151B — Homework 1

Ricardo J. Acuna

(862079740)

1 Find an alternative proof to the Bolzano–Weierstrass Theorem as follows: Let B be the set of points such that $f(t) > c$. Show that the greatest lower bound of B exists. Let x be the glb of B. Prove that $f(x) = c$.

pf.

Given $[a, b] \subset \mathbb{R}$, and a continuous map f defined on $[a, b]$. Want to show,

$$f(a) < f(b) \implies \forall c \in (f(a), f(b)) : \exists x \in (a, b) : f(x) = c.$$

$\forall c \in (f(a), f(b))$, define $B_c := \{t \in [a, b] | f(t) > c\}$.

$f(b) \in B_c \implies B_c \neq \emptyset$.

$f^{-1}(B_c) \subset [a, b] \implies B_c = f(f^{-1}(B_c)) \subset f([a, b])$.

f is continuous $\implies f([a, b])$ is an interval.

Now, B_c is a bounded, not empty set in $\mathbb{R}$. By the completeness of $\mathbb{R}$, the inf B_c exists.

Let $x = \inf B_c$. Want to show by contradiction that $f(x) = c$,

Ass. $f(x) > c$,

Put $\epsilon = f(x) - c \geq 0$. By the continuity of f,

$$\exists \delta > 0 : |t - x| < \delta \implies |f(t) - f(x)| < \epsilon.$$

$\implies |f(t) - f(x)| < f(x) - c.$

$\implies -(f(x) - c) < f(t) - f(x) < f(x) - c.$

$\implies c - f(x) < f(t) - f(x).$

$\implies c < f(t).$

$\implies c < f(t_0)$. whenever $x - \delta < t_0 < x$

$\implies x$ is a not a lower bound of B_c.

Ass. $f(x) < c$,

Put $\epsilon = c - f(x) > 0$. By the continuity of f,

$$\exists \delta > 0 : |t - x| < \delta \implies |f(t) - f(x)| < \epsilon.$$

$\implies |f(t) - f(x)| < c - f(x).$

$\implies f(t) - f(x) < c - f(x).$

$\implies f(t) < c.$

$\implies t \notin B_c$ and $t \in (x - \delta, x + \delta)$.

For some t_1, $x + \delta > t_1 > x$. t_1 is a bigger bound than x.

$\implies x$ is not the greatest lower bound of B_c.

So, by contradiction $f(x) = c$ $\blacksquare$

2 Let $E = x_n$ be a countable subset of $[a, b]$ and c_n be a sequence of positive numbers such that c_n converge and is finite. Define

$$f(x) = \sum_{x_n < x} c_n.$$

Prove the following:

(1) f is monotonically increasing on $[a, b]$.

(2) f is discontinuous at every points of E with $f(x_n+) - f(x_n-) = c_n$.

(3) f is continuous at every other point of (a, b).

pf. of (a)

We want to show $x < y \implies f(x) < f(y)$.

Let $x, y \in [a, b]$, without loss of generality ass. $x < y$.

By definition, $f(x) = \sum_{x_n<x} c_n$, and $f(y) = \sum_{x_n<y} c_n$.

$f(x) = \sum_{x_n<x} c_n \leq \sum_{x_n<x} c_n + \sum_{y<x} c_n = \sum_{x_n<y} c_n = f(y)$.

We can do this because $\forall n : c_n > 0$ ■

pf. of (b) and (c)

Since f is increasing by 4.29

$f(x+) = \text{sup}_{a<t<x} f(t) < f(x) < \text{inf}_{x<t<b} f(t) = f(x-)$

Since every bounded sequence has a convergent subsequence,

$\{x_n\} \subset [a, b] \implies \exists \{x_{n_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = x$, furthermore we can choose the subsequence such that $\forall x_i \in \{x_n\} \backslash \{x_{n_k}\}$. Such that, $x_i \geq x$. Then we get a subsequence that has a matching c_{n_k}, for each x_{n_k}, and it covers all of the values between a, and x, not including x.

So since all the values of the sequence are all smaller than x, and we chose the sup, it must be that,

$f(x-) = \sum_{x_n<x} c_n = f(x)$.

