1 The Real and Complex Number Systems

1. Exercise 1: if r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

    Solution: (*ghostofgarborg*)
    Note that $\mathbb{Q}$ is closed under the arithmetic operations of addition, subtraction, multiplication and taking multiplicative inverses. If $r + x$ were rational, so would $(r + x) - r = x$ be, a contradiction. If rx were rational, so would $\frac{1}{r}rx = x$ be, a contradiction.

2. Exercise 2: Prove that there is no rational number whose square is 12.

    Solution: (*ghostofgarborg*)
    We will assume without proof that $\mathbb{Z}$ has unique prime factorizations. (This follows from $\mathbb{Z}$ being a primary ideal domain, and is covered in an abstract algebra course.) This implies that p is a prime iff $p \mid ab \Rightarrow p \mid a$ or $p \mid b$.

    Assume there are m, n coprime such that $(\frac{m}{n})^2 = 12$. This implies that $12n^2 = m^2$. Consequently, $3 \mid m^2$, and since 3 is prime, $3 \mid m$. This means that there is a p such that $m = 3p$, so that $12n^2 = 9p^2$, and consequently $4n^2 = 3p^2$. Using the primality of 3, we can conclude that since $3 \nmid 4$, $3 \mid n^2$, and therefore $3 \mid n$. This contradicts the coprimality of m and n. Therefore there cannot be any such m, n.

    Solution: (*analambanomenos*) (This is the same without assuming any algebra)
    If r is a rational number whose square is 12, then $(r/2)^2 = 3$, so this is equivalent to showing there is no rational number whose square is 3. So suppose $(m/n)^2 = 3$ where m, n are integers with no common factors. Then $m^2 = 3n^2$. Either $m = 3p$, or $m = 3p \pm 1$, and since $(3p \pm 1)^2 = 3(3p^2 \pm 2p) + 1$, m must be a multiple of 3. But then we have $(3p)^2 = 3n^2$, or $3p^2 = n^2$. Hence n is also a multiple of 3, contradicting the assumption that m and n have no common factors.

    Comment: (*ghostofgarborg*) This is a good proof, and reducing it to the existence of a square root of 3 is great. It is debatable if the solution does not assume any algebra. The main fact that was needed above is one which is assumed here, namely that if $3p^2 = n^2$, then n is a multiple of 3. To understand why that is not a a priori given, take the ring $\mathbb{Z}[\sqrt{5}]$, consisting of numbers of the form $a + \sqrt{5}b$, where a and b are integers, and the operations ordinary addition and multiplication. Then 3 divides $(1 + \sqrt{5})(1 - \sqrt{5})$, but 3 neither divides $1 + \sqrt{5}$ or $1 - \sqrt{5}$. The assumption that the irreducibility of 3 allows us to conclude that 3 divides at least one of the factors in a product in which it occurs is an algebraic result that is called upon in this proof as well. Of course, it is a well known result, and it is implicitly assumed by Rudin, but it is always good to be aware of which presuppositions we are relying on.

3. Exercise 3: Show

    (a) If $x \neq 0$ and $xy = xz$, then $y = z$.

    (b) If $x \neq 0$ and $xy = x$ then $y = 1$.

    (c) If $x \neq 0$ and $xy = 1$, then $y = 1/x$.

    (d) If $x \neq 0$ then $1/(1/x) = x$.

Solution: (*ghostofgarborg*)
(a): Solution 1: $x \neq 0$ implies that x has a multiplicative inverse. Multiply by $1/x$.

However, we can prove this without resorting to the existence of inverses, and get a solution that carries over to a larger class of rings, i.e. integral domains like $\mathbb{Z}$. Solution 2: Note that proposition 1.16b implies that $xy = 0$ implies that $x = 0$ or $y = 0$. If $xy = xz$, then $x(y - z) = 0$. If $x \neq 0$, then by prop. 1.16b, $(y - z) = 0$. The claim follows.

(b): Follows from (a) by letting $z = 1$.

(c): Follows from (a) by letting $z = x^{-1}$.

(d): Follows by applying (a) to

$$\left(\frac{1}{x}\right)\frac{1}{\left(\frac{1}{x}\right)} = \left(\frac{1}{x}\right)x$$

4. Exercise 4: Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E, and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Solution: (*ghostofgarborg*)
Let $x \in E$. By definition of lower and upper bounds, $\alpha \leq x \leq \beta$.

5. Exercise 5: Let A be a nonempty set of real numbers, which is bounded below. Let $-A$ be the set of all numbers $-x$ where $x \in A$. Prove that:

$$\inf A = -\sup(-A)$$

Solution: (*ghostofgarborg*)
Assume α is the greatest lower bound of A. If $x \in (-A)$ then $-x \in A$, so $\alpha \leq -x$, and therefore $-\alpha \geq x$. This implies that $-\alpha$ is an upper bound for $(-A)$. If $\beta < -\alpha$ then $-\beta > \alpha$, and there is an $x \in A$ such that $x < -\beta$. Then $-x \in -A$, and $-x > \beta$. This shows that $-\alpha$ is the least upper bound of $(-A)$, and we are done.

6. Exercise 6: Fix $b > 1$.
   (a) If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that
   $$(b^m)^{1/n} = (b^p)^{1/q}.$$
   (b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.
   (c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that
   $$b^r = \sup B(r)$$
   when r is rational. Hence it makes sense to define
   $$b^x = \sup B(x)$$
   for every real x.
   (d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

Solution: (*analambanomenos*)
(a): Since $np = qm$,
$$((b^p)^{1/q})^{np} = ((b^p)^{1/q})^{qm} = b^{mp}.$$
Hence by Theorem 1.21,
$$((b^p)^{1/q})^n = b^m.$$
A second application of Theorem 1.21 gives us
$$(b^p)^{1/q} = (b^m)^{1/n}.$$

(b): Let $r = m/n$, $s = p/q$.
$$(b^r b^s)^{nq} = ((b^m)^{1/n}(b^p)^{1/q})^{nq} = b^{mq}b^{np}$$
$$(b^{r+s})^{nq} = ((b^{mq+np})^{1/nq})^{nq} = b^{mq}b^{np}$$

Hence by Theorem 1.21, $b^r b^s = b^{r+s}$.

(c): Note that the positive rational powers of b are greater than 1 since the positive integral powers and roots of real numbers greater than 1 are also greater than 1. Hence if $t < r$,
$$b^t < b^t b^{r-t} = b^{t+r-t} = b^r.$$
Hence, since $b^r \in B(r)$, we have $b^r = \sup B(r)$.

(d): We need to show that $\sup B(x+y) = \sup B(x) \sup B(y)$. If S and T are sets of positive real numbers, it isn't hard to show that $\sup ST = \sup S \sup T$, where ST is the set of products of elements of S with elements of T. Hence we want to show that $\sup B(x+y) = \sup(B(x)B(y))$.

