151B — 6

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1 Consider the $f(x)$ on $[0,1]$ such that $f(x)=0$ if $x=\frac{1}{2^n}$ for some positive integer n and $f(x)=1$ otherwise. Prove that $f(x)$ is integrable and compute its integral.

pf.

$$\begin{aligned}
\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty} &\subset \bigcup_{n=1}^{\infty}\left(\frac{1}{2^n}-\frac{1}{2^{n+2}},\frac{1}{2^n}+\frac{1}{2^{n+2}}\right)\\
&=\bigcup_{n=1}^{\infty}\left(\frac{2^{n+2}-2^n}{2^{2n+2}},\frac{2^{n+2}+2^n}{2^{2n+2}}\right)\\
&=\bigcup_{n=1}^{\infty}\left(\frac{2^n(4-1)}{2^{2n+2}},\frac{2^n(4+1)}{2^{2n+2}}\right)\\
&=\bigcup_{n=1}^{\infty}\left(\frac{3}{2^{n+2}},\frac{5}{2^{n+2}}\right)\\
&=\bigcup_{n=1}^{N-1}\left(\frac{3}{2^{n+2}},\frac{5}{2^{n+2}}\right)\bigcup\bigcup_{n=N}^{\infty}\left(\frac{3}{2^{n+2}},\frac{5}{2^{n+2}}\right)\\
&\subset\bigcup_{n=1}^{N-1}\left(\frac{3}{2^{n+2}},\frac{5}{2^{n+2}}\right)\bigcup\left[0,\frac{5}{2^{N+2}}\right)
\end{aligned}$$

f is bounded, so put $M=\sup f=1$ on $[0,1]$.

Then Let $P=\{0,\frac{5}{2^{N+2}},\frac{1}{2^n}-\frac{5}{2^{N+2}},\frac{1}{2^n}+\frac{5}{2^{N+2}},1\}_{n=1}^{N-1}$.

$\frac{1}{2^n}\in\left(\frac{1}{2^n}-\frac{5}{2^{N+2}},\frac{1}{2^n}+\frac{5}{2^{N+2}}\right)\implies\Delta x_n=\frac{1}{2^n}+\frac{5}{2^{N+2}}-\left(\frac{1}{2^n}-\frac{5}{2^{N+2}}\right)=\frac{5}{2^{N+1}}$

Choose N so large such that, $\Delta x_n=\frac{5}{2^{N+1}}<\frac{\varepsilon}{MN}$.

$$\frac{1}{4}<\frac{1}{2}\implies\frac{5}{2^{N+2}}<\frac{5}{2^{N+1}}\implies\Delta x_0=\frac{5}{2^{N+2}}<\varepsilon$$

$M_n=\sup f$ for $\frac{1}{2^n}+\frac{5}{2^{N+2}}\le x\le\frac{1}{2^n}+\frac{5}{2^{N+2}}$, and $M_0=\sup f$ for $0\le x\le\frac{1}{2^{N+2}}$.

$\sum_{n=0}^{N-1}M_n\Delta x_n<MN\frac{\varepsilon}{MN}=\varepsilon$

So, now we've isolated all of the discontinuities of f.

Refine P to P^* such that $\Delta x_{n_i}=\frac{5}{2^{N+1}}$, for all $\frac{1}{2^n}+\frac{5}{2^{N+2}}\le x\le\frac{1}{2^{n+1}}-\frac{5}{2^{N+2}}$.

On each of the intervals corresponding to Δx_{n_i}, and $M_{n_i}=m_{n_i}=1$.

Then $U(P,f)-L(P,f)<\varepsilon$ since the only difference is the intervals corresponding to the n-index.

Now, since we have that $f\in\mathcal{R}$.

Then, we can pass to sample points. Since the integral over the discontinuities can be made arbitrarily small it's obvious that,

$\int_0^1 f dx=\int_0^1 dx=1$

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2 Is f as in the problem integrable? pf. f is continuous at s_n, so f satisfies the requirements of (6.15) at each s_n

(6.15) $\implies \int f dI(x-s_n) = f(s_n)$

(6.12) $\implies \int fd\sum_{n=1}^{N} c_n I(x-s_n) = \sum_{n=1}^{N} c_n f(s_n)$

$\alpha_1(x) = \sum_{n=1}^{N} c_n I(x-s_n)$ and $\alpha_2(x) = \sum_{n=N+1}^{\infty} c_n I(x-s_n)$

Since, $\sum_{n=1}^{\infty} c_n$ converges $\implies \sum_{n=1}^{\infty} c_n I(x-s_n)$ converges.

$\implies \sum_{n=N}^{\infty} c_n I(x-s_n) < \varepsilon$ for sufficiently large N.

Let P be any partition of $[a,b]$.

f is bounded on $(a,b) \implies |f| \leq M \implies \sum_{n=N}^{\infty} f(s_n)\Delta\alpha_{2i} < M\varepsilon$

Let $\alpha(x) = \alpha_1(x) + \alpha_2(x) \implies |\int fd\alpha - \sum_{n=1}^{N} f(s_n)| < M\varepsilon$

Take the limit as $N \to \infty$ and the result follows.

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3 As in the problem.

pf.

Since ϕ is a continuous one to one map from $[a,b]$ onto $[c,d]$, and $\phi(c)a$. ϕ is surjective. By that $\phi([c,d]) = [a,b]$. Since ϕ is continuous and surjective it is a homeomorphism so it admits a continuous inverse.$\gamma_2 = \gamma_1 \circ \phi$

($\implies$) Ass. γ_2 is rectifiable $\Lambda(\gamma_2) < \infty$

So,

4 By hint:

$$\begin{aligned} g(t_i)\Delta x_i = G(x_i) - G(x_{i-1}) &\implies (\alpha(x_i) - \alpha(x_{i-1}))g(t_i)\Delta x_i = (\alpha(x_i) - \alpha(x_{i-1}))(G(x_i) - G(x_{i-1})) \\ &\implies \alpha(x_i)g(t_i)\Delta x_i - \alpha(x_{i-1})g(t_i)\Delta x_i = G(x_i)(\alpha(x_i) - \alpha(x_{i-1})) - G(x_{i-1})(\alpha(x_i) - \alpha(x_{i-1})) \end{aligned}$$