Math 151B (Advanced Calculus)
Homework 06 Solutions

Throughout this document, referenced definition, theorems, and other results are from the course text, Rudin's *Principles of Mathematical Analysis*, unless otherwise cited.

Problem 1. Consider f on $[0,1]$ such that $f(x) = 0$ if $x = 1/2^n$ for some positive integer n and $f(x) = 1$ otherwise. Prove that f is integrable and compute its integral.

Proof. The essential idea behind this proof is that the set of points where the function vanishes can be covered by a finite number of balls (intervals) with very small radius. Away from these balls, the function is constant and equal to 1, hence the infimum and supremum of f agree on these intervals. In these balls, the infimum and supremum differ by 1, but the balls can be chosen to be small enough to make the upper and lower sums arbitrarily close to each other. A version of the pedantic details follows.

Fix $\varepsilon > 0$, let $N > 2/\varepsilon$, and define

$$\delta = \frac{\varepsilon}{4N}.$$

For each $n \leq N$, define the points

$$a_n = \frac{1}{2^n} - \delta \qquad \text{and} \qquad b_n = \frac{1}{2^n} + \delta.$$

To unify notation, further define $a_0 = 0$ and $b_0 = \varepsilon/2$. Observe that if $j \neq k$, then

$$[a_j, b_j] \cap [a_k, b_k] = \varnothing.$$

The points

$$\left\{0, \frac{\varepsilon}{2}, 1\right\} \cup \{a_n, b_n\}_{n=1}^{N}$$

form a a partition of the interval $[0,1]$. Relabel this partion

$$P_\varepsilon = \{0 = x_0 < x_1 < x_2 < \cdots < x_m = 1\}.$$

For each $j = 1, 2, \ldots, m$, define

$$M_j = \sup\{f(x) \mid x \in [x_{j-1}, x_j]\} \qquad \text{and} \qquad m_j = \inf\{f(x) \mid x \in [x_{j-1}, x_j]\}.$$

If $[x_{j-1}, x_j]$ is not equal to $[a_n, b_n]$ for some $n < N$, then

$$M_j - m_j = 1 - 1 = 0.$$

On the other hand, if $[x_{j-1}, x_j]$ is equal to $[a_n, b_n]$ for some $n < N$, then $M_j - m_j = 1$. Therefore

$$U(P_\varepsilon, f) - L(P_\varepsilon, f) = \sum_{j=1}^{m} (M_j - m_j)\Delta x_j$$

$$
\begin{aligned}
&= \sum_{n=0}^{N}(b_n - a_n) \\
&= (b_0 - a_0) + \sum_{n=1}^{N}(b_n - a_n) \\
&= \frac{\varepsilon}{2} + \sum_{n=1}^{N} 2\delta \\
&= \frac{\varepsilon}{2}\frac{\varepsilon}{2N}N \\
&= \varepsilon.
\end{aligned}
$$

For any $\varepsilon > 0$, it is therefore possible to find a partition P of $[0,1]$ such that $U(P, f) - L(P, f) < \varepsilon$. Hence, by Theorem 6.6, f is integrable. Moreover, as noted above, $M_j = 1$ for all j, and so

$$\int_0^1 f(x)\,\mathrm{d}x = \overline{\int_0^1} f(x)\,\mathrm{d}x = \inf_P U(P, f) \leq U(P_\varepsilon, f) = 1.$$

The first two identities are precisely Definition 6.2, while the inequality follows from the definition of the infimum. □

Problem 2. Suppose that $c_n \geq 0$ such that $\sum C_n$ converges and s_n is a sequence of distinct points in (a, b). Let

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n),$$

where I is the unit step function

$$I(x) = \begin{cases} 0 & \text{if } x \leq 0, \text{ and} \\ 1 & \text{if } x > 0. \end{cases}$$

Let f be bounded on (a, b) and continuous at s_n for each n. Is it true that $f \in \mathscr{R}(\alpha)$?

Proof. Yes. The basic intuition of the proof is that the integrator α is a step function which does most of its "jumping" at a finite number of points. We partition the interval $[a, b]$ so that the continuity of f can be used to control the terms of Riemann sums near the big jumps. On the remainder of the interval, α is very close to being constant, so those terms don't add much more to the Riemann sums. The actual argument, which is rather technical, is below.

