The following simple computation yields a good approximation to Stirling's formula.

For $m = 1, 2, \dots$, define

$$f(x) = (m + 1 - x)\log m + (x - m)\log(m + 1)$$

if $m \le x \le m + 1$, and define

$$g(x) = \frac{x}{m} - 1 + \log m$$

if $m - \frac{1}{2} \le x \le m + \frac{1}{2}$. Draw the graphs of f and g.Note that $f(x) \le \log(x) \le g(x)$ if $x \ge 1$ and that

$$\int_1^n f(x)dx = \log(n!) - \frac{1}{2}\log n > -\frac{1}{8} + \int_1^n g(x)dx.$$

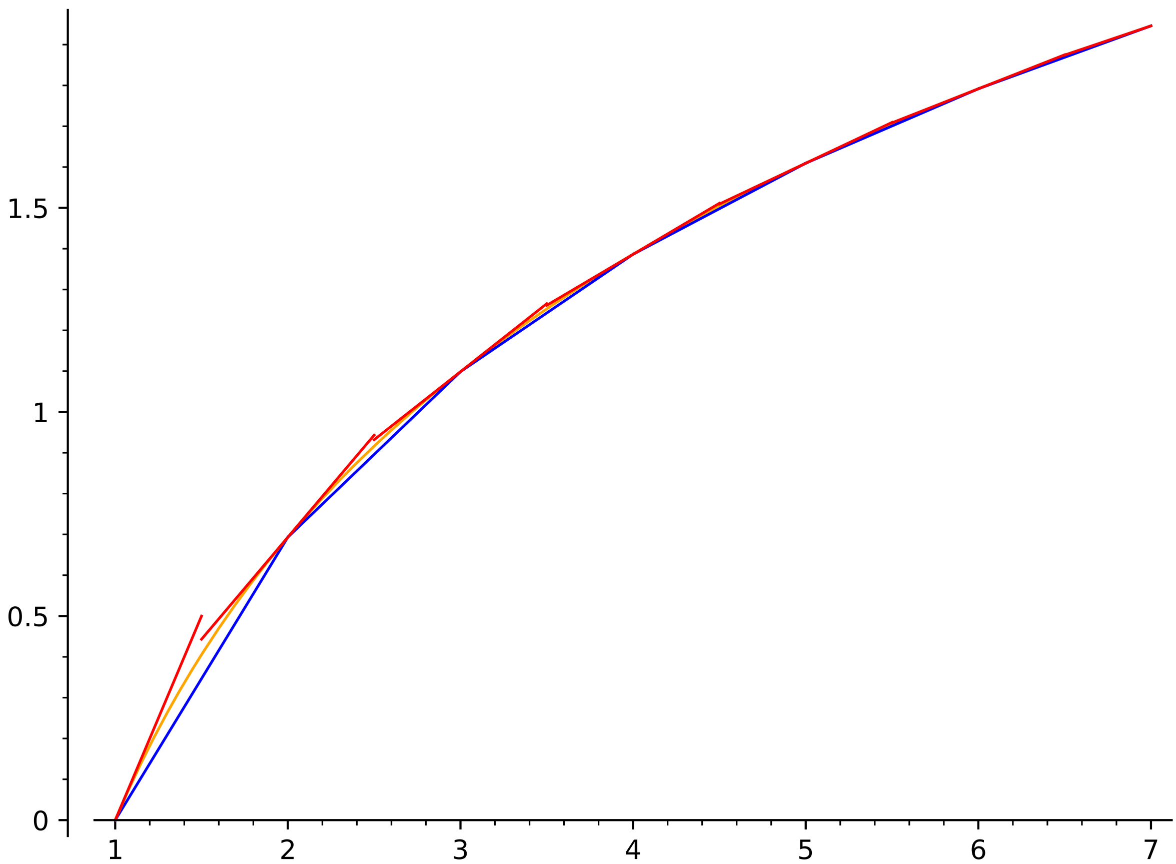
Integrate $\log x$ over $[1, n]$. Conclude that

$$\frac{7}{8} < \log(n!) - (n + \frac{1}{2})\log n + n < 1$$

for $n = 2, 3, 4, \dots$. (*Note* $\log\sqrt{2\pi} \sim 0.918\dots$.) Thus

$$e^{7/8} < \frac{n!}{(n/e)^n\sqrt{n}} < e$$

Note, Rudin is a fucking genius because the agreement is uncanny, if I graph the interval $[0, 100]$ my eyes can't tell the difference. So $f < \log < g$ for $x \ge 1$. The graphs correspond to the colors, and the range is $[1, 7]$ for emphasis. After 5 the agreement is so good you can't even see log function.



