Math 151B (Advanced Calculus)
Homework 05 Solutions

Throughout this document, referenced definition, theorems, and other results are from the course text, Rudin's *Principles of Mathematical Analysis*, unless otherwise cited.

Problem 1. Suppose that f is a bounded real function on $[a, b]$ and f^2 is Riemann integrable. Does it follow that f is Riemann integrable? Does the answer change if we assume that f^3 is Riemann integrable?

Proof. No, it is not true that if f is bounded and f^2 is Riemann integrable then f must be Riemann integrable. A counterexample to this claim is the function $f : [a, b] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in [a, b] \cap \mathbb{Q}, \text{ and} \\ -1 & \text{if } x \in [a, b] \setminus \mathbb{Q}. \end{cases}$$

In Problem 4 of Homework 4, it was shown that the function

$$\chi_{\mathbb{Q}} : [a, b] \to \mathbb{R} : x \mapsto \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \text{ and} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is not Riemann integrable. Note that $f = 2\chi_{\mathbb{Q}} - 1$, hence it follows from Theorem 6.12 (which describes the "algebra" of integrals) that f is not Riemann integrable. However, f is bounded (since $|f(x)| = 1$ for all $x \in [a, b]$), and

$$f^2(x) = (f(x))^2 = 1$$

for all $x \in [a, b]$, which defines a Riemann integrable function.

On the other hand, if f is bounded and f^3 is Riemann integrable, then so too is f. This is a consequence of Theorem 6.11. Let α be the identity function on $[a, b]$ (that is, $\alpha(x) = x$ for all x). By hypothesis, $f^3 \in \mathscr{R}(\alpha)$ and the boundedness of f implies the boundedness of f^3. Additionally, the function

$$\varphi : [m, M] \to \mathbb{R} : x \mapsto \sqrt[3]{x}$$

is continuous on any interval $[m, M]$. Observe that

$$f = \varphi \circ f^3.$$

The hypotheses of Theorem 6.11 are satisfied, and so $f \in \mathscr{R}(\alpha)$, which is the desired result. □

Problem 2. Let $\mathscr{C}$ be the Cantor set constructed in Section 2.44. Let f be a bounded real function on $[0, 1]$ which is continuous at every point outside of E. Prove that f is Riemann integrable.

Proof. Fix $\varepsilon > 0$ and let $M = \sup |f(x)|$. Choose a collection $\mathscr{U} = \{u_j, v_j\}$ of open intervals such that

$$\bigcup_{(u_j,v_j)\in\mathscr{U}} (u_j, v_j) \qquad \text{and} \qquad \sum_{(u_j,v_j)\in\mathscr{U}} v_j - u_j < \frac{\varepsilon}{4M}. \tag{1}$$

Such a covering of $\mathscr{C}$ is always possible: recall that the Cantor set can be expressed as

$$\bigcap_{n=0}^{\infty} \mathscr{C}_n$$

where $\mathscr{C}_k \supseteq \mathscr{C}_{k+1}$ for each k, and each set $\mathscr{C}_k$ consists of exactly 2^k closed intervals of length 3^{-k}. Making a rough approximation, this implies that $\mathscr{C}_k$ can be covered by 2^k slightly larger open intervals, e.g. intervals of length 3^{1-k}. Choose m so that

$$m > \frac{\log(\varepsilon/12M)}{\log(2/3)}.$$

and suppose that each interval in $\mathscr{U}$ is of length 3^{1-m}. Then

$$\begin{aligned}
\sum_{(u_j,v_j)\in\mathscr{U}} v_j - u_j &\le \operatorname{card}(\mathscr{U})3^{1-m} && (\operatorname{card}(\mathscr{U}) \text{ denotes cardinality}) \\
&= 3\cdot\left(\frac{2}{3}\right)^m && (\operatorname{card}(\mathscr{U}) = 2^m) \\
&= 3e^{m\log(2/3)} \\
&< 3e^{\log(\varepsilon/12M)} && (\text{watch the signs here}) \\
&= \frac{\varepsilon}{4M}.
\end{aligned}$$

Thus, as claimed, it is always possible to find an open cover $\mathscr{U} = \{(u_j, v_j)\}$ of the Cantor set so that (1) is satisfied.

Define

$$K = [0,1] \setminus \bigcup_{(u_j,v_j)\in\mathscr{U}} (u_j, v_j).$$

As K is the complement of an open set in $[0,1]$, it is both closed and bounded and therefore compact (by the Heine-Borel Theorem, Th. 2.41). By hypothesis, f is continuous on the compact set K which implies that f is uniformly continuous on that set (Th. 4.19). By uniform continuity, choose $\delta > 0$ so that

$$|s - t| < \delta \implies |f(s) - f(t)| < \frac{\varepsilon}{2}.$$

Let $\{w_j\}$ be a partition of $[0,1]$ with mesh smaller than δ. That is, assume

$$0 = w_0 < w_1 < w_2 < \cdots < w_m = 1$$

and that $w_j - w_{j-1} < \delta$ for all $j = 1, 2, \ldots, n$.

