151B — 5

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1 Suppose f is a bounded real function on $[a, b]$ and f^2 is Riemman integrable. Does it follow that f is Riemann integrable? does the answer change if we assume that f^3 is Riemann integrable?

pf.

Consider,

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & \text{else} \end{cases}$$

$f \notin \mathcal{R} \impliedby \underline{\int} f dx \neq \overline{\int} f dx \impliedby \forall$ partitions P of $[a, b]$, $U(P, f) = 1$, and $L(P, f) = -1$.

Notice, $f^2 \equiv 1 \implies f^2 \in \mathcal{R}$, since constant functions are integrable.

Therefore, f is a counterexample to the assumptions of the question.

When we exchange f^2 by f^3. We have the statement,

$$f \text{ is a bounded real function on } [a, b] \text{ and } f^3 \in \mathcal{R} \implies f \in \mathcal{R}$$

Since the cube root is continuous on $\mathbb{R}$ it is integrable, so the composition $(f^3)^{1/3} = f \in \mathcal{R}$.

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2 That E be the Cantor set constructed in Sec. 2.44. Let f be a bounded real function on $[0, 1]$ which is continuous at every point outside of E. Prove that f is Riemann integrable. **Hint** Cover E by finitely many segments whose total length can be made as small as desired. Proceed as in Theorem 6.10.

pf.

$$E = \bigcap_{n=1}^{\infty} \bigcap_{k=0}^{3^{(n-1)}-1} \left(\left[0, \frac{3k+1}{3^n}\right] \cup \left[\frac{3k+2}{3^n}, 1\right] \right)$$

We can see that

$$E \subset K = \left\{ \left(\frac{4-3n}{3^n}, \frac{3n+4}{3^n} \right) \right\}_{k=1}^{\infty}$$

.

E is compact so, we can choose a finite subcover K_k of E such that the total length of $K_k < \varepsilon$.

Now, $L = [0, 1]_k$ is compact, so since f is continuous on L, f is uniformly continuous on L, so f is integrable on each of the sub-intervals of L.

Since, f is bounded, if follows that $f < M < \infty$. Therefore on K_k, the $U(P, f) - L(P, f) < M\varepsilon$, for each of the closed sets that are the left and right endpoints of each of the open sets that make up K_k. Therefore, f is integrable on K_k, and if K_k is composed of l many sets, the total value of the sums of the integrals is less than $lM\varepsilon$.

So adding up everything gives us that $f \in \mathcal{R}$ on $[0, 1]$, as $lM\varepsilon$ is a constant multiple of ε and ε was arbitrary.

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3 Suppose f is Riemann integrable on $[a, b]$ for all $b > a$ where a is fixed. Define

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

if the limit exists and is finite. Assume that $f(x) \geq 0$ and f decreases monotonically on $[1, \infty)$. Prove that

$$\int_1^\infty f(x)dx$$

Converges if and only if

$$\sum_{n=1}^{\infty} f(n)$$

Converges.

pf.

($\implies$) Ass. $\int_1^\infty f dx$ converges.

$f \geq 0 \implies 0 \leq \int_1^\infty f dx = C \in \mathbb{R}$

$\implies 0 \leq \int_1^N f dx < C$

Put, $P = \{1, 2, 3, \dots, N\}$ a partition of $[1, N] \implies \Delta x_n = 1$

$\implies 0 \leq \sum_{n=1}^N f(n) = \sum_{n=1}^N f(n)\Delta x_n = U(P, f) < \int_1^N f dx < C.$

So, $\sum_{n=1}^N f(n)$ converges for every finite N.

($\impliedby$) Ass. $\sum_{n=1}^\infty f(n)$ converges.

$f \geq 0 \implies 0 < \int_1^\infty f dx$

Consider, again the same partition P of $[1, N]$

$\sum_{n=1}^\infty f(n) = D \in \mathbb{R} \implies \sum_{n=1}^N f(n)\Delta x_n + \sum_{n=N+1}^\infty f(n)\Delta x_n = D$

Now, for each $N \in \mathbb{N}, \sum_{n=1}^N f(n)\Delta x_n = U(P, f)$

$\implies 0 < \int_1^N f(x)dx < U(P, f) < D.$

We can choose N sufficiently large such that, $\sum_{n=N+1}^\infty f(n)\Delta x_n < \varepsilon$

For the same reasons we have $\int_N^\infty f dx < \varepsilon$

Combining we get $0 < \int_1^N f dx + \int_N^\infty f dx = \int_1^\infty f dx < D + \varepsilon.$

Since ε, was arbitrary we have that $\int_1^\infty f dx$ converges.

