Math 151B (Advanced Calculus)
Homework 01 Solutions

Problem 1. Find an alternative proof to the Bolzano-Weierstrass theorem (i.e. the intermediate value theorem) as follows: let B be the set of points such that $f(t) > c$. Show that the greatest lower bound of B exists. Let x be the greatest lower bound of B. Prove that $f(x) = c$.

Proof. Suppose that f is continuous on $[a, b]$, that $f(a) < f(b)$, and that c is a real number such that $f(a) < c < f(b)$. Define

$$B = \{t \in \mathbb{R} \mid f(t) > c\}.$$

Note that a is a lower bound of B. Since $\mathbb{R}$ has the least upper bound property (Theorem 1.19 in Rudin), it follows that

$$-B = \{-b \mid b \in B\}$$

has a least upper bound, say $\operatorname{lub}(-B) = -x$. But then x is the greatest lower bound of B. It is a lower bound: if $b \in B$, then $-b < -x$ since $-x$ is an upper bound of $-B$, and so $b > x$. It is the greatest lower bound: if x' is another lower bound, then $-x'$ is an upper bound of $-B$, which implies that $-x' > -x$. But then $x' < x$. Therefore $x = \operatorname{glb}(B) = -\operatorname{lub}(-B)$.

Observe that $x \notin B$. If $x \in B$, then $f(x) > 0$. But f is continuous, so with $\varepsilon = f(x) - c > 0$, there exists some $\delta > 0$ such that $|x - y| < \delta$ implies that

$$\begin{aligned} |f(x) - f(y)| < (f(x) - c &\implies -(f(x) - c) < f(x) - f(y) < f(x) - c \\ &\implies -f(y) < -c \\ &\implies f(y) > c. \end{aligned}$$

In particular, if $y \in (x - \delta, x)$, then $f(y) > c$, which implies that $y \in B$. But $x = \operatorname{glb}(B)$ and $y < x$, which is a contradiction. Therefore $x \notin B$, which implies that $f(x) \leq c$.

Finally, for each $k \in \mathbb{N}$, there exists some $t_k \in (x, \frac{1}{k})$ such that $t_k \in B$. If not, then there exists some k such $B \cap (x, t_k) = \varnothing$, which implies that t_k is a lower bound of B. But $x = \operatorname{glb}(B)$, which is a contradiction. Observe that $t_k \in B$ for each k, hence

$$c < f(t_k)$$

for all $k \in \mathbb{N}$. On the other hand, $t_k \to x$, which implies that

$$c \leq \lim_{t_k \to x} f(t_k) = f(x)$$

by the continuity of f. As previously observed, $f(x) \leq c$. Combining these inequalities,

$$c \leq \lim_{t_k \to x} f(t_k) = f(x) \leq c,$$

which implies that $f(x) = c$. ∎

Problem 2. Let $\{x_n\}$ be a countable subset of $[a, b]$ and $\{c_n\}$ a sequence of positive real numbers such that

$$\sum_{n=1}^{\infty} c_n$$

converges to some finite value. Define

$$f(x) := \sum_{x_n < x} c_n.$$

Prove the following:

(a) f is monotonically increasing on $[a, b]$.

(b) f is discontinuous at every point of E, and

$$f(x_n+) - f(x_n-) = c_n.$$

(c) f is continuous at every other point of (a, b).

Proof of (*2a*). Fix arbitrary $x, y \in [a, b]$ with $x < y$. Then

$$f(x) = \sum_{x_n < x} c_n \leq \sum_{x_n < x} c_n + \underbrace{\sum_{x \leq x_n < y} c_n}_{=:S} = \sum_{x_n < y} c_n = f(y).$$

The inequality holds because each c_n is postive, which implies that the sum S is nonnegative (S may be zero, if $[x, y) \cap E = \varnothing$). Hence if $x, y \in [a, b]$ with $x < y$, then $f(x) \leq f(y)$. Therefore f is monotonically increasing on $[a, b]$. □

Proof of (*2b*). As per Rudin (p. 94), a function g has a discontinuity of the first kind at a point x of its domain (a "jump" discontinuity in the language of introductory calculus) if both $g(x+)$ and $g(x-)$ exist, but $g(x+) \neq g(x-)$. Hence to show that f is discontinuous at each point of E, it is sufficient to prove the claim that

$$f(x_n+) - f(x_n-) = c_n$$

for each $x_n \in E$.