Let p, q be rational numbers such that $p \leq x$ and $q \leq y$. Then $b^p b^q = b^{p+q} \leq \sup B(x+y)$, so $\sup(B(x)B(y)) \leq \sup B(x+y)$.

To get the reverse inequality, let t be any rational number such that $t < x+y$, and let $\epsilon > 0$ such that $t < x + y - \epsilon$. By Theorem 1.20(b), there are rational numbers $p \leq x$, and $q \leq y$ such that $x - \epsilon/2 < p$ and $y - \epsilon/2 < q$. Then $t < x + y - \epsilon < p + q$ so that $b^t < b^{p+q} = b^p b^q \leq \sup B(x)B(y)$. Hence $\sup B(x+y) \leq \sup(B(x)B(y))$. *(thanks to Matt Lundy for pointing out errors)*

7. Exercise 7: Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline.

   (a) For any positive integer n, $b^n - 1 \geq n(b-1)$.

   (b) Hence $b - 1 \geq n(b^{1/n} - 1)$.

   (c) If $t > 1$ and $n > (b-1)/(t-1)$, then $b^{1/n} < t$.

   (d) If w is such that $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n; to see this, apply part (c) with $t = y \cdot b^{-w}$.

   (e) If w is such that $b^w < y$, then $b^{w-(1/n)} > y$ for sufficiently large n.

   (f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.

   (g) Prove that this x is unique.

Solution: (*analambanomenos*)
(a): Using the binomial expansion, for any positive integer n,

$$b^n = (1 + (b-1))^n = 1^n + n1^{n-1}(b-1) + A$$

where A is a non-negative sum of terms involving higher powers of $(b-1)$. Hence

$$b^n - 1 = n(b-1) + A \geq n(b-1).$$

(b): Replace b in case (a) with $b^{1/n}$ to get $b - 1 \geq n(b^{1/n} - 1)$.

(c): From case (b) we have

$$\begin{aligned} \frac{b-1}{t-1}(b^{1/n} - 1) < n(b^{1/n} - 1) &\leq b - 1 \\ b^{1/n} - 1 &< t - 1 \\ b^{1/n} &< t \end{aligned}$$

(d): By case (c), if $n > (b-1)/(yb^{-w} - 1)$, then $b^{1/n} < yb^{-w}$, or $b^{w+(1/n)} < y$.

(e): Applying case (c) to $t = y^{-1}b^w > 1$, if $n > (b-1)/(y^{-1}b^w - 1)$, then $b^{1/n} < y^{-1}b^w$, or $y < b^{w-(1/n)}$.

(f): If $b^x < y$, then from case (d) there is a sufficiently large integer n such that $b^{x+(1/n)} < y$, that is, $x + (1/n) \in A$, contradicting $x = \sup A$. And if $b^x > y$, then from case (e) there is a sufficiently large integer n such that $b^{x-(1/n)} > y$, so that $x - (1/n)$ is an upper bound of A, contradicting $x = \sup A$.

(g): Suppose there are real numbers $x_1 < x_2$ such that $b^{x_1} = b^{x_2}$. Then $b^{x_1}b^{x_2-x_1} = b^{x_2} = b^{x_1}$, so that $b^{x_2-x_1} = 1$. Using the definition of real powers given in exercise 6, this means there are positive integers m, n such that $(b^m)^{1/n} \leq 1$. However, since the positive integeral powers of numbers greater than 1 are also greater than 1, this is impossible.

8. Exercise 8: Prove that no order can be defined in the complex field that turns it into an ordered field.

Solution: (*ghostofgarborg*)
By proposition 1.18d, an ordering $<$ that makes $\mathbb{C}$ an ordered field would have to satisfy $-1 = i^2 > 0$, contradicting $1 > 0$.

9. Exercise 9: Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$ or $a = c$ and $b < d$. Prove that this is an order. Does it have the least upper bound property?

Solution: (*ghostofgarborg*)
Let $z_i = a_i + b_i$.

Property 1.5(i): Assume $z_1 \neq z_2$. If $a_1 \neq a_2$, either $z_1 < z_2$ or $z_2 < z_1$. Otherwise $b_1 \neq b_2$, in which case $z_1 < z_2$ or $z_2 < z_1$.

Property 1.5(ii): Assume $z_1 < z_2$ and $z_2 < z_3$. Then $a_1 \leq a_2 \leq a_3$. If either of the inequalities is strict, $a_1 < a_3$, and $z_1 < z_3$. Otherwise, $b_1 \leq b_2 \leq b_3$ and all inequalities are strict, so that $z_1 < z_3$.

This proves that the order is well-defined.

The set does not have the least upper bound property. Consider the set $\mathbb{R}i = \{0 + xi : i \in \mathbb{R}\}$. The set is bounded above by 1. Assume for contradiction that $\alpha = a + bi$ is a least upper bound. It is clear that we must have $a \geq 0$. If $a > 0$, then $\frac{1}{2}\alpha$ is another upper bound that is strictly smaller than α, contradicting minimality. Therefore, we must have $a = 0$. Then $\alpha + i \in \mathbb{R}i$ is greater than α, contradicting α being an upper bound.

10. Exercise 10: Suppose $z = a + ib$, $w = u + iv$, and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2}, \qquad b = \left(\frac{|w| - u}{2}\right)^{1/2}.$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\overline{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

Solution: (*analalambanomenos*)
We have

$$(a^2 - b^2) = \frac{|w| + u}{2} - \frac{|w| - u}{2} = u$$

$$2ab = (|w| + u)^{1/2}(|w| - u)^{1/2} = (|w|^2 - u^2)^{1/2} = (v^2)^{1/2} = |v|.$$

Hence $z^2 = (a^2 - b^2) + 2abi = u + |v|i = w$ if $v \geq 0$, and $(\overline{z}^2) = (a^2 - b^2) - 2abi = u - |v|i = w$ if $v \leq 0$. Hence every nonzero w has two square roots $\pm z$ or $\pm\overline{z}$. Of course, 0 has only one square root, itself.

11. Exercise 11: If $z \in \mathbb{C}$, prove that there exists an $r \geq 0$ and $w \in \mathbb{C}$ subh that $z = rw$. Are w and r always uniquely determined by z?

Solution: (*ghostofgarborg*)
There is a solution, and it is unique whenever $z \neq 0$. Assume $z = rw$, $r \geq 0$ and $|w|^2 = w\bar{w} = 1$.

$$r = \sqrt{r^2} = \sqrt{(rw)(\overline{rw})} = \sqrt{z\bar{z}} = |z|$$

If $r = 0$, we can take any w with $|w| = 1$, e.g. $w = e^{i\theta}$. Otherwise

$$w = \frac{z}{r} = \frac{z}{|z|}$$

It is easy to check that the uniquely determined r and w have the desired properties.