$$
\begin{aligned}
\int_1^n f(x)dx &= \sum_{k=1}^{n}\int_k^{k+1} f(x)dx \\
&= \sum_{k=1}^{n}\int_k^{k+1} (k+1-x)\log k + (x-k)\log(k+1)dx \\
&= \sum_{k=1}^{n}\int_k^{k+1} (k+1)\log k - x\log k + x\log(k+1) - k\log(k+1)dx \\
&= \sum_{k=1}^{n}[(k+1)\log k - k\log(k+1)][k+1-k] + [\log(k+1)-\log k]\int_k^{k+1} xdx \\
&= \sum_{k=1}^{n}[(k+1)\log k - k\log(k+1)][k+1-k] + [\log(k+1)-\log k]\frac{1}{2}[(k+1)^2-k^2] \\
&= \sum_{k=1}^{n}[(k+1)\log k - k\log(k+1)][k+1-k] + [\log(k+1)-\log k]\frac{1}{2}[k^2+2k+1-k^2] \\
&= \sum_{k=1}^{n}[(k+1)\log k - k\log(k+1)][k+1-k] + [\log(k+1)-\log k][k+\frac{1}{2}] \\
&= \sum_{k=1}^{n}\log\left(\frac{k^{k+1}}{(k+1)^k}\right) + \log\left(\left(\frac{k+1}{k}\right)^{k+\frac{1}{2}}\right) = \sum_{k=1}^{n}\log\left(\frac{k^{k+1}}{(k+1)^k}\frac{(k+1)^{k+\frac{1}{2}}}{k^{k+\frac{1}{2}}}\right) \\
&= \sum_{k=1}^{n}\log\left(k^{k+1-k-\frac{1}{2}}(k+1)^{k+\frac{1}{2}-k}\right) = \sum_{k=1}^{n}\log\left(k^{\frac{1}{2}}(k+1)^{\frac{1}{2}}\right) = \sum_{k=1}^{n}\log\left(k(k+1)\right)^{\frac{1}{2}} \\
&= \sum_{k=1}^{n}\frac{1}{2}\log\left(k(k+1)\right) = \frac{1}{2}\sum_{k=1}^{n}\log k + \log(k+1) \\
&= \frac{1}{2}\sum_{k=1}^{n}\log k + \frac{1}{2}\sum_{k=1}^{n}\log(k+1) \\
&= \frac{1}{2}\log(1) + \frac{1}{2}\sum_{k=2}^{n}\log k + \frac{1}{2}\sum_{k=1}^{n-1}\log(k+1) + \frac{1}{2}\log(n+1)
\end{aligned}
$$

if $l = k+1$ then $l \in [2,n] \implies \frac{1}{2}\sum_{k=1}^{n-1}\log(k+1) = \frac{1}{2}\sum_{l=2}^{n}\log l$

$$
\begin{aligned}
\text{Thus } \int_1^n f(x)dx &= \frac{1}{2}\cancel{\log(1)} + \sum_{k=2}^{n}\log k + \frac{1}{2}\log(n+1) \\
&= \log\prod_{k=2}^{n} k + \frac{1}{2}\log(n+1) \\
&= \log(n!) + \frac{1}{2}\log(n+1)
\end{aligned}
$$

So, Rudin isn't that smart... I even checked by wolframalpha.com that he's wrong. By quite a bit, but whatever.

