$\mathcal{Dynamical\ Systems}$—$\mathcal{Homework}$ 6

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1 (1)

$\tau = I\alpha$, and $I = ml^2$, and $\alpha = \frac{d^2\theta}{dt^2} = \theta''$

Let $m = l = g = 1$

$\implies \tau = \theta''$

Assume $\tau = -\sin\theta - \theta'$

$\implies \theta'' = -\sin\theta - \theta'$

Which is the second order differential equation for this dampened system.

(2)

Let $\theta' = \omega$

$\implies \theta'' = \omega'$

$\implies \omega' = -\sin\theta - \theta' = -\sin\theta - \omega$

So, the non-linear planar system in terms of (θ, ω), where ω is the angular velocity is as follows:

$$(*) \begin{cases} \theta' = \omega \\ \omega' = -\sin\theta - \omega \end{cases}$$

(3)

Let $\theta' = \omega' = 0$

$$\implies \begin{cases} 0 = \omega \\ 0 = -\sin\theta \end{cases} \implies \begin{cases} \omega = 0 \\ \theta = k\pi, k \in \mathbb{Z} \end{cases} \implies \begin{pmatrix} k\pi \\ 0 \end{pmatrix} \text{ are the equilibria of } (*)$$

$$(*) \implies \begin{pmatrix} \theta' \\ \omega' \end{pmatrix} = \begin{pmatrix} \omega \\ -\sin\theta - \omega \end{pmatrix} = \begin{pmatrix} f(\theta,\omega) \\ g(\theta,\omega) \end{pmatrix} = \mathcal{F}(\theta,\omega)$$

$$\implies \mathrm{D}\mathcal{F}(k\pi, 0) = \begin{pmatrix} f_\theta(k\pi,0) & f_\omega(k\pi,0) \\ g_\theta(k\pi,0) & g_\omega(k\pi,0) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\cos k\pi & -1 \end{pmatrix}$$

$$\implies \mathrm{Tr}(\mathrm{D}\mathcal{F}(k\pi,0)) = -1 \text{ and } |\mathrm{D}\mathcal{F}(k\pi,0)| = \begin{cases} 1, k \text{ even} \\ -1, k \text{ odd} \end{cases}$$

So none of the systems is a center. So, all of the points, are hyperbolic.

Therefore, we can apply *linearization* to all of the points.

$\mathrm{p}(\lambda) = \lambda^2 - \mathrm{Tr}(\mathrm{D}\mathcal{F}(k\pi,0))\lambda + |\mathrm{D}\mathcal{F}(k\pi,0)|$

$\mathrm{p}(\lambda) = 0 \implies \lambda = \frac{\mathrm{Tr}(\mathrm{D}\mathcal{F}(k\pi,0)) \pm \sqrt{\mathrm{Tr}(\mathrm{D}\mathcal{F}(k\pi,0))^2 - 4|\mathrm{D}\mathcal{F}(k\pi,0)|}}{2}$

For k even, $\lambda = \frac{-1\pm\sqrt{(-1)^2-4(1)}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

$$\implies \underline{X} = e^{-\frac{1}{2}t}\begin{pmatrix}\cos(\frac{\sqrt{3}}{2}t) & \sin(\frac{\sqrt{3}}{2}t)\\ -\sin(\frac{\sqrt{3}}{2}t) & \cos(\frac{\sqrt{3}}{2}t)\end{pmatrix} = e^{-\frac{t}{2}}R_{-\frac{\sqrt{3}}{2}t}$$

It's clear from $(*)$ that it rotates clockwisely, because the rate of change of the angle is direclty proportional to the angular velocity—i.e: In QI of the (θ, ω)-plane, θ' is positive. And, $t \to \infty \implies e^{-\frac{t}{2}} \to 0$, so $t \to \infty \implies \overline{X} \to 0$. So $\vec{0}$ is a spiral sink of $\mathrm{D}\mathcal{F}(k\pi, 0)$ when k is even. By, *linearization* $\exists h_1 : \mathbb{R}^2 \to \mathbb{R}^2$, a local homomorphism: $\forall k \in \mathbb{Z} : k$ is even and $h_1^{-1}(\vec{0}) = \begin{pmatrix}k\pi\\0\end{pmatrix}$ is locally a spiral sink of $\mathcal{F}$.

