}(x,y) \\ y' = \frac{\partial H}{\partial x}(x,y), \end{cases} \qquad (1)$$

where $H: \mathbb{R}^2 \to \mathbb{R}$ is a nice function in (x,y). Here nice means H is C^2, i.e. all its second derivatives are continuous & (1) defines a smooth dynamical system on $\mathbb{R}^2$. H is called a Hamiltonian function or Hamiltonian.

Hamiltonian system is a particular type of dynamical systems that has very deep & rich phenomena. Moreover, they lie at the origin of ODEs, dynamical systems & mechanics as many physical models can be formulated as Hamiltonian systems such as motions of celestial objects.

In general, a Hamiltonian system can be defined on any even dimensional space like $\mathbb{R}^{2n}$. Suppose we write points in $\mathbb{R}^{2n}$ as $(\vec{p}, \vec{q})$ where

$\vec{p} = (p_1, \dots, p_n)$ & $\vec{q} = (q_1, \dots, q_n) \in \mathbb{R}^n$.

Consider a nice Hamiltonian $H: \mathbb{R}^{2n} \to \mathbb{R}$. Then a general Hamiltonian system on $\mathbb{R}^{2n}$ is a system of $2n$ ODEs as

$$\begin{cases} p_i' = -\frac{\partial H}{\partial q_i} \\ q_i' = \frac{\partial H}{\partial p_i} \end{cases} \quad i=1,\dots,n \quad \text{or simply} \quad \begin{cases} \vec{p}\,' = -\frac{\partial H}{\partial \vec{q}} \\ \vec{q}\,' = \frac{\partial H}{\partial \vec{p}} \end{cases}$$

We will mainly focus on planar Hamiltonian systems, i.e. $n=1$. as we defined in (1) of Def 4.1.

First, we note the following simple but beautiful fact regarding Hamiltonian systems.

- Theorem 4.1: For a Hamiltonian system

$$(1)\begin{cases} x' = -\frac{\partial H}{\partial y}(x,y) \\ y' = \frac{\partial H}{\partial x}(x,y) \end{cases},$$

the Hamiltonian function $H: \mathbb{R}^2 \to \mathbb{R}$ stays constant along any trajectories of the system. In other word, if $(x(t), y(t))$ is a solution of the system (1), then $H(x(t), y(t)) \equiv C$ for all t, for some constant C.

Proof: To show that $H(x(t), y(t)) \equiv C$ for all t, we just need to show that as a function in t, $H(x(t), y(t))$ has zero derivative for all t. By chain rule,

$$\frac{d}{dt} H(x(t), y(t)) = \frac{\partial H}{\partial x}(x(t), y(t))\, x'(t) + \frac{\partial H}{\partial y}(x(t), y(t)) \cdot y'(t)$$

by equation (1) & the fact $(x(t), y(t))$ are its solution

$$= \frac{\partial H}{\partial x}(x(t), y(t)) \cdot \left(-\frac{\partial H}{\partial y}(x(t), y(t))\right) + \frac{\partial H}{\partial y}(x(t), y(t)) \cdot \frac{\partial H}{\partial x}(x(t), y(t))$$

$$= -\frac{\partial H}{\partial x} \cdot \frac{\partial H}{\partial y} + \frac{\partial H}{\partial y} \cdot \frac{\partial H}{\partial x}$$

$$= 0 \quad \text{for all } t.$$

as desired. $\square$

Definition 4.2 : Consider an ODE system $\vec{x}' = F(\vec{x})$ on $\mathbb{R}^n$. Suppose there is a non-constant smooth function

$$g : \mathbb{R}^n \to \mathbb{R}$$

s.t. for any solution $\vec{x}(t)$ of $\vec{x}' = F(x)$, there is a $c \in \mathbb{R}$ with $g(\vec{x}(t)) \equiv c$ for all t. Then we say such a g is a first integral of the ODE system $\vec{x}' = F(\vec{x})$.

Basically, Theorem 4.1 says that Hamiltonian functions are alway first integrals of their Hamiltonian system.

Why do we care about first? It reduces dimension of the phase space.