12. Exercise 12: If $z_1, \cdots, z_n \in \mathbb{C}$, prove that

$$|z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|$$

Solution: (*ghostofgarborg*)
Note that by the triangle inequality, $|z_1 + z_2| \leq |z_1| + |z_2|$. Assume the statement holds for $n - 1$. Then

$$|z_1 + \cdots + z_{n-1} + z_n| \leq |z_1 + \cdots + z_{n-1}| + |z_n| \leq |z_1| + \cdots + |z_n|$$

which establishes the claim by induction.

13. Exercise 13: If x, y are complex, prove that

$$||x| - |y|| \leq |x - y|$$

Solution: (*ghostofgarborg*)
By the triangle inequality

$$|x| = |(x - y) + y| \leq |x - y| + |y|$$

so that

$$|x| - |y| \leq |x - y|$$

Interchanging the roles of x and y in the above, we also have

$$|y| - |x| \leq |x - y|$$

so that

$$||x| - |y|| \leq |x - y|$$

14. Exercise 14: If $z \in \mathbb{C}$, $|z| = 1$, compute $|1 + z|^2 + |1 - z|^2$.

Solution: (*ghostofgarborg*)

$$\begin{aligned} |1+z|^2 + |1-z|^2 &= (1+z)(\overline{1+z}) + (1-z)(\overline{1-z}) \\ &= (1+z)(1+\bar{z}) + (1-z)(1-\bar{z}) \\ &= (1 + z + \bar{z} + z\bar{z}) + (1 - z - \bar{z} + z\bar{z}) \\ &= 2 + 2z\bar{z} \\ &= 4 \end{aligned}$$

15. Exercise 15: Under what conditions does equality hold in the Schwarz inequality?

Solution: (*ghostofgarborg*)
We observe that $(AB - |C|^2) > 0$ iff $S = \sum |Ba_j - Cb_j|^2 > 0$. If $S = 0$, then $a_j = -\frac{C}{B} b_j = z b_j$. Assume $a_j = z b_j$. Then $C = zB$, so that $S = \sum |Bzb_j - zBb_j|^2 = 0$. I.e., equality holds iff there is a $z \in \mathbb{C}$ such that $a_j = z b_j$.

16. Exercise 16: Suppose $k \geq 3$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$, and $r > 0$. Prove:
    (a) If $2r > d$, there are infinitely many $\mathbf{z} \in \mathbb{R}^k$ such that

    $$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r.$$

    (b) If $2r = d$, there is exactly one such $\mathbf{z}$.
    (c) If $2r < d$, there is no such $\mathbf{z}$.

    How must these statements be modified if k is 2 or 1?

Solution: (*analambanomenos*)
(a): To simplify the calculations, we may assume without losing generality that $\mathbf{x} = (0, \ldots, 0)$ and $\mathbf{y} = (d, 0, \ldots, 0)$. (Translate $\mathbb{R}^k$ by $-\mathbf{x}$, then rotate it to place $\mathbf{y}$ on the positive x-axis. Both transformations preserve distances and angles.)

For $0 \leq \theta < 1$ let

$$\mathbf{z}_\theta = (d/2, \cos\theta\sqrt{4r^2 - d^2}/4, \sin\theta\sqrt{4r^2 - d^2}/2, 0, \ldots, 0).$$

Then

$$|\mathbf{z}_\theta - \mathbf{x}|^2 = |\mathbf{z}_\theta - \mathbf{y}|^2 = \frac{d^2}{4} + (\cos^2\theta + \sin^2\theta)\frac{4r^2 - d^2}{4} = r^2.$$

For $k = 2$, the hyperplane of equidistant points is the line $(d/2, x_2)$, so there are two points with distance $r > d/2$ from $\mathbf{x}$ and $\mathbf{y}$, $(d/2, \pm\sqrt{4r^2 - d^2}/2)$. For $k = 1$, the hyperplane reduces to the single point $z = d/2$, so there are no points with distance $r > d/2$ from $\mathbf{x}$ and $\mathbf{y}$.

(b): If $d = 2r$, then $|\mathbf{z} - \mathbf{x}| + |\mathbf{z} - \mathbf{x}| = |\mathbf{x} - \mathbf{y}|$, that is, we have equality in Theorem 1.37(f). Looking at the proof, this only happens if $(\mathbf{z} - \mathbf{x}) \cdot (\mathbf{z} - \mathbf{y}) = |\mathbf{z} - \mathbf{x}||\mathbf{z} - \mathbf{y}|$. By the solution to exercise 15, this only happens if $\mathbf{z} - \mathbf{x} = t(\mathbf{z} - \mathbf{y})$ for some real number t. Since $|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}|$, t must be ± 1. If $t = 1$, then $\mathbf{x} = \mathbf{y}$, so $t = -1$ and $\mathbf{z}$ is the midpoint $(\mathbf{x} - \mathbf{y})/2$.

(c): $|\mathbf{z} - \mathbf{x}| + |\mathbf{z} - \mathbf{y}| = 2r < |\mathbf{x} - \mathbf{y}|$ contradicts Theorem 1.37(f).

17. Exercise 17: Prove that

$$|x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2$$

if $x \in R^k$ and $y \in R^k$. Interpret this geometrically, as a statement about parallelograms.

Solution: (*Matt "frito" Lundy*)
This solution will use some linear algebra. In particular, for any inner product $\langle \cdot, \cdot \rangle$ the norm is

$$|\cdot| = \sqrt{\langle \cdot, \cdot \rangle}$$

or

$$|\cdot|^2 = \langle \cdot, \cdot \rangle.$$

In this real vector space R^k, for any $x, y, z \in R^k$ we also have the following properties:

$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$$

$$\langle x, y \rangle = \langle y, x \rangle.$$

Then

$$\begin{aligned}
|x + y|^2 + |x - y|^2 &= \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle + \langle x, x \rangle - 2\langle x, y \rangle + \langle y, y \rangle \\
&= 2\langle x, x \rangle + 2\langle y, y \rangle \\
&= 2|x|^2 + 2|y|^2
\end{aligned}$$

Geometrically, this says that for any parallelogram in Euclidian Space, the sum of the lengths of the diagonals is equal to the perimeter.

18. Exercise 18: If $k \geq 2$ and $x \in R^k$, prove that there exists $y \in R^k$ such that $y \neq 0$ but $x \cdot y = 0$ ($\langle x, y \rangle = 0$). Is this also true if $k = 1$?

