□

1.6 Fourier series

Roughly speaking, a Fourier series is an infinite sum of the form

$$a_0 + \sum_{n=0}^{\infty}(a_n \cos nx + b_n \sin nx)$$

which can be viewed as the limit of partial sums of the form

$$a_0 + \sum_{n=0}^{N}(a_n \cos nx + b_n \sin nx). \tag{6}$$

(6) is called a trigonometric polynomial. Since $e^{ix} = \cos x + i \sin x$, (6) can also be written in the form

$$\sum_{-N}^{N} c_n e^{inx}$$

and a Fourier sequence, or trigonometric series, can be written in the form

$$\sum_{-\infty}^{\infty} c_n e^{inx}.$$

We notice that the system $\{e^{inx}\}_{-\infty}^{\infty}$ is orthogonal:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imx} \cdot \overline{e^{imx}} dx = \begin{cases} 1 & (\text{if } m = n) \\ 0 & (\text{otherwise}) \end{cases} \tag{7}$$

(7) suggests that we use our intuition from complex linear space with a Hermitian inner product as follows. Our complex linear space is going to be all (Riemann) integrable functions defined on $[-\pi, \pi]$, and the Hermitian inner product $\langle f, g\rangle$ of two functions f and g is defined as

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)} dx \tag{8}$$

and the norm with respect to this inner product is

$$\|f\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx\right)^{1/2}. \tag{9}$$

Notice that, unlike the linear space we work with in a standard linear algebra course, the linear space above is infinite dimensional, and the orthonormal system $\{e^{inx}\}_{-\infty}^{\infty}$ is infinite.

We will be interested in expressing functions by Fourier series. According to our intuition from linear algebra, we define

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{e^{inx}}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \tag{10}$$

for every integrable function f and every $n \in \mathbb{Z}$ and call them Fourier coefficients. The Fourier sequence corresponding to f is defined as

$$\sum_{-\infty}^{\infty} c_n e^{inx} \tag{11}$$

where c_n are defined by (10). We denote the relation between the function f and its Fourier series as

$$f \sim \sum_{-\infty}^{\infty} c_n e^{inx}.$$

From intuition of linear algebra, we propose the following theorems.

Theorem (8.11)**.** *Among different choices of the coefficients, the choices given by* (10) *provides the best approximation of f (in terms of square integral).*

Theorem (8.12)**.** *The sum of the squares of $|c_n|$ does not exceed the square integral of $|f|$.*

Idea of the proof: Compare our case with the case of a finite dimensional linear space. In case of a finite dimensional linear space with a Hermitian inner product, any orthonormal system spaces a linear subspace S. Given any vector v, the "Fourier series" of v is the orthonormal projection of v into S, which is the best approximation of v among vectors in S. This is Theorem 8.11. Theorem 8.12 also follows as the projection is always "shorter" than v. □

Remark 1.5. The above two theorems can be generalized to general orthonormal systems, as done by the textbook. However, we will not need this generalization.

1.7 Approximation by Fourier sequences

Now consider a periodic function $f(x)$ integrable on $[-\pi, \pi]$ with period 2π (which is the same as extending a function defined on $[-\pi, \pi]$ by periodicity but the current setting is more convenient for our purpose) and its Fourier series defined by (10) and (11). Let us first consider under what condition does the this series converges to f pointwise. It is natural to look at partial sum

$$s_N(f, x) = \sum_{-N}^{N} c_n e^{inx}.$$

Using (10), we can rewrite $s_N(f, x)$ as

$$s_N(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\sum_{-N}^{N} e^{in(x-t)}\right)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\left(\sum_{-N}^{N} e^{int}\right)dt.$$

