Math 151B (Advanced Calculus)
Homework 04 Solutions

Throughout this document, referenced definition, theorems, and other results are from the course text, Rudin's *Principles of Mathematical Analysis*, unless otherwise cited.

Problem 1. Suppose f and g are complex differentiable functions on $(0,1)$,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0, \qquad \lim_{x\to 0} f'(x) = A \qquad \text{and} \qquad \lim_{x\to 0} g'(x) = B,$$

where A and B are complex numbers with $B \neq 0$. Prove that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

Proof. As per Remark 5.16, a function γ is complex differentiable if and only if its real and imaginary parts are differentiable as single-variable functions, and

$$\gamma'(x) = (\Re\gamma)'(x) + i(\Im\gamma)'(x),$$

where $\Re\gamma$ and $\Im\gamma$ represent the real and imaginary parts of γ, respectively. By hypothesis,

$$\lim_{x\to 0}(\Re f)(x) = \lim_{x\to 0}(\Im f)(x) = 0, \quad \lim_{x\to 0}(\Re f)'(x) = \Re A, \quad \text{and} \quad \lim_{x\to 0}(\Im f)'(x) = \Im A.$$

Thus, by application of L'Hospital's rule for real-valued functions and the arithmetic of limits over a metric space,

$$\begin{aligned}
\lim_{x\to 0} \frac{f(x)}{x} &= \lim_{x\to 0} \left[\frac{(\Re f)(x)}{x} + i\frac{(\Im f)(x)}{x}\right] \\
&= \lim_{x\to 0} \left[\frac{(\Re f)(x)}{x}\right] + i \lim_{x\to 0} \left[\frac{(\Im f)(x)}{x}\right] && \text{(Th. 4.4)} \\
&= \lim_{x\to 0} \left[\frac{(\Re f)'(x)}{1}\right] + i \lim_{x\to 0} \left[\frac{(\Im f)'(x)}{1}\right] && \text{(Th. 5.13)} \\
&= \Re A + i\Im A \\
&= A.
\end{aligned}$$

A similar argument applies to g, and so

$$\lim_{x\to 0} \frac{f(x)}{x} = A \qquad \text{and} \qquad \lim_{x\to 0} \frac{g(x)}{x} = B. \tag{1}$$

It further follows from Theorem 4.4 that

$$\lim_{x\to 0} \frac{x}{g(x)} = \frac{1}{B}, \tag{2}$$

since $B \neq 0$.

Observe that

$$\frac{f(x)}{g(x)} = \left(\frac{f(x)}{x} - A\right)\frac{x}{g(x)} + \frac{Ax}{g(x)}. \tag{3}$$

This is non-intuitive, and seems unnecessary. Such a manipulation is required, however, as the goal is to apply L'Hospital's rule, and the proof of this rule (given as Theorem 5.13) only applies to real-valued functions. Indeed, this exercise is a proof of L'Hospital's rule for complex-valued functions. Taking limits in 3,

$$\begin{aligned}
\lim_{x\to 0}\frac{f(x)}{g(x)} &= \lim_{x\to 0}\left[\left(\frac{f(x)}{x} - A\right)\frac{x}{g(x)} + \frac{Ax}{g(x)}\right] && \text{(by (3), above)}\\
&= \left(\lim_{x\to 0}\left[\frac{f(x)}{x}\right] - A\right)\lim_{x\to 0}\left[\frac{x}{g(x)}\right] + A\lim_{x\to 0}\left[\frac{x}{g(x)}\right] && \text{(by Th. 4.4)}\\
&= (A - A)\frac{1}{B} + A\cdot\frac{1}{B} && \text{(by (1) \& (2), above)}\\
&= \frac{A}{B},
\end{aligned}$$

which is the desired result. □

Problem 2. Suppose that α is an increasing function on $[a,b]$ which is continuous at $\xi \in [a,b]$. Let f be the function

$$f(x) = \begin{cases} 1 & \text{if } x = \xi, \text{ and} \\ 0 & \text{if } x \neq \xi. \end{cases}$$