Another way to that g stays constant along solutions $\vec{x}(t)$ of $\vec{x}' = F(\vec{x})$ is that $\vec{x}(t)$ stays on the level set of g. By definition, level sets $g^{-1}(c)$ of g are $g^{-1}(c) = \{\vec{x} \in \mathbb{R}^n : g(\vec{x}) = c\}$ for some $c \in \mathbb{R}$.

Clearly, if $g(\vec{x}(t)) \equiv c$, then

$$\vec{x}(t) \in g^{-1}(c) \quad \text{for all } t.$$

Thus, if we pick a initial state $\vec{x}_0 \in g^{-1}(c)$, then the whole trajectory $\vec{x}(t)$ of $\vec{x}_0$ stays in $g^{-1}(c)$. Such a subset $g^{-1}(c)$ of $\mathbb{R}^n$ is called an invariant set of $\vec{x}' = F(\vec{x})$.

Now instead of studying $\vec{x}' = F(\vec{x})$ as a system in $\mathbb{R}^n$, we may consider their behaviors on $g^{-1}(c)$, for each $c \in \mathbb{R}$.

Generally speaking, if g is smooth & nonconstant & nice in some sense, then for most $c \in \mathbb{R}$

$$g^{-1}(c) \text{ is a nice } n-1 \text{ dimensional}$$

object of $\mathbb{R}^n$. For example, consider

$$g: \mathbb{R}^3 \to \mathbb{R} \quad \text{as } g(x,y,z) = x^2+y^2+z^2$$

Then $g^{-1}(c) = \begin{cases} \phi, & c<0 \\ \vec{0}, & c=0 \\ \text{sphere } \{(x,y,z) : x^2+y^2+z^2 = c\}, & c>0 \end{cases}$.

In particular, spheres are 2 dimensional.

If in addition, $\vec{x}' = F(\vec{x})$ has two first integrals g & f that are independent in some sense, then $\forall c_1, c_2 \in \mathbb{R}$ $\quad g^{-1}(c_1) \cap f^{-1}(c_2)$ is also invariant subset of $\vec{x}' = F(\vec{x})$, which generally is a $n-2$ dimensional object in $\mathbb{R}^n$.

One could image, for a n-dimensional system, if we can find $n-1$ independent first integrals

$$g_1, \ldots, g_{n-1}$$

Then $g_1^{-1}(c_1) \cap g_2^{-1}(c_2) \cap \cdots \cap g_{n-1}^{-1}(c_{n-1})$ is again invariant which is a $n-(n-1)=1$ dimensional object, in other words, curves. Thus since trajectories stay on those curves, those curves are typically just trajectories.

In fact, in old times, when mathematicians and physicists were talking about solving ODEs, what they really meant was to find enough first integrals.

Back to planar Hamiltonian system $\begin{cases} x' = -\frac{\partial H}{\partial y}(x,y) \\ y' = \frac{\partial H}{\partial x}(x,y). \end{cases}$

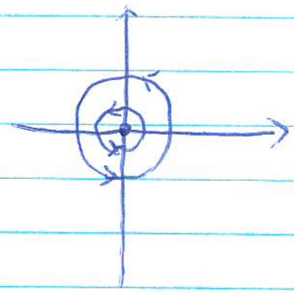
clearly, $H^{-1}(c)$, $c \in \mathbb{R}$ are invariant sets of the system since Hamiltonian is a first integral, Since H is a two variable function, $H^{-1}(c)$ is generally a curve, i.e. trajectories. Thus it's relatively easy to find trajectories of planar Hamiltonian system.

Example ①: $\begin{cases} x' = -y \\ y' = x. \end{cases}$ Let $H(x,y) = \frac{1}{2}(x^2+y^2)$, Then

$\frac{\partial H}{\partial x} = x$ & $\frac{\partial H}{\partial y} = y$, hence $\begin{cases} x' = -\frac{\partial H}{\partial y} \\ y' = \frac{\partial H}{\partial x} \end{cases}$, i.e. it's a Hamiltonian system. Level sets are $\{(x,y) : \frac{1}{2}(x^2+y^2) = c\}$

$$\Rightarrow H^{-1}(c) = \begin{cases} \phi, & c<0 \\ \vec{0}, & c=0 \longrightarrow \text{equilibrium} \\ \text{circles } \{(x,y) : x^2+y^2 = 2c\}, & c>0 \longrightarrow \text{periodic solutions} \end{cases}$$

$\Rightarrow$ Phase portraits



$\vec{0}$ is a center.