Fix some $x_n \in E$. By Theorem 4.29 of Rudin, since f is monotonically increasing, both $f(x_n+)$ and $f(x_n-)$ exist. Moreover

$$f(x_n+) - f(x_n-) = \inf_{x_n < t < b} f(t) - \sup_{a < s < x_n} f(s)$$

As the sum $\sum c_n$ converges, for each $k \in \mathbb{N}$ there exists natural number J_k such that

$$\sum_{j=J_k}^{\infty} c_j < \frac{1}{k}.$$

As the set $\{x_j \mid 1 \leq j < J_k\}$ is finite,

$$d_k := \min\{|x_n - x_j| \mid 1 \leq j < J_k, x_j < x_n\} > 0.$$

So for each k, it is possible choose $s_k \in [a, x_n)$ such that

$$|x_n - s_k| \leq \min\left\{d_k, \frac{1}{2}|x_n - s_{k-1}|\right\}.$$

This condition guarantees that $s_k \to x_n$ and that $\{x_j \mid 1 \leq j < J_k\} \cap [s_k, x_n) = \varnothing$. This second condition then implies that

$$f(x_n) - f(s_k) = \sum_{s_k \leq x_j < x_n} c_j < \sum_{j=J_k}^{\infty} c_j < \frac{1}{k} \implies f(x_n) - \frac{1}{k} < f(s_k).$$

As f is monotonically increasing and $t_k < x_n$,

$$f(x_n) - \frac{1}{k} < f(s_k) \leq f(x_n),$$

from which it follows that

$$f(x_n-) = \sup_{a<s<x_n} f(s) = \lim_{s_k \to x_n} f(s_k) = f(x_n) = \sum_{x_j < x_n} c_j. \tag{1}$$

Similarly, for each k, choose $t_k \in (x_n, b]$ such that

$$|x_n - t_k| \leq \min\left\{d_k, \frac{1}{2}|x_n - t_{k-1}|\right\}.$$

As above, this guarantees that $t_k \to x_n$ and that $\{x_j \mid 1 \leq j < J_k\} \cap [x_n, t_k) = \varnothing$. From this second condition,

$$f(t_k) - \sum_{x_j < x_n} c_j = \sum_{x_n \leq x_j \leq t_k} c_j < \sum_{j=J_k}^{\infty} c_j < \frac{1}{k}$$

Hence, as $t_k > x_n$,

$$\sum_{x_j \leq x_n} c_j \leq f(t_k) < \sum_{x_j \leq x_n} c_j + \frac{1}{k}.$$

Therefore

$$f(x_n+) = \inf_{x_n<t<b} f(t) = \lim_{t_k \to x_n} f(t_k) = \sum_{x_j \leq x_n} c_j. \tag{2}$$

Finally, combine the identities at (1) and (2) in order to obtain

$$f(x_n+) - f(x_n-) = \sum_{x_j \leq x_n} c_j - \sum_{x_j < x_n} c_j = c_n,$$

which is the desired result. □

Proof of (2c). Fix $x \in [a,b] \setminus E$ and $\varepsilon > 0$. As $\sum c_n$ converges, there exists some N so large that

$$\sum_{n=N}^{\infty} c_n < \varepsilon.$$

Choose $\delta > 0$ so that

$$(x-\delta, x+\delta) \cap \{x_n \mid 1 \leq n < N\} = \emptyset.$$

Note then that

$$f(x-\delta) = \sum_{x_n < x-\delta} c_n = \sum_{x_n < x} c_n - \sum_{x-\delta \leq x_n < x} c_n \geq f(x) - \sum_{n=N}^{\infty} c_n > f(x) - \varepsilon.$$

Similarly,

$$f(x+\delta) = \sum_{x_n < x+\delta} c_n = \sum_{x \leq x_n < x+\delta} c_n + \sum_{x_n < x} c_n \leq \sum_{n=N}^{\infty} c_n + f(x) < f(x) + \varepsilon.$$

Therefore if $y \in (x-\delta, x+\delta)$, then

$$\begin{aligned} x-\delta < y < x+\delta &\implies f(x-\delta) \leq f(y) \leq f(x+\delta) && \text{(monotonicity of } f\text{)} \\ &\implies f(x) - \varepsilon < f(y) < f(x) + \varepsilon && \text{(by the previous estimates)} \\ &\implies -\varepsilon < f(y) - f(x) < \varepsilon \\ &\implies |f(x) - f(y)| < \varepsilon. \end{aligned}$$

Thus for any $\varepsilon > 0$ it is possible to choose $\delta > 0$ such that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$. Therefore f is continuous at x. But x was an arbitrarily chosen point of $[a,b] \setminus E$, therefore f is continuous everywhere in $[a,b]$ except at the points of E. □

Problem 3. Suppose that f and g are defined on $\mathbb{R}$ and that $f(t) \to A$ and $g(t) \to B$ as $t \to +\infty$, where A and B are real numbers. Prove that $(f+g)(t) \to A+B$ and that $(fg)(t) \to AB$ as $t \to +\infty$.

