Math 151B (Advanced Calculus)
Homework 02 Solutions

Throughout this document, referenced definition, theorems, and other results are from the course text, Rudin's *Principles of Mathematical Analysis*, unless otherwise cited.

Problem 1. Let f be defined for all real x, and suppose that

$$|f(x) - f(y)| < (x - y)^2$$

for all real x and y. Prove that f is constant.

Proof. Observe that

$$|f(x) - f(y)| < (x - y)^2 \iff \left|\frac{f(x) - f(y)}{x - y}\right| < |x - y|.$$

Fix $x \in \mathbb{R}$ and note that, by taking limits on both sides of this second inequality,

$$0 \leq |f'(x)| = \left|\lim_{y \to x} \frac{f(x) - f(y)}{x - y}\right| = \left|\lim_{y \to x} x - y\right| = 0.$$

Therefore $f'(x) = 0$ for all real x. Thus by Th. 5.11(b), f is constant. $\square$

Problem 2. If

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0 \tag{1}$$

for real constants $C_0, C_1, \ldots, C_n$, prove that the equation

$$C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n \tag{2}$$

has at least one real root between 0 and 1.

Proof. Define a function $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = C_0 x + \frac{C_1}{2} x^2 + \frac{C_2}{3} x^3 + \cdots + \frac{C_{n-1}}{n} x^n + \frac{C_n}{n+1} x^{n+1}.$$

Observe that $f(0) = 0$ (as $f(x)$ is a polynomial divisible by x), and $f(1) = 0$ (since $f(1)$ is equal to the expression on the right-hand side of (1)). By the mean value theorem (Th. 5.10), there is some $c \in (0, 1)$ such that

$$f(1) - f(0) = (1 - 0) f'(c) \implies 0 = f'(c).$$

But

$$f'(c) = C_0 + C_1 c + C_2 c^2 + \cdots + C_n c^n = 0,$$

hence $c \in (0, 1)$ is a root of the polynomial in (2). $\square$

Problem 3. Suppose that f is continuous for $x \geq 0$, that $f'(x)$ exists for $x > 0$, that $f(0) = 0$, and that f' is monotonically increasing. For $x > 0$, define

$$g(x) = \frac{f(x)}{x}.$$

Prove that g is monotonically increasing.

Proof. By Th. 5.11(a), it is sufficient to show that $g'(x) > 0$ for all $x > 0$. By the quotient rule (Th. 5.3(c))

$$g'(x) = \frac{xf'(x) - f(x)}{x^2}$$

for all $x \geq 0$. As the denominator is positive for any $x > 0$, it remains only to show that $xf'(x) - f(x) > 0$. To that end, fix some $x > 0$. Then, by the mean value theorem (Th. 5.10), there is some $c \in (0, x)$ such that

$$f(x) - f(0) = (x - 0)f'(c).$$

By hypothesis $f(0) = 0$, and f' is monotonically increasing and so $f'(c) \leq f'(x)$. Thus

$$f(x) = xf'(c) \leq xf'(x) \implies xf'(x) - f(x) \geq 0,$$

which is the desired result. □

Problem 4. Suppose that f is differentiable on (a, b) and $f'(x) > 0$. Prove that f is strictly increasing on (a, b). Let g be the inverse of f. Prove that g is differentiable and that

$$g'(f(x)) = \frac{1}{f'(x)}$$

for all $x > 0$.

Proof. To show that f is increasing, fix $x, y \in \mathbb{R}$ with $x < y$. By the mean value theorem (Th. 5.10), there is some $c \in (x, y)$ such that

$$f(y) - f(x) = (y - x)f'(c) > 0.$$

The inequality follows from the assumption that $x < y$ and the hypothesis that $f'(c) > 0$ for all $c \in (a, b)$. But then

$$x < y \implies f(x) < f(y),$$

and so f is strictly increasing.

Differentiable functions are continuous (Th. 5.2), and the inverse of a continuous, injective function on a metric space is continuous (Th. 4.17), hence g is continuous on its domain (the set $f((a, b))$). Note that if $s = f(x)$ and $t = f(y)$ (where s, t, x, and y are drawn from the appropriate sets), then

$$y \to x \iff t \to s.$$

The forward implication is by the continuity of f, while the backward implication is by the continuity of g. Thus

$$\begin{aligned}
\lim_{t\to s}\frac{g(t)-g(s)}{t-s} &= \lim_{y\to x}\frac{g(f(y))-g(f(x))}{f(y)-f(x)} && \text{(by definition of } s \text{ and } t\text{)}\\
&= \lim_{y\to x}\frac{y-x}{f(y)-f(x)} && \text{(since } g=f^{-1}\text{)}\\
&= \frac{1}{\lim_{y\to x}\frac{y-x}{f(y)-f(x)}}\\
&= \frac{1}{f'(x)}.
\end{aligned}$$

The last expression is well-defined, as it has been assumed that $f'(x) > 0$ for all $x \in (a,b)$. However, by definition of the derivative (Defn. 5.1), the initial limit is the derivative of g at $s = f(x)$. Therefore

$$g'(f(x)) = \frac{1}{f'(x)}$$

for all $x \in (a,b)$, as claimed. □