In fact, this is a linear system. So we say solve it as a linear system.

$\vec{x}' = A\vec{x}$, $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. $\Rightarrow p(\lambda) = \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$

$\Rightarrow \vec{0}$ is indeed a center. Moreover, in this case let's instead use $\lambda_2 = -i$ & $\vec{v}_2$. Clearly

$$A + iI_2 = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \to \begin{bmatrix} i & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot i$$

Since we used $-\beta$, the fundamental matrix is

$$\bar{X}(t) = e^{\alpha t} \cdot \cancel{R_{\beta t}} \cdot [\vec{p} \; \vec{q}\,] \cdot R_{\beta t} \quad \text{where } \vec{p} + i\vec{q} = \vec{v}_2$$

$\alpha = 0$, $\beta = 1$, $[\vec{p} \; \vec{q}\,] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

$\Rightarrow \bar{X}(t) = R_t \Rightarrow$ general solutions are $\vec{x}(t) = \bar{X}(t) \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$\Rightarrow \vec{x}(t) = \cancel{R_{-t}}\, R_t \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, where R_t is just rotation counter-clockwisely as t increases. Thus $\vec{x}(t)$ stays on the circle of $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ & rotates ~~clock~~ counter clock-wisely, which matches the previous description.

In general, for a planar system $\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$, how can we determine if it's Hamiltonian? In other words, how do we know if there is a $H: \mathbb{R}^2 \to \mathbb{R}$ s.t.

$$\frac{\partial H}{\partial x} = g(x,y) \quad \& \quad \frac{\partial H}{\partial y} = -f(x,y).$$

It turns out that in this case, we just need to verify the mixed partial matches or not, i.e.

" $\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial^2 H}{\partial y \partial x}$ if such H exists "

implies that "if $\frac{\partial g}{\partial y} = \frac{-\partial f}{\partial x}$, then such H exists"

If $\frac{\partial g}{\partial y} = -\frac{\partial f}{\partial x}$, then one can find H via

$$\begin{cases} \frac{\partial H}{\partial x} = g \\ \frac{\partial H}{\partial y} = -f \end{cases} \Rightarrow \left\{ \begin{array}{l} H(x,y) = \int g(x,y)\,dx + h(y) \\ \frac{\partial H}{\partial y} = -f \end{array} \right\}$$

$\Rightarrow$ One can find h via $\frac{\partial \int g(x,y)\,dx}{\partial y} + h'(y) = -f$.

There is a special case that will be easily determined to be Hamiltonian, i.e.

$$\begin{cases} x' = f(y) \\ y' = g(x) \end{cases}$$ where f is a single variable function in x & g a single variable function in x.

Because $\frac{\partial g(x)}{\partial y} = 0 = \frac{\partial f(y)}{\partial x}$ & clearly

$H(x,y) = \int g(x)\,dx - \int f(y)\,dy + C, \quad C \in \mathbb{R}$.

Example ②: $\begin{cases} x' = y \\ y' = x(x-4) \end{cases}$ is Hamiltonian with

Hamiltonian function $H(x,y) = \int x(x-4)\,dx - \int y\,dy$

$\Rightarrow H(x,y) = \frac{1}{3}x^3 - 2x^2 - \frac{1}{2}y^2$

Thus trajectories are curves $\frac{1}{3}x^3 - 2x^2 - \frac{1}{2}y^2 = C$.

How to find such curves ① & connect them to trajectories ② of the Hamiltonian system?

Let's rewrite it as $y^2 = \frac{2}{3}x^3 - 4x^2 - 2C$.