Solution. To show that the limit of the sum is the sum of the limits, fix $\varepsilon > 0$. By hypothesis, there is a real number T_1 so large that $t > T_1$ implies that

$$|f(t) - A| < \frac{\varepsilon}{2},$$

and a real number T_2 so large that $t > T_2$ implies that

$$|g(t) - B| < \frac{\varepsilon}{2}.$$

Set $T = \max\{T_1, T_2\}$, and observe that if $t > T$, then

$$|(f+g)(t) - (A+B)| = |(f(t) - A) + (g(t) - B)| \leq |f(t) - A| + |g(t) - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore
$$\lim_{t \to +\infty} (f+g)(t) = A + B,$$
as claimed.

To show that the limit of the products is the product of the limits, I am going to take a somewhat more pedagogical approach. The goal is to show that we can make $|(fg)(t) - AB|$ "small" by choosing t "large enough". We know that for any $\varepsilon' > 0$, we can find $T > 0$ so that
$$|f(t) - A| < \varepsilon' \qquad \text{and} \qquad |g(t) - B| < \varepsilon'.$$
It is difficult to see what can be done with $|(fg)(t) - AB|$ directly, so we are going to have to manipulate this expression before we can may any progress. One possibility is to add zero in order to introduce an intermediate point of comparison so that we can do some algebra. In this particular case, $Ag(t) - Ag(t) = 0$, and so
$$\begin{aligned}
|(fg)(t) - AB| &= |(fg)(t) + (Ag(t) - Ag(t)) - AB| && \text{(add zero)}\\
&= |(f(t) - A)(g(t)) + A(g(t) - B)| && \text{(rearrange and factor)}\\
&\leq |f(t) - A||g(t)| + |A||g(t) - B| && \text{(triangle inequality)}\\
&< |g(t)|\varepsilon' + |A|\varepsilon'\\
&= (*).
\end{aligned}$$
We can make $|A|\varepsilon'$ as small as we like by choosing ε' to be small. However, it is not clear that the other term can be made small, because we don't yet have any control over $|g(t)|$. One possibility is to expand out the inequality $|g(t) - B| < \varepsilon'$:
$$\begin{aligned}
|g(t) - B| < \varepsilon' &\implies -\varepsilon' < g(t) - B < \varepsilon'\\
&\implies B - \varepsilon' < g(t) < B + \varepsilon'\\
&\implies |g(t)| < \begin{cases} |B + \varepsilon'| & \text{if } g(t) \geq 0, \text{ and}\\ |B - \varepsilon'| & \text{if } g(t) < 0. \end{cases}\\
&\implies |g(t)| < |B| + \varepsilon'. && \text{(triangle inequality)}
\end{aligned}$$
Continuing the earlier estimates, we now have
$$\begin{aligned}
(*) &= |g(t)|\varepsilon' + |A|\varepsilon'\\
&< (|B| + \varepsilon')\varepsilon' + |A|\varepsilon'\\
&= (**).
\end{aligned}$$
At this point, we might be satisfied that the proof is complete, because $|A|$ and $|B|$ are fixed constants, so we can make this last quantity as small as we like by selecting ε' to be very small. However, the proof will almost never be presented in this manner. Instead, most authors will make the further observation that we can assume that $\varepsilon' < 1$. Thus
$$\begin{aligned}
(**) &= (|B| + \varepsilon')\varepsilon' + |A|\varepsilon'\\
&< (|B| + 1)\varepsilon' + |A|\varepsilon'\\
&= (|A| + |B| + 1)\varepsilon'.
\end{aligned}$$

Remember that the goal is to find some T such that $t > T$ implies that $|fg(t) - AB| < \varepsilon$. This last quantity—$(|A| + |B| + 1)\varepsilon'$—is ε. Hence

$$\varepsilon' = \frac{\varepsilon}{|A| + |B| + 1}.$$

So the usual presentation of the proof is as follows:

Proof. Fix $\varepsilon > 0$, assuming without loss of generality that $\varepsilon < 1$. Choose $T > 0$ so that

$$|f(t) - A| < \frac{\varepsilon}{|A| + |B| + 1} \qquad \text{and} \qquad |g(t) - B| < \frac{\varepsilon}{|A| + |B| + 1} < 1.$$

