(a)

$$(e-(1+x)^{1/x})' = -((1+x)^{1/x})'$$

$$(1+x)^{1/x} = e^{\frac{1}{x}\log(1+x)} \implies \log(1+x)^{1/x} = \frac{1}{x}\log(1+x) \implies \left(\log(1+x)^{1/x}\right)' = \left(\frac{1}{x}\log(1+x)\right)'$$

$$\begin{aligned}\left(\log(1+x)^{1/x}\right)' &= \frac{1}{(1+x)^{1/x}}\left((1+x)^{1/x}\right)' \\ &= -\frac{1}{x^2}\log(1+x) + \frac{1}{x}\frac{1}{(1+x)} \\ &= \left(\frac{1}{x}\log(1+x)\right)'\end{aligned}$$

So,

$$\frac{1}{(1+x)^{1/x}}\left((1+x)^{1/x}\right)' = -\frac{1}{x^2}\log(1+x) + \frac{1}{x}\frac{1}{(1+x)} \implies \left((1+x)^{1/x}\right)' = (1+x)^{1/x}\left(-\frac{1}{x^2}\log(1+x) + \frac{1}{x}\frac{1}{(1+x)}\right)$$

$$\lim_{x\to 0}(1+x)^{1/x}\left(-\frac{1}{x^2}\log(1+x) + \frac{1}{x}\frac{1}{(1+x)}\right) = \lim_{x\to 0}(1+x)^{1/x}\lim_{x\to 0}\left(-\frac{1}{x^2}\log(1+x) + \frac{1}{x}\frac{1}{(1+x)}\right)$$

Note,

$$-\frac{1}{x^2}\log(1+x) + \frac{1}{x}\frac{1}{(1+x)} = \frac{x-(1+x)\log(1+x)}{x^2(1+x)} = \frac{x-(1+x)\log(1+x)}{x^2+x^3}$$

Taking derivatives gives,

$$(x-(1+x)\log(1+x))' = 1-\log(1+x)-1 = -\log(1+x)$$

and,

$$(x^2+x^3)' = 2x+3x^2$$

Taking derivatives again,

$$(-\log(1+x))' = -\frac{1}{1+x}$$

and,

$$(2x+3x^2)' = 2+6x\ .$$

Now,

$$\lim_{x\to 0}\frac{-\frac{1}{1+x}}{2+6x} = -\frac{1}{2}$$

Since as $x \longrightarrow 0$,

$$-\log(1+x)\to 0 \text{ and } 2x+3x^2 \longrightarrow 0$$

Also,

$$x-(1+x)\log(1+x) \longrightarrow 0 \text{ and } x^2(1+x) \longrightarrow 0$$

So,

$$\lim_{x\to 0} -\frac{1}{x^2}\log(1+x) + \frac{1}{x}\frac{1}{(1+x)} = -\frac{1}{2}$$

And since

$$\lim_{x\to 0}(1+x)^{1/x} = e$$

We have

$$\lim_{x\to 0} -((1+x)^{1/x})' = -\lim_{x\to 0}((1+x)^{1/x})' = -1\cdot e\cdot(-\frac{1}{2}) = \frac{e}{2}$$

As $x \to 0$,

$$e-(1+x)^{1/x} \to 0$$

So finally by the applications of L'Hospital's rule above,

$$\lim_{x\to 0}\frac{e-(1+x)^{1/x}}{x} = \frac{e}{2}$$

(b)

Note, $\frac{n}{\log n}[n^{1/n}-1] = \frac{[n^{1/n}-1]}{\frac{\log n}{n}}$

$$n^{1/n} = e^{\frac{1}{n}\log n} \implies \log n^{1/n} = \frac{1}{n}\log n \implies (\log n^{1/n})' = \frac{1}{n^{1/n}}\left(n^{1/n}\right)' = \left(\frac{1}{n}\log n\right)'$$