(68)

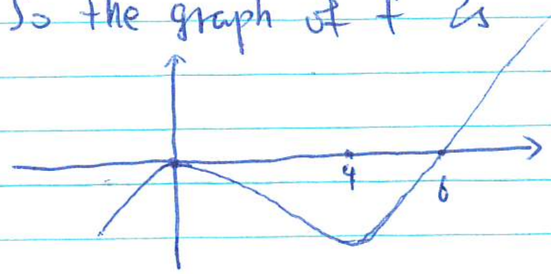
- Clearly, $y^2 \geq 0$ implies that we must
$$\frac{2}{3}x^3 - 4x^2 - 2c \geq 0,$$
i.e. not all x are allowed since it goes to $-\infty$ as $x \to -\infty$.

- Secondly, it gives two functions
$$y = \pm\sqrt{\frac{2}{3}x^3 - 4x^2 - 2c}$$
one in the upper half plane & ~~ond~~ the other in the lower half plane. Moreover, they are reflection w.r.t. x-axis. In particular, if they meet, they can only meet when they are both zero.

To consider which x's are ~~add~~ admissible, we need to consider the function
$$f(x) = \frac{2}{3}x^3 - 4x^2$$
Clearly, $f'(x) = 2x^2 - 8x = 0$ gives $x(x-4) = 0$, i.e. $x = 0$, $x = 4$. Moreover, $f''(x) = 4x - 8 \Rightarrow f''(0) = -8$, $f''(4) = 8$
$\Rightarrow 0$ is a local maximum & 4 a local minimum pt.

Since $f'(x) = 2x^2 - 8x \begin{cases} > 0, & x < 0 \\ < 0, & 0 < x < 4 \\ > 0, & x > 4 \end{cases} \Rightarrow f(x)$

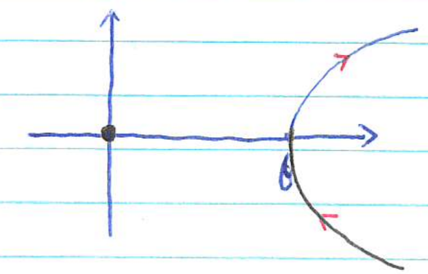
is ~~decreasing~~ increasing on $(-\infty, 0)$ & on $(4, +\infty)$ & is decreasing on $(0, 4)$. So the graph of f is



Note $y = \pm\sqrt{f(x)+c}$ (we may replace $-2c$ by c since it's arbitrary constanst anyway).

Basically, we take the part of graph ~~this~~ that is on the upper half plane, then we take square root of it. Then we get the part $y=\sqrt{f(x)+c}$. Reflecting w.r.t. x-axis, we get the other part $y=-\sqrt{f(x)+c}$.

(i) Take $c=0$ for example, the curve corresponds to the graph of $y=\sqrt{f(x)}$ is roughly: ~~Consider also the part $y=-\sqrt{f(x)}$~~

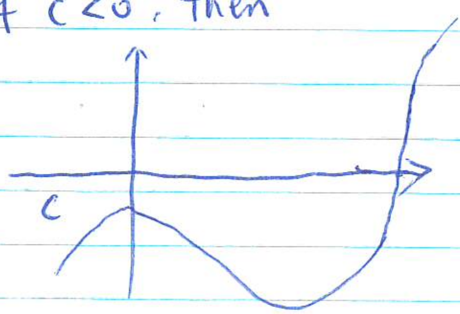


Since $(0,0)$ is isolated & some trajectory are on it, it must be an equilibrium pt.

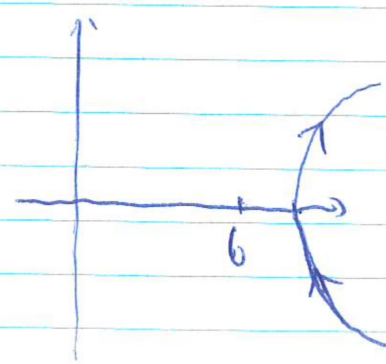
The other part is another trajectory. Using the vector field $\begin{pmatrix} y \\ x(x-4) \end{pmatrix}$, we can easily determine the direction of motion ($x'=y>0$ on the upper half plane).

