151B — 4

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1 Suppose f and g are complex differentiable function on $(0,1), f(x) \to 0, g(x) \to 0, f'(x) \to A, g'(x) \to B$ as $x \to 0$ where A and B are complex numbers and $B \neq 0$. Prove that,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{A}{B}$$

pf.

Both $z = x + iy$ and $h(z) = u(x,y) + iv(x,y)$

On the interval $(0,1) \subset \mathbb{C}$, $\Im z = 0 \implies z = x$.

So, $u(x,y) = u(x,0) = u(x)$ and $v(x,y) = v(x,0) = v(x)$.

Write, $h(z)$ as,

$$h(x) = u(x) + iv(x)$$

Now, in the complex plane the Cauchy Riemann equations give that if the derivative exists,

$$h'(x) = u_x(x) + iv_x(x)$$

The partial of any $w(x,y)$ with respect to x, give the regular derivative if w doesn't depend on y.

So, we can write

$$f(x) = u_1(x) + iv_1(x), \text{ and } g(x) = u_2(x) + iv_2(x)$$

and

$$f'(x) = u_1'(x) + iu_1'(x), \text{ and } g'(x) = u_2'(x) + iv_2'(x)$$

We have that $f'(x) \to A$ and $g'(x) \to B$, as $x \to 0$.

And, also $A, B \in \mathbb{C}$ and $\mathbb{C}$ being a vector space with basis $\{1, i\}$ allows us to write.

$$A = a + bi, \text{ and } B = c + di$$

So we have both,

$$u_1'(x) + iv_1'(x) \to a + bi$$
$$u_2'(x) + iv_2'(x) \to c + di$$
$$\text{as } x \to 0$$

Since, $\mathbb{C}$ is a vector space, we have that the component functions converge to the components of the limits. So we have,

$$u_1'(x) \to a, \text{ and } u_2'(x) \to b, \text{ and } u_2'(x) \to c, \text{ and } v_2(x) \to d$$
$$\text{as } x \to 0$$

$x' = 1$, and $1 \to 1$ as $x \to 0$. Therefore,

$$u_1'(x) = \frac{u_1'(x)}{1} \to \frac{a}{1} = a, \text{ and } u_1'(x) = \frac{u_1'(x)}{1} \to \frac{b}{1} = b, \text{ and } u_2(x) = \frac{u_2'(x)}{1} \to \frac{c}{1} = c, \text{ and } v_2'(x) = \frac{v_2'(x)}{1} \to \frac{d}{1} = d$$

Also we have,

$$f(x) = u_1(x) + iv_1(x) \to 0 + 0i = 0$$
$$g(x) = u_2(x) + iv_2(x) \to 0 + 0i = 0$$
$$\text{as } x \to 0$$

So,

$$u_1(x) \to 0, \text{ and } v_1(x) \to 0, \text{ and } u_2(x) \to 0, \text{ and } u_2(x) \to 0$$
$$\text{as } x \to 0$$

Now, it may seem silly to note, but $x \to 0$, as $x \to 0$.

Therefore all of our component functions give us limits of real numbers, and all of them satisfy the hypotheses of L'Hospital's rule. Therefore,

$$\frac{u_1(x)}{x} \to a, \text{ and } \frac{v_1(x)}{x} \to b, \text{ and } \frac{u_2(x)}{x} \to c, \text{ and } \frac{u_2(x)}{x} \to d$$
$$\text{as } x \to 0$$

Notice,

$$\frac{f(x)}{g(x)} = \left(\frac{f(x)}{x} - A\right)\frac{x}{g(x)} + A\frac{x}{g(x)}$$

$$\frac{f(x)}{x} = \frac{u_1(x)}{x} + i\frac{v_1(x)}{x}, \text{ and } \frac{g(x)}{x} = \frac{u_2(x)}{x} + i\frac{v_2(x)}{x}$$

Since the component functions converge to A and B, respectively as $x \to 0$, we have.