Solution: (*Matt "frito" Lundy*)
This solution uses some linear algebra. If $x = 0$ then any non-zero y will suffice, so assume that $x \neq 0$. Also choose any $z \in R^k$ such that x and z are linearly indepedent (this is possible because $k \geq 2$). Then let

$$y = z - \frac{\langle z, x\rangle}{\langle x, x\rangle} x.$$

Notice that $y \neq 0$ because x and z are linearly independent and y is a linear combination of x and z with not all coefficients equal to zero. And so we have

$$\begin{aligned}
\langle x, y\rangle &= \langle x, z - \frac{\langle z, x\rangle}{\langle x, x\rangle} x\rangle \\
&= \langle x, z\rangle - \frac{\langle z, x\rangle\langle x, x\rangle}{\langle x, x\rangle} \\
&= \langle x, z\rangle - \langle x, z\rangle \\
&= 0
\end{aligned}$$

There is trouble when $k = 1$ because any two vectors in R^1 are linearly dependent.

19. Exercise 19: Suppose $\mathbf{a} \in \mathbb{R}^k$, $\mathbf{b} \in \mathbb{R}^k$. Find $\mathbf{c} \in \mathbb{R}^k$ and $r > 0$ such that

$$|\mathbf{x} - \mathbf{a}| = 2|\mathbf{x} - \mathbf{b}|$$

if and only if $|\mathbf{x} - \mathbf{c}| = r$.

Solution: (*analambanomenos*)

$$\begin{aligned}
4|\mathbf{x} - \mathbf{b}|^2 &= |\mathbf{x} - \mathbf{a}|^2 \\
4|\mathbf{x}|^2 - 8\mathbf{x}\cdot\mathbf{b} + 4|\mathbf{b}|^2 &= |\mathbf{x}|^2 - 2\mathbf{x}\cdot\mathbf{a} + |\mathbf{a}|^2 \\
|\mathbf{x}|^2 - 2\mathbf{x}\cdot\big((1/3)(4\mathbf{b} - \mathbf{a})\big) &= (1/3)|\mathbf{a}|^2 - (4/3)|\mathbf{b}|^2 \\
|\mathbf{x}|^2 - 2\mathbf{x}\cdot\big((1/3)(4\mathbf{b} - \mathbf{a})\big) + \big|(1/3)(4\mathbf{b} - \mathbf{a})\big|^2 &= (1/3)|\mathbf{a}|^2 - (4/3)|\mathbf{b}|^2 + \big|(1/3)(4\mathbf{b} - \mathbf{a})\big|^2 \\
\big|\mathbf{x} - (1/3)(4\mathbf{b} - \mathbf{a})\big|^2 &= (1/9)(3|\mathbf{a}|^2 - 12\mathbf{b}^2 + |\mathbf{a}|^2 - 8\mathbf{a}\cdot\mathbf{b} + 16|\mathbf{b}|^2) \\
\big|\mathbf{x} - (1/3)(4\mathbf{b} - \mathbf{a})\big|^2 &= (1/9)(4|\mathbf{a}|^2 - 8\mathbf{a}\cdot\mathbf{b} + 4|\mathbf{b}|^2) \\
\big|\mathbf{x} - (1/3)(4\mathbf{b} - \mathbf{a})\big|^2 &= (4/9)\big|\mathbf{a} - \mathbf{b}\big|^2
\end{aligned}$$

This describes a sphere with center $(1/3)(4\mathbf{b} - \mathbf{a})$ and radius $(2/3)|\mathbf{a} - \mathbf{b}|$.

20. Exercise 20: Suppose that we omitted property (III) from the definition of a cut. Keep the same definitions of order and addition. Show that the resulting set has the least-upper-bound property, that addition satisfies axioms (A1) to (A4) (with a slightly different zero element!) but that (A5) fails.

Solution: (*Jack Gallagher*)
Proof is omitted whenever the proof in the book does not use property (III).

In this case, a cut is defined as any inhabited set $\alpha \subset Q$ such that

1. $\alpha \neq Q$

2. If $p \in \alpha$, $q \in Q$, and $q < p$, then $q \in \alpha$

I'll adopt the same convention as the book, labelling rationals with Roman letters and cuts with Greek.

The proof that R' has the least-upper-bound property is the same as that given in the book.

The proofs of axioms (A1-3) are again the same. For (A4), we define

$$0_R = \{n \mid n \in Q,\ n \leq 0\}$$

(A4). For any $a \in \alpha$ and $s \in 0_r$, we have either that

$$a + s = a \text{ or } a + s < a$$

In either case we have that $a + s \in \alpha$, either by equality or by (II). Thus $\alpha + 0_{R'} \subset \alpha$.

Similarly, if $b \in \alpha + 0_{R'}$, we have that $b = a + s$ for some $a \in \alpha$ and $s \in 0_{R'}$, and the same analysis applies. ■

Finally, to negate (A5), we shall first show that any addition with an open cut will produce another open cut.

Proof. Consider two cuts $\alpha \in R$, $\beta \in R'$. For any $r \in \alpha + \beta$, we have that $r = a + b$ for some $a \in \alpha$, $b \in \beta$. But, because α is open, we can pick some $a' \in \alpha$ such that $a' > a$, and therefore we have $a' + b > a + b \in \alpha + \beta$. ■

(A5) fails! Suppose we had a valid negation operation. Then we would have

$$0^* - 0^* = 0_{R'}$$

But this leads to a contradiction, as with the given definition of addition we cannot add any number to the open 0^* to produce the closed $0_{R'}$. ■

2 Basic Topology

21. Exercise 1: Prove that the empty set is a subset of every set.

Solution: (*ghostofgarborg*)
Let S be any set. The empty set $\emptyset$ is a subset of S iff for all $x \in \emptyset$, $x \in S$. This is vacuously true, so we are done.

22. Exercise 2: Prove that the set of algebraic numbers is countable.

Solution: (*ghostofgarborg*)
Let $\mathbb{A}$ be the set of algebraic numbers. Let p be a polynomial over $\mathbb{C}$. Using the division algorithm for polynomials over a field, we observe that $(z - \alpha) \mid p(x)$ iff α is a root of p. We can deduce that a polynomial of degree n has at most n roots.

Let P_n be the set of polynomials p in $\mathbb{Z}[x]$ for which the coefficients a_i satisfy $\sum_{i=1}^{m} |a_i| = n - \deg p$. This is a finite set, and $\bigcup P_n = \mathbb{Z}[x]$. Let V_n be the corresponding set of all roots of the polynomials in P_n. By the above observation, this set is finite. Consequently, $\mathbb{A} = \bigcup_n V_n$ is at most countable.

Since $\mathbb{Z} \subset \mathbb{A}$, it is also at least countable.

23. Exercise 3: Prove that there exist real numbers which are not algebraic.

Solution: (*ghostofgarborg*)
This must be the case, as the set of real numbers otherwise would be countable.

24. Exercise 4: Is the set of all irrational real numbers countable?

Solution: (*ghostofgarborg*)
No. We know the set of rational numbers is countable. Since $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ and $\mathbb{R}$ is uncountable, we must have that $(\mathbb{R} \setminus \mathbb{Q})$ is uncountable by theorem 2.12.