By the triangle inequality,

$$|g(t)| = |g(t) - B + B| \le |g(t) - B| + |B| \le 1 + |B|.$$

Then

$$\begin{aligned} |(fg)(t) - AB| &= |(fg)(t) - Ag(t) + Ag(t) - AB| \\ &\le |f(t) - A||g(t)| + |A||g(t) - B| \\ &< \frac{1 + |B|}{|A| + |B| + 1}\varepsilon + \frac{|A|}{|A| + |B| + 1}\varepsilon \\ &= \varepsilon, \end{aligned}$$

which completes the proof. □

Problem 4. A function f is *one-to-one* (or *injective*) on E if $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$ for all $x_1, x_2 \in E$. If f is one-to-one and continuous on $[a, b]$ and $f(a) < f(b)$, prove that f is strictly increasing. That is, show that $x_1 < x_2$ implies that $f(x_1) < f(x_2)$.

Discussion. The problem is actually not all that difficult. The basic idea is that you choose some $x_1 < x_2$ so that $f(x_1) > f(x_2)$, then use the intermediate value theorem to arrive at a contradiction. This works because no matter how the values of $f(a)$, $f(b)$, $f(x_1)$, and $f(x_2)$ are arranged relative to each other, the intervals $f((a, x_1))$ and $f((x_2, b))$ will overlap, so we will be able to find some value C in that overlap which is the image of two distinct points in the domain of f (one in (a, x_1), and the other in (x_2, b)). This contradicts the injectivity of f, and thereby completes the proof. The actual proof requires some rather tedious case work.

Proof. For contradiction, suppose that there are $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$ but $f(x_1) \geq f(x_2)$. If $f(x_1) = f(x_2)$, then f is not one-to-one, which is a contradiction. So suppose that $f(x_1) > f(x_2)$. There are six cases to consider:

(a) Suppose that $f(a) < f(b) < f(x_2) < f(x_1)$. Fix some value

$$C \in \big(f(b), f(x_2)\big).$$

By the intermediate value theorem, there are $c_1 \in (a, x_1)$ and $c_2 \in (x_2, b)$ such that

$$f(c_1) = f(c_2) = C.$$

But $c_1 < x_1 < x_2 < c_2$, and so f is not one-to-one, which is a contradiction.

(b) Suppose that $f(a) < f(x_2) < f(b) < f(x_1)$. Fix some value

$$C \in \big(f(x_2), f(b)\big).$$

By the intermediate value theorem, there are $c_1 \in (a, x_1)$ and $c_2 \in (x_2, b)$ such that

$$f(c_1) = f(c_2) = C.$$

But $c_1 < x_1 < x_2 < c_2$, and so f is not one-to-one, which is a contradiction.

(c) Suppose that $f(x_2) < f(a) < f(b) < f(x_1)$. Fix some value

$$C \in \big(f(a), f(b)\big).$$

By the intermediate value theorem, there are $c_1 \in (a, x_1)$ and $c_2 \in (x_2, b)$ such that

$$f(c_1) = f(c_2) = C.$$

But $c_1 < x_1 < x_2 < c_2$, and so f is not one-to-one, which is a contradiction.

(d) Suppose that $f(a) < f(x_2) < f(x_1) < f(b)$. Fix some value

$$C \in \big(f(x_2), f(x_2)\big).$$

By the intermediate value theorem, there are $c_1 \in (a, x_1)$ and $c_2 \in (x_2, b)$ such that

$$f(c_1) = f(c_2) = C.$$

But $c_1 < x_1 < x_2 < c_2$, and so f is not one-to-one, which is a contradiction.

(e) Suppose that $f(x_2) < f(a) < f(x_1) < f(b)$. Fix some value

$$C \in \big(f(a), f(x_1)\big).$$

By the intermediate value theorem, there are $c_1 \in (a, x_1)$ and $c_2 \in (x_2, b)$ such that

$$f(c_1) = f(c_2) = C.$$

But $c_1 < x_1 < x_2 < c_2$, and so f is not one-to-one, which is a contradiction.

(f) Suppose that $f(x_2) < f(x_1) < f(a) < f(b)$. Fix some value

$$C \in \big(f(x_1), f(a)\big).$$

By the intermediate value theorem, there are $c_1 \in (a, x_1)$ and $c_2 \in (x_2, b)$ such that

$$f(c_1) = f(c_2) = C.$$

But $c_1 < x_1 < x_2 < c_2$, and so f is not one-to-one, which is a contradiction.

In each case there is a contradiction, therefore it is not possible to choose $x_1 < x_2$ so that $f(x_1) > f(x_2)$. Therefore f is strictly increasing. $\square$