Math 151B (Advanced Calculus)
Homework 03 Solutions

Throughout this document, referenced definition, theorems, and other results are from the course text, Rudin's *Principles of Mathematical Analysis*, unless otherwise cited.

Problem 1. Let f be a continuous real function on $\mathbb{R}$. If $f'(x)$ exists for all $x \neq 0$ and $f'(x) \to 3$ as $x \to 0$, does it follow that $f'(0)$ exists?

Solution. Yes, the given hypotheses imply that $f'(0)$ exists. By Definition 5.1, $f'(0)$ exists if and only if the limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} \tag{1}$$

exists. As f is continuous,

$$\lim_{x\to 0} \big(f(x) - f(0)\big) = 0,$$

hence the limit in (1) is of the form

$$\lim_{x\to 0} \frac{g(x)}{h(x)}, \qquad \text{where } g(x), h(x) \to 0 \text{ as } x \to 0.$$

Moreover, it has been assumed that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Hence

$$3 = \lim_{x\to 0} f'(x) = \lim_{x\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}x}(f(x) - f(0))}{\frac{\mathrm{d}}{\mathrm{d}x}(x)}.$$

Thus the hypotheses of Theorem 5.13 (L'Hospital's rule) hold, and so

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{f'(x)}{1} = 3.$$

Therefore $f'(0)$ exists (and is equal to 3). □

Problem 2. Let f be a twice-differentiable real function on $\mathbb{R}$ and M_0, M_1, and M_2 be the least upper bounds of $|f(x)|$, $|f'(x)|$, and $|f''(x)|$, respectively. Further assume that $M_2 > 0$. Prove that

$$M_1^2 \leq 4M_0M_2.$$

Note that the statement here is slightly different from the original statment in the homework—I have made the additional assumption that $M_2 > 0$. Something like this is necessary. To see this, suppose that $f(x) = ax + b$, where $a, b \in \mathbb{R}$ are constants and $a \neq 0$. Then

$$M_0 = \infty, \qquad M_1 = |a|, \qquad \text{and} \qquad M_2 = 0.$$

The statement of the result is then

$$|a|^2 \leq 4 \cdot \infty \cdot 0,$$

which is a nonsensical statement ($\infty \cdot 0$ is not well-defined, particularly in a setting where no limits are being taken). In order to avoid this situation, it is sufficient to assume that $M_2 > 0$. Once we have proved the fundamental theorem of calculus, we will be able to state with greater certainty that functions of the from $f(x) = ax + b$ with $a \neq 0$ are the only exceptions.

The argument for this problem involves a rabbit being pulled out of a hat in a somewhat contrived way. If you know what you are trying to prove (i.e. that $M_1^2 \leq 4M_0M_2$), one can work backwards to find the correct rabbit, but it is not altogether obvious why one might suspect this result to be true in the first place. In the following argument, I will use γ to represent the magical lagomorph, as it has pointy ears and looks a bit like a rabbit.

Proof. Fix some arbitrary $x \in \mathbb{R}$ and set

$$\gamma = 2\frac{\sqrt{M_0}}{\sqrt{M_2}} > 0.$$

By Taylor's theorem (Theorem 5.15), there is some $\xi \in (x, x+\gamma)$ such that

$$\begin{aligned} f(x+\gamma) &= f(x) + f'(x)\gamma + \frac{1}{2}f''(\xi)\gamma^2 \\ &\implies f'(x)\gamma = f(x+\gamma) - f(x) - \frac{1}{2}f''(\xi)\gamma^2 \\ &\implies f'(x) = \frac{1}{\gamma}(f(x+\gamma) - f(x)) - \frac{\gamma}{2}f''(\xi) \end{aligned}$$

Taking absolute values, applying the triangle inequality, and invoking the definitions of M_0 and M_2 as suprema, this implies that

$$\begin{aligned} |f'(x)| &\leq \frac{1}{\gamma}|f(x+\gamma)| + \frac{1}{\gamma}|f(x)| + \frac{\gamma}{2}|f''(\xi)| \\ &\leq \frac{2}{\gamma}M_0 + \frac{\gamma}{2}M_2 \\ &= \frac{2}{2\frac{\sqrt{M_0}}{\sqrt{M_2}}}M_0 + \frac{2\frac{\sqrt{M_0}}{\sqrt{M_2}}}{2}M_2 \\ &= 2\sqrt{M_0M_2}. \end{aligned}$$

As this holds for any choice of $x \in \mathbb{R}$, it follows that $2\sqrt{M_0M_2}$ is an upper bound of $|f'(x)|$. But M_1 is, by definition, the least upper bound of $|f'(x)|$, and so any other upper bound must be larger. Hence

$$M_1 = \operatorname{lub}|f'(x)| \leq 2\sqrt{M_0M_2} \implies M_1^2 \leq 4M_0M_2,$$

which is the desired result. □

Gerardo proposed a slightly different argument. The basic outline is the same, but his argument (perhaps) sheds some light on where the estimate comes from, or why we might expect it to look as it does.