25. Exercise 5: Construct a bounded set of real numbers with exactly three limit points.

Solution: (*ghostofgarborg*)
Take e.g. the set

$$\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{1 + \frac{1}{n} : n \in \mathbb{N}\} \cup \{2 + \frac{1}{n} : n \in \mathbb{N}\}$$

26. Exercise 6: Prove that E' is closed. Prove that E and $\bar{E}$ have the same limit points. Do E and E' always have the same limit points?

Solution: (*ghostofgarborg*)
Let x be a limit point of E'. Any neighborhood N of x contains an $x' \in E'$ for which $x' \neq x$. By theorem 2.22, $N \setminus \{x\} = N \cap \{x\}^c$ is open, and therefore contains a neighborhood N' of x'. N' intersects E since $x' \in E'$, and consequently so does N. This implies that $x \in E'$, and since this holds for any limit point, E' is closed.

Since $E \subseteq \bar{E}$, a limit point of E is a limit point of $\bar{E}$. If x is a limit point of $\bar{E} = E \cup E'$, $N_{\frac{1}{n}}(x)$, must intersect either E for an infinite number of values of $n \in \mathbf{N}$, or it must intersect E' for an infinite number of values of n. (Otherwise, there would be an N for which $N_{\frac{1}{n}}(x)$ did not intersect either whenever $n > N$, a contradiction.) Consequently, since $N_\epsilon(x) \subseteq N_{\epsilon'}(x)$ whenever $\epsilon < \epsilon'$, all neighborhoods of x intersect either E or E'. In both cases, $x \in E'$. This finishes the proof.

Solution: (*Matt "frito" Lundy*)
Let $x \in (E')^c$. Then $x \notin E'$, so x is not a limit point of E. Hence there exists a neighborhood N of x that contains no points of E, except possibly x itself. If N contained a limit point y of E, then there would exist a neighborhood N_y of y that contains a point of E other than y such that $N_y \subset N$. This would contradict the fact that x is a limit point of E, so N must not contain any limits points of E. So $N \subset (E')^c$, and $(E')^c$ is open, thus E' is closed.

If x is a limit point of E, then $E \subset \bar{E}$ implies that x is also a limit point of $\bar{E}$. If x is a limit point of $\bar{E}$ but not a limit point of E, then there is a neighborhood N of x such that N contains no points of E other than possibly x itself. But x is a limit point of $\bar{E}$ means that there is a $q \in N$ such that $q \neq x$ and $q \in \bar{E}$. If $q \in E$, we have arrived at a contradiction. If $q \in E'$, then q is a limit point of

E, and there is some neighborhood of N_q of q such that $N_q \subset N$ that contains a point of E, which is also a contradiction. Therefore, E and $\bar{E}$ share the same limit points.

E and E' need not share the same limit points. Consider any segment $E = (a, b)$ in R^1.

Solution: (*analambanomenos*)
Let $x \notin E'$. Then there is a neighborhood N of x which contains no elements of E (other than possibly x). Hence every element of $y \neq x$ of N has has a small neighborhood contained in N containing no points of E, and so $y \notin E'$ also. Hence E'^c is open, and so E' is closed.

Since $E \subset \bar{E}$, $E' \subset \bar{E}'$. Since $\bar{E}$ is closed by Theorem 2.27(a), $x \in \bar{E}'$ is an element of $\bar{E} = E \cup E'$. Suppose $x \notin E'$. Then it has a neighborhood with no elements of E other than x itself, and since E' is closed, x has a neighborhood which contains no elements of E'. This contradicts the assumption that $x \in \bar{E}' = (E \cup E')'$, so $x \in E'$.

E and E' do not always have the same limit points. Take the answer to Exercise 5, for example. E' is a set of 3 points, so $E'' = \emptyset$.

27. Exercise 7: Let $B_n = \bigcup_{i=1}^n A_i$, $B = B_\infty$. (a) Show that $\bar{B}_n = \bigcup^n \bar{A}_i$. (b) Show that $\bar{B} \supset \bigcup^\infty \bar{A}_i$.

Solution: (*ghostofgarborg*)
(a) Assume $1 \leq i \leq n$. Since $A_i \subseteq B_n$, a limit point of A_i is a limit point of B_n. Let x be a limit point of B_n, and consider the neighborhoods $N_k = N_{\frac{1}{k}}(x)$ for $k \in \mathbf{N}$. There must be a j for which there is no K such that $k > K$ implies that N_k does not intersect A_j, as the negation of that statement would imply that x is not a limit point of B_n. Since $N_\ell \subset N_k$ whenever $\ell > k$, this implies that N_k intersects A_j for all $k \in \mathbb{N}$. Given any neighborhood $N_\epsilon(x)$ of x, we can find a k such that $N_k \subseteq N_\epsilon(x)$, so all neighborhoods of x intersect A_j, and $x \in \bar{A}_j$.

(b) It is clear that a limit point of A_i is a limit point of B, so that the inclusion holds. To see that the inclusion is proper, let $A_i = (\frac{1}{i}, 1)$. Then $\bar{A}_i = [\frac{1}{i}, 1]$, so that $\bigcup_i \bar{A}_i = (0, 1]$. However, $B = \bigcup_i (\frac{1}{i}, 1) = (0, 1)$, so that $\bar{B} = [0, 1]$, and $\bar{B} \supsetneq \bigcup \bar{A}_i$.

Solution: (*Matt "frito" Lundy*)
Lemma: For any collection $\{C_i\}$ of sets in a metric space, we have:

$$\bigcup_{i=1}^{n} C_i' = \left[\bigcup_{i=1}^{n} C_i\right]' \tag{1}$$

$$\bigcup_{i=1}^{\infty} C_i' \subset \left[\bigcup_{i=1}^{\infty} C_i\right]'. \tag{2}$$

To prove (1), first take $x \in \bigcup_{i=1}^n C_i'$, then $x \in C_i'$ for some i, so x is a limit point of some C_i. So x is a limit of point of $\bigcup_{i=1}^n C_i$ and $x \in [\bigcup_{i=1}^n C_i]'$. Hence $\bigcup_{i=1}^n C_i' \subset [\bigcup_{i=1}^n C_i]'$. (Note, a similar argument proves (2)).

Now take $x \in [\bigcup_{i=1}^n C_i]'$. x is a limit point of $\bigcup_{i=1}^n C_i$ means that for any neighborhood N of x, there is some $q \in N$, such that $q \neq x$ and $q \in \bigcup_{i=1}^n C_i$. But this means that $q \in C_i$ for some i, and x must be a limit point of some C_i, so $x \in \bigcup_{i=1}^n C_i'$. Hence $[\bigcup_{i=1}^n C_i]' \subset \bigcup_{i=1}^n C_i'$, completing the proof of (1).

