151B — 7

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**1** If the sequences $\{f_n\}$ and $\{g_n\}$ converge uniformly on $E$, prove that $\{f_n + g_n\}$ converges uniformly on $E$.

pf.

Let $\{f_n\}$ and $\{g_n\}$ converge uniformly on $E \subseteq \mathbb{R}$.

So $\{f_n\}$ and $\{g_n\}$ are Cauchy sequences of functions $\forall \varepsilon > 0$ :

$f_n \to f \implies \exists M \in \mathbb{N}, \forall m_1 \geq M, m_2 \geq M, |f_{m_1} - f_{m_2}| < \frac{\varepsilon}{2}$

$g_n \to g \implies \exists K \in \mathbb{N}, \forall k_1 \geq K, k_2 \geq K, |g_{k_1} - g_{k_2}| < \frac{\varepsilon}{2}$

Let $N = \max\{M, K\}$, $\forall n \geq N, l \geq N$ we have both,

$|f_n - f_l| < \frac{\varepsilon}{2}$ and $|g_n - g_l| < \frac{\varepsilon}{2}$

$|f_n + g_n - f_l - g_l| \leq |f_n - f_l| + |g_n - g_l| \leq \varepsilon$

So, $\{f_n + g_n\}$ converges uniformly on $E$ ■

**2** If the sequences $f_n$ and $g_n$ defined on $E$ such that

(1) $\sum f_n$ converges uniformly on $E$

(2) $g_n$ converges to $0$ uniformly on $E$.

(3) $g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots$ for every $x \in E$

Prove that $\sum f_n g_n$ converge uniformly on E.

pf.

Let $S_n = \sum_{n=1}^{N} f_n$

$\sum f_n$ converges uniformly on $E \implies \{S_n\}$ is bounded.

So, $\exists M \in \mathbb{R} : |S_n| \leq M$ for all $n$.

$g_n \to 0$ uniformly on $E$.

Let $\varepsilon > 0$, there is an $N$ such that $g_N(x) \leq \frac{\varepsilon}{2M}$

From the third condition $g_n - g_{n+1} \geq 0$.

For $N \leq p \leq q$, we have

$$\begin{aligned}
\left|\sum_{n=1}^{q} f_n g_n - \sum_{n=1}^{p-1} f_n g_n\right| &= \left|\sum_{n=p}^{q} f_n g_n\right| \\
&= \left|\sum_{n=p}^{q-1} S_n(g_n - g_{n+1}) + S_q g_q - S_{q-1} g_p\right| \\
&\leq M\left|\sum_{n=p}^{q-1}(g_n - g_{n+1}) + g_q + g_p\right| \\
&= 2Mg_p \leq 2Mg_N \leq \varepsilon
\end{aligned}$$

So, $\{\sum_{n=1}^{k} f_n g_n\}$ converges uniformly by the Cauchy criterion. Thus, $\sum_{n=1}^{\infty} f_n g_n$ ■

**3** If $\sum |c_n| \leq \infty$ and $x_n$ is a sequence of distinct points in $(a, b)$, show that

$$f(x) = \sum_{k=1}^{\infty} c_k I(x - x_k)$$

converges uniformly and that $f$ is continuous for every $x \neq x_n$.

pf.

Let $f(x) = \sum_{k=1}^{\infty} c_k I(x - x_k)$

$|c_n I(x - x_n)| \leq |c_n|$, and $\sum |c_n| \leq \infty$

By, theorem $7.10$, it follows that $f$ converges.

Let $S_n = \sum_{k=1}^{n} c_n I(x - x_n)$, and $\{S_n\}$ be the sequence of partial sums of $f$.

Each $S_n$ is a linear combination of functions with discontinuities at $\{x_k\}_{k=1}^{n}$, so $S_n$ is discontinuous precisely there.

So, all of the discontinuities of $\{S_n\}$ is the sequence $\{x_n\}$ of distinct points.

let $E = (a, b) \backslash \{x_n\}$, now since the discontinuities of $\{S_n\}$ are not on $E$, $\forall n \in \mathbb{N}, S_n$ is continuous on $E$.

So, $\{S_n\}$ is a sequence of continous functions on $E$, that converges uniformly to $f$ in $E$.

So, by theorem $7.12$ $f$ is continuous on $E$.

Which is precisely the set of points $x \in (a, b)$, where $x \neq x_n$ for all $n \in \mathbb{N}$ ■

**4** Suppose $f_n$ is an equicontinuous sequence of functions on $[a, b]$ and $\{f_n\}$ converge pointwise on $[a, b]$. Prove that it converges uniformly.

pf.

$\{f_n\}$ converges pointwise $\implies \forall x \in [a, b], \{f_n(x)\}$ converges.

So by the Cauchy criterion for sequences of real numbers,

$\exists N \in \mathbb{N} : n \geq N, m \geq N \implies |f_n(x) - f_m(x)| \leq \frac{\varepsilon}{3}$

$\{f_n\}$ is equicontinuous $\implies \forall \varepsilon > 0, \exists \delta > 0 : |x - y| < \delta \implies |f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$

$$\begin{aligned}
|f_n(x) - f_m(x)| &= |f_n(x) - f_m(x) + f_n(y) - f_n(y) + f_m(y) - f_m(y)| \\
&= |f_n(x) - f_n(y) + f_n(y) - f_m(y) - f_m(x) + f_m(y)| \\
&\leq |f_n(x) - f_n(y)| + |f_n(y) - f_m(y)| + |f_m(x) - f_m(y)| \\
&\leq |f_n(x) - f_n(y)| + |f_n(y) - f_m(y)| + |f_m(x) - f_m(y)| \\
&\leq \frac{2\varepsilon}{3} + |f_n(y) - f_m(y)| \quad \text{by equicontinuity of } \{f_n\} \\
&\leq \varepsilon \qquad\qquad\qquad\qquad\quad \text{by pointwise convergence of } \{f_n\}
\end{aligned}$$

Since, $[a, b]$ so there exists a finite subcover by open balls of radius $\delta/2$. Then $x$ lies in one of the balls of radius $\delta/2$, so the previous inequality shows the Cauchy condition for uniform convergence of functions holds for all $x \in [a, b]$.

Therefore, $\{f_n\}$ converges uniformly ■