Proof. Let $x \in \mathbb{R}$ be arbitrary. Then for any $\gamma > 0$, Taylor's theorem (Theorem 5.15) implies that there is some $\xi \in (x, x+\gamma)$ such that

$$f(x+\gamma) = f(x) + f'(x)\gamma + \frac{1}{2}f''(\xi)\gamma^2.$$

As in the previous proof, this implies that for any arbitrary $x \in \mathbb{R}$,

$$|f'(x)| \leq \frac{2}{\gamma}M_0 + \frac{\gamma}{2}M_2.$$

At this point, we would like to accomplish two goals: (1) the estimate on the right depends on γ, and we would prefer to have a result which is independent of this arbitrary variable, and (2) we would like to obtain the "best" bound; ideally, we want to make the right-hand side as small as possible.

Addressing the second point, define $g : (0, \infty) \to \mathbb{R}$ by

$$g(\gamma) = \frac{2}{\gamma}M_0 + \frac{\gamma}{2}M_2.$$

Observe that g is continuous and differentiable on $(0, \infty)$, with

$$g'(\gamma) = -\frac{2}{\gamma^2}M_0 + \frac{1}{2}M_2.$$

Then

$$\begin{aligned} g'(\gamma) < 0 &\iff -\frac{2}{\gamma^2}M_0 + \frac{1}{2}M_2 < 0 \\ &\iff \frac{1}{2}M_2 < \frac{2}{\gamma^2}M_0 \\ &\iff \gamma^2 < 4\frac{M_0}{M_2} \\ &\iff \gamma < 2\frac{\sqrt{M_0}}{\sqrt{M_2}}, \end{aligned}$$

where the last inequality follows from the fact that we have assumed that $\gamma > 0$. Similarly,

$$g'(\gamma) = 0 \iff \gamma = 2\frac{\sqrt{M_0}}{\sqrt{M_2}} \qquad \text{and} \qquad g'(\gamma) > 0 \iff \gamma > 2\frac{\sqrt{M_0}}{\sqrt{M_2}}.$$

Therefore (by Theorem 5.11) g is monotonically decreasing on $(0, 2\sqrt{M_0}/\sqrt{M_2})$ and increasing on $(2\sqrt{M_0}/\sqrt{M_2})$. From this, it follows that g has a global minimum at

$$\gamma = 2\frac{\sqrt{M_0}}{\sqrt{M_2}}.$$

Therefore for any $\gamma > 0$

$$|f'(x)| \leq \frac{2}{\gamma} M_0 + \frac{\gamma}{2} M_2 = g(\gamma) \leq g\left(2\frac{\sqrt{M_0}}{\sqrt{M_2}}\right) = 2\sqrt{M_0 M_2}.$$

From here, the argument proceeds in the same manner as the original proof. $\square$

Problem 3. Suppose that f is defined in a neighborhood of x and that $f''(x)$. Show that

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

This identity is actually *incredibly* useful, and shows up in applied mathematical problems quite frequently, as it allows us to compute second derivatives numerically without needing to find explicit first derivatives. A straightforward proof of this result via L'Hospital's rule is presented below. However, this proof is somewhat unenlightening, as it doesn't shed any light on *why* this result might be true. So before I give the proof, let's discuss the problem a bit first.