Let $M = \sup |f(x)|$ and

$$C = \sum_{n=1}^{\infty} c_n = \alpha(b) - \alpha(a).$$

Fix $\varepsilon > 0$ and choose $N > 0$ so large that

$$\sum_{n=N}^{\infty} c_n < \frac{\varepsilon}{4M}.$$

As f is continuous at each s_n, for each $n < N$, there exists some $\delta_n > 0$ such that

$$|x - s_n| < \delta_n \implies |f(x) - f(s_n)| < \frac{\varepsilon}{4CN}$$

Fix some δ such that

$$\delta < \min\left(\left\{\delta_n \,\middle|\, n < N\right\} \cup \left\{\tfrac{1}{2}|s_m - s_n| \,\middle|\, m \neq n < N\right\} \cup \left\{s_n - a, b - s_n \,\middle|\, n < N\right\}\right).$$

The goal of this choice of δ is to cover the set $\{s_n\}_{n=1}^{N}$ with a collection of non-intersecting δ-balls which are contained in (a, b), and on which we have control over the variation of f. Hence δ is chosen to be smaller than half the gap between distinct s_n (so that the balls are disjoint), smaller than the distance from any s_n to the endpoints (so that the balls are contained in (a, b)), and smaller than any δ_n (so that f doesn't vary too much). For each $n < N$, define

$$u_n = s_n - \delta \qquad \text{and} \qquad v_n = s_n + \delta$$

The collection of points $\{u_n, v_n\}$, along with the endpoints a and b, form a partition of $[a, b]$. Let P denote this partition, and relabel the points $\{x_j\}$ so that

$$P = \{u_n, v_n\}_{n=1}^{N} \cup \{a, b\} = \{a = x_0 < x_1 < x_2 < \cdots < x_{2n} < x_{2n+1} = b\}.$$

For each $j = 1, 2, \ldots, 2n + 1$, define

$$M_j = \sup\left\{f(x) \,\middle|\, x \in [x_{j-1}, x_j]\right\} \qquad \text{and} \qquad m_j = \inf\left\{f(x) \,\middle|\, x \in [x_{j-1}, x_j]\right\}.$$

Observe that if j is even, then there exists some $n < N$ such that

$$[x_{j-1}, x_j] = [u_n, v_n].$$

Then

$$\left[M_j - m_j\right]\left[\alpha(x_j) - \alpha(x_{j-1})\right] \leq 2\frac{\varepsilon}{4CN}(\alpha(b) - \alpha(a)) \leq \frac{\varepsilon}{2N}.$$

With $\Delta\alpha_j = \alpha(x_j) - \alpha(x_{j-1})$, this implies that

$$\sum_{j=1}^{N}\left[M_{2j} - m_{2j}\right]\Delta\alpha_{2j} \leq \sum_{j=1}^{N}\frac{\varepsilon}{2N} = \frac{\varepsilon}{2}. \tag{1}$$

On the other hand, if j is odd, then

$$\Delta\alpha_j = \alpha(x_{j+1}) - \alpha(x_j) = \sum_{\{n \mid s_n \in [x_j, x_{j+1}]\}} c_n.$$

By construction of the partition,

$$\{s_n\}_{n=1}^{\infty} \cap \bigcup_{j=0}^{N}[x_{2j}, x_{2j+1}] \subseteq \{s_n\}_{n=N}^{\infty}.$$

It therefore follows that

$$\sum_{n=0}^{N} \Delta\alpha_{2j+1} = \sum_{n=0}^{N} \left[\sum_{\{n \mid s_n \in [x_j, x_{j+1}]\}} c_n \right] \leq \sum_{n=N}^{\infty} c_n.$$

Thus

$$\sum_{n=0}^{N} [M_{2n+1} - m_{2n+1}] \, \Delta\alpha_{2n+1} \leq 2M \sum_{n=0}^{N} \Delta\alpha_{2n+1} \leq 2M \sum_{n=N}^{\infty} c_n \leq \frac{\varepsilon}{2}. \tag{2}$$

Combining the inequalities at (1) and (2) gives

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &\leq \sum_{m=1}^{2n+1} [M_m, m_m] \, [\alpha(x_m) - \alpha(x_{m-1})] \\ &= \sum_{n=1}^{N} [M_{2n} - m_{2n}] \, \Delta\alpha_{2n} + \sum_{n=0}^{N} [M_{2n+1} - m_{2n+1}] \, \Delta\alpha_{2n+1} \\ &< \varepsilon. \end{aligned}$$

Hence for any $\varepsilon > 0$, there exists some partition P of $[a, b]$ so that

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

Therefore $f \in \mathscr{R}(\alpha)$ by Theorem 6.6. □

Problem 3. Let γ_1 be a curve defined on $[a, b]$ and φ a continuous one-to-one mapping of $[c, d]$ onto $[a, b]$ such that $\varphi(c) = a$. Define $\gamma_2(s) = \gamma_1(\varphi(s))$. Prove that γ_2 is rectifiable if and only if γ_1 is rectifiable. In that case, prove that they have the same length.