For k odd, $\lambda = \frac{-1\pm\sqrt{(-1)^2-4(-1)}}{2} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$

$$\implies \lambda_1 = -\frac{1}{2} + \frac{\sqrt{5}}{2} \text{ and} \lambda_2 = -\frac{1}{2} - \frac{\sqrt{5}}{2}$$

$$\mathrm{D}\mathcal{F}(k\pi, 0) = \begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix}$$

$$(\mathrm{D}\mathcal{F}(k\pi, 0) - \lambda_1 I) = \begin{pmatrix}\frac{1}{2} - \frac{\sqrt{5}}{2} & 1\\ 1 & -\frac{1}{2} - \frac{\sqrt{5}}{2}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1-\sqrt{5}}{2} & 1\\ 1 & -\frac{1+\sqrt{5}}{2}\end{pmatrix} \to \begin{pmatrix}1 & \frac{2}{1-\sqrt{5}}\frac{1+\sqrt{5}}{1+\sqrt{5}}\\ 1 & -\frac{1+\sqrt{5}}{2}\end{pmatrix} \to \begin{pmatrix}1 & \frac{2(1+\sqrt{5})}{1-5}\\ 1 & -\frac{1+\sqrt{5}}{2}\end{pmatrix} \to \begin{pmatrix}1 & -\frac{1+\sqrt{5}}{2}\\ 1 & -\frac{1+\sqrt{5}}{2}\end{pmatrix} \to \begin{pmatrix}1 & -\frac{1+\sqrt{5}}{2}\\ 0 & 0\end{pmatrix}$$

$$\implies v_1 = \begin{pmatrix}\frac{1+\sqrt{5}}{2}\\ 1\end{pmatrix}$$

$$(\mathrm{D}\mathcal{F}(k\pi, 0) - \lambda_2 I) = \begin{pmatrix}\frac{1}{2} + \frac{\sqrt{5}}{2} & 1\\ 1 & -\frac{1}{2} + \frac{\sqrt{5}}{2}\end{pmatrix}$$

$$\begin{pmatrix}\frac{\sqrt{5}+1}{2} & 1\\ 1 & \frac{\sqrt{5}-1}{2}\end{pmatrix} \to \begin{pmatrix}\frac{\sqrt{5}+1}{2} & 1\\ \frac{2}{\sqrt{5}-1}\frac{\sqrt{5}+1}{\sqrt{5}+1} & 1\end{pmatrix} \to \begin{pmatrix}\frac{\sqrt{5}+1}{2} & 1\\ \frac{2(\sqrt{5}+1)}{5-1} & 1\end{pmatrix} \to \begin{pmatrix}\frac{\sqrt{5}+1}{2} & 1\\ \frac{\sqrt{5}+1}{2} & 1\end{pmatrix} \to \begin{pmatrix}\frac{\sqrt{5}+1}{2} & 1\\ 0 & 0\end{pmatrix}$$

$$\implies v_2 = \begin{pmatrix}1\\ -\frac{1+\sqrt{5}}{2}\end{pmatrix}$$

$\implies x(t) = e^{\lambda_1 t}v_1 + e^{\lambda_2 t}v_2$ is the generalized solution to $\mathrm{D}\mathcal{F}(k\pi, 0)$ when k is odd.

$\lambda_1 = -\frac{1}{2} + \frac{\sqrt{5}}{2} \approx 0.6880339887$ and$\lambda_2 = -\frac{1}{2} - \frac{\sqrt{5}}{2} \approx -1.6180339887$

$\implies$

$$\lim_{t\to-\infty} x(t) = v_2, \quad \text{since } e^{-\lambda_1\infty} << e^{-\lambda_2\infty}$$

and

$$\lim_{t\to\infty} x(t) = v_1, \quad \text{since } e^{\lambda_2\infty} << e^{\lambda_1\infty}$$

So, $\vec{0}$ is a saddle of $\mathrm{D}\mathcal{F}(k\pi, 0)$, when k is odd, with trajectories coming from the v_2 direction to the v_1 direction. By, *linearization* $\exists h_2 : \mathbb{R}^2 \to \mathbb{R}^2$, a local homomorphism: $\forall k \in \mathbb{Z} : k$ is odd and $h_2^{-1}(\vec{0}) = \begin{pmatrix}k\pi\\0\end{pmatrix}$ is locally a saddle of $\mathcal{F}$.

(4)

The dampened pendulum will eventually stop if the angle start's between strictly between $-\pi$ and $k2pi + \pi$ that's the meaning of even multiples of π being sinks. If, the angular momentum is too big, or too small, that is outside the bounds outside of the diamonds determined by the saddle points, then it will get stuck about the odd multiples of π. The uniqueness part of the system being nice, won't let the angle increase more that an odd multiple of π, and as infinite angular velocity makes no sense, the only reasonable explanation is that it gets stuck. Actually, what I think would happen for the low energies is that it gets stuck and falls, and in the high energies, the string breaks.