The intuition follows from Taylor's theorem. Suppose that f is three-times continuously differentiable in a neighborhood of x. Then Taylor's theorem implies that for any sufficiently small h, there exists some $\xi(h^+) \in (x, x+h)$ such that

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2} f''(x)h^2 + \frac{1}{6} f'''(\xi(h^+))h^3.$$

Similarly, there is some $\xi(h^-) \in (x-h, x)$ such that

$$f(x-h) = f(x) - f'(x)h + \frac{1}{2} f''(x)h^2 - \frac{1}{6} f'''(\xi(h^-))h^3.$$

Adding these two estimates and doing a little bit of algebraic manipulation gives

$$\begin{aligned} f(x+h) + f(x-h) &= 2f(x) + h^2 f''(x) + \frac{h^3}{6}(f'''(\xi(h^+)) - f'''(\xi(h^-))) \\ \implies \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} &= f''(x) + \frac{h}{6}(f'''(\xi(h^+)) - f'''(\xi(h^-))). \end{aligned}$$

Observe that both $\xi(h^+)$ and $\xi(h^-)$ are real numbers which depend on the choice of h. However, for any $h > 0$, we have

$$x - h < \xi(h^-) < x < \xi(h^+) < x + h,$$

which means that both $\xi(h^+)$ and $\xi(h^-)$ tend to x as $h \to 0$. Since it has been assumed that f is three times continuously differentiable, this implies that f''' is continuous at x, and so

$$\lim_{h \to 0} f'''(\xi(h^+)) - f'''(\xi(h^-)) = f'''(x) - f'''(x) = 0.$$

Therefore

$$
\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2} &= \lim_{h\to 0}\left(f''(x)+\frac{h}{6}(f'''(\xi(h^+))-f'''(\xi(h^-)))\right)\\
&= f''(x)+\underbrace{\lim_{h\to 0}\frac{h}{6}(f'''(\xi(h^+))-f'''(\xi(h^-)))}_{=0}\\
&= f''(x),
\end{aligned}
$$

which is precisely the result we want to prove. Unfortunately, this argument (which is, actually, a pretty rigorous proof) assumes more than we are given in this problem. Specifically, it requires that f be three-times continuously differentiable, whereas we are only given *two* continuous derivatives. On the bright side, as is demonstrated by the following proof, the theorem remains true even if f is only twice-differentiable near x.

Proof. Starting on the left-hand side,

$$
\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2} &= \lim_{h\to 0}\frac{\frac{\mathrm{d}}{\mathrm{d}h}f(x+h)+\frac{\mathrm{d}}{\mathrm{d}h}f(x-h)-2\frac{\mathrm{d}}{\mathrm{d}h}f(x)}{\frac{\mathrm{d}}{\mathrm{d}h}h^2}\\
&\qquad\qquad\text{(Th. 5.13; L'Hospital's rule)}\\
&= \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h} \qquad \text{(Th. 5.5; chain rule)}\\
&= \lim_{h\to 0}\frac{\frac{\mathrm{d}}{\mathrm{d}h}f(x+h)-\frac{\mathrm{d}}{\mathrm{d}h}f(x-h)}{2\frac{\mathrm{d}}{\mathrm{d}h}h} \qquad \text{(Th. 5.13)}\\
&= \lim_{h\to 0}\frac{f''(x+h)+f''(x-h)}{2} \qquad \text{(Th. 5.5)}\\
&= f''(x),
\end{aligned}
$$

which is the desired result. □

Problem 4. Recall that a continuous function f on (a,b) is convex if and only if for all $x, y \in (a,b)$,

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(f(x)+f(y)). \tag{2}$$

Let f be a differentiable real function on (a,b). Prove that f is convex if f' is monotonically increasing.

Proof. Fix $x, y \in \mathbb{R}$ and, without loss of generality assume that $x < y$. It follows from the mean value theorem (Theorem 5.10) that there is some $\xi \in (x, \frac{1}{2}(x+y))$ such that

$$f\left(\frac{x+y}{2}\right)-f(x) = \left(\frac{x+2}{2}-x\right)f'(\xi) \implies f\left(\frac{x+y}{2}\right) = \left(\frac{y-x}{2}\right)f'(\xi)+f(x).$$

Similarly, there is some $\eta \in (\frac{1}{2}(x+y), y)$ such that

$$f\left(\frac{x+y}{2}\right) = \left(\frac{x-y}{2}\right) f'(\eta) + f(y).$$

Add these two identies to get

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y) + \left(\frac{y-x}{2}\right)(f'(\xi) - f'(\eta)).$$

It has been assumed that $x < y$, and $\xi < \frac{1}{2}(x+y) < \eta$ hence $f'(\xi) < f'(\eta)$ by the assumption that f' is monotonically increasing. Therefore

$$\left(\frac{y-x}{2}\right)(f'(\xi) - f'(\eta)) < 0.$$

The desired result follows immediately. □