If $f(0,0) = 0$ and

$$f(x,y) = \frac{xy}{x^2+y^2}, \quad \text{if } (x,y) \neq (0,0),$$

Prove that $(D_1 f)(x,y)$ and $(D_2 f)(x,y)$ exist at every point of R^2, although f is not continuous at $(0,0)$.

pf.

In $\mathbb{R}^2$,

$$(x,y) \neq (0,0) \iff x^2 + y^2 \neq 0$$

Thus $\forall (x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)$ gives us that,

$$(D_1 f)(x,y) = \frac{y^3 - yx^2}{(x^2+y^2)^2} \tag{1}$$

and

$$(D_2 f)(x,y) = \frac{x^3 - xy^2}{(x^2+y^2)^2} \tag{2}$$

both exist, since their common denominator is never zero.

If $(x,y) = (0,0)$, formula (25) gives,

$$(D_1 f)(0,0) = \lim_{t \to 0} \frac{f(0+t,0) - f(0,0)}{t} = \lim_{t \to 0} \frac{f(t,0) - 0}{t} = \lim_{t \to 0} \frac{\frac{t \cdot 0}{t^2+0^2}}{t} = 0$$

and by symmetry of f,

$$(D_2 f)(0,0) = \lim_{t \to 0} \frac{f(0,0+t) - f(0,0)}{t} = 0$$

So,

$$\forall (x,y) \in \mathbb{R}^2, \exists (D_1 f)(x,y) \text{ and } (D_2 f)(x,y)$$

Along the diagonal $x = y$,

$$\lim_{(x,y) \to (0,0)} f(x,y) = \lim_{x \to 0} \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

And along the x-axis,

$$\lim_{(x,y) \to (0,0)} f(x,y) = \lim_{x \to 0} \frac{x \cdot 0}{x^2 + 0^2} = 0$$

It follows that the limit doesn't exist, so f is not continuous at $(0,0)$ ■