$$
\begin{aligned}
\int_1^n g(x)dx &= \int_1^{3/2} g(x)dx + \sum_{k=2}^{n-1} \int_{k-1/2}^{k+1/2} g(x)dx + \int_{n-1/2}^{n} g(x)dx \\
&= \int_1^{3/2} x - 1\, dx + \sum_{k=2}^{n-1} \int_{k-1/2}^{k+1/2} \frac{x}{k} - 1 + \log k\, dx + \int_{n-1/2}^{n} \frac{x}{n} - 1 + \log n\, dx \\
&= \frac{x^2}{2}\bigg|_1^{3/2} - \frac{1/2}{+} \sum_{k=2}^{n-1} \int_{k-1/2}^{k+1/2} \frac{x}{k} - 1 + \log k\, dx + \frac{x^2}{2n}\bigg|_n^{n-1/2} + [\log n - 1]\left(n - n + \frac{1}{2}\right) \\
&= \frac{(3/2)^2 - 1}{2} - \frac{1}{2} + \sum_{k=2}^{n-1} \int_{k-1/2}^{k+1/2} \frac{x}{k} - 1 + \log k\, dx + \frac{x^2}{2n}\bigg|_{n-1/2}^{n} + \frac{\log n - 1}{2} \\
&= \frac{1}{8} + \frac{n^2 - (n-1/2)^2}{2n} - \frac{\log n - 1}{2} + \sum_{k=2}^{n-1} \int_{k-1/2}^{k+1/2} \frac{x}{k} - 1 + \log k\, dx \\
&= \frac{1}{8} + \frac{n^2 - n^2 + n - 1/4}{2n} + \frac{\log n - 1}{2} + \sum_{k=2}^{n-1} \int_{k-1/2}^{k+1/2} \frac{x}{k} - 1 + \log k\, dx \\
&= \frac{1}{8} + \frac{n}{2n} - \frac{1}{8n} + \frac{\log n - 1}{2} + \sum_{k=2}^{n-1} \int_{k-1/2}^{k+1/2} \frac{x}{k} - 1 + \log k\, dx \\
&= \frac{1}{8} + \frac{1}{2} - \frac{1}{8n} + \frac{\log n - 1}{2} + \sum_{k=2}^{n-1} \frac{x^2}{2k}\bigg|_{k-1/2}^{k+1/2} + (k + 1/2 - (k - 1/2))(\log k - 1) \\
&= \frac{1}{8} - \frac{1}{8n} + \frac{\log n}{2} + \sum_{k=2}^{n-1} \frac{(k+1/2)^2 - (k-1/2)^2}{2k} + \log k - 1 \\
&= \frac{1}{8} - \frac{1}{8n} + \frac{\log n}{2} + \sum_{k=2}^{n-1} \frac{k^2 + k + 1/4 - (k^2 - k + 1/4)}{2k} + \log k - 1 \\
&= \frac{1}{8} - \frac{1}{8n} + \frac{\log n}{2} + \sum_{k=2}^{n-1} \frac{2k}{2k} + \log k - 1 \\
&= \frac{1}{8} - \frac{1}{8n} + \frac{1}{2}\log n + \sum_{k=2}^{n-1} \log k \\
&= \frac{1}{8} - \frac{1}{8n} - \frac{1}{2}\log n + \sum_{k=2}^{n-1} \log k + \log n \\
&= \frac{1}{8} - \frac{1}{8n} - \frac{1}{2}\log n + \sum_{k=2}^{n} \log k \\
&= \frac{1}{8} - \frac{1}{8n} - \frac{1}{2}\log n + \log(n!)
\end{aligned}
$$

Now,

$$
\int_1^n \log x\, dx = x(\log x - 1)\bigg|_1^n = n(\log n - 1) - 1(\log 1 - 1) = n\log n - n + 1
$$

Now $f \leq \log \leq g \implies \int_1^n f dx \leq \int_1^n \log dx \leq \int g dx$. So,

$$\begin{aligned}
&\log(n!) + \frac{1}{2}\log(n+1) \leq n\log n - n + 1 \leq \frac{1}{8} - \frac{1}{8n} - \frac{1}{2}\log n + \log(n!) \\
\implies &\frac{1}{2}\log(n+1) \leq n\log n - \log(n!) - n + 1 \leq \frac{1}{8} - \frac{1}{8n} - \frac{1}{2}\log n \\
\implies &\frac{1}{2}\log(n+1) - 1 \leq n\log n - \log(n!) - n \leq -\frac{7}{8} - \frac{1}{8n} - \frac{1}{2}\log n \\
\implies &1 > 1 - \frac{1}{2}\log(n+1) \geq \log(n!) + n - n\log n \geq \frac{7}{8} + \frac{1}{8n} + \frac{1}{2}\log n > \frac{7}{8} \\
\implies &e > n! e^{n - n\log n} > e^{\frac{7}{8}} \\
\implies &e > \frac{n!}{(n/e)^n} > e^{\frac{7}{8}}
\end{aligned}$$

Now this is the best I can do, however this formula is false, and doesn't follow from f and g. The counter example is ridiculously small...

$$\log(2!) + \frac{1}{2}\log(2+1) \leq 2\log 2 - 2 + 1 \leq \frac{1}{8} - \frac{1}{16} - \frac{1}{2}\log 2 + \log(2!)$$

$$\implies 1.24\cdots \leq 0.38\cdots \leq 0.40\cdots$$

g is fine. However, the counterexample contradicts the assumption $f < \log$ in $[1, n]$.