Similarly, we can choose a subsequence that converges to x that covers all the values between x, and b, not including x.

So, since the values of the sequence are all bigger than x, and we chose the inf, it must be that,

$f(x+) = \sum_{x_n \leq x} c_n$.

Now, $\forall x_i \in E : f(x_i+) - f(x_i-) = \sum_{x_n \leq x_i} c_n - \sum_{x_n<x_i} c_n = c_i$.

Which shows f is discontinuous for all $x_i \in E$.

Now if $x \notin E \implies x_n < x < x_{n+1}$ for some $n \in \mathbb{N}$.

$\implies x_n < x \implies x_n \neq x$.

Now,$x_n \leq x$ and $x_n \neq x \implies x_n < x$, which reduces $f(x+)$ to $f(x)$.

$\implies f(x+) - f(x-) = f(x) - f(x) = 0 \implies f(x+) = f(x-)$

So, f is continuous for all $x \notin E$. ■

3 Suppose f and g are defined and that $f(t) \to A$ and $g(t) \to B$ as $t \to +\infty$ where A and B are real numbers. Prove that $(f+g)(t) \to A+B$ and $(fg)(t) \to AB$ as $t \to +\infty$.

pf. $f(t) \to A$ as $t \to +\infty$, and $g(t) \to B$ as $t \to +\infty$

$\implies \forall \epsilon > 0 : \exists M, N \in \mathbb{N} : m > M$, and $n > N$ and sequences $\{t_n\}$ and $\{t_k\}$ such that,

$|g(t_m) - B| < \frac{\epsilon}{2}$ and $|f(t_n) - A| < \frac{\epsilon}{2}$

Put $K = \max\{N, M\}, k > K \implies |f(t_k) + g(t_k) - (A+B)| < |f(t_k) - A| + |g(t_k) - B| < \epsilon$.

So, $(f+g)(t) \to A+B$ as $t \to +\infty$

ϵ was arbitrary so put,

$|g(t_k) - B| < \frac{\epsilon}{2|B|}$ and $|f(t_k) - A| < \frac{\epsilon}{2|A|}$

We can see,

$|f(t_k)g(t_k) - AB| = |f(t_k)g(t_k) - f(t_k)B + f(t_k)B - AB| < |f(t_k)g(t_k) - f(t_k)B| + |f(t_k)B - AB| = |f(t_k)(g(t_k) - B)| + |B(f(t_k) - A)| = |f(t_k)||g(t_k) - B| + |B||f(t_k) - A| \leq |f(t_k)|\frac{\epsilon}{2|A|} + |B|\frac{\epsilon}{2|B|}$

Taking the limit $k \to +\infty$ gives,

$|f(t_k)g(t_k) - AB| < |A|\frac{\epsilon}{2|A|} + |B|\frac{\epsilon}{2|B|} = \epsilon$

So $(fg)(t) \to AB$ as $t \to +\infty$. ■

4 We say that f is one-to-one on E if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$ for all $x_1, x_2 \in E$. If f is one-to-one and continuous on $[a, b]$ and $f(a) < f(b)$, prove that f is strictly increasing. That is, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

pf.

$\forall x_1, x_2 \in E$.

Suppose $x_1 < x_2$ and $f(x_1) = f(x_2)$,

f is one to one $\implies x_1 = x_2$, which is false, so $f(x_1) \neq f(x_2)$.

Suppose $x_1 < x_2$ and $f(x_1) > f(x_2)$,

Let $\epsilon = f(x_1) - f(x_2) > 0$

f is continuous $\implies \exists \delta > 0 : |x_2 - x_1| < \delta \implies |f(x_2) - f(x_1)| < \epsilon$.

$\implies |f(x_2) - f(x_1)| < f(x_1) - f(x_2)$

$\implies -(f(x_1) - f(x_2)) < f(x_2) - f(x_1)$

$\implies -f(x_1) + f(x_2) < f(x_2) - f(x_1)$

$\implies 0 < 0$ which is false.

So, we arrive to a contradiction.

Therefore, $x_1 < x_2 \implies f(x_1) < f(x_2)$ ■