Note $f(x)+c$ are just translations of f. We may get the following different cases:

(ii) if $c<0$, then

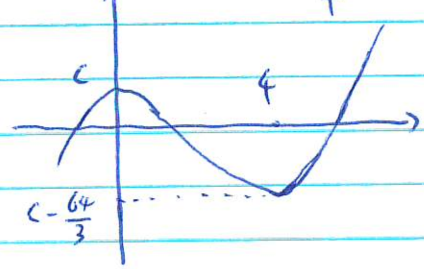


Take the part on the upper-plane, take square root, reflection w.r.t. x-axis

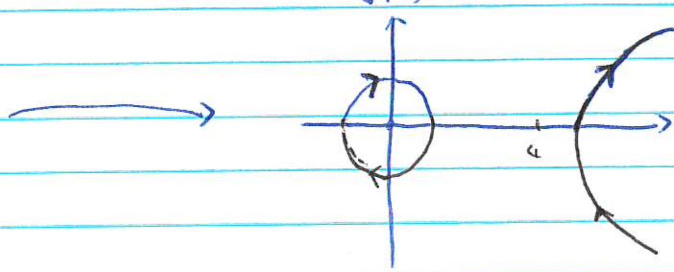


The larger the $|c|$ is, the ~~curve~~ more to the left the curve is.

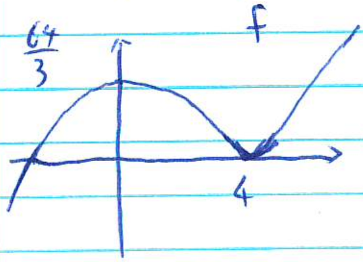
(iii). Note $f(4) = -\frac{64}{3}$ is a local minum. If $0 < c < \frac{64}{3}$, then the graph of f is

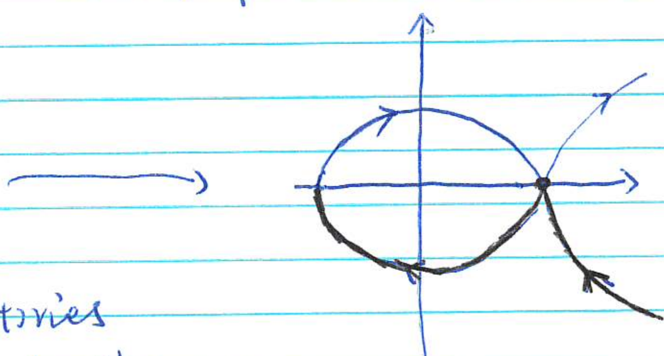


$\pm\sqrt{f(x)+c}$ are



(iv). If $c = \frac{64}{3}$, then the graph of f is

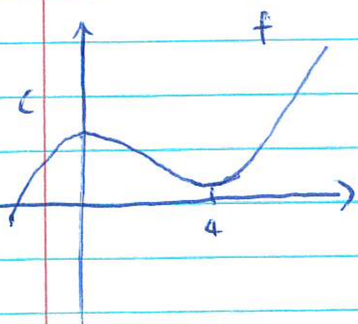


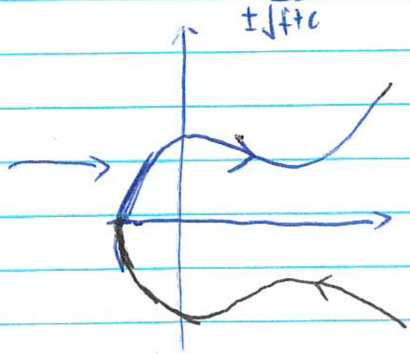


Note different trajectories may not cross over each other.