Now, suppose there exists a piecewise function h such that $h < \log$ in $[1, n]$, and such that:

$$\int_1^n h dx = \log(n!) - \frac{1}{2}\log(n)$$

Then,

$$\begin{aligned}
&\log(n!) - \frac{1}{2}\log(n) \leq n\log n - n + 1 \leq \frac{1}{8} - \frac{1}{8n} - \frac{1}{2}\log n + \log(n!) \\
\implies &-\frac{1}{2}\log(n) \leq -\log(n!) + n\log n - n + 1 \leq \frac{1}{8} - \frac{1}{8n} - \frac{1}{2}\log n \\
\implies &0 \leq -\log(n!) + \left(n + \frac{1}{2}\right)\log n - n + 1 \leq \frac{1}{8} - \frac{1}{8n} \\
\implies &-1 \leq -\log(n!) + \left(n + \frac{1}{2}\right)\log n - n \leq -\frac{7}{8} - \frac{1}{8n} \\
\implies &1 \geq \log(n!) - \left(n + \frac{1}{2}\right)\log n + n \geq \frac{7}{8} + \frac{1}{8n} \\
\implies &1 \geq \log(n!) - \left(n + \frac{1}{2}\right)\log n + n \geq \frac{7}{8} + \frac{1}{8n} > \frac{7}{8} \\
\implies &e \geq n! n^{-(n+\frac{1}{2})} e^n > e^{\frac{7}{8}} \\
\implies &e \geq \frac{n!}{(n/e)^n \sqrt{n}} > e^{\frac{7}{8}}
\end{aligned}$$

The preceding calculation was easy. However, how do I know that,

$$\log(n!) - \frac{1}{2}\log(n) \leq n\log n - n + 1 \text{ ?}$$

$n, \log(n!) - \frac{1}{2}\log n, n\log n - n + 1$
$1, 0.00000, 0.00000$
$2, 0.34657, 0.38629$
$3, 1.2425, 1.2958$

$4, 2.4849, 2.5452$
$5, 3.9828, 4.0472$
$6, 5.6834, 5.7505$
$7, 7.5522, 7.6214$
$8, 9.5649, 9.6355$
$9, 11.703, 11.775$
$10, 13.953, 14.026$
$11, 16.303, 16.377$
$12, 18.745, 18.819$
$13, 21.270, 21.344$
$14, 23.872, 23.947$
$15, 26.545, 26.621$
$16, 29.286, 29.361$
$17, 32.088, 32.165$
$18, 34.950, 35.027$
$19, 37.868, 37.944$
$20, 40.838, 40.915$
$21, 43.858, 43.935$
$22, 46.926, 47.003$
$23, 50.039, 50.116$
$24, 53.196, 53.273$
$25, 56.394, 56.472$
$26, 59.633, 59.710$
$27, 62.910, 62.988$
$28, 66.224, 66.302$
$29, 69.573, 69.651$
$30, 72.958, 73.036$
$31, 76.375, 76.454$
$32, 79.825, 79.903$
$33, 83.306, 83.385$
$34, 86.818, 86.896$
$35, 90.359, 90.437$
$36, 93.928, 94.007$
$37, 97.525, 97.604$
$38, 101.15, 101.23$
$39, 104.80, 104.88$
$40, 108.48, 108.56$
$41, 112.18, 112.26$
$42, 115.90, 115.98$
$43, 119.65, 119.73$
$44, 123.43, 123.50$
$45, 127.22, 127.30$
$46, 131.04, 131.12$
$47, 134.88, 134.96$
$48, 138.74, 138.82$
$49, 142.62, 142.70$
$50, 146.52, 146.60$
$51, 150.44, 150.52$
$52, 154.39, 154.46$
$53, 158.35, 158.43$
$54, 162.33, 162.41$
$55, 166.32, 166.40$
$56, 170.34, 170.42$
$57, 174.37, 174.45$
$58, 178.43, 178.51$
$59, 182.50, 182.57$
$60, 186.58, 186.66$
$61, 190.68, 190.76$