Thus,

$$(n^{1/n}-1)' = (n^{1/n})' = n^{1/n}\left(\frac{\log n}{n}\right)'$$

So,

$$\lim_{n\to\infty}\frac{\left(n^{1/n}-1\right)'}{\left(\frac{\log n}{n}\right)'} = \lim_{n\to\infty}\frac{n^{1/n}\left(\frac{\log n}{n}\right)'}{\left(\frac{\log n}{n}\right)'} = \lim_{n\to\infty} n^{1/n} = 1$$

Now, $n = n^1$, and $1 \neq -1$. So, by formula (45) in p. 181,

$$\lim_{n\to\infty}\frac{\log n}{n} = 0$$

Note, as $n\to\infty$,

$$n^{1/n}-1 \to 0$$

So, by L'Hospital's rule,

$$\lim_{n\to\infty}\frac{n}{\log n}[n^{1/n}-1] = 1$$

(c)

Compute the power series for $\tan x$,

$$\begin{aligned}
(\tan x)^{(0)} &= \tan x \implies a_0 = 0\\
(\tan x)^{(1)} &= \sec^2 x \implies a_1 = 1\\
(\tan x)^{(2)} &= 2\sec^2 x\tan x \implies a_2 = 0\\
(\tan x)^{(3)} &= 6\sec^4 x - 4\sec^2 x \implies a_3 = \frac{2}{3!} = \frac{1}{3}\\
(\tan x)^{(4)} &= 8\sec^2(x)\tan(x)(2\sec^2(x)+\tan^2(x)) \implies a_4 = 0\\
(\tan x)^{(5)} &= 8(2\sec^6(x)+11\sec^4(x)\tan^2(x)+2\sec^2(x)\tan^4(x)) \implies a_5 = \frac{16}{5!} = \frac{2}{15}
\end{aligned}$$

Then,

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \implies \tan x - x = \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$$

Now,

$$\cos x = (1-\frac{1}{2}x^2+\frac{1}{4!}x^4+\cdots) \implies x(1-\cos x) = x(1-1+\frac{1}{2}x^2-\frac{1}{4!}x^4\cdots) = \frac{1}{2}x^3-\frac{1}{4!}x^5+\cdots$$

So,

$$\frac{\tan x - x}{x(1-\cos x)} = \frac{\frac{1}{3}x^3+\frac{2}{15}x^5+\cdots}{\frac{1}{2}x^3-\frac{x^5}{4!}+\cdots} = \frac{1/x^3}{1/x^3}\frac{\frac{1}{3}x^3+\frac{2}{15}x^5+\cdots}{\frac{1}{2}x^3-\frac{x^5}{4!}+\cdots} = \frac{\frac{1}{3}+\frac{2}{15}x^2+\cdots}{\frac{1}{2}-\frac{x^2}{4!}+\cdots}$$

Then,

$$\lim_{x\to 0}\frac{\tan x - x}{x(1-\cos x)} = \lim_{x\to 0}\frac{\frac{1}{3}+\frac{2}{15}x^2+\cdots}{\frac{1}{2}-\frac{x^2}{4!}+\cdots} = \frac{2}{3}$$

(d)

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots \implies x - \sin x = \frac{1}{6}x^3 - \frac{1}{120}x^5 + \cdots$$

So,

$$\frac{x-\sin x}{\tan x - x} = \frac{\frac{1}{6}x^3-\frac{1}{120}x^5+\cdots}{\frac{1}{3}x^3+\frac{2}{15}x^5+\cdots} = \frac{1/x^3}{1/x^3}\frac{\frac{1}{6}x^3-\frac{1}{120}x^5+\cdots}{\frac{1}{3}x^3+\frac{2}{15}x^5+\cdots} = \frac{\frac{1}{6}-\frac{1}{120}x^2+\cdots}{\frac{1}{3}+\frac{2}{15}x^2+\cdots}$$

Then,

$$\lim_{x\to 0}\frac{x-\sin x}{\tan x - x} = \lim_{x\to 0}\frac{\frac{1}{6}-\frac{1}{120}x^2+\cdots}{\frac{1}{3}+\frac{2}{15}x^2+\cdots} = \frac{1}{2}$$