Let H be a compact convex set in R^k, with nonempty interior. Let $f \in \mathscr{C}(H)$, put $f(\mathbf{x}) = 0$ in the complement of H, and define $\int_H f$ as in Definition 10.3.

Prove that $\int_H f$ is independent of the order in which the k integrations are carried out.

Hint: Approximate f by functions that are continuous on R^k and whose supports are in H, as was done in Example 10.4.

pf.

By, Proposition 5.1, in *Introduction to Topological Manifolds*, by J.M. Lee, GTM 202, pp. 128-129. $H \subset \mathbb{R}^k$ is a compact convex set with non-empty interior it is homeomorphic to $\overline{B}^k$ the closed unit ball centered at the origin, with the homeomorphism ϕ mapping $\partial H \to S^{k-1} = \partial B^k$. Let $I^k = \{\mathrm{x} \in \mathbb{R}^k |,\ -1 \leq x_i \leq 1,\ 1 \leq i \leq k\}$, then since I^k is also a compact convex set with non-empty interior it is homeomorphic to $\overline{B}^k$, with the homeomorphism ψ mapping $\partial I^k \to S^{k-1} = \partial B^k$. Then $\psi^{-1} \circ \phi$ is a homeomorphism between H and I^k, that is a contiuous bijection from ∂H to ∂I^k.

Since $\forall \varepsilon > 0 : S^{k-1} \subset D^{k-1}_{1+\varepsilon}(0) \backslash D^{k-1}_{1-\varepsilon}(0) \implies 0 \leq \mu(S^{k-1}) \leq \mu(D^{k-1}_{1+\varepsilon}(0) \backslash D^{k-1}_{1-\varepsilon}(0)) = 0.$

So, $\mu(S^{k-1}) = 0$, see *Fractals and Self-similarity*, Hutchinson, 1981, pp. 719-720.

Therefore, by the continuity of ψ^{-1}, $\mu(\partial I^k) = 0$.

Then for a function f that's continuous on H, we can define

$$\int_H f = \int_{I^k} (\phi^{-1} \circ \psi \circ f)|J_{\phi^{-1} \circ \psi}|$$

And, we can approximate f by a continuous function F on $\mathbb{R}^k$, so that, for all $\varepsilon > 0$,

$$\left| \int_H F - \int_H f \right| < \varepsilon.$$

Then, we can apply theorem 10.4, as

$$\int_H F = \int_{I^k} (\phi^{-1} \circ \psi \circ F)|J_{\phi^{-1} \circ \psi}|,$$

is independent of the order of integration.

So that $\int_H f$ is independent of the order of integration. ■