Proof. Recall the definitions: a *curve* is any continuous function $\gamma : [a, b] \to \mathbb{R}^k$ (more precisely, the curve is the image of the interval $[a, b]$ with respect to the mapping γ, which is a parameterization of that curve; however, most authors use the term "curve" to mean either the continuous mapping itself and the range of that mapping without distinction). For any partition $P = \{x_j\}_{j=0}^{n}$ of $[a, b]$, define

$$\Lambda(P, \gamma) = \sum_{j=1}^{n} |\gamma(x_j) - \gamma(x_{j-1})|.$$

By way of explanation, the value $\Lambda(P, \gamma)$ is an approximation of the "length" of γ with respect to the partion P. The *length* of γ is given by

$$\Lambda(\gamma) = \sup \Lambda(P, \gamma),$$

where the supremum is taken over all partitions of $[a, b]$. If $\Lambda(\gamma) < \infty$, then γ is said to be *rectifiable*.

By assumption, φ is one-to-one and $\varphi(c) = a$. This implies that φ is strictly increasing. Hence there is a one-to-one correspondence between partitions of $[a, b]$ and partitions of $[c, d]$ via the mapping φ: if $Q = \{t_j\}$ is any partition of $[c, d]$, then

$$P = \{s_j = \varphi(t_j)\}$$

is a partition of $[a, b]$. Conversely, if $P = \{s_j\}$ is any partition of $[a, b]$, then

$$Q = \{t_j = \varphi^{-1}(s_j)\}$$

is a partition of $[c, d]$. If P and Q are any two partitions of $[a, b]$ and $[c, d]$, respectively, which are in correspondence as described above, then

$$P = \left\{s_j = \varphi(t_j)\right\} \iff Q = \left\{t_j = \varphi^{-1}(s_j)\right\}.$$

Hence

$$\begin{aligned}
\Lambda(Q, \gamma_2) &= \sum_j \left|\gamma_2(t_j) - \gamma_2(t_{j-1})\right| \\
&= \sum_j \left|\gamma_1(\varphi(t_j)) - \gamma_1(\varphi(t_{j-1}))\right| && \text{(definition of } \gamma_2\text{)} \\
&= \sum_j \left|\gamma_1(s_j) - \gamma_1(s_{j-1})\right| \\
&= \Lambda(P, \gamma_1).
\end{aligned}$$

From this, it follows that $\Lambda(\gamma_1) = \Lambda(\gamma_2)$: let P be any partition of $[a, b]$ and take Q to be the corresponding partition of $[c, d]$. Then

$$\Lambda(P, \gamma_1) = \Lambda(Q, \gamma_2) \leq \Lambda(\gamma_2),$$

which implies that $\Lambda(\gamma_2)$ is an upper bound of $\Lambda(P, \gamma_1)$. But $\Lambda(\gamma_1)$ is the least upper bound, and so

$$\Lambda(\gamma_1) \leq \Lambda(\gamma_2).$$

By an identical argument, exchanging the roles of γ_1 and γ_2, it is possible to obtain the bound $\Lambda(\gamma_1) \geq \Lambda(\gamma_2)$, which implies equality. That is,

$$\Lambda(\gamma_1) = \Lambda(\gamma_2).$$

These are finite together, hence γ_1 is rectifiable if and only if γ_2 is rectifiable. Moreover, the two curves have the same length. □

Problem 4. Suppose that α is increasing monotonically on $[a, b]$, g is continuous, and $g(x) = G'(x)$. Prove that

$$\int_a^b G \, \mathrm{d}\alpha = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b g(x)\alpha(x) \, \mathrm{d}x.$$

Proof. Fix $\varepsilon > 0$. Since G is differentiable on (a, b), it must also be continuous on this interval (Th. 5.2). This implies, via Theorem 6.8, that $G \in \mathscr{R}(\alpha)$. Then by Theorem 6.7 there is a partition $P_1 = \{x_j^1\}_{j=0}^{n_1}$

$$\left| \int_a^b G(x)\,\mathrm{d}\alpha(x) - \sum_{j=1}^{n_1} G(x_{j-1}^1)\Delta\alpha_j^1 \right| < \frac{\varepsilon}{4}. \tag{3}$$

Note that the superscript is an index, and not an exponent.

