151B — 3

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1 Let f be a continuous real function on $\mathbb{R}$. If $f'(x)$ exists for all $x \neq 0$ and $f'(x) \to 3$ as $x \to 0$, does it follow that $f'(0)$ exists?

pf.

$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$

$(f(h) - f(0))' = f'(h) \to 3$ as $h \to 0$

$h' = 1 \to 1$ as $h \to 0$

$\frac{(f(h)-f(0))'}{h'} \to \frac{3}{1} = 3$, as $h \to 0$

$f(h) - f(0) \to 0$, as $h \to 0$

Then by L'Hospital's rule,

$\frac{f(h)-f(0)}{h} \to 3$, as $h \to 0$.

So, $f'(0) = 3$ ■

2 Let f be a twice-differentiable real function on R and M_0, M_1 and M_2 be the least upper bounds of $|f(x)|$, $|f'(x)|$ and $|f''(x)|$. Prove that

$$M_1^2 \leq 4M_0M_2$$

.

Hint: Use the Taylor's theorem with $n = 2$.

pf. $\forall \alpha, \beta \in [a, b] : \exists x \in (\alpha, \beta)$ by Taylor's theorem,

$$f(\beta) = f(\alpha) + f'(\alpha)(\beta - \alpha) + \frac{f''(x)}{2}(\beta - \alpha)^2$$

$$\implies f'(\alpha) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha} - \frac{f''(x)}{2}(\beta - \alpha)$$

$$\implies |f'(\alpha)| = \left|\frac{f(\beta) - f(\alpha)}{\beta - \alpha} - \frac{f''(x)}{2}(\beta - \alpha)\right| \leq \left|\frac{f(\beta) - f(\alpha)}{\beta - \alpha}\right| + \left|\frac{f''(x)}{2}(\beta - \alpha)\right|$$

$$\implies |f'(\alpha)| \leq \frac{|f(\beta)| + |f(\alpha)|}{|\beta - \alpha|} + \left|\frac{f''(x)}{2}\right||\beta - \alpha| \implies M_1 \leq \frac{2M_0}{|\beta - \alpha|} + \frac{M_2}{2}|\beta - \alpha|$$

$$\implies M_1 \leq \frac{1}{2}(\frac{4M_0}{|\beta - \alpha|} + M_2|\beta - \alpha|)$$

By the inequality of arithmetic geometric means,

$\frac{1}{2}(\frac{4M_0}{|\beta-\alpha|} + M_2|\beta - \alpha|) > \sqrt{\frac{4M_0}{|\beta-\alpha|} \cdot M_2|\beta - \alpha|} = 2\sqrt{M_0M_2}$

Since $\frac{4M_0}{|\beta-\alpha|} + M_2|\beta - \alpha|$ bounds M_1 for arbitrary $|\beta - \alpha|$, it must be at least $2\sqrt{M_0M_2}$ by the inequality above. And M_1 is the least upper bound the equation $M_1 \leq 2\sqrt{M_0M_2}$ still holds.

$$\implies M_1^2 \leq 4M_0M_2 \quad ■$$

3 Suppose f is defined in a neighborhood of x and suppose $f''(x)$ exists. Show that

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f''(x)$$

pf.

$f''(y)$ exists. So,

$$f''(y) = \lim_{h\to 0} \frac{f'(y+h)-f'(y)}{h} =$$

$$\lim_{h\to 0} \frac{\frac{f(y+h+h)-f(y+h)}{h} - \frac{f(y+h)-f(y)}{h}}{h} =$$

f is defined and differentiable on a neighborhood, so let $y = x-h \in (x-\delta, x+\delta)$.

$$f''(x) = \lim_{h\to 0} \frac{\frac{f(x-h+h+h)-f(x-h+h)}{h} - \frac{f(x-h+h)-f(x-h)}{h}}{h} =$$

$$\lim_{h\to 0} \frac{\frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h}}{h} =$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)-(f(x)-f(x-h))}{h^2} =$$

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2}$$

■

4 Recall that a continuous function f on (a,b) is convex if and only if for all $x,y \in (a,b)$

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(f(x)+f(y))$$

Let f be a differentiable real function on (a,b). Prove that f is convex if and only if f' is monotonically increasing.

pf.

($\Longrightarrow$) Ass. f is convex,

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(f(x)+f(y))$$

.

$\Longrightarrow\ f(x) = f\left(\frac{(x+h)+(x-h)}{2}\right) \leq \frac{1}{2}(f(x+h)+f(x-h))$.

$\Longrightarrow\ 2f(x) \leq f(x+h)+f(x-h)$.

$\Longrightarrow\ 0 = 2f(x)-2f(x) \leq f(x+h)+f(x-h)-2f(x)$.

$\Longrightarrow\ 0 = \lim_{h\to 0} \frac{2f(x)-2f(x)}{h^2} \leq \lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f''(x)$, by 3.

So, if f is convex and f'' exists for all $x \in (a,b)$, then f' is increasing.

($\Longleftarrow$) Ass. f' is increasing. Let $x,y \in (a,b)$.

$$\begin{aligned}
x < y &\implies f'(x) \leq f'(y) \\
&\implies \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \leq \lim_{h\to 0} \frac{f(y+h)-f(y)}{h}
\end{aligned}$$

$$\begin{aligned}
x-h < x &\implies \lim_{h\to 0} \frac{f(x-h+h)-f(x-h)}{h} \leq \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&\implies \lim_{h\to 0} \frac{2f(x)}{h} \leq \lim_{h\to 0} \frac{f(x-h)+f(x+h)}{h} \\
&\implies \lim_{h\to 0} \frac{f(x)}{h} \leq \lim_{h\to 0} \frac{f(x-h)+f(x+h)}{2h} \\
&\implies \frac{f(x)}{\lim_{h\to 0} h} \leq \frac{f(x-h)+f(x+h)}{\lim_{h\to 0} 2h} \\
&\implies f(x) \leq \frac{f(x-h)+f(x+h)}{\lim_{h\to 0} 2h} \lim_{h\to 0} h \\
&\implies f(x) \leq \frac{f(x-h)+f(x+h)}{2} \\
&\implies f\left(\frac{x+h+x-h}{2}\right) \leq \frac{f(x-h)+f(x+h)}{2}
\end{aligned}$$

Put $x = x-h$ and $y = x+h$,

$$\implies f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

Now, letting $x \in (a,b)$, $0 < h < \min\{|a-x|, |b-x|\}$, covers all of (a,b) ■

I still don't know what to do when f'' doesn't exist.