(a)

$b^x = e^{x \log b} (b > 0)$ by formula (43) in p. 181.

So, $(b^x - 1)' = (e^{x \log b} - 1)' = e^{x \log b}(x \log b)' + (-1)' = e^{x \log b} \log b + 0 = e^{x \log b} \log b$

Since $x' = 1$, we have

$$\lim_{x\to 0} e^{x \log b} \log b = \lim_{x\to 0} \sum_{k=0}^{\infty} \frac{(x \log b)^k}{k!} = \sum_{k=0}^{\infty} \frac{\lim_{x\to 0} (x \log b)^k}{k!} = \sum_{k=0}^{\infty} \frac{(\lim_{x\to 0} x \log b)^k}{k!} = \sum_{k=0}^{\infty} \frac{0^k}{k!} = e^0 \log b = \log b$$

As $x \to 0$

$$b^x - 1 \longrightarrow b^0 - 1 = e^{0 \log b} - 1 = 1 - 1 = 0$$

So by L'Hospital's rule,

$$\lim_{x\to 0} \frac{b^x - 1}{x} = \log b$$

(b)

Since, $\log(1 + x)' = \frac{1}{1+x}$ and $x' = 1$. $\lim_{x\to 0} = \frac{1}{1+0} = 1$. As $x \to 0$,

$$\log(1 + x) \longrightarrow \log(1 + 0) = \log 1 = 0$$

So by L'Hospital's rule,

$$\lim_{x\to 0} \frac{\log(1 + x)}{x} = 1$$

(c)

We know from Theorem 3.31 that,

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

Put $x = \frac{1}{n}$, then $n = 1/x$, and

$$x \longrightarrow 0 \iff n \longrightarrow \infty$$

So,

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e = \lim_{x\to 0} (1 + x)^{1/x}$$

(d)

$$\begin{aligned}
\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n &= \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} \frac{x^k}{n^k} \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{x^k}{n^k} \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!} \frac{n!}{n^k(n-k)!} \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!} \frac{(n-1)(n-2)\cdots(n-(k+1))}{n^{k-1}} \\
&= \sum_{k=0}^{n} \frac{x^k}{k!} \lim_{n\to\infty} \frac{n^{k-1} + P(n)}{n^{k-1}} \text{ where } \deg P(n) < k - 1 \\
&= \sum_{k=0}^{n} \frac{x^k}{k!} \\
&= e^x
\end{aligned}$$

Note, we could've used (d) to prove (c) instead of invoking 3.31