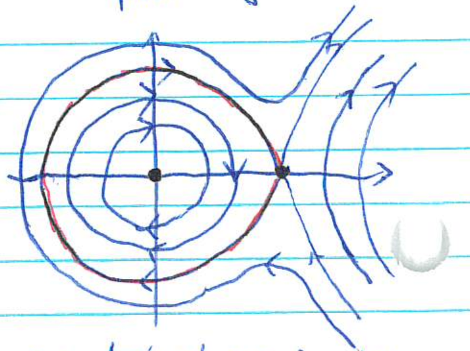
Thus in this case the graph must be divided into 4 different trajectories separated by a equilibrium pt $(4,0)$

(V) If $c > \frac{64}{3}$



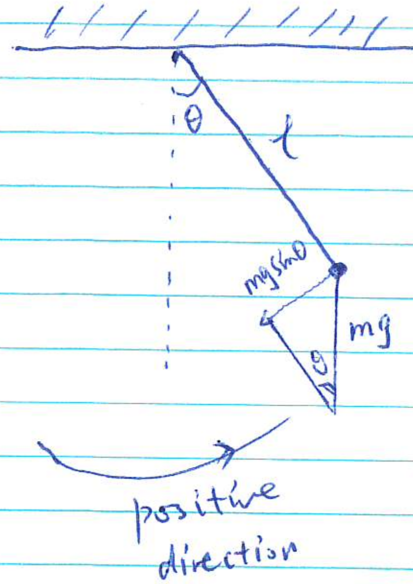


Put together iii-(V):



as claimed previously.

Example ③ (Pendulum, undamped)



Newton's 2nd Law for rotation: $\tau = I \cdot \alpha$

τ: net external torque

I: moment of inertia

α: angular acceleration

$\tau = -mg\sin\theta \cdot l$

$I = m \cdot l^2$

$\alpha = \frac{d^2\theta}{dt^2}$ or θ''

$\Rightarrow -mg\sin\theta \cdot l = ml^2\theta''$. For simplicity, set $m=l=g=1$

$\Rightarrow \theta'' = -\sin\theta$ (*) second order non-linear ODE, which in general is impossible to find its solutions.

Here is what we can do:

Set $v = \theta'$, angular velocity. The we may rewrite (*) as

$$\begin{cases} \theta' = v \\ v' = -\sin\theta \end{cases} \quad (**)$$

In particular, it's a Hamiltonian system with Hamiltonian

$E(\theta, v) = \int v\,dv + \int \sin\theta\,d\theta + c$ for some suitable choice of c

$= \frac{1}{2}v^2 + 1 - \cos\theta$ we set $c=1$.

In particular, E is a first integral of (**). In fact, E is the total energy $\frac{1}{2}v^2 = \frac{1}{2}mv^2$ kinetic energy.

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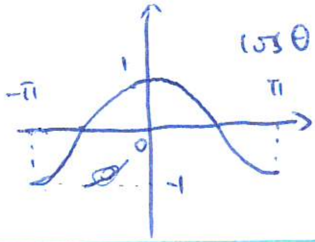
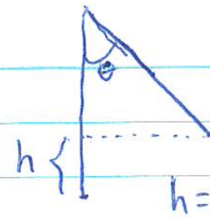


$$\begin{cases} \theta' = v \\ v' = -\sin\theta \end{cases}$$



$1-\cos\theta = mgl(1-\cos\theta) = mgh$ is the potential energy. $h = l - l\cos\theta$

E is a first integral means E stays constant along trajectories which in physics means conservation of ~~eneg~~ energy, which is indeed our case since we assumed there is no damping force.

Since E is a first integral, its level sets $E^{-1}(c) = \{(\theta, v) : E(\theta, v) = c\}$ corresponds to trajectories.

Thus we consider

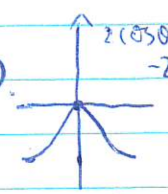
$E(\theta, v) = \frac{1}{2}v^2 + 1 - \cos\theta = c$ Note we must have $c \geq 0$.

Rewrite it as $v^2 = 2\cos\theta + c$, $c \geq -2$.

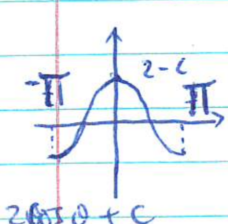
$\Rightarrow 2\cos\theta + c \geq 0$ put restrictions on the choice of θ

Then for admissible θ's, $v = \pm\sqrt{2\cos\theta + c}$. We focus on $-\pi \leq \theta \leq \pi$.

