Prove that if a norm $||\,.\,||$ on a $\mathbb{C}-$vector space V satisfies the following parallelogram identity:

$$\|u+v\|^2+\|u-v\|^2=2\|u\|^2+2\|v\|^2 \text{ for all } u,v\in V, \tag{1}$$

Then $\|\,.\,\|$ is induced by a Hermitian inner product.

pf.

The following equations will be useful, for all $u, v \in V$

$$\|u+v\|^2-\|u-v\|^2=2\left(\|u\|^2+\|v\|^2-\|u-v\|^2\right) \tag{2}$$

$$-i\|u+iv\|^2+i\|u-iv\|^2=-2i\left(\|u\|^2+\|v\|^2-\|u-iv\|^2\right) \tag{3}$$

Let $\langle .,. \rangle : V^2 \to \mathbb{C}$ defined by,

$$\langle x,\ y\rangle := \frac{1}{4}\left(\|x+y\|^2-\|x-y\|^2-i\|x-iy\|^2+i\|x+iy\|^2\right)\ \forall\ x,y\in V \tag{4}$$

With (2) and (3) we can rewrite (4)

$$\langle x,\ y\rangle = \frac{1}{2}\left(\|x\|^2+\|y\|^2-\|x-y\|^2\right)-\frac{1}{2}i\left(\|x\|^2+\|y\|^2-\|x-iy\|^2\right) \tag{5}$$

Henceforth $u, v, w \in V$,

$$\begin{aligned}
\overline{\langle v,u\rangle} &= \overline{\frac{1}{4}\left(\|v+u\|^2-\|v-u\|^2-i\|v-iu\|^2+i\|v+iu\|^2\right)}\\
&= \frac{1}{4}\left(\|v+u\|^2-\|v-u\|^2+i\|v-iu\|^2-i\|v+iu\|^2\right)\\
&= \frac{1}{4}\left(\|u+v\|^2-\|u-v\|^2+i\|-i(iv+u)\|^2-i\|i(-iv+u)\|^2\right)\\
&= \frac{1}{4}\left(\|u+v\|^2-\|u-v\|^2+i(|-i|\|u+iv\|)^2-i(|i|\|u-iv\|)^2\right)\\
&= \frac{1}{4}\left(\|u+v\|^2-\|u-v\|^2+i\|u+iv\|^2-i\|u-iv\|^2\right)\\
&= \frac{1}{4}\left(\|u+v\|^2-\|u-v\|^2-i\|u-iv\|^2+i\|u+iv\|^2\right)\\
&= \langle u,v\rangle
\end{aligned}$$

So, $\langle .,. \rangle$ has conjugate symmetry.

$$\begin{aligned}
\langle u,u\rangle &= \frac{1}{2}\left(\|u\|^2+\|u\|^2-\|u-u\|^2\right)-\frac{1}{2}i\left(\|u\|^2+\|u\|^2-\|u-iu\|^2\right)\\
&= \|u\|^2-\frac{1}{2}i\left(2\|u\|^2-\|(1-i)u\|^2\right)\\
&= \|u\|^2-\frac{1}{2}i\left(2\|u\|^2-|1-i|^2\|u\|^2\right)\\
&= \|u\|^2-\frac{1}{2}i\left(2\|u\|^2-2\|u\|^2\right)\\
&= \|u\|^2>0
\end{aligned}$$

So, $\langle .,. \rangle$ is positive definite.

The following equations will also be useful, for all $x, y \in V$

$$\|x+y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right) - \|x-y\|^2 \tag{6}$$

$$\|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right) - \|x+y\|^2 \tag{7}$$

$$\|x\|^2 + \|y\|^2 = \frac{1}{2}\left(\|x+y\|^2 + \|x-y\|^2\right) \tag{8}$$

By repeated applications of (6), (7), and (8) we get,

$$\begin{aligned}
&\frac{1}{4}\left(\|u+v+w\|^2 - \|u+v-w\|^2\right)\\
&= \frac{1}{4}\left(\|u+v+w\|^2 + \|w\|^2 - \|u+v\|^2 - (\|u+v-w\|^2 + \|w\|^2 - \|u+v\|^2)\right)\\
&= \frac{1}{4}\left(\|u+v+w\|^2 + \|w\|^2 - \|u+v+w-w\|^2 - (\|u+v-w\|^2 + \|-w\|^2 - \|u+v+w-w\|^2)\right)\\
&= \frac{1}{4}\left(\frac{1}{2}\left(2(\|u+v+w\|^2 + \|w\|^2 - \|u+v+w-w\|^2\right) - (\frac{1}{2}\left(2(\|u+v-w\|^2 + \|-w\|^2 - \|u+v+w-w\|^2\right)))\right)\\
&= \frac{1}{4}\left(\frac{1}{2}\left(\|u+v+w+w\|^2\right) - (\frac{1}{2}\left(\|u+v-w-w\|^2\right))\right)\\
&= \frac{1}{4}\left(\frac{1}{2}\left(\|u+v+w+w\|^2\right) - (\frac{1}{2}\left(\|u+v-w-w\|^2\right))\right)\\
&= \frac{1}{4}\left(\frac{1}{2}\left(\|u+v+w+w\|^2 + \|u-v\|^2\right) - (\frac{1}{2}\left(\|u+v-w-w\|^2 + \|u-v\|^2\right))\right)\\
&= \frac{1}{4}\left(\frac{1}{2}\left(\|u+w+v+w\|^2 + \|u+w-(v+w)\|^2\right) - (\frac{1}{2}\left(\|u-w+v-w\|^2 + \|u-w-(v-w)\|^2\right))\right)\\
&= \frac{1}{4}\left(\|u+w\|^2 + \|v+w\|^2 - (\|u-w\|^2 + \|v-w\|^2)\right)\\
&= \frac{1}{4}\left(\|u+w\|^2 - \|u-w\|^2 + \|v+w\|^2 - \|v-w\|^2\right)\\
&= \frac{1}{4}\left(2(\|u\|^2 + \|w\|^2 - \|u-w\|^2) + 2(\|v\|^2 + \|w\|^2 - \|v-w\|^2)\right)\\
&= \frac{1}{2}\left(\|u\|^2 + \|w\|^2 - \|u-w\|^2 + \|v\|^2 + \|w\|^2 - \|v-w\|^2\right)\\
&= \frac{1}{2}\left(\|u\|^2 + \|w\|^2 - \|u-w\|^2\right) + \frac{1}{2}\left(\|v\|^2 + \|w\|^2 - \|v-w\|^2\right)
\end{aligned}$$