Note that $\sum_{-N}^{N} e^{int}$ is the sum a geometric progression, and thus can be computed as follows:

$$\sum_{-N}^{N} e^{int} = \frac{\sin[(N+1/2)t]}{\sin(t/2)}.$$

Therefore,

$$s_N(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\frac{\sin[(N+1/2)t]}{\sin(t/2)}dt$$

and

$$\begin{aligned} s_N(f,x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\sin[(N+1/2)t]dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}[g(t)\cos(t/2)]\sin(Nt)dt + \frac{1}{2\pi}\int_{-\pi}^{\pi}[g(t)\sin(t/2)]\cos(Nt)dt, \end{aligned}$$

where

$$g(t) = \frac{f(x-t)-f(x)}{\sin(t/2)}.$$

Since $\sin(Nt) = (e^{iNt} - e^{-iNt})/(2i)$ and $\cos(Nt) = (e^{iNt} + e^{-iNt})/2$, we see that $s_N(f,x) - f(x)$ is the sum of the Nth and $-N$th Fourier coefficients of $g(t)$. If $g(t)$ is integrable, then these coefficients tend to 0 as $N \to \infty$. We thus obtain Theorem 8.14 in the textbook.

Note that for the Fourier series to approximate f pointwise, it is not enough that f is continuous. However, if we consider general trigonometric polynomials, we can approximate f uniformly provided f is continuous. This is Theorem 8.15 and its reason is follows. Instead of thinking f as a periodic function on $\mathbb{R}$, we can think of f as a function on the unit circle T. Trigonometric polynomials form a self-adjoint algebra $\mathscr{A}$, which by the Stone-Weierstrass theorem is dense in the algebra oc continuous function $\mathscr{T}$ since $\mathscr{A}$ separates points.

Interestingly, Theorems 8.15 provides a approximation of f by its Fourier series under the norm $\|\cdot\|_2$. We first note that since f is integrable on $[-\pi,\pi]$, we can approximate f in terms of integral by a periodic continuous function h. Indeed, f being integrable on $[-\pi,\pi]$ means that if we partition $\mathbb{R}$ sufficiently fine, the supremum and infimum of f on each small interval are very closed to each other, and we can define h to be the function which assigns the average of the supremum and infimum to the midpoint of the small interval and then extend the definition of h by connecting those points by line segments.

We have seen that the integral of $|f-h|^2$ on $[-\pi,\pi]$ is very small. Theorem 8.15 provides us with a trigonometric polynomial P such that P approximate h uniformly and in particular, the integral of $|h-P|^2$ on $[-\pi,\pi]$ is very small. Therefore, $\|f-h\|_2$ and $\|h-P\|_2$ are both small. Since $\|\cdot\|_2$ satisfies triangle inequality (which is true for every norm given by an inner product), $\|f-P\|_2$ will be very small. But $\|f-s_N(f,x)\|_2$ is smaller than $\|f-P\|_2$ by Theorem 8.11. Thus, $\|f-s_N(f,x)\|_2$ is also very small, which means $s_N(f,x)$ approximates f under the norm $\|\cdot\|_2$. This is the first part of Theorem 8.16.

We next observe that triangle inequality implies

$$\|f\|_2 \leqslant \|f - s_N(f,x)\|_2 + \|s_N(f,x)\|_2.$$

By orthonormality, we have

$$\|s_N(f,x)\|_2 = (\sum_{-N}^{N} |c_n|^2)^{1/2}.$$

Since $\|f - s_N(f,x)\|_2 \to 0$ as $n \to \infty$, we have

$$\|f\|_2 \leqslant (\sum_{-\infty}^{\infty} |c_n|^2)^{1/2}.$$

The converse inequality is the content of Theorem 8.12. Therefore,

$$\|f\|_2 = (\sum_{-\infty}^{\infty} |c_n|^2)^{1/2},$$

which is another assertion of Theorem 8.16.

The remaining assertion of Theorem 8.16 can be seen as follows. First, we notice that if we write the Fourier coefficients of f as a bi-infinite sequence $(c_n)_{n\in\mathbb{Z}}$, then this sequence is square sum-able, which suggests we look at the linear space V of all square sum-able bi-infinite sequences. V comes with a natural Hermitian inner product

$$\langle (a_n)_{n\in\mathbb{Z}}, (b_n)_{n\in\mathbb{Z}} \rangle = \sum_{-\infty}^{\infty} a_n \overline{b_n},$$

which induces a norm

$$\|(a_n)_{n\in\mathbb{Z}}\|_2 = \left(\sum_{-\infty}^{\infty} |a_n|^2 \right)^{1/2}.$$

Denote the inner product space of all 2π-periodic $[-\pi,\pi]-$integrable functions as W (with inner product and norm given by (8) and (9), respectively) and let F be the map with sends a function in W to the sequence of its Fourier coefficients. Then F is a linear map from W to V and the previous proved part of Theorem 8.16 says that F preserves norm. It is a general fact that the inner product of an inner product space can be obtained from its norm and thus F also preserves the inner product, which is the last part of Theorem 8.16.