(a)

$$\begin{aligned}\bar{B}_n &= B_n \cup B_n' \\ &= (\cup_{i=1}^n A_i) \cup \left([\cup_{i=1}^n A_i]'\right) \\ &= (\cup_{i=1}^n A_i) \cup (\cup_{i=1}^n A_i') \\ &= \cup_{i=1}^n (A_i \cup A_i') \\ &= \cup_{i=1}^n \bar{A}_i\end{aligned}$$

(b)

$$\begin{aligned}\bar{B} &= B \cup B' \\ &= (\cup_{i=1}^\infty A_i) \cup \left([\cup_{i=1}^\infty A_i]'\right) \\ &\supset (\cup_{i=1}^\infty A_i) \cup (\cup_{i=1}^\infty A_i') \\ &= \cup_{i=1}^\infty (A_i \cup A_i') \\ &= \cup_{i=1}^\infty \bar{A}_i\end{aligned}$$

See solution by *ghostofgarborg* for an example of a proper subset in (b).

Solution: (*analambanomenos*)
Recall Theorem 1.27(c): $\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

(a) Since $\cup_1^n \bar{A}_i$ is a closed set that contains B_n, we have $\bar{B}_n \subset \cup_1^n \bar{A}_i$. Conversely, since $\bar{B}_n$ is a closed set that contains A_i, $i = 1, \dots, n$, we have $\bar{A}_i \subset \bar{B}_n$, $i = 1, \dots, n$, so $\cup_1^n \bar{A}_i \subset \bar{B}_n$. Hence $\bar{B}_n = \cup_1^n \bar{A}_i$.

(b) Since $\bar{B}$ is a closed set containing each A_i, we have $\bar{A}_i \subset \bar{B}$ for each i, so $\cup_1^\infty \bar{A}_i \subset \bar{B}$.

(These are fast and cute, but ghostofgarborg's and frito's solutions give you a better idea of what's going on.)

28. Exercise 8: Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? What about closed sets?

Solution: (*ghostofgarborg*)
Open sets: Yes. Any point x in an open set E is contained in a neighborhood $N_\epsilon(x) \subset E$. Any point y such that $d(y, x) < \epsilon$ is contained in E. It is clear that any neighborhood of x contains such a point y.

Closed sets: No. A single point set $E = \{x\}$ is closed, but is not a limit point, since no neighborhood of x contains a point $y \in E$ such that $y \neq x$.

29. Exercise 9: (a) Prove that E° is always open. (b) E is open iff $E = E^\circ$. (c) If $G \subset E$ is open, then $G \subset E^\circ$. (d) Prove that $(E^\circ)^c = \overline{E^c}$. (e) Do E and $\bar{E}$ always have the same interiors? (f) Do E and E° always have the same closures?

Solution: (*ghostofgarborg*)
By definition, if $x \in E^\circ$, then x in an open subset $U \subset E$. Let $U \subset E$ be open. Then any point of

U has a neighborhood inside E, so that $U \subset E^\circ$. This establishes that

$$E^\circ = \bigcup_{\substack{U \subset E \\ U \text{ open}}} U$$

(a) E° is a union of open sets, and is therefore open. (Note: This is also true if the union is empty.)

(b) If $E = E^\circ$, then we know from (a) that E is open. For the converse, begin by noting that $E^\circ \subseteq E$, since it is a union of subsets of E. If E is open, then E is in the union, so that $E \subseteq E^\circ$ as well, from which equality follows.

(c) This is clear, as G is included in the union.

(d) By theorem 2.27 (a) and (e)

$$(E^\circ)^c = \bigcap_{\substack{U \subset E \\ U \text{ open}}} U^c = \bigcap_{\substack{V \supset E^c \\ V \text{ closed}}} V = \overline{E^c}$$

(e) No. Let $E = \mathbb{Q}$ in $\mathbb{R}$. Then $E^\circ = \emptyset$, but $(\bar{E})^\circ = \mathbb{R}^\circ = \mathbb{R}$.

(f) No. Let $E = \mathbb{Q}$ in $\mathbb{R}$. Then $\bar{E} = \mathbb{R}$, but $\overline{E^\circ} = \bar{\emptyset} = \emptyset$.

30. Exercise 10: Define $d(x, y) = 1$ if $x \neq y$ and 0 otherwise. Show that this is a metric. Identify which subsets are open, closed, compact.

Solution: (*ghostofgarborg*)
We see that $d(x, x) = 0$, and $d(x, y) \neq 0$ whenever $x \neq y$. We also have $d(x, y) = d(y, x)$. Lastly, consider

$$d(x, z) \leq d(x, y) + d(y, z)$$

The inequality is obvious if $x = z$, and otherwise either $y \neq x$ or $y \neq z$, so that the inequality still holds. The function d is thus a metric.

Note that every point set is open, since $\{x\} = N_{1/2}(x)$. Consequently, every subset of X is open, and therefore every subset of X is closed. Let S be a subset of X, and consider the open cover $\{x\}_{x \in S}$. This cover has a finite subcover iff S is finite. Any compact set must therefore be finite, and conversely, it is easy to see that any finite set is compact. It is thus clear that the compact sets are precisely those that are finite.

31. Exercise 11: For $x \in \mathbf{R}$ and $y \in \mathbf{R}$, define

$$\begin{aligned} d_1(x, y) &= (x - y)^2, \\ d_2(x, y) &= \sqrt{|x - y|}, \\ d_3(x, y) &= |x^2 - y^2|, \\ d_4(x, y) &= |x - 2y|, \\ d_5(x, y) &= \frac{|x - y|}{1 + |x - y|}. \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

Solution: (*ghostofgarborg*)
d_2 and d_5 are metrics, the others are not. We note that $d_3(1,-1) = 0$, and therefore violates condition (a). $d_4(0,1) \neq d_4(1,0)$, and violates condition (b). We further note that

$$d_1(1,-1) = 4 > 2 = d_1(1,0) + d_1(0,-1)$$

which violates (c).

We now note the following:

Let $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be a strictly increasing function such that $f(0) = 0$, which is subadditive, i.e.:

$$f(a+b) \leq f(a) + f(b)$$

and let d be a metric. Then $f \circ d$ is a metric. That $f \circ d$ satisfies condition (a) follows from the injectivity of f, and from the fact that $f(0) = 0$. That it satisfies (b) follows because d does. Lastly, if

$$d(a,c) \leq d(a,b) + d(b,c)$$

then by virtue of being increasing and subadditive:

$$f \circ d(a,c) \leq f(d(a,b) + d(b,c)) \leq f \circ d(a,b) + f \circ d(b,c)$$

which establishes (c).