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4 $f \in \mathcal{R}(\alpha_1)$ and $f \in \mathcal{R}(\alpha_2)$, then $f \in \mathcal{R}(\alpha_1 + \alpha_2)$ and

$$\int_a^b fd(\alpha_1 + \alpha_2) = \int_a^b fd(\alpha_1) + \int_a^b fd(\alpha_2)$$

pf.

Let $\alpha = \alpha_1 + \alpha_2$, and $P = \{x_0, \dots, x_n\}$ be any partition of $[a, b]$.

$$\begin{aligned}
\Delta\alpha_i &= \alpha_i(x_i) - \alpha(x_{i-1}) \\
&= \alpha_{1i}(x_i) + \alpha_{2i}(x_i) - (\alpha_{1i}(x_{i-1}) + \alpha_{2i}(x_{i-1})) \\
&= \alpha_{1i}(x_i) - \alpha_{1i}(x_{i-1}) + \alpha_{2i}(x_i) - \alpha_{2i}(x_{i-1}) \\
&= \Delta\alpha_{1i} + \Delta\alpha_{2i}
\end{aligned}$$

So,

$$L(P, f, \alpha_1) = \sum_{i=1}^{n} m_i \Delta\alpha_{1i} \text{ and } L(P, f, \alpha_2) = \sum_{i=1}^{n} m_i \Delta\alpha_{2i}$$

Notice,

$$\begin{aligned}
L(P, f, \alpha) &= \sum_{i=1}^{n} m_i \Delta\alpha_i \\
&= \sum_{i=1}^{n} m_i(\Delta\alpha_{1i} + \Delta\alpha_{2i}) \\
&= \sum_{i=1}^{n} m_i \Delta\alpha_{1i} + m_i \Delta\alpha_{2i} \\
&= \sum_{i=1}^{n} m_i \Delta\alpha_{1i} + \sum_{i=1}^{n} m_i \Delta\alpha_{2i} \\
&= L(P, f, \alpha_1) + L(P, f, \alpha_2)
\end{aligned}$$

Similarly,

$$U(P, f, \alpha) = U(P, f, \alpha_1) + U(P, f, \alpha_2)$$

Since $f \in \mathcal{R}(\alpha_1) \implies \forall \varepsilon > 0, \ \exists$ a partition P_1 of $[a, b]$:

$$U(P_1, f, \alpha_1) - L(P_1, f, \alpha_1) < \frac{\varepsilon}{2}$$

Also $f \in \mathcal{R}(\alpha_2) \implies \forall \varepsilon > 0, \ \exists$ a partition P_2 of $[a, b]$:

$$U(P_2, f, \alpha_2) - L(P_2, f, \alpha_2) < \frac{\varepsilon}{2}$$

Since passing to their common refinement $P = P_1 \cup P_2$ maintains the inequalities we can write,

$$U(P, f, \alpha_1) - L(P, f, \alpha_1) < \frac{\varepsilon}{2} \text{ and } U(P, f, \alpha_2) - L(P, f, \alpha_2) < \frac{\varepsilon}{2}$$

Adding the preceding inequalities yields,

$$U(P, f, \alpha_1) + U(P, f, \alpha_2) - (L(P, f, \alpha_1) + L(P, f, \alpha_2)) < \varepsilon$$

So by the second and third equalities we have that,

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon \implies f \in \mathcal{R}(\alpha_1 + \alpha_2)$$

Since $U(P, f, \alpha) = U(P, f, \alpha_1) + U(P, f, \alpha_2)$ taking the infimum over all P, we get that $\overline{\int} f(\alpha_1 + \alpha_2) = \overline{\int} f d\alpha_1 + \overline{\int} f d\alpha_2$. Similarly, taking the equality of the lower Riemann sums and applying the supremum over all P gives us that $\underline{\int} f(\alpha_1 + \alpha_2) = \underline{\int} f d\alpha_1 + \underline{\int} f d\alpha_2$. Since $f \in \mathcal{R}\alpha_1$, and $f \in \mathcal{R}(\alpha_2)$, it follows that those upper and the lower Riemann integrals are equal respectively. So,

$$\int_a^b f d(\alpha_1 + \alpha_2) = \int_a^b f d(\alpha_1) + \int_a^b f d(\alpha_2)$$

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