Prove that $f \in \mathscr{R}(\alpha)$ and that

$$\int_a^b f \,\mathrm{d}\alpha = 0.$$

Proof. First, note that if $[x_{j-1}, x_j]$ is any interval contained in $[a,b]$, then

$$m_j = \inf\{f(x) \mid x \in [x_{j-1}, x_j]\} = 0.$$

Therefore, for any fixed partition $P^* = \{x_0 = a, x_1, \ldots, x_n = b\}$ of $[a,b]$,

$$\underline{\int_a^b} f \,\mathrm{d}\alpha = \sup L(P,f,\alpha) \geq L(P^*,f,\alpha) = \sum_{j=1}^n m_j(\alpha(x_j) - \alpha(x_{j-1})) = 0.$$

Note that we could also have invoked the monotonicity of the lower Riemann integral to obtain this result (if $f \geq g$, then $\underline{\int} f \leq \underline{\int} g$; us the fact that $f \geq 0$ to bound the integral from below by zero).

To show that the upper integral is also zero, fix some $\varepsilon > 0$ and choose $\delta > 0$ so that $|x - \xi| < \delta$ implies that $|\alpha(x) - \alpha(\xi)| < \frac{1}{2}\varepsilon$. Note that such a choice is possible, as α is continuous at ξ. Then define the partition P_ε by

$$P_\varepsilon = \left\{x_0 = a, x_1 = \xi - \frac{\delta}{2}, x_2 = \xi + \frac{\delta}{2}, x_3 = b\right\}.$$

Then

$$\begin{aligned}
\overline{\int_a^b} f \,\mathrm{d}\alpha &= \inf U(P, f, \alpha) \le U(P_\varepsilon, f, \alpha) \\
&= 0 \cdot \big(\alpha(\xi - \tfrac{\delta}{2}) - \alpha(a)\big) + 1 \cdot \big(\alpha(\xi + \tfrac{\delta}{2}) - \alpha(\xi - \tfrac{\delta}{2})\big) + 0 \cdot \big(\alpha(b) - \alpha(\xi - \tfrac{\delta}{2}\big) \\
&= \alpha(\xi + \tfrac{\delta}{2}) - \alpha(\xi - \tfrac{\delta}{2}) \\
&= \big(\alpha(\xi + \tfrac{\delta}{2}) - \alpha(\xi)\big) + \big(\alpha(\xi) - \alpha(\xi - \tfrac{\delta}{2})\big) \\
&< \varepsilon.
\end{aligned}$$

As ε was arbitrary, it follows that

$$\overline{\int_a^b} f \,\mathrm{d}\alpha = 0.$$

But then

$$\underline{\int_a^b} f \,\mathrm{d}\alpha = \overline{\int_a^b} f \,\mathrm{d}\alpha = 0,$$

which implies that $f \in \mathscr{R}(\alpha)$, and integrates to zero over the interval $[a, b]$. □

NB: There are a couple of places where the proof above skips over some details. For example, if $\xi = a$ or if $\xi = b$, then we have to choose a slightly different partition (e.g. $P_\varepsilon = \{x_0 = a = \xi, x_1 = \xi + \delta, x_2 = b\}$). We have also implicitly assumed that the interval $(\xi - \delta/2, \xi + \delta/2)$ is contained in the interval $[a, b]$—this assumption is not *a priori* justified, though we can always choose δ small enough so that the assumption holds. In any event, there are some niggling details which one might want to iron out, though these edge cases don't actually cause problems for the proof. Similar comments apply to the proof given for the next problem.