By assumption, g is continuous and α is monotonically increasing and therefore has (at worst) countably many jump discontinuities. By an argument similar to that used to show that a function which is continuous off of the Cantor set is integrable, the function $g\alpha$ is integrable (it has at most countably many jump discontinuities; treat these points of discontinuity as we did the Cantor set last week). Again applying Theorem 6.7, choose a partition $P_2 = \{x_j^2\}_{j=1}^{n_2}$ so that

$$\left| \int_a^b g(x)\alpha(x)\,\mathrm{d}x - \sum_{j=1}^{n_2} g(x_j^2)\alpha(x_j^2)\Delta x_j^2 \right| < \frac{\varepsilon}{4}. \tag{4}$$

Let $P = \{x_j\}_{j=1}^{n}$ be a common refinement of P_1 and P_2.

As g is continuous on $[a, b]$, it is uniformly continuous. Thus there is some $\delta > 0$ such that

$$|x - y| < \delta \implies |g(x) - g(y)| < \frac{\varepsilon}{3M_\alpha(b-a)}, \tag{5}$$

where

$$M_\alpha := \sup|\alpha(x)| = \max\{|\alpha(a)|, |\alpha(b)|\}. \tag{6}$$

By taking a further refinement, if necessary, assume that the mesh of P is smaller than δ. That is, assume that

$$|x_j - x_{j-1}| < \delta.$$

By the mean value theorem, for each j there is some $t_j \in (x_{j-1}, x_j)$ such that

$$G'(t_j) = \frac{G(x_j) - G(x_{j-1})}{x_j - x_{j-1}} \implies g(t_j)\Delta x_j = G'(t_j)(x_j - x_{j-1}) = G(x_j) - G(x_{j-1}).$$

Hence, multiplying both sides by $\alpha(x_j)$, for any $j = 1, 2, \ldots, n$,

$$g(t_j)\alpha(x_j)\Delta x_j = \left[G(x_j) - G(x_{j-1})\right]\alpha(x_j)$$

Summing over j, this implies that

$$\sum_{j=1}^{n} g(t_j)\alpha(x_j)\Delta x_j$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \left[G(x_j) - G(x_{j-1})\right] \alpha(x_j) \\
&= \sum_{j=1}^{n} G(x_j)\alpha(x_j) - \sum_{j=0}^{n-1} G(x_j)\alpha(x_{j+1}) \\
&= G(b)\alpha(b) + \sum_{j=1}^{n-1} G(x_j) \left[\alpha(x_j) - \alpha(x_{j+1})\right] - G(a)\alpha(a) \\
&= G(b)\alpha(b) - G(a)\alpha(x_1) - \sum_{j=2}^{n} G(x_{j-1})\alpha(x_j) \\
&= G(b)\alpha(b) - G(a)\alpha(a) + G(a)\alpha(a) - G(a)\alpha(x_1) - \sum_{j=2}^{n} G(x_{j-1})\Delta\alpha_j \\
&= G(b)\alpha(b) - G(a)\alpha(a) + G(x_0)\Delta\alpha_1 - \sum_{j=2}^{n} G(x_{j-1})\Delta\alpha_j \\
&= G(b)\alpha(b) - G(a)\alpha(a) - \sum_{j=1}^{n} G(x_{j-1})\Delta\alpha_j .
\end{aligned}
$$

In short,

$$
\sum_{j=1}^{n} g(t_j)\alpha(x_j)\Delta x_j = G(b)\alpha(b) - G(a)\alpha(a) - \sum_{j=1}^{n} G(x_{j-1})\Delta\alpha_j . \tag{7}
$$

Combining the various estimates above, we obtain the (frankly disgusting) series of increasingly refined estimates:

$$
\begin{aligned}
&\left| \int_a^b G(x)\,\mathrm{d}\alpha - \left(G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b g(x)\alpha(x)\,\mathrm{d}\alpha(x) \right) \right| \\
&\quad \le \left| \int_a^b G(x)\,\mathrm{d}\alpha - \sum_{j=1}^{n} G(x_j)\Delta\alpha_j \right| \\
&\qquad\qquad + \left| \sum_{j=1}^{n} G(x_j)\Delta\alpha_j - G(b)\alpha(b) + G(a)\alpha(a) + \int_a^b g(x)\alpha(x)\,\mathrm{d}\alpha(x) \right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(add zero, apply the triangle inequality)} \\
&\quad \le \frac{\varepsilon}{3} + \left| \sum_{j=1}^{n} G(x_j)\Delta\alpha_j - G(b)\alpha(b) + G(a)\alpha(a) + \int_a^b g(x)\alpha(x)\,\mathrm{d}\alpha(x) \right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (3) and Theorem 6.7, as } P \text{ refines } P_1\text{)} \\
&\quad \le \frac{\varepsilon}{3} + \left| \sum_{j=1}^{n} G(x_j)\Delta\alpha_j - G(b)\alpha(b) + G(a)\alpha(a) + \sum_{j=1}^{n} g(x_j)\alpha(x_j)\Delta x_j \right|
\end{aligned}
$$

$$+\left|\int_a^b g(x)\alpha(x)\,\mathrm{d}\alpha(x)-\sum_{j=1}^n g(x_j)\alpha(x_j)\Delta x_j\right|$$

(add zero, apply the triangle inequality)

$$\le \frac{2\varepsilon}{3}+\left|\sum_{j=1}^n G(x_j)\Delta\alpha_j - G(b)\alpha(b)+G(a)\alpha(a)+\sum_{j=1}^n g(x_j)\alpha(x_j)\Delta x_j\right|$$

(by (4) and Theorem 6.7, as P refines P_1)

$$\le \frac{2\varepsilon}{3}+\left|\sum_{j=1}^n G(x_j)\Delta\alpha_j - G(b)\alpha(b)+G(a)\alpha(a)+\sum_{j=1}^n g(t_j)\alpha(x_j)\Delta x_j\right|$$

$$+\left|\sum_{j=1}^n g(x_j)\alpha(x_j)\Delta x_j-\sum_{j=1}^n g(t_j)\alpha(x_j)\Delta x_j\right|$$

(add zero, apply the triangle inequality)

$$\le \frac{2\varepsilon}{3}+\left|\sum_{j=1}^n G(x_j)\Delta\alpha_j - G(b)\alpha(b)+G(a)\alpha(a)+\sum_{j=1}^n g(t_j)\alpha(x_j)\Delta x_j\right|$$

$$+\left|\sum_{j=1}^n \left[g(t_j)-g(x_j)\right]\alpha(x_j)\Delta x_j\right|$$

$$\le \frac{2\varepsilon}{3}+\left|\sum_{j=1}^n G(x_j)\Delta\alpha_j - G(b)\alpha(b)+G(a)\alpha(a)+\sum_{j=1}^n g(t_j)\alpha(x_j)\Delta x_j\right|$$

$$+\sum_{j=1}^n \left|g(t_j)-g(x_j)\right|\left|\alpha(x_j)\right|\Delta x_j$$

(triangle inequality)

$$\le \frac{2\varepsilon}{3}+\left|\sum_{j=1}^n G(x_j)\Delta\alpha_j - G(b)\alpha(b)+G(a)\alpha(a)+\sum_{j=1}^n g(t_j)\alpha(x_j)\Delta x_j\right|$$

$$+\sum_{j=1}^n \frac{\varepsilon}{3M_\alpha(b-a)}M_\alpha\Delta x_j$$

(the mesh of P is smaller than δ, so $|t_j-x_j|<\delta$; apply (5) and (6))

$$\le \frac{2\varepsilon}{3}+\left|\sum_{j=1}^n G(x_j)\Delta\alpha_j - G(b)\alpha(b)+G(a)\alpha(a)+\sum_{j=1}^n g(t_j)\alpha(x_j)\Delta x_j\right|$$

$$+\frac{\varepsilon}{3(b-a)}\sum_{j=1}^n \Delta x_j$$

$$
\begin{aligned}
&\le \varepsilon + \left| \sum_{j=1}^{n} G(x_j)\Delta\alpha_j - G(b)\alpha(b) + G(a)\alpha(a) + \sum_{j=1}^{n} g(t_j)\alpha(x_j)\Delta x_j \right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left(\text{since } \textstyle\sum \Delta x_j = b - a\right) \\
&= \varepsilon. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (7))}
\end{aligned}
$$

In other words, for any $\varepsilon > 0$,

$$
\left| \int_a^b G(x)\,\mathrm{d}\alpha - \left(G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b g(x)\alpha(x)\,\mathrm{d}\alpha(x) \right) \right| < \varepsilon.
$$

This implies that

$$
\int_a^b G(x)\,\mathrm{d}\alpha - \left(G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b g(x)\alpha(x)\,\mathrm{d}\alpha(x) \right) = 0,
$$

which is the desired result. $\square$