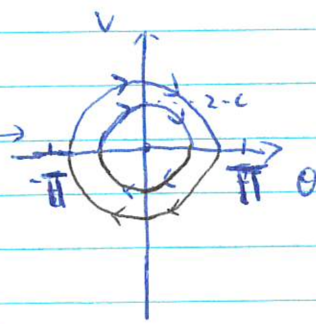
(i) If $c = -2$, the $\theta = 0$ is the only choice which forces $v = 0$. Thus $(0,0)$ is the graph of $v(\theta)$.

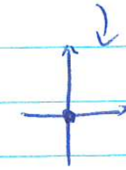


(ii) If $-2 < c < 2$, then



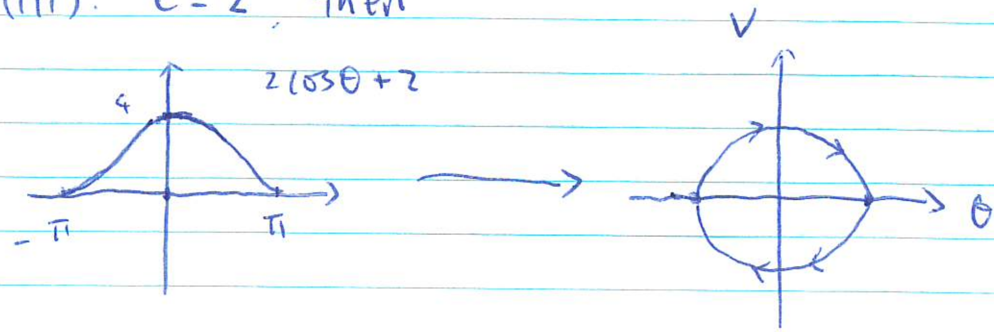
take the upper half part & reflecting w.r.t. θ-axis



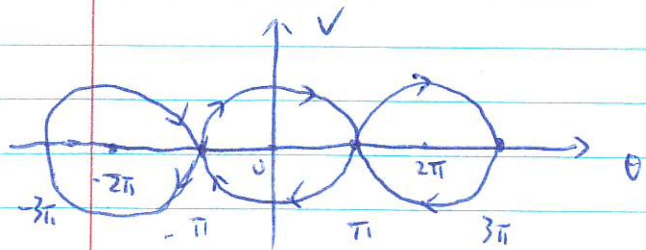


Note the range of admissible θ's are strictly in between $-\pi$ & π.

(iii). $c = 2$ then



It looks like in this we got a cycle as well. However, don't forget the graph extends periodically to all $\theta \in \mathbb{R}$ which is



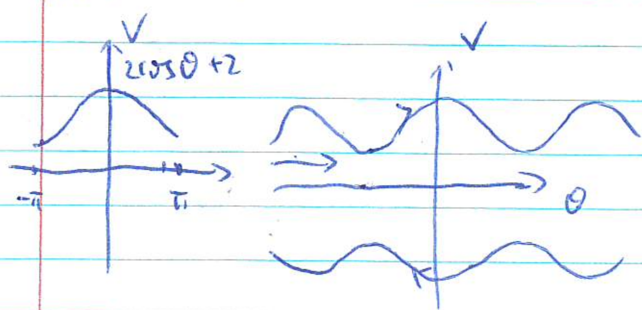
Again, the intesecting property of the graph forces $(\pm\pi, 0)$ to be an equilibrium pt, in fact saddles.

Moreover, two heteroclinic orbits connecting $(\pm\pi, 0)$.

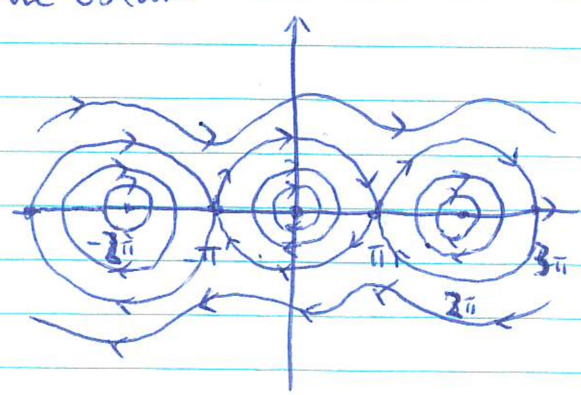
In fact, $\begin{cases} V = 0 \\ -\sin\theta = 0 \end{cases}$ indeed yields $\begin{cases} \theta = \pm k\pi, \; k \in \mathbb{Z} \\ V = 0 \end{cases}$

as equilibrium

(iv) $c > 2$, then



Put together everything we obtain



One may easily translate it into the motion of the pendulum, depending on energy level.