The point of Theorem 8.16 is that, as long as $\|\cdot\|_2$ is being considered, instead of working with periodic integrable functions, we can look as there Fourier coefficients and consider square sum-able sequences, which is discrete and easier to deal with in some context.

1.8 Gamma function

The Gamma function stems from the attempt to extend the definition of the factorial function. As a function defined only on natural numbers, the factorial function rejects techniques from calculus, which require the function being studied to be differentiable. The point of extending the factorial function is thus to make it vulnerable to calculus, and the definition should have as much smoothness as possible. The successful candidate was due to Euler and is called the Gamma function, which is defined as the following integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \text{ for } x > 0.$$

We first notice that

$$\begin{aligned}\Gamma(x+1) &= \int_0^\infty t^x e^{-t}dt = \int_0^\infty t^x d(-e^{-t}) \\ &= [-t^x e^{-t}]_0^\infty + \int_0^\infty e^{-t}d(t^x) = x\Gamma(x).\end{aligned}$$

Since

$$\Gamma(1) = \int_0^\infty e^{-t}dt = 1,$$

we have

$$\Gamma(n+1) = n!.$$

Thus, Γ indeed extends the factorial function. Γ has nice properties as it is not just differentiable but one can also expend Γ locally as a power series. But we will not prove this statement.

We also notice that if $p, q > 0$ satisfy $1/p + 1/q = 1$, then

$$\begin{aligned}\Gamma(\frac{x}{p} + \frac{y}{q}) &= \int_0^\infty t^{x/p+y/q}e^{-t/p-t/q}dt \\ &= \int_0^\infty \left(t^{x/p}e^{-t/p}\right)\left(t^{y/q}e^{-t/q}\right)dt \\ &\leqslant \left(\int_0^\infty t^{x/p}e^{-t/p}dt\right)^{1/p}\left(\int_0^\infty t^{y/q}e^{-t/q}dt\right)^{1/q} \\ &= \Gamma\left(\frac{x}{p}\right)^{1/p}\Gamma\left(\frac{y}{q}\right)^{1/q},\end{aligned}$$

which means that $\log\Gamma$ is convex. Interestingly, Theorem 8.19 asserts that the above properties characterizes the Gamma function.

The above trick of Hölder's inequality is also used in the proof of Theorem 8.20.

One can estimate the growth of the Gamma function and obtain the Stirling formula:

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{(x/e)^x\sqrt{2\pi x}} = 1$$

Idea of the proof: The idea is to take the limit $x \to \infty$ in the integral

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt$$

and commute the limit operator with the integral. Of course, the limit of the function inside is just infinite and thus we modify this idea by first single out the large factors before taking limit. Consider the expansion

$$\ln(1+u) = u - \frac{1}{2}u^2 + \cdots$$

We rewrite the definition of $\Gamma(x+1)$ as follows.

$$\Gamma(x+1) = \int_0^\infty e^{-t+x\ln t} dt = \int_0^\infty e^{-t+x\ln(1+\frac{t-x}{x})+x\ln x} dt.$$

Morally, we think of

$$\ln(1+\frac{t-x}{x})$$

as

$$\frac{t-x}{x} - \frac{(t-x)^2}{2x} + \cdots$$

and thus

$$\Gamma(x+1) \sim \int_0^\infty e^{x\ln x - x - \frac{(t-x)^2}{2}+\cdots} dt \sim x^x e^{-x} \int_0^\infty e^{-\frac{(t-x)^2}{2}+\cdots} dt$$

which suggests the substitution

$$s^2 = \frac{(t-x)^2}{2x}.$$

□