$$\frac{f(x)}{x} \to a + bi = A, \quad \text{and} \quad \frac{g(x)}{x} \to c + di = B$$
$$\text{as } x \to 0$$

We are given $B \neq 0$, so $\frac{1}{B}$ exists in $\mathbb{C}$.

$$\begin{aligned}
\lim_{x\to 0}\frac{f(x)}{g(x)} &= \lim_{x\to 0}\left(\frac{f(x)}{x} - A\right)\frac{x}{g(x)} + A\frac{x}{g(x)} \\
&= (A - A)\frac{1}{B} + A\frac{1}{B} \\
&= \frac{A}{B}
\end{aligned}$$

■

2 Suppose α is increasing on $[a, b]$ and continuous at $x_0 \in [a, b]$. Let f be the function $f(x_0) = 1$, and $f = 0$ otherwise. Prove that $f \in \mathfrak{R}(\alpha)$ and

$$\int_a^b f d\alpha = 0$$

pf.

Let P be any partition of $[a, b]$.

Then, be cause f can only take the values 0or 1. It follows that $M_j = 1$ for precisely one index j such that $x_0 \in (x_{j-1}, x_j)$, and $M_i = 0$, for all $i \neq j$. Furthermore, $m_i = 0$ for all i.

$$U(P, f, \alpha) = \sum_{x_i} M_i \Delta\alpha_i = 1\Delta\alpha_j = \Delta\alpha_j, \text{ and } L(P, f, \alpha) = 0$$

Now, α is continuous at x_0 so,

$$\forall \varepsilon > 0 : \exists \delta > 0 : |x_i - x_{i-1}| < \delta \implies |\Delta\alpha_i| = |\alpha(x_i) - \alpha(x_{i-1})| < \varepsilon$$

Since α is increasing,

$$x_{i-1} < x_i \implies \alpha(x_{i-1}) < \alpha(x_i) \implies \Delta\alpha_i > 0 \implies \Delta\alpha_i < \varepsilon$$
$$\implies U(P, f, \alpha) - L(P, f, \alpha) = \Delta\alpha_j < \varepsilon$$

Since the partition was arbitrary, therefore this works for all partitions, therefore it works for at least one. It follows that there exist one partition that works for all epsilon,

$$\text{so by } (6.6) \quad f \in \mathfrak{R}(\alpha).$$

$$f \in \mathfrak{R}(\alpha) \implies \int_a^b f d\alpha = \overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha)$$

Now, the conclusion follows noticing $\forall P : L(P, f, \alpha) = 0$. The sup of $\{0\}$ is 0. Therefore,

$$\int_a^b f d\alpha = 0$$

■

3 Suppose $f \geq 0$ is continuous on $[a, b]$ and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

pf.

Suppose for a contradiction that f is not constantly 0.

$$\exists c \in [a, b] : f(c) \neq 0 \implies \frac{f(c)}{2} > 0$$

.

$$L(P, f) \leq \int_{\underline{b}}^{a} f(x)dx = \int f(x)dx = 0$$

Now, $\frac{f(c)}{2} = \epsilon > 0$, by continuity,

$\exists \delta > 0 : |c - x| < \delta \implies |f(c) - f(x)| < \frac{f(c)}{2} \implies f(c) - f(x) < \frac{f(c)}{2} \implies \frac{f(c)}{2} = f(c) - \frac{f(c)}{2} < f(x)$.

So, there is an interval where $L(P, f) > 0$.

So, $0 < L(P, f) < \int f(x)dx = 0$ gives us a contradiction. Therefore $f = 0$ ■

4 Suppose $f(x) = 0$ for all irrational x and $f(x) = 1$ for all rational x, prove that f is not Riemann integrable on any $[a, b]$ for any $a < b$.

pf.

Let P, be any partition of $[a, b]$

Then $M_i = 1$, and $m_i = 0$ for all x_i.

$a < b \implies 0 < b - a$. So,

$$\underline{\int}_a^b f(x)dx = 0 < (b - a) = \overline{\int}_a^b f(x)dx$$

So, $f \notin R$ ■