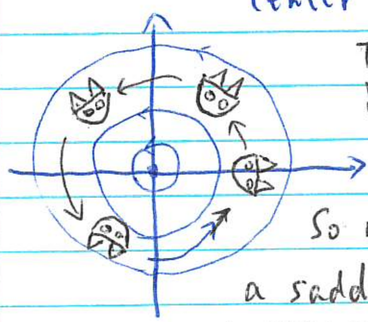
One may observe that in a Hamiltonian system, we only see saddle or center equilibria so far. In fact, it's a general fact that a Hamiltonian system may not have sink, source, or even limit cycles. It's from the following deep result, which is usually referred to as Liouville's Theorem:

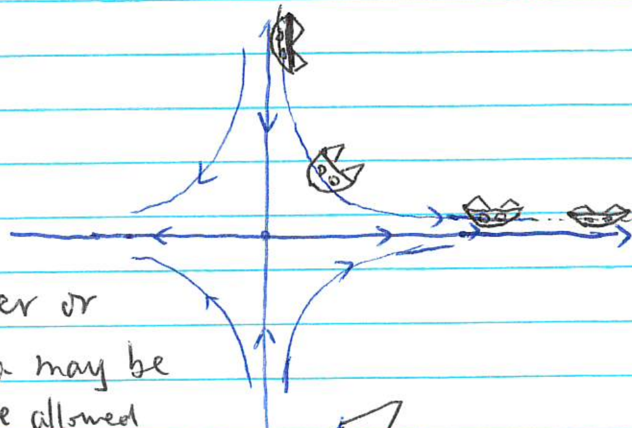
- Theorem: Let $\phi: \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ be a dynamical system defined by Hamiltonian systems. Then the flow $\phi_t: \mathbb{R}^2 \to \mathbb{R}^2$ is area preserving. Precisely, let's take a plane region $\mathcal{D}$ (a circle, retangle, oval, ...) & let $\phi_t(\mathcal{D})$ denotes the image of $\mathcal{D}$ under the time t map. Then the area of the region $\phi_t(\mathcal{D})$ is equal to the are of $\mathcal{D}$ for all $t \in \mathbb{R}$.

center

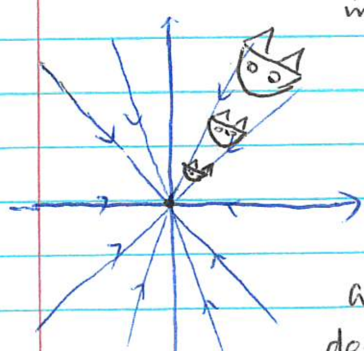
$\mathcal{D}$

Take $\mathcal{D}$ to be the cat.

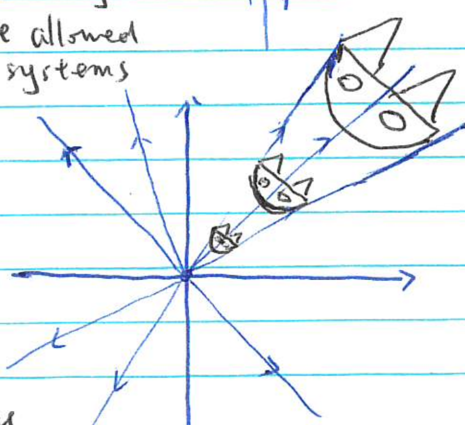




So near a center or a saddle, the area may be preserved. So they are allowed in a Hamiltonian systems



near a sink or source the area will decrease or increas along flows, thus may not appear in a Hamiltonian system



Similar idea applies to limit cycles, which are not allowed.