2 $(\#)\begin{cases} x' = -y \\ y' = x^3 - x. \end{cases}$

(1)

$H(x,y) = -\int -y dy + \int x^3 - x dx = \frac{1}{2}y^2 + \frac{1}{4}x^4 - \frac{1}{2}x^2 + C$

Let $C = \frac{1}{4} : \frac{1}{4}x^4 - \frac{1}{2}x^2 + \frac{1}{4} = \frac{1}{4}(x^4 - 2x^2 + 1) = \frac{1}{4}(x^2-1)^2$

$\implies H(x,y) = \frac{1}{2}y^2 + \frac{1}{4}(x^2-1)^2$

(2)

$X = \{(x,y) \in \mathbb{R}^2 | H(x,y) \leq C\}$ for $0 \leq C$

Is X is bounded for each C because the C for each (x,y), the least upper bound of the set is C.

For each C, $X = \cup_{c\in[0,C]} H^{-1}(c)$, the level sets of H for each c, where $0 \leq c \leq C$. Individually each of the level sets are trajectories, since H is hamiltonian. So X is invariant, as any point in the set will stay in one of the level sets and X is the union of all those level sets.

In particular, the level sets are curves, so they're closed subsets of R^2, so their union is closed. So, for each C, X is a closed and bounded invariant set.

(3)

Letting $x' = y' = 0$ in $(\#) \implies \begin{cases} 0 = -y \\ 0 = x^3 - x. \end{cases}$

$\implies y = 0$ and $0 = x^3 - x = x(x^2-1) = x(x+1)(x-1)$

$\implies y = 0$ and $(x = 0$ or $x = 1$ or $x = -1)$

$\implies \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ are the equilibria of $(\#)$

(4)

Consider $\frac{1}{2}y^2 + \frac{1}{4}(x^2-1)^2 \leq C$

$\implies \frac{1}{2}y^2 \leq C - \frac{1}{4}(x^2-1)^2$

$\implies y^2 \leq 2C - \frac{1}{2}(x^2-1)^2$

$\implies y \leq \pm\sqrt{2C - \frac{1}{2}(x^2-1)^2}$

Case 0: $C < 0$

Then $C = -D$ some $D > 0$

$\implies y \leq \pm\sqrt{-2D - \frac{1}{2}(x^2-1)^2}$

$\implies y \leq \pm i\sqrt{2D + \frac{1}{2}(x^2-1)^2}$

$y \in \mathbb{R} \implies X = \emptyset$

Case 1: $C = 0$

Reconsider $\frac{1}{2}y^2 + \frac{1}{4}(x^2-1)^2 \leq 0$

y^2 and $(x^2-1)^2$ are squares, so they're positive, so a linear combination with positive coefficients is positive. They have no negative solutions, so the only way their linear combination is equal to 0 is if they're both 0.

So, $y^2 = 0 \implies y = 0$

and, $(x^2-1)^2 = 0 \implies x^2 - 1 = 0 \implies x = \pm 1$

So, if $C = 0$, X is the same as two of the equilibrium points we already had.

Case 3: $C = \frac{1}{4}$

Reconsider $\frac{1}{2}y^2 + \frac{1}{4}(x^2-1)^2 \leq \frac{1}{4}$

$\overset{*4}{\implies} 2y^2 + (x^2-1)^2 \leq 1$

$\implies 2y^2 \leq 1-(x^2-1)^2 = 1-(x^4-2x^2+1) = -x^4+2x^2 = x^2(2-x^2) = x^2(\sqrt{2}-x)(\sqrt{2}+x)$

$\implies 2y^2 \leq x^2(\sqrt{2}-x)(\sqrt{2}+x)$

For the same reasons, if $x = \pm\sqrt{2}$ or $x = 0$, then $y = 0$

So, at $\begin{pmatrix}\sqrt{2}\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\end{pmatrix}$ and $\begin{pmatrix}-\sqrt{2}\\0\end{pmatrix}$ the trajectory crosses the x-axis.

Next we consider the function:

$y = \frac{x\sqrt{(\sqrt{2}-x)(\sqrt{2}+x)}}{\sqrt{2}} = \frac{x\sqrt{2-x^2}}{\sqrt{2}}$

$y' = (\frac{x\sqrt{2-x^2}}{\sqrt{2}})' = \frac{x}{\sqrt{2}}'\sqrt{2-x^2} + \frac{x}{\sqrt{2}}(\sqrt{2-x^2})'$

$= \frac{1}{\sqrt{2}}\sqrt{2-x^2} + \frac{x}{\sqrt{2}}\frac{-2x}{2\sqrt{2-x^2}} = \frac{1}{\sqrt{2}}(\sqrt{2-x^2} - \frac{x^2}{\sqrt{2-x^2}})$

$= \frac{1}{\sqrt{2}}\frac{2-2x^2}{\sqrt{2-x^2}} = \frac{1}{\sqrt{2}}^2\frac{2(1-x^2)}{\sqrt{1-\frac{x^2}{2}}}$

$= \frac{1-x^2}{\sqrt{1-\frac{x^2}{2}}}$

y' is clearly undefined if $x \geq \sqrt{2}$ or $x \leq \sqrt{2}$. Since $1-\frac{x^2}{2} < 0$, whenever $x^2 > 2$

For the values of x where y' is defined Let $y' = 0$

$\implies 0 = \frac{1-x^2}{\sqrt{1-\frac{x^2}{2}}} \implies 0 = 1-x^2 \implies x = 1$ or $x = -1$

So y' attains it's bounds at $x = \pm 1$

$y'' = \left(\frac{1-x^2}{\sqrt{1-\frac{x^2}{2}}}\right)'$

$= \frac{(1-x^2)'\sqrt{1-\frac{x^2}{2}}-(1-x^2)\left(\sqrt{1-\frac{x^2}{2}}\right)'}{1-\frac{x^2}{2}} = \frac{-2x\sqrt{1-\frac{x^2}{2}}-(1-x^2)\frac{\frac{-2x}{2}}{2\sqrt{1-\frac{x^2}{2}}}}{1-\frac{x^2}{2}} = \frac{-2x\sqrt{1-\frac{x^2}{2}}-(1-x^2)\frac{-x}{2\sqrt{1-\frac{x^2}{2}}}}{1-\frac{x^2}{2}} = \frac{-2x\sqrt{1-\frac{x^2}{2}}+\frac{x-x^3}{2\sqrt{1-\frac{x^2}{2}}}}{1-\frac{x^2}{2}}$

$= \frac{\frac{-4x(1-\frac{x^2}{2})+x-x^3}{2\sqrt{1-\frac{x^2}{2}}}}{\frac{2-x^2}{2}} = \frac{-4x(1-\frac{x^2}{2})+x-x^3}{(2-x^2)\sqrt{1-\frac{x^2}{2}}} = \frac{-4x+2x^3+x-x^3}{(2-x^2)\sqrt{1-\frac{x^2}{2}}} = \frac{x(x^2-3)}{(2-x^2)\sqrt{1-\frac{x^2}{2}}} = \frac{x(x^2-3)}{(2-x^2)\frac{1}{\sqrt{2}}\sqrt{2-x^2}} = \frac{\sqrt{2}\ \ x(x^2-3)}{(2-x^2)^{\frac{3}{2}}}$

By the second derivative test:

$y''(-1) = \frac{\sqrt{2}\ \ -1((-1)^2-3)}{(2-(-1)^2)^{\frac{3}{2}}} = 2\sqrt{2} > 0 \implies x = -1$ is a local minimum of y

$\implies x = -1$ is a local maximum of $-y$

$y''(1) = \frac{\sqrt{2}\ \ 1(1^2-3)}{(2-1^2)^{\frac{3}{2}}} = -2\sqrt{2} < 0 \implies x = 1$ is a local maximum of y

$\implies x = 1$ is a local minimum of $-y$

So $H^{-1}(\frac{1}{4})$ has 3 orbits, because $C = \frac{1}{4}$ is the only orbit that includes the equilibrium point $\begin{pmatrix}0\\0\end{pmatrix}$. That's the homoclinic point. And there are two homoclinic orbits, one to the right through $x = 1$, and one through the left through $x = -1$.

The direction is counterclockwise, since $(\#) \implies x' = -y$, and on QI of the $(x, y)-$plane, y is posititive, so x is decreasing and on QIV y is negative so, x is increasing.

Case 2: $0 < C < \frac{1}{4}$

Consider, $2y^2 < x^2(\sqrt{2} - x)(\sqrt{2} + x)$, we have 2 cycles for each C because it is the area inside the homoclinic orbits of case 3. And it doesn't contract to a point because that's case 1. So, each of the left loops will cross the $x-$axis on points greater than $-\sqrt{2}$ and smaller than 0. Similarly for the right loops, they'll cross the $x-$axis on points greater than 0 and smaller than $\sqrt{2}$.

The $y-$bounds of each loop will be between the bounds of our y function. So, reconsidering y of case 3:

$y(-1) = \frac{-1\sqrt{2-(-1)^2}}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$ and $y(1) = \frac{1\sqrt{2-1^2}}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

So, the loops will be between $-\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$.

Case 4: $\frac{1}{4} < C$

Consider, $2y^2 > x^2(\sqrt{2} - x)(\sqrt{2} + x)$, we have 1 cycle for each C because it is the area outside the homoclinic orbits of case 3. So, the loops will cross the $x-$axis on points smaller than $-\sqrt{2}$ and greater than $\sqrt{2}$.

So, the loops will be between values of $-\frac{1}{\sqrt{2}} - \varepsilon$ and $\frac{1}{\sqrt{2}} + \varepsilon$, some $\varepsilon > 0$.