$$\begin{aligned}
\langle u+v, w\rangle &= \frac{1}{4}\left(\|u+v+w\|^2 - \|u+v-w\|^2\right) + \frac{i}{4}\left(\|u+v+iw\|^2 - \|u+v-iw\|^2\right)\\
&= \frac{1}{2}\left(\|u\|^2 + \|w\|^2 - \|u-w\|^2\right) + \frac{1}{2}\left(\|v\|^2 + \|w\|^2 - \|v-w\|^2\right)\\
&- \frac{i}{2}\left(\|u\|^2 + \|w\|^2 - \|u-iw\|^2\right) - \frac{i}{2}\left(\|v\|^2 + \|w\|^2 - \|v-iw\|^2\right)\\
&= \langle u, w\rangle + \langle v, w\rangle
\end{aligned}$$

$$\begin{aligned}
\langle u+w,v\rangle = \langle u,v\rangle + \langle w,v\rangle &\implies \langle 2u,v\rangle = 2\langle u,v\rangle \\
\text{do induction} &\implies \langle nu,v\rangle = n\langle u,v\rangle \forall n\in\mathbb{N} \\
&\implies \langle \frac{p}{q}u,v\rangle = p\langle \frac{1}{q}u,v\rangle \forall p\in\mathbb{Z}, q\in\mathbb{N} \\
&\implies q\langle \frac{p}{q}u,v\rangle = pq\langle \frac{1}{q}u,v\rangle = p\langle \frac{q}{q}u,v\rangle = p\langle u,v\rangle \\
&\implies \langle \frac{p}{q}u,v\rangle = \frac{p}{q}\langle u,v\rangle \\
&\implies \langle ru,v\rangle = r\langle u,v\rangle \forall r\in\mathbb{Q}
\end{aligned}$$

Since, V is a $\mathbb{C}-$vector space and $\langle .,.\rangle \in \mathbb{C}$, it follows that $\{\langle r_i u, v\rangle\}_{i=1}^{\infty}, u, v \in V$, converges if it's a Cauchy sequence.

Since every $\alpha \in \mathbb{R}$ is the limit of a Cauchy sequence $\{r_i\}_{i=1}^{\infty}$.

Let $\varepsilon > 0 : N \in \mathbb{N} : i,j > N \implies |r_i - r_j| < \frac{\varepsilon}{|\langle u,v\rangle|}$.

$$\implies |\langle r_i u,v\rangle - \langle r_j u,v\rangle| = |r_i\langle u,v\rangle - r_j\langle u,v\rangle| = |r_i - r_j||\langle u,v\rangle| < \frac{\varepsilon}{|\langle u,v\rangle|}|\langle u,v\rangle| = \varepsilon$$

Thus

$$\forall \alpha \in \mathbb{R} \quad \langle \alpha u,v\rangle = \alpha\langle u,v\rangle \tag{9}$$

$$\begin{aligned}
\langle iu,v\rangle &= \frac{1}{4}\left(\|iu+v\|^2 - \|iu-v\|^2 - i\|iu-iv\|^2 + i\|iu+iv\|^2\right) \\
&= \frac{1}{4}\left(\|i(u-iv)\|^2 - \|i(u+iv)\|^2 - i\|u-v\|^2 + i\|u+v\|^2\right) \\
&= \frac{1}{4}\left((|i|\|u-iv\|)^2 - (|i|\|u+iv\|)^2 - i\|u-v\|^2 + i\|u+v\|^2\right) \\
&= \frac{i}{4}\left(\|u+v\|^2 - \|u-v\|^2 - i\|u-iv\|^2 + i\|u+iv\|^2\right) \\
&= i\langle u,v\rangle
\end{aligned}$$

Let $z \in \mathbb{C} : \exists \alpha, \beta \in \mathbb{R} : z = \alpha + i\beta$,

$$\begin{aligned}
\langle zu,v\rangle &= \langle (\alpha+i\beta)u,v\rangle \\
&= \alpha\langle u,v\rangle + \beta\langle iu,v\rangle \\
&= \alpha\langle u,v\rangle + i\beta\langle u,v\rangle \\
&= (\alpha+i\beta)\langle u,v\rangle \\
&= z\langle u,v\rangle
\end{aligned}$$

Thus, $\langle .\,, .\rangle$ is linear in its first entry. So it is a Hermitian inner product $\blacksquare$