Let P be the partition of $[0,1]$ consisting of all of the u_j, v_j, and w_j, where the indices are drawn from the appropriate sets, i.e.

$$P = \{u_j, v_j\}_{j=1}^n \cup \{w_j\}_{j=1}^m.$$

To simplify notation, let $P = \{x_k\}_{k=1}^K$, where

$$0 = x_0 < x_1 < x_2 < \cdots < x_K = 1 \qquad \text{and} \qquad \Delta x_k := x_k - x_{k-1}.$$

For each k, define

$$m_k = \inf\{f(x) \mid x \in [x_{k-1}, x_k] \qquad \text{and} M_k = \sup\{f(x) \mid x \in [x_{k-1}, x_k].$$

Finally, let $\mathscr{K}$ denote the set of indices k such that the closed interval $[x_{k-1}, x_j]$ is contained in an interval $[u_j, v_j]$ for some $(u_j, v_j) \in \mathscr{U}$. That is

$$\mathscr{K} := \left\{k \,\middle|\, \exists j \in \{1,2,\ldots,n\} \text{ s.t. } [x_{k-1}, x_k] \subseteq [u_j, v_j]\right\}$$

Then

$$U(f,P) - L(f,P) = \sum_{k=1}^K (M_k - m_k)\Delta x_k = \sum_{k\in\mathscr{K}} (M_j - m_j)\Delta x_k + \sum_{k\notin\mathscr{K}} (M_j - m_j)\Delta x_k. \quad (2)$$

Note that if $k \in \mathscr{K}$, then the triangle inequality implies that

$$M_j - m_j \leq 2M.$$

Moreover, by construction of the partition P and the set $\mathscr{K}$,

$$\bigcup_{k\in\mathscr{K}} (x_{k-1}, x_k) = \bigcup_{(u_j,v_j)\in\mathscr{U}} (u_j, v_j).$$

Therefore

$$\sum_{k\in\mathscr{K}} (M_j - m_j)\Delta x_k \leq \sum_{k\in\mathscr{K}} 2M\Delta x_k < 2M\sum_{j=1}^n (v_j - u_j) < \frac{\varepsilon}{2}. \quad (3)$$

Additionally $x_k - x_{k-1} < \delta$ for any $k \notin \mathscr{K}$, and so

$$\sum_{k\notin\mathscr{K}} (M_j - m_j)\Delta x_k < \sum_{k\notin\mathscr{K}} \frac{\varepsilon}{2}\Delta_k < \sum_{k=1}^K \frac{\varepsilon}{2}\Delta_k = \frac{\varepsilon}{2}. \quad (4)$$

Substituting the results in (3) and (4) into (2) renders

$$U(f,P) - L(f,P) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus for any $\varepsilon > 0$, there exists a partition P of $[0,1]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

Therefore, by Theorem 6.6, $f \in \mathscr{R}$ (in the language of that theorem, α is the identity function, i.e. $\alpha(x) = x$ for all $x \in [0,1]$). □

Problem 3. Suppose that f is Riemann integrable on $[a, b]$ for every $b > a$ where a is fixed. Define

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x$$

if the limit exists and is finite. Assume that $f \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that

$$\int_1^\infty f(x)\,\mathrm{d}x \tag{5}$$

converges if and only if

$$\sum_{k=1}^\infty f(k) \tag{6}$$

converges.

Proof. To reduce notation a bit, define

$$S(n) = \sum_{k=1}^n f(n) \qquad \text{and} \qquad I(b) = \int_1^b f(x)\,\mathrm{d}x.$$

As f is nonnegative, both S and I are monotonically increasing functions.

For the forward implication, suppose that (5) converges. That is, suppose that there is some $L \in \mathbb{R}$ such that $I(b) \to L$. It is implicit in this assumption that $I(b)$ exists for each $b > 1$. To show that (6) converges, it is sufficient to show that it is bounded above (this follows from Theorem 3.14). As f is decreasing, for any $k \in \mathbb{N}$,

$$f(k) \leq f(x) \qquad \text{for all } x \in [k-1, k].$$

But then

$$S(n) = \sum_{k=1}^n f(k) = f(1) + \sum_{k=2}^n \inf_{x\in[k-1,k]} f(x)\cdot 1 = L(P, f),$$

where $P = \{1 < 2 < 3 < \cdots < n\}$ is a partition of the interval $[1, n]$. By definition of the [lower] Riemann integral,

$$L(P, f) \leq \overline{\int_1^n} f(x)\,\mathrm{d}x = \int_1^n f(x)\,\mathrm{d}x = I(n).$$

As I is monotonically increasing, $I(n) \leq L$ for each $n \in \mathbb{N}$. Combining the above results,

$$S(n) \leq I(n) \leq L.$$

Therefore the sequence of partial sums $S(n)$ is monotonically increasing and bounded which, by Theorem 3.14, gives the desired result.