$62, 194.80, 194.88$
$63, 198.94, 199.02$
$64, 203.09, 203.17$
$65, 207.26, 207.33$
$66, 211.44, 211.52$
$67, 215.63, 215.71$
$68, 219.85, 219.93$
$69, 224.07, 224.15$
$70, 228.31, 228.40$
$71, 232.57, 232.65$
$72, 236.84, 236.92$
$73, 241.12, 241.20$
$74, 245.42, 245.50$
$75, 249.73, 249.81$
$76, 254.06, 254.14$
$77, 258.39, 258.47$
$78, 262.74, 262.82$
$79, 267.11, 267.19$
$80, 271.48, 271.56$
$81, 275.87, 275.95$
$82, 280.27, 280.35$
$83, 284.68, 284.76$
$84, 289.11, 289.19$
$85, 293.55, 293.62$
$86, 297.99, 298.07$
$87, 302.45, 302.53$
$88, 306.93, 307.01$
$89, 311.41, 311.49$
$90, 315.90, 315.98$
$91, 320.41, 320.49$
$92, 324.92, 325.00$
$93, 329.45, 329.53$
$94, 333.99, 334.07$
$95, 338.54, 338.62$
$96, 343.10, 343.18$
$97, 347.67, 347.75$
$98, 352.25, 352.33$
$99, 356.84, 356.92$
$100, 361.44, 361.52$
$101, 366.05, 366.13$
$102, 370.67, 370.75$
$103, 375.30, 375.38$
$104, 379.94, 380.02$
$105, 384.59, 384.67$
$106, 389.24, 389.32$
$107, 393.91, 393.99$
$108, 398.59, 398.67$
$109, 403.28, 403.36$
$110, 407.97, 408.05$
$111, 412.68, 412.76$
$112, 417.39, 417.47$
$113, 422.11, 422.19$
$114, 426.85, 426.93$
$115, 431.59, 431.67$
$116, 436.34, 436.42$
$117, 441.09, 441.17$
$118, 445.86, 445.94$
$119, 450.64, 450.72$

$120, 455.42, 455.50$
$121, 460.21, 460.29$
$122, 465.01, 465.09$
$123, 469.82, 469.90$
$124, 474.63, 474.71$
$125, 479.46, 479.54$
$126, 484.29, 484.37$
$127, 489.13, 489.21$
$128, 493.98, 494.06$
$129, 498.84, 498.92$
$130, 503.70, 503.78$
$131, 508.57, 508.65$
$132, 513.45, 513.53$
$133, 518.34, 518.42$
$134, 523.23, 523.31$
$135, 528.13, 528.21$
$136, 533.04, 533.12$
$137, 537.96, 538.04$
$138, 542.88, 542.96$
$139, 547.81, 547.89$
$140, 552.75, 552.83$
$141, 557.69, 557.78$
$142, 562.65, 562.73$
$143, 567.61, 567.69$
$144, 572.57, 572.65$
$145, 577.55, 577.63$
$146, 582.53, 582.61$
$147, 587.51, 587.59$
$148, 592.51, 592.59$
$149, 597.51, 597.59$
$150, 602.52, 602.60$
$151, 607.53, 607.61$
$152, 612.55, 612.63$
$153, 617.58, 617.66$
$154, 622.61, 622.69$
$155, 627.65, 627.73$
$156, 632.70, 632.78$
$157, 637.75, 637.83$
$158, 642.81, 642.89$
$159, 647.88, 647.96$
$160, 652.95, 653.03$
$161, 658.03, 658.11$
$162, 663.11, 663.19$
$163, 668.20, 668.28$
$164, 673.30, 673.38$
$165, 678.40, 678.48$
$166, 683.51, 683.59$
$167, 688.62, 688.71$
$168, 693.75, 693.83$
$169, 698.87, 698.95$
$170, 704.00, 704.08$
$171, 709.14, 709.22$
$172, 714.29, 714.37$
$173, 719.44, 719.52$

So, it's a reasonable assumption. Let's ask the computer if it holds for $1 \leq n \leq 20000$, else give me the first n that fails. My computer says yes it holds. However, for $1 \leq n \leq 100000$ my computer was taking more than 5 minutes which is longer than I'm willing to wait.

Now, let's see if there's an h, that satisfies those properties. We know,

$$\int_1^n f\,dx = \log(n!) + \frac{1}{2}\log(n+1) \quad \text{and} \quad \int_1^n h\,dx = \log(n!) - \frac{1}{2}\log(n)$$

$$\implies \int_1^n f\,dx - \int_1^n h\,dx = \log(n!) + \frac{1}{2}\log(n+1) - \left(\log(n!) - \frac{1}{2}\log n\right)$$

$$\implies \int_1^n f - h\,dx = \frac{1}{2}\log(n+1) + \frac{1}{2}\log n$$

Now, I was going to give up there. However, wolframalpha.com came to the rescue and found that,

$$\int_1^n \frac{1+2nx}{2x-2nx^2}dx = \frac{1}{2}\log(n+1) + \frac{1}{2}\log n$$

Then

$$f - h = \frac{1+2nx}{2x-2nx^2} \implies h(n,x) = f(x) - \frac{1+2nx}{2x-2nx^2}$$

If we could find a piecewise representation of $f - h$, for $m \leq x \leq m+1$, then the definition of h wouldn't depend on n.

However, we only need to define h in the interval $[1, n]$. We need to show $h < \log$ there. Actually, we also need to show $\log < g$ there. But, since Rudin didn't bother(and was wrong) I won't either.