For $t \geq 0$, put

$$\varphi(x,t) = \begin{cases} x & (0 \leq x \leq \sqrt{t}) \\ -x + 2\sqrt{t} & (\sqrt{t} \leq x \leq 2\sqrt{t}) \\ 0 & (\text{otherwise}), \end{cases}$$

and put $\varphi(x,t) = -\varphi(x,|t|)$ if $t < 0$.

Show that φ is continuous on R^2, and

$$(D_2\varphi)(x,0) = 0$$

for all x. Define

$$f(t) = \int_{-1}^{1} \varphi(x,t)\,dx.$$

Show that $f(t) = t$ if $|t| < \frac{1}{4}$. Hence

$$f'(0) \neq \int_{-1}^{1} (D_2\varphi)(x,0)\,dx.$$

slu.

If $t \geq 0$, and $x = \sqrt{t}$, $\sqrt{t} = -\sqrt{t} + 2\sqrt{t}$.

If $t \geq 0$, and $x = 2\sqrt{t}$, $0 = -2\sqrt{t} + 2\sqrt{t}$.

If $t < 0$, and $-x = -\sqrt{|t|}$, $-\sqrt{|t|} = \sqrt{|t|} - 2\sqrt{|t|}$.

If $t < 0$, and $x = 2\sqrt{|t|}$, $0 = -2\sqrt{|t|} + 2\sqrt{|t|}$.

Let $\varepsilon > 0$, for $|t| < \varepsilon$, $|x| < \varepsilon$, $|\phi(x,t)| < \sqrt{\varepsilon}$, as the maximum value of ϕ for each t is $\pm\sqrt{t}$.

So, f is continuous at the origin because ε was arbitrary, and away from the origin all the pieces agree.

So, f is continuous on $\mathbb{R}^2$.

Since $\lim_{h\to 0} \frac{\sqrt{h}}{h} = \lim_{\to 0} \frac{1}{\sqrt{h}} = 0$, we have both that,

$$\lim_{h\to 0^+} \frac{\begin{cases} x & 0 \leq x \leq \sqrt{h} \\ -x + 2\sqrt{h} & \sqrt{h} \leq x \leq 2\sqrt{h} \\ 0 & \text{otherwise} \end{cases} - \begin{cases} x & 0 \leq x \leq \sqrt{0} \\ -x + 2\sqrt{0} & \sqrt{0} \leq x \leq 2\sqrt{0} \\ 0 & \text{otherwise} \end{cases}}{h} = \lim_{h\to 0^+} \begin{cases} \frac{x}{h} & 0 \leq x \leq \sqrt{h} \\ \frac{-x+2\sqrt{h}}{h} & \sqrt{h} \leq x \leq 2\sqrt{h} \\ \frac{0}{h} & \text{otherwise} \end{cases} = 0,$$

and

$$\lim_{h\to 0^-} \frac{\begin{cases} -x & 0 \leq x \leq \sqrt{|h|} \\ x - 2\sqrt{|h|} & \sqrt{|h|} \leq x \leq 2\sqrt{|h|} \\ 0 & \text{otherwise} \end{cases}}{h} = \lim_{h\to 0^-} \begin{cases} -\frac{x}{h} & 0 \leq x \leq \sqrt{|h|} \\ \frac{x-2\sqrt{|h|}}{h} & \sqrt{|h|} \leq x \leq 2\sqrt{|h|} \\ \frac{0}{h} & \text{otherwise} \end{cases} = 0.$$

So $\forall x \in \mathbb{R}, (D_2 f)(x,0) = 0$.

By, the definition of $f(t)$ in the problem we have, for $1/4 > t \geq 0$

$$
\begin{aligned}
f(t) = \int_{-1}^{1} \phi(x,t)\,\mathrm{d}x = \int_{0}^{1} \phi(x,t)\,\mathrm{d}x \\
&= \int_{0}^{\sqrt{t}} x\,\mathrm{d}x + \int_{\sqrt{t}}^{2\sqrt{t}} -x + 2\sqrt{t}\,\mathrm{d}x \\
&= \frac{t}{2} + \left(-\frac{x^2}{2} + 2\sqrt{t}x\right)\Big|_{\sqrt{t}}^{2\sqrt{t}} \\
&= \frac{t}{2} - 2t + 4t + \frac{t}{2} - 2t = t.
\end{aligned}
$$

for, $-1/4 < t \leq 0$ we have

$$
\begin{aligned}
f(t) = \int_{-1}^{1} -\phi(x,|t|)\,\mathrm{d}x = \int_{0}^{1} -\phi(x,|t|)\,\mathrm{d}x \\
&= \int_{0}^{\sqrt{|t|}} -x\,\mathrm{d}x + \int_{\sqrt{|t|}}^{2\sqrt{|t|}} x - 2\sqrt{|t|}\,\mathrm{d}x \\
&= -\frac{|t|}{2} + \left(\frac{x^2}{2} - 2\sqrt{|t|}x\right)\Big|_{\sqrt{|t|}}^{2\sqrt{|t|}} \\
&= -\frac{|t|}{2} - 2|t| + 4|t| - \frac{|t|}{2} - 2|t| = -|t| = -(-t) = t.
\end{aligned}
$$

So,

$$
f'(t) = 1 \implies f'(0) = 1 \neq 0 = \int_{-1}^{1} 0\,\mathrm{d}x = \int_{-1}^{1} (D_2\phi)(x,0)\,\mathrm{d}x \quad \Diamond
$$