The following simple computation yields a good approximation to Stirling's formula.

For $m = 1, 2, \dots$, define

$$f(x) = (m + 1 - x)\log m + (x - m)\log(m + 1)$$

if $m \leq x \leq m + 1$, and define

$$g(x) = \frac{x}{m} - 1 + \log m$$

if $m - \frac{1}{2} \leq x \leq m + \frac{1}{2}$. Draw the graphs of f and g.Note that $f(x) \leq \log(x) \leq g(x)$ if $x \geq 1$ and that

$$\int_1^n f(x)dx = \log(n!) - \frac{1}{2}\log n > -\frac{1}{8} + \int_1^n g(x)dx.$$

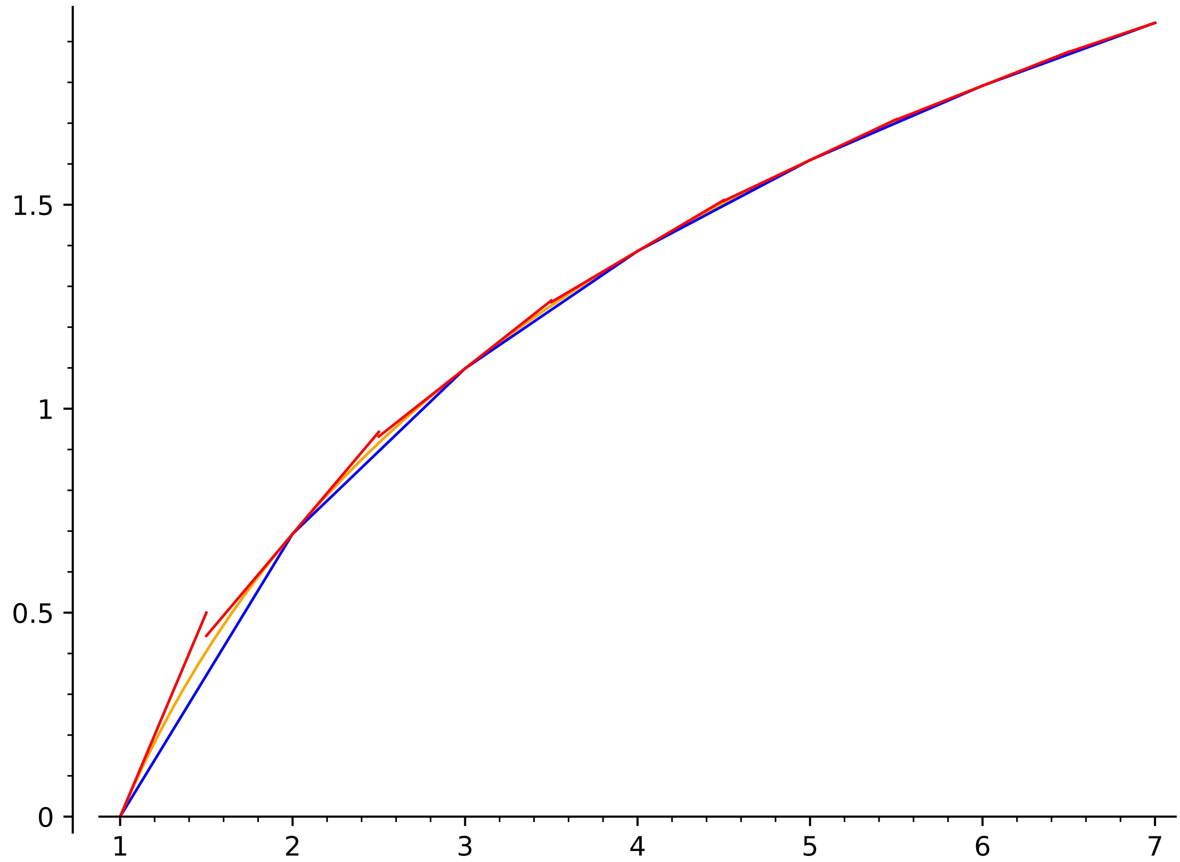
Integrate $\log x$ over $[1, n]$. Conclude that

$$\frac{7}{8} < \log(n!) - (n + \frac{1}{2})\log n + n < 1$$

for $n = 2, 3, 4, \dots$. (*Note* $\log\sqrt{2\pi} \sim 0.918\dots$.) Thus

$$e^{7/8} < \frac{n!}{(n/e)^n\sqrt{n}} < e$$

Note, Rudin is a fucking genius because the agreement is uncanny, if I graph the interval $[0, 100]$ my eyes can't tell the difference. So f < log < g for $x \geq 1$. The graphs correspond to the colors, and the range is $[1, 7]$ for emphasis. After 5 the agreement is so good you can't even see log function.