To show that d_2 and d_5 are metrics, it therefore suffices to show that $\sqrt{x}$ and $\frac{x}{1+x}$ satisfy the criteria on f above.

We know from chapter 1 that $\sqrt{x}$ is strictly increasing, and that $\sqrt{0} = 0$. Subadditivity follows by noting that

$$a + b \leq a + b + 2\sqrt{ab} = (\sqrt{a} + \sqrt{b})^2$$

Therefore, d_2 is a metric.

The function

$$\frac{x}{1+x} = 1 - \frac{1}{1+x}$$

is clearly strictly increasing, and $0/(1+0) = 0$. Moreover,

$$\frac{a+b}{1+a+b} = \frac{a}{1+a+b} + \frac{b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b}$$

Consequently, d_5 is a metric.

32. Exercise 12: Let $K \subset \mathbf{R}$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

Solution: (*ghostofgarborg*)
Let $\mathcal{U}$ be an open cover for K. Then there exists a $U \in \mathcal{U}$ for which $0 \in U$. By openness, there exists an $\epsilon > 0$ such that $N_\epsilon(0) \subset U$. By the archimedean property, there exists an N such that $N\epsilon > 1$, i.e. such that $\frac{1}{n} < \epsilon$ for $n > N$. Therefore, $N_\epsilon(0)$ contains all but a finite number of elements of K. We can now pick a set in $\mathcal{U}$ for each of the remaining points, and obtain a finite cover.

33. Exercise 13: Construct a compact set of real numbers whose limit points form a countable set.

Solution: (*ghostofgarborg*)
Note that $\left|\frac{1}{n} - \frac{1}{n-1}\right| = \frac{1}{n(n-1)} > \frac{1}{n^2}$, so that the sets

$$S_n = \{\frac{1}{n} + \frac{1}{m} : m \geq n^2\} \cup \{\frac{1}{n}\}$$

are disjoint. Consider the set $S = \bigcup_n S_n \cup \{0\}$. The set is clearly bounded, so we need to show that it is closed, and that the limit points are countable.

We claim that the limit points are precisely $L = \{\frac{1}{n} : n \in \mathbb{N}\} \cup 0$. Since L is contained in S and is countable, that will suffice to prove our claim. It is easy to see that these points are limit points. To see that there are no other, observe that any point not in $[0, 2]$ or the previously mentioned set is either strictly greater than the lub or strictly smaller than the glb of all but at most one of the disjoint sets S_j. It would consequently have to be a limit point of S_j, but all such limit points are contained in L. Lastly, we can easily see that 0 is a limit point, but 2 is not.

34. Exercise 14: Give an example of a cover of $(0, 1)$ which does not have a finite subcover.

Solution: (*ghostofgarborg*)
Let $U_n = (\frac{1}{n}, 1 - \frac{1}{n})$. This collection evidently covers $(0, 1)$, but has no finite subcover.

35. Exercise 15: Show that Thm 2.36 and its corollary become false if "compact" is replaced with "closed" or "bounded"

Solution: (*ghostofgarborg*)
Possible examples from $\mathbb{R}$:
Closed: Let $S_n = [n, \infty)$. Then $\bigcap_n S_n$ is empty.
Bounded: Let $S_n = (0, \frac{1}{n})$. Then $\bigcap S_n$ is empty.

36. Exercise 16: Show that E is closed and bounded, but not compact. Is it open?

Solution: (*ghostofgarborg*)
Note that $E = [\sqrt{2}, \sqrt{3}] \cap \mathbb{Q}$. It follows from thm. 2.23 and thm. 2.30 that E is closed. Boundedness is clear.

Let a_n be a sequence in $\mathbb{Q}$ that converges from above to $\sqrt{2}$ (as described in chapter 1). Let $A_n = (a_n, \sqrt{3}) \cap \mathbb{Q}$. This is an open cover of E, but clearly has no finite subcover. E is consequently not compact.

E is open by thm 2.30, since $E = (\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$.

37. Exercise 17: Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0, 1]$? Is E compact? Is E perfect?

Solution: (*ghostofgarborg*)
Let A be the set of sequences on the digits 0 and 1. We get an injection $A \to E$ by associating with a sequence a_n the number whose nth decimal is 4 if $a_n = 1$ and 7 otherwise. By thm. 2.14, this makes E uncountable.

$E \subset [0.4, 0.8]$, and is therefore not dense in $[0, 1]$.

Since E is bounded, compactness is equivalent to closedness. Assume $x \notin E$, and let x_n be the nth digit in its decimal expansion. For some N, x_N is not equal to 4 or 7. Choose Δ such that the Nth decimal digit of all x in a Δ-neighborhood of x is x_n. (Note that this can be done.) This is a neighborhood of x that does not intersect E. Therefore E^c is open and E is closed, hence compact.

Let $x \in E$, choose $\epsilon > 0$, and choose n such that $10^{-n} < \epsilon$. The number obtained by changing the nth digit of x from 4 to 7 or vice versa, is contained in an ϵ-neighborhood of x. This means that x is a limit point, so E is perfect.

38. Exercise 18: Is there a nonempty perfect set in $\mathbb{R}$ which contains no rational number?

Solution: (*ghostofgarborg*)
Yes. Here is one example: Choose two irrational numbers, e.g. $\alpha = \sqrt{2}$ and $\beta = \sqrt{3}$, and let $\{a_n\}$ be an enumeration of the rational numbers in $E_1 = [\alpha, \beta]$.

Let r_1 be the first element of $\{a_n\}$ that is in E_1, and pick irrational numbers $\alpha_1 < r_1$, $\beta_1 > r_1$ such that $\alpha < \alpha_1 < r_1 < \beta_1 < \beta$. By removing the interval (α_1, β_1), we end up with a union of closed intervals

$$E_2 = [\alpha, \alpha_1] \cup [\beta_1, \beta]$$

By repeating this process for each closed interval in this union and continuing this process in the same vein as the construction of the Cantor set, we end up with a sequence of compact sets

$$E_1 \supset E_2 \supset \cdots$$

Now consider $E = \bigcup_{n=1}^{\infty} E_n$. The reasoning on p. 41 carries through with small modifications, so that we can deduce that E is perfect. By construction, there are no rational numbers in E.

39. Exercise 19: (a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.

    (b) Prove the same for disjoint open sets.

    (c) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.

    (d) Prove that every connected metric space with at least two points is uncountable.

Solution: (*ghostofgarborg*)
(a) Assume A and B are closed and disjoint

$$\bar{A} \cap B = A \cap B = \emptyset$$

$$A \cap \bar{B} = A \cap B = \emptyset$$

so that they are separated.

(b) A^c is closed, so by thm 2.27 (c), $\bar{B} \subset A^c$, and $A \cap \bar{B} = \emptyset$. Similarly by interchanging A and B.

