Suppose that f is a differentiable mapping of a connected open set $E \subset R^n$ into R^m, and if $f'(\mathrm{x}) = 0$ for every $\mathrm{x} \in E$, prove that f is constant in E.

pf.

By Chapter 2, Exercises 23, and 23, $\mathbb{R}^n$ has a countable base by open balls. So every open set can be covered by countably many open balls. Since E is open it follows that for any sequence $\{\mathrm{x}_i\}_{i=1}^{\infty}$ of elements of E, there exists a sequence $\{\delta_i\}_{i=1}^{\infty}$ of positive real numbers such that,

$$\begin{aligned}
E = \bigcup_{i=1}^{\infty} B_{\delta_i}(\mathrm{x}_i) &\implies \forall \mathrm{x} \in E, \exists i \in \mathbb{N} : \mathrm{x} \in B_{\delta_i}(\mathrm{x}_i) \\
&\implies \forall \mathrm{y} \in B_{\delta_i}(\mathrm{x}_i),\ \mathrm{y} \in E \\
&\implies \forall \mathrm{y} \in B_{\delta_i}(\mathrm{x}_i),\ f'(\mathrm{y}) = 0 \\
&\implies \forall \mathrm{y} \in B_{\delta_i}(\mathrm{x}_i),\ \|f'(\mathrm{y})\| \le 0 \\
\forall i \in \mathbb{N},\ B_{\delta_i}(\mathrm{x}_i) \text{ is a convex open set } &\overset{9.19}{\implies} \forall \mathrm{a}, \mathrm{b} \in B_{\delta_i}(\mathrm{x}_i),\ |f(\mathrm{b}) - f(\mathrm{a})| \le 0|\mathrm{b} - \mathrm{a}| = 0 \\
&\implies \forall \mathrm{a}, \mathrm{b} \in B_{\delta_i}(\mathrm{x}_i),\ f(\mathrm{b}) = f(\mathrm{a}) \qquad (0)
\end{aligned}$$

E is connected, which by definition 2.45 means that E is not a union of two non empty separated set. A pair of separated set A, and B, in $\mathbb{R}^n$ satisfies both $A \cap \overline{B} = \emptyset$, and $\overline{A} \cap B = \emptyset$. So, since E is already contained in a union and E is connected. It means it is not the union of a pair of separated sets.It follows that, since $\mathrm{x}_1 \in E$, both

$$B_{\delta_1}(\mathrm{x}_1) \bigcap \overline{\bigcup_{i=2}^{\infty} B_{\delta_i}(\mathrm{x}_i)} \neq \emptyset \tag{1}$$

and

$$\overline{B_{\delta_1}(\mathrm{x}_1)} \bigcap \bigcup_{i=2}^{\infty} B_{\delta_i}(\mathrm{x}_i) \neq \emptyset \tag{2}$$

hold.Then by (2),

$$\overline{B_{\delta_1}(\mathrm{x}_1)} \bigcap \bigcup_{i=2}^{\infty} B_{\delta_i}(\mathrm{x}_i) = \bigcup_{i=2}^{\infty} \overline{B_{\delta_1}(\mathrm{x}_1)} \bigcap B_{\delta_i}(\mathrm{x}_i) \neq \emptyset \implies \exists j > 1 : \overline{B_{\delta_1}(\mathrm{x}_1)} \bigcap B_{\delta_j}(\mathrm{x}_j) \neq \emptyset$$

Then $\exists \mathrm{y}_0 \in \overline{B_{\delta_1}(\mathrm{x}_1)} \bigcap B_{\delta_j}(\mathrm{x}_j) \implies \mathrm{y}_0 \in \overline{B_{\delta_1}(\mathrm{x}_1)}$ and $\mathrm{y}_0 \in B_{\delta_j}(\mathrm{x}_j)$

$$\begin{aligned}
j > 1 &\implies B_{\delta_j}(\mathrm{x}_j) \subset \bigcup_{i=2}^{\infty} B_{\delta_i}(\mathrm{x}_i) \subset \overline{\bigcup_{i=2}^{\infty} B_{\delta_i}(\mathrm{x}_i)} \\
&\implies \mathrm{y}_0 \in \overline{\bigcup_{i=2}^{\infty} B_{\delta_i}(\mathrm{x}_i)} \\
&\overset{(1)}{\implies} \mathrm{y}_0 \in B_{\delta_1}(\mathrm{x}_1) \implies \mathrm{y}_0 \in B_{\delta_1}(\mathrm{x}_1) \cap B_{\delta_j}(\mathrm{x}_j) \\
&\overset{(0)}{\implies} \forall \mathrm{a} \in B_{\delta_1}(\mathrm{x}_1), \text{ and } \forall \mathrm{b} \in B_{\delta_j}(\mathrm{x}_j), \quad f(\mathrm{a}) = f(\mathrm{y}_0) = f(\mathrm{b})
\end{aligned}$$

Since $B_{\delta_1}(\mathrm{x}_1)$ and $B_{\delta_j}(\mathrm{x}_j)$ are open balls. So, their union and their intersection is open. Furthermore $B_{\delta_1}(\mathrm{x}_1) \cap B_{\delta_j}(\mathrm{x}_j) \neq \emptyset \implies \overline{B_{\delta_1}(\mathrm{x}_1)} \cap B_{\delta_j}(\mathrm{x}_j) \neq \emptyset$ and $B_{\delta_1}(\mathrm{x}_1) \cap \overline{B_{\delta_j}(\mathrm{x}_j)} \neq \emptyset$. So, $B_{\delta_1}(\mathrm{x}_1)$, and $B_{\delta_1}(\mathrm{x}_1)$ are not separated, therefore $B_{\delta_1}(\mathrm{x}_1) \cup B_{\delta_1}(\mathrm{x}_1)$ is a connected open set. We can re-index as follows $\{\mathrm{x}_k\}_{k=1}^{\infty}$, and $\{\delta_k\}_{k=1}^{\infty}$, such that the j gets swapped with 2. And, then we follow the process above to find y_2 noticing that with the re-indexing, both

$$\overline{B_{\delta_1}(\mathrm{x}_1) \cup B_{\delta_2}(\mathrm{x}_1)} \bigcup_{k=3}^{\infty} B_{\delta_k}(\mathrm{x}_k) \neq \emptyset \quad \text{and } B_{\delta_1}(\mathrm{x}_1) \cup B_{\delta_2}(\mathrm{x}_1) \overline{\bigcup_{k=3}^{\infty} B_{\delta_k}(\mathrm{x}_k)} \neq \emptyset$$

Then it follows by induction that can find a sequence $\{y_n\}_{n=1}^{\infty}$, and a re-indexing, such that,

$$\forall \mathrm{a}, \mathrm{b} \in E \quad f(\mathrm{a}) = f(\mathrm{b})$$

Therefore, f is constant in E ■