151B — Homework 2

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1 Let f be defined for all real x, and suppose that

$$|f(x) - f(y)| < (x - y)^2$$

for all real x and y. Prove that f is constant.

pf.

$|f(x) - f(y)| < (x - y)^2 \iff -(x - y)^2 < f(x) - f(y) < (x - y)^2$.

Case 1: $x - y > 0$. $-(x - y) < \frac{f(x)-f(y)}{x-y} < x - y$

Case 2: $x - y < 0$. $-(x - y) > \frac{f(x)-f(y)}{x-y} > x - y$

In both cases, taking the limit as $x \to y \implies x - y \to 0$.

We squeeze the difference quotient in between 0 and 0.

So $f'(x) = 0 \implies f$ is constant ■

2 If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$

for real constants $C_0, \cdots, C_n$, prove that the equation

$$C_0 + C_1 x + \cdots + C_n x^n = 0$$

has at least one real root between 0 and 1.

pf.

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0 \implies C_0 = -(\frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1})$$

Let $P(x) = C_0 + C_1 x + \cdots + C_n x^n$

If $C_0 < 0$, then $P(0) = C_0 < 0$ and $P(1) = C_0 + C_1 + \cdots + C_n = -(\frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1}) + C_1 + \cdots + C_n$

$C_0 < 0 \implies -C_0 > 0 \implies (\frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1}) > 0$

$\frac{C_i}{i+1} < C_i \forall i \in \mathbb{N}$ and $C_i \in \mathbb{R}$. And $\forall a, b, c \in \mathbb{R} a < b \implies a + c < b + c$.

Repeated applications of the previous line yield, $P(1) > 0$

P is a polynomial, therefore it is continuous.

Since P satisfies the hypotheses of the intermediate value theorem,

$$\exists x \in (0, 1) : P(x) = 0$$

Rinse and repeat, $C_0 > 0 \implies P(0) > 0$ and $P(1) < 0 \implies \exists x \in (0, 1) : P(x) = 0$.

If $C_0 = 0$, then $P(x) = C_1 x + \cdots + C_n x^n$

We want to find, if P has a root between 0 and 1. That is we want to solve,

$$0 = C_1 x + \cdots + C_n x^n$$

$x \in (0, 1) \implies x \neq 0$, so we can divide by x,

$$0 = C_1 + \cdots + C_n x^{n-1}$$

Now, we look at the zeroes of $Q(x) = C_1 + \cdots + C_n x^{n-1}$

Then, by the same argument as before $C_1 < 0$ and $C_1 > 0$, imply there is a root of $Q(x)$, between 0 and 1, which in turn implies there is a root of $xQ(x) = P(x)$.

Now, since there are finitely many C_i, there are finitely many applications of this procedure which yield a real zero in between 0 and 1.

Note that, if $C_0, \cdots, C_{n-1}$ are all zero. Then,

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0 \implies C_n = 0$$

So, $P(x) = Z(x)$ the zero polynomial, which clearly has a zero between 0 and 1.

So,

$$C_0 + C_1 x + \cdots + C_n x^n = 0$$

has a real root between 0 and 1 ■

3 Suppose f is continuous for $x \geq 0$, $f'(x)$ exists for $x > 0$, $f(0) = 0$, and f' is monotonically increasing. Define, for $x > 0$

$$g(x) = \frac{f(x)}{x}$$

.

Prove that g is monotonically increasing.

pf.

Compute $g'(x) = \frac{xf'(x)-f(x)}{x^2}$ by the quotient rule. Which exists because f' exists for $x > 0$.

So, g is differentiable on $(-\infty, 0)$.

Want to show $g'(x) = \frac{xf'(x)-f(x)}{x^2} \geq 0 \iff xf'(x) - f(x) \geq 0$ since $x^2 > 0 \forall x \neq 0$.

f' is increasing, so $x < y \implies f'(x)'(y)$

f is defined on $[0, \infty] \implies f$ is defined on $[0, x]$.

f is differentiable on $(0, \infty) \implies f$ is differentiable on $(0, x)$.

By the mean value theorem, $\exists t \in (0, x) : f(x) - f(0) = (x - 0)f'(t)$

$f(0) = 0 \implies f(x) = xf'(t)$

$t \in (0, x) \implies t < x \implies f'(t) < f'(x)$

$\implies f(x) = xf'(t) < xf'(x)$ ■

4 If f is differentiable in (a, b) and $f'(x) > 0$. Prove that f is strictly increasing in (a, b). Let g be its inverse function. Prove that g is differentiable and that,

$$g'(f(x)) = \frac{1}{f'(x)}$$

pf.

$f'(x) > 0 \implies$ f is strictly increasing on (a, b).

Fix $s \in \text{dom}(g)$. Let $\phi(t) = \frac{g(t)-g(s)}{t-s}$.

Since, dom$(g) = f((a, b))$, $\exists x, y \in (a, b) : f(x) = t$ and $f(y) = s$.

Now, $\phi(f(x)) = \frac{g(f(x))-g(f(y))}{f(x)-f(y)} = \frac{x-y}{f(x)-f(y)} = \frac{1}{\frac{f(x)-f(y)}{x-y}}$.

Since f is differentiable it is continuous, and since f has an inverse it is $1 \to 1$. So it follows g is continuous.

$\lim_{s \to t} \phi(t) = g'(f(x)) = \lim_{f(x) \to f(y)} \phi(f(x)) \overset{g \text{ is continuous}}{=} \lim_{x \to y} \phi(f(x)) = \frac{1}{\lim_{x \to y} \frac{f(x)-f(y)}{x-y}} \overset{\text{definition of} f'}{=} \frac{1}{f'(x)}$

■