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\begin{aligned}
\int_1^n f(x)dx &= \sum_{k=1}^{n-1} \int_k^{k+1} f(x)dx \\
&= \sum_{k=1}^{n-1} \int_k^{k+1} (k+1-x)\log k + (x-k)\log(k+1)dx \\
&= \sum_{k=1}^{n-1} \int_k^{k+1} (k+1)\log k - x\log k + x\log(k+1) - k\log(k+1)dx \\
&= \sum_{k=1}^{n-1} [(k+1)\log k - k\log(k+1)][k+1-k] + [\log(k+1) - \log k]\int_k^{k+1} xdx \\
&= \sum_{k=1}^{n-1} [(k+1)\log k - k\log(k+1)][k+1-k] + [\log(k+1) - \log k]\frac{1}{2}[(k+1)^2 - k^2] \\
&= \sum_{k=1}^{n-1} [(k+1)\log k - k\log(k+1)][k+1-k] + [\log(k+1) - \log k]\frac{1}{2}[k^2 + 2k + 1 - k^2] \\
&= \sum_{k=1}^{n-1} [(k+1)\log k - k\log(k+1)][k+1-k] + [\log(k+1) - \log k][k + \frac{1}{2}] \\
&= \sum_{k=1}^{n-1} \log\left(\frac{k^{k+1}}{(k+1)^k}\right) + \log\left(\left(\frac{k+1}{k}\right)^{k+\frac{1}{2}}\right) = \sum_{k=1}^{n-1} \log\left(\frac{k^{k+1}}{(k+1)^k}\frac{(k+1)^{k+\frac{1}{2}}}{k^{k+\frac{1}{2}}}\right) \\
&= \sum_{k=1}^{n-1} \log\left(k^{k+1-k-\frac{1}{2}}(k+1)^{k+\frac{1}{2}-k}\right) = \sum_{k=1}^{n-1} \log\left(k^{\frac{1}{2}}(k+1)^{\frac{1}{2}}\right) = \sum_{k=1}^{n-1} \log\left(k(k+1)\right)^{\frac{1}{2}} \\
&= \sum_{k=1}^{n-1} \frac{1}{2}\log\left(k(k+1)\right) = \frac{1}{2}\sum_{k=1}^{n-1} \log k + \log(k+1) \\
&= \frac{1}{2}\sum_{k=1}^{n-1} \log k + \frac{1}{2}\sum_{k=1}^{n-1} \log(k+1) \\
&= \frac{1}{2}\log(1) + \frac{1}{2}\sum_{k=2}^{n-1} \log k + \frac{1}{2}\sum_{k=1}^{n-2} \log(k+1) + \frac{1}{2}\log n
\end{aligned}
$$

if $l = k+1$ then $l \in [2, n-1] \implies \frac{1}{2}\sum_{k=1}^{n-2} \log(k+1) = \frac{1}{2}\sum_{l=2}^{n-1} \log l$

$$
\begin{aligned}
\text{Thus } \int_1^n f(x)dx &= \frac{1}{2}\cancel{\log(1)} + \sum_{k=2}^{n-1} \log k + \frac{1}{2}\log n \\
&= \prod_{k=2}^{n-1} \log k + \frac{1}{2}\log n + \frac{1}{2}\log n - \frac{1}{2}\log n \\
&= \prod_{k=2}^{n} \log k - \frac{1}{2}\log n \\
&= \log(n!) - \frac{1}{2}\log(n)
\end{aligned}
$$

