(a) Assume f is differentiable.

$$f(x)f(y) = f(x+y) \implies f(0)f(1) = f(0+1) = f(1) \implies f(0) = 1$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x)f(h) - f(x)}{h} = f(x) \lim_{h\to 0} \frac{f(h) - 1}{h}$$

$$f(0) = 1 \implies \lim_{h\to 0} \frac{f(h) - 1}{h} = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = f'(0)$$

$$f'(0) = \lim_{h\to 0} \frac{f(h) - 1}{h} \implies f'(x) = f'(0)f(x)$$

$g(x) = e^{cx}$, when $f'(x) = c$, satisfies the same properties.Therefore,

$$\left(\frac{f}{g}\right)' = \frac{cfg - cfg}{g^2} = 0 \implies \frac{f}{g} = M \in \mathbb{R} \implies \forall x \in \mathbb{R}, f(x) = Mg(x)$$

$g(0) = 1 = f(0) \implies M = 1$, therefore $f(x) = e^{cx}$.

(b) Assume f is continuous,

Let $x = y = 1$ in $f(x)f(y) = f(x+y) \implies f(2) = f(1)^2$. Assume $f(k) = f(1)^k$ for all $k \in \mathbb{N} : k \leq n$,

$$f(n)f(1) = f(1)^n f(1) \implies f(n+1) = f(1)^{n+1}$$

So, by induction $\forall n \in \mathbb{N}, f(n) = f(1)^n$.

$$f(x)f(y) = f(x+y) \implies f(-x)f(x) = f(-x+x) = f(0) = 1 \implies \forall x \in \mathbb{R}, f(-x) = \frac{1}{f(x)}$$

$$f(-1) = \frac{1}{f(1)} \implies \forall n \in \mathbb{N}, f(-n) = \frac{1}{f(n)} = \frac{1}{f(1)^n}$$

So,

$$\forall n \in \mathbb{Z}, f(n) = f(1)^n$$

From $f(x)f(y) = f(x+y)$ an induction argument and extension to $\mathbb{Z}$ like above shows,

$$\forall p \in \mathbb{Z}, x \in \mathbb{R}, f(px) = f\left(x \sum_{k=1}^{p} 1\right) = f(x)^p \quad (1)$$

So,

$$f(px) = f(x)^p \implies \log f(px) = p \log f(x) \implies \frac{1}{p} \log f(px) = \log f(x) \implies f(x) = e^{\frac{1}{p} \log f(px)} = f(px)^{1/p}$$

Let $x = \frac{1}{p}$ then,

$$f(x) = f(px)^{1/p} \implies f\left(\frac{1}{p}\right) = f\left(p\frac{1}{p}\right)^{1/p} = f(1)^{1/p}$$

So,

$$\forall p \in \mathbb{Z}, f\left(\frac{1}{p}\right) = f(1)^{1/p}$$

Thus by equation (1),

$$\forall r \in \mathbb{Q}, \quad f(r) = f(1)^r$$

Since, $\mathbb{Q}$ is dense in $\mathbb{R}$ and f agrees with $g(x) = e^{cx}$ at all rational numbers, $f = g$ by a previous exercise.