Problem 3. Suppose that $f \ge 0$ is continuous on $[a, b]$ and that $\int_a^b f(x) \,\mathrm{d}x = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Proof. Suppose for contradiction that there is some $\xi \in (a, b)$ with $f(\xi) > 0$. As f is continuous, there exists some $\delta > 0$ such that

$$|x - \xi| < \delta \implies |f(x) - f(\xi)| < \frac{f(\xi)}{2} \implies \frac{f(\xi)}{2} < f(x).$$

Let

$$P^* = \left\{x_0 = a, x_1 = \xi - \tfrac{\delta}{2}, x_2 = \xi + \tfrac{\delta}{2}, x_3 = b\right\}.$$

For each $j = 1, 2, 3$, define m_j as in Definition 6.1, i.e. $m_j = \inf\{f(x) \mid x_{j-1} \le x \le x_j\}$. The choice of δ ensures that

$$m_2 = \inf\left\{f(x) \,\middle|\, \xi - \frac{\delta}{2} \le x \le \xi + \frac{\delta}{2}\right\} \ge \frac{f(\xi)}{2}.$$

Moreover, as $f \geq$, both m_1 and m_2 are nonnegative. Therefore

$$\begin{aligned}
0 &= \int_a^b f(x)\,\mathrm{d}x && \text{(by hypothesis)} \\
&= \underline{\int_a^b} f(x)\,\mathrm{d}x && \text{(defn. 6.1)} \\
&= \sup L(P, f) && \text{(notation as in defn. 6.1)} \\
&\geq L(P^*, f) && \text{(defn. of sup)} \\
&= m_1 \cdot \left(\xi - \frac{\delta}{2} - a\right) + m_2 \cdot \left(\left(\xi + \frac{\delta}{2}\right) - \left(\xi - \frac{\delta}{2}\right)\right) + m_3 \cdot \left(b - \xi - \frac{\delta}{2}\right) \\
&\geq \frac{f(\xi)}{2} \cdot \left(\left(\xi + \frac{\delta}{2}\right) - \left(\xi - \frac{\delta}{2}\right)\right) \\
&= \frac{f(\xi)}{2}\delta \\
&> 0.
\end{aligned}$$

But then $0 > 0$, which is a contradiction. Therefore there is no $\xi \in [a, b]$ such that $f(\xi) > 0$. Hence $f \leq 0$ which, when combined with the hypothesis that $f \geq 0$, implies that $f(x) = 0$ for all $x \in [a, b]$. □

Problem 4. Define f by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \text{ and} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Prove that f is not Riemann integrable on any interval $[a, b]$, where $a < b$.

Proof. Fix $a < b$ and let $P = \{x_0 = a, x_1, \dots, x_n = b\}$ be any partition of $[a, b]$. As both the rationals the irrationals are dense in $\mathbb{R}$, for each j there are ξ_j and η_j such that

$$\xi_j \in [x_{j-1}, x_j] \cap \mathbb{Q} \qquad \text{and} \qquad \eta_j \in [x_{j-1}, x_j] \setminus \mathbb{Q}.$$

That is, ξ_j and η_j are, respectively, rational and irrational elements of the interval $[x_{j-1}, x_j]$. But then

$$\begin{aligned}
m_j &= \inf\{f(x) \mid x_{j-1} \leq x \leq x_j\} \leq f(\xi_j) = 0, \\
M_j &= \sup\{f(x) \mid x_{j-1} \leq x \leq x_j\} \geq f(\eta_j) = 1.
\end{aligned}$$

Indeed, equality holds throughout, since $0 \leq f(x) \leq 1$ for all x. Hence

$$\begin{aligned}
U(P, f) &= \sum_{j=1}^{n} M_j(x_j - x_{j-1}) \\
&= \sum_{j=1}^{n} (x_j - x_{j-1}) \\
&= (x_1 - a) + (x_2 - x_1) + (x_3 - x_2) + \cdots + (b - x_{n-1}) \\
&= b - a.
\end{aligned}$$

Since P is arbitrary, this implies that

$$\overline{\int_a^b} f(x)\,\mathrm{d}x = \inf U(P,f) = \inf\{b-a\} = b-a.$$

On the other hand,

$$L(P,f) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}) = \sum_{j=1}^{n} 0(x_j - x_{j-1}) = 0.$$

Again, P is arbitrary, and so

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \sup L(P,f) = \sup\{0\} = 0.$$

Therefore

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = 0 \neq b-a = \overline{\int_a^b} f(x)\,\mathrm{d}x.$$

Hence f is not Reimann integrable on any interval $[a,b]$. □