For the backward implication, suppose that (6) converges. That is, suppose that there is some $L \in \mathbb{R}$ such that $S(n) \to L$. To show that (5) converges, it is sufficient to show that $I(b)$ is bounded. Then for any $b > 1$,

$$I(b) \le I(\lceil b \rceil) = \int_1^{\lceil b \rceil} f(x)\,\mathrm{d}x \le \overline{\int_1^{\lceil b \rceil}} f(x)\,\mathrm{d}x \le U(P, f),$$

where $\lceil \cdot \rceil$ denotes the *greatest integer function* (or the ceiling function; this function rounds a number up to the next nearest integer), and P is any partition of the interval $[1, \lceil b \rceil]$. Taking $P = \{1, 2, 3, \ldots, \lceil b \rceil\}$ and again observing that

$$f(x) \le f(k) \qquad \text{for all } x \in [k, k+1]$$

for any $k \in \mathbb{N}$, it follows that

$$U(P, f) = \sum_{k=2}^{\lceil b \rceil} \sup_{x \in [k-1,k]} f(x) \cdot 1 = \sum_{k=2}^{\lceil b \rceil} f(k) \le \sum_{k=1}^{\lceil b \rceil} f(k) = S(\lceil b \rceil).$$

But S is monotonically increasing, and so $S(\lceil b \rceil) \le L$. Combining the above results,

$$I(b) \le S(\lceil b \rceil) \le L.$$

Therefore I is a monotonically increasing function which is bounded above. It follows (more or less directly from Theorem 3.14 and Definition 4.2) that $I(b)$ converges. □

Problem 4. Prove that if $f \in \mathscr{R}(\alpha_1)$ and $f \in \mathscr{R}(\alpha_2)$, then $f \in \mathscr{R}(\alpha_1 + \alpha_2)$ and

$$\int_a^b f\,\mathrm{d}(\alpha_1 + \alpha_2) = \int_a^b f\,\mathrm{d}\alpha_1 + \int_a^b f\,\mathrm{d}\alpha_2.$$

Proof. To show that f is integrable with respect to the Riemann-Stieltjes integrator $\alpha_1 + \alpha_2$, first observe that if $P = \{x_k\}_{k=0}^n$ is any partition of $[a, b]$, then

$$\begin{aligned}
&U(P, f, \alpha_1 + \alpha_2) \\
&\quad = \sum_{k=1}^{n} \sup_{x \in [k-1,k]} f(x)\big((\alpha_1 + \alpha_2)(x_k) - (\alpha_1 + \alpha_2)(x_{k-1})\big) \\
&\quad = \sum_{k=1}^{n} \sup_{x \in [k-1,k]} f(x)\big((\alpha_1(x_k) - \alpha_1(x_{k-1})) + (\alpha_2(x_k) - \alpha_2(x_{k-1}))\big) \\
&\quad = \sum_{k=1}^{n} \sup_{x \in [k-1,k]} f(x)(\alpha_1(x_k) - \alpha_1(x_{k-1})) + \sum_{k=1}^{n} \sup_{x \in [k-1,k]} f(x)(\alpha_2(x_k) - \alpha_2(x_{k-1})) \\
&\quad = U(P, f, \alpha_1) + U(P, f, \alpha_2).
\end{aligned}$$

Fix $\varepsilon > 0$. Choose partitions P_1 and P_2 so that

$$U(P_j, f, \alpha_j) - L(P_j, f, \alpha_j) < \frac{\varepsilon}{2}$$

for each $j = 1, 2$. But then, taking P to be a common refinement of P_1 and P_2,

$$\begin{aligned} &U(P, f, \alpha_1 + \alpha_2) - L(P, f, \alpha_1 + \alpha_2) \\ &\qquad\qquad\qquad = U(P, f, \alpha_1) + U(P, f, \alpha_2) - L(P, f, \alpha_1) - L(P, f, \alpha_2) < \varepsilon. \end{aligned}$$

As ε is arbitrary, Theorem 6.6 implies that $f \in \mathscr{R}(\alpha_1 + \alpha_2)$. Moreover, applying computations from the proof of Theorem 6.6,

$$U(P, f, \alpha_j) < \int_a^b f \, \mathrm{d}\alpha_j + \frac{\varepsilon}{2},$$

where $j = 1, 2$. Thus

$$\int_a^b f \, \mathrm{d}(\alpha_1 + \alpha_2) \leq U(P, f, \alpha_1 + \alpha_2) \leq \int_a^b f \, \mathrm{d}\alpha_1 + \int_a^b f \, \mathrm{d}\alpha_2 + \varepsilon$$

By similar arguments,

$$\int_a^b f \, \mathrm{d}\alpha_1 + \int_a^b f \, \mathrm{d}\alpha_2 - \varepsilon \leq L(P, f, \alpha_1 + \alpha_2) \leq \int_a^b f \, \mathrm{d}(\alpha_1 + \alpha_2).$$

But then, as ε is arbitrary, the desired identity is shown. $\square$