$$
\begin{aligned}
\int_1^n g(x)dx &= \int_1^{3/2} g(x)dx + \sum_{k=2}^{n-1} \int_{k-1/2}^{k+1/2} g(x)dx + \int_{n-1/2}^{n} g(x)dx \\
&= \int_1^{3/2} x - 1\,dx + \sum_{k=2}^{n-1} \int_{k-1/2}^{k+1/2} \frac{x}{k} - 1 + \log k\,dx + \int_{n-1/2}^{n} \frac{x}{n} - 1 + \log n\,dx \\
&= \frac{x^2}{2}\bigg|_1^{3/2} - \frac{1/2}{+} \sum_{k=2}^{n-1} \int_{k-1/2}^{k+1/2} \frac{x}{k} - 1 + \log k\,dx + \frac{x^2}{2n}\bigg|_n^{n-1/2} + [\log n - 1]\left(n - n + \frac{1}{2}\right) \\
&= \frac{(3/2)^2 - 1}{2} - \frac{1}{2} + \sum_{k=2}^{n-1} \int_{k-1/2}^{k+1/2} \frac{x}{k} - 1 + \log k\,dx + \frac{x^2}{2n}\bigg|_{n-1/2}^{n} + \frac{\log n - 1}{2} \\
&= \frac{1}{8} + \frac{n^2 - (n-1/2)^2}{2n} - \frac{\log n - 1}{2} + \sum_{k=2}^{n-1} \int_{k-1/2}^{k+1/2} \frac{x}{k} - 1 + \log k\,dx \\
&= \frac{1}{8} + \frac{n^2 - n^2 + n - 1/4}{2n} + \frac{\log n - 1}{2} + \sum_{k=2}^{n-1} \int_{k-1/2}^{k+1/2} \frac{x}{k} - 1 + \log k\,dx \\
&= \frac{1}{8} + \frac{n}{2n} - \frac{1}{8n} + \frac{\log n - 1}{2} + \sum_{k=2}^{n-1} \int_{k-1/2}^{k+1/2} \frac{x}{k} - 1 + \log k\,dx \\
&= \frac{1}{8} + \frac{1}{2} - \frac{1}{8n} + \frac{\log n - 1}{2} + \sum_{k=2}^{n-1} \frac{x^2}{2k}\bigg|_{k-1/2}^{k+1/2} + (k + 1/2 - (k - 1/2))(\log k - 1) \\
&= \frac{1}{8} - \frac{1}{8n} + \frac{\log n}{2} + \sum_{k=2}^{n-1} \frac{(k+1/2)^2 - (k-1/2)^2}{2k} + \log k - 1 \\
&= \frac{1}{8} - \frac{1}{8n} + \frac{\log n}{2} + \sum_{k=2}^{n-1} \frac{k^2 + k + 1/4 - (k^2 - k + 1/4)}{2k} + \log k - 1 \\
&= \frac{1}{8} - \frac{1}{8n} + \frac{\log n}{2} + \sum_{k=2}^{n-1} \frac{2k}{2k} + \log k - 1 \\
&= \frac{1}{8} - \frac{1}{8n} + \frac{1}{2}\log n + \sum_{k=2}^{n-1} \log k \\
&= \frac{1}{8} - \frac{1}{8n} - \frac{1}{2}\log n + \sum_{k=2}^{n-1} \log k + \log n \\
&= \frac{1}{8} - \frac{1}{8n} - \frac{1}{2}\log n + \sum_{k=2}^{n} \log k \\
&= \frac{1}{8} - \frac{1}{8n} - \frac{1}{2}\log n + \log(n!)
\end{aligned}
$$

Now,

$$\int_1^n \log x \, dx = x(\log x - 1)\Big|_1^n = n(\log n - 1) - 1(\log 1 - 1) = n \log n - n + 1$$

Since $f < \log < g$ for $x \geq 1$. It follows that $\int_1^n f \, dx \leq \int_1^n \log \, dx \leq \int_1^n g \, dx$

$$\begin{aligned}
&\log(n!) - \frac{1}{2}\log(n) \leq n \log n - n + 1 \leq \frac{1}{8} - \frac{1}{8n} - \frac{1}{2}\log n + \log(n!) \\
\implies & -\frac{1}{2}\log(n) \leq -\log(n!) + n \log n - n + 1 \leq \frac{1}{8} - \frac{1}{8n} - \frac{1}{2}\log n \\
\implies & 0 \leq -\log(n!) + \left(n + \frac{1}{2}\right)\log n - n + 1 \leq \frac{1}{8} - \frac{1}{8n} \\
\implies & -1 \leq -\log(n!) + \left(n + \frac{1}{2}\right)\log n - n \leq -\frac{7}{8} - \frac{1}{8n} \\
\implies & 1 \geq \log(n!) - \left(n + \frac{1}{2}\right)\log n + n \geq \frac{7}{8} + \frac{1}{8n} \\
\implies & 1 \geq \log(n!) - \left(n + \frac{1}{2}\right)\log n + n \geq \frac{7}{8} + \frac{1}{8n} > \frac{7}{8} \\
\implies & e \geq n! n^{-(n+\frac{1}{2})} e^n > e^{\frac{7}{8}} \\
\implies & e^{7/8} < \frac{n!}{(n/e)^n \sqrt{n}} < e
\end{aligned}$$