(c) A and B are open and disjoint sets. They are separated by (b).

(d) Fix a point x of a connected metric space M, and assume for contradiction that the set of other points is non-empty and at most countable. Then $D = \{d(x, y)\}_{y \neq x}$ is countable, and we can pick $r \in \mathbb{R} \setminus D$ such that $r > 0$ and such that there is a y with $d(x, y) > r$. By the choice of r,

$$M = \{y \in M : d(x, y) < r\} \cup \{y \in M : d(x, y) > r\}$$

which by (c) would imply that M is disconnected, a contradiction.

40. Exercise 20: Are closures and interiors of connected sets always connected?

Solution: (*Dan "kyp44" Whitman*)
We claim first that closures of connected sets are always connected. To show this consider any connected set E and consider nonempty sets A and B where $\bar{E} = A \cup B$. Now, let $\breve{E} = \bar{E} - E$, $C = A - \breve{E}$, and $D = B - \breve{E}$.

First we show that that $E = C \cup D$. So consider any $x \in E$. Then $x \in E \cup E' = \bar{E}$, from which it follows that $x \in A$ or $x \in B$ since $\bar{E} = A \cup B$. Since $x \in E$ it follows that $x \notin \breve{E}$. Then $x \in A - \breve{E} = C$ or $x \in B - \breve{E} = D$ so that $x \in C \cup D$. Thus $E \subseteq C \cup D$. Now consider any $x \in C \cup D$. If $x \in C$ then $x \in A$ and $x \notin \breve{E}$. It follows that $x \notin \bar{E}$ or $x \in E$, which is logically equivalent to $x \in \bar{E} \to x \in E$. But since $x \in A$, $x \in \bar{E} = A \cup B$ so that indeed $x \in E$. A similar argument shows that $x \in E$ if $x \in D$. Therefore $C \cup D \subseteq E$, hence $E = C \cup D$ as claimed.

Now since E is connected and $E = C \cup D$, by the definition of connectedness $\bar{C} \cap D \neq \emptyset$ and $C \cap \bar{D} \neq \emptyset$. Thus there is an $x \in \bar{C} \cap D$ so that $x \in \bar{C}$ and $x \in D$. Since $x \in D$, $x \in B$ by definition, and since $x \in \bar{C}$, $x \in C \cup C'$. If $x \in C$ then $x \in A$ by definition so that $x \in A \cup A' = \bar{A}$. On the other hand if $x \in C'$ then x is a limit point of C. In this case consider any neighborhood $N_r(x)$. Then there is a $y \in N_r(x)$ where $y \neq x$ and $y \in C$. So since $C = A - \breve{E}$, $y \in A$. From this it follows that x is also a limit point of A since $N_r(x)$ was arbitrary and $y \neq x$. Thus $x \in A'$ so that $x \in A \cup A' = \bar{A}$. Hence in either case $x \in \bar{A}$ so since also $x \in B$, $x \in \bar{A} \cap B$ so that $\bar{A} \cap B \neq \emptyset$. A similar argument shows that $A \cap \bar{B} \neq \emptyset$, thereby showing that $\bar{E}$ is connected.

However there is an example of a set in $\mathbb{R}^2$, that is connected but whose interior is not. Such an example can be constructed from the following simple sets:

$$A = N_1((-2, 0))$$

$$B = N_1((2, 0))$$

$$C = \{(x, 0) : x \in (-2, 2)\}$$

Then $E = A \cup B \cup C$ is clearly connected and yet its interior is not since the interior of C in $\mathbb{R}^2$ is the empty set, thereby separating the interiors of A and B.

Solution: (*ghostofgarborg*)
Closures of connected sets are connected: Let E be connected, and assume $\bar{E} = A \sqcup B$, both non-empty, such that $\bar{A} \cap B = A \cap \bar{B} = \emptyset$. Since E is connected,

$$\overline{(A \cap E)} \cap (B \cap E) \subset \bar{A} \cap B = \emptyset$$
$$(A \cap E) \cap \overline{(B \cap E)} \subset A \cap \bar{B} = \emptyset$$

force $A \cap E = \emptyset$ or $B \cap E = \emptyset$, so that $A \subset E' \setminus E$ or $B \subset E' \setminus E$. Without loss of generality, assume it is the former, so that $E \subset B$. Then $A \cap \bar{B} \supset A \cap \bar{E} = A$, a contradiction.

Interiors of connected sets need not be connected: Let

$$E = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\} \cup \{(x, y) \in \mathbb{R}^2 : x \leq 0, y \leq 0\}$$

Given two points $\mathbf{x}_1 \neq \mathbf{x}_2 \in E$, the map $f : [0, 1] \to E$

$$f(t) = \begin{cases} \mathbf{x}_1 - 2t\mathbf{x}_1 & 0 \leq t \leq \frac{1}{2} \\ 2t\mathbf{x}_2 - \mathbf{x}_2 & \frac{1}{2} < t \leq 1 \end{cases}$$

is a continuous path between them, showing connectedness. However,

$$E^\circ = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\} \cup \{(x, y) \in \mathbb{R}^2 : x < 0, y < 0\}$$

which is the union of two open, disjoint sets and therefore disconnected.

41. Exercise 21: Let A and B be separated subsets of some $\mathbf{R}^k$, suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1-t)\mathbf{a} + t\mathbf{b}$$

for $t \in \mathbf{R}$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$.

(a) Prove that A_0 and B_0 are separated subsets of $\mathbf{R}$.

(b) Prove that there exists $t_0 \in (0, 1)$ such that $\mathbf{p}(t_0) \notin A \cup B$.

(c) Prove that every convex subset of $\mathbf{R}^k$ is connected.

Solution: (*ghostofgarborg*)
(a) We note that when $\epsilon > 0$

$$\|\mathbf{p}(t \pm \epsilon) - \mathbf{p}(t)\| = \epsilon \|\mathbf{b} - \mathbf{a}\|$$

and if $|t - t'| < \epsilon / \|\mathbf{b} - \mathbf{a}\|$, then $\|\mathbf{p}(t) - \mathbf{p}(t')\| < \epsilon$. Therefore, if $\mathbf{p}(t) = \mathbf{y}$, then

$$N_{\epsilon/\|\mathbf{b}-\mathbf{a}\|}(t) \subset \mathbf{p}^{-1}(N_\epsilon(\mathbf{y}))$$

Consequently, the inverse image under $\mathbf{p}$ of an open set is open, and the inverse image of a closed set is closed. Clearly A_0 and B_0 are disjoint and non-empty, so this allows us to conclude that

$$\bar{A}_0 \cap B_0 \subseteq \mathbf{p}^{-1}(\bar{A}) \cap \mathbf{p}^{-1}(B) = \mathbf{p}^{-1}(\bar{A} \cap B) = \emptyset$$

And similarly when interchