MATH 151B HOMEWORK 1

Problem 1

Find an alternative proof to the Bolzano–Weierstrass Theorem as follows: Let B be the set of points such that $f(t) > c$. Show that the greatest lower bound of B exists. Let x be the glb of B. Prove that $f(x) = c$.

Problem 2

Let $E = \{x_n\}$ be a countable subset of $[a, b]$ and $\{c_n\}$ be a sequence of positive number such that $\sum c_n$ converge and is finite. Define

$$f(x) = \sum_{x_n < x} c_n.$$

Prove the following:

(1) f is monotonically increasing on $[a, b]$.
(2) f is discontinuous at every points of E with

$$f(x_n+) - f(x_n-) = c_n.$$

(3) f is continuous at every other point of (a, b).

Problem 3

Suppose f and g are defined and that $f(t) \to A$ and $f(t) \to B$ as $t \to +\infty$ where A and B are real numbers. Prove that $(f + g)(t) \to A + B$ and $(fg)(t) \to AB$ as $t \to +\infty$.

Problem 4

We say that f is one-to-one on E if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$ for all $x_1, x_2 \in E$. If f is one-to-one and continuous on $[a, b]$ and $f(a) < f(b)$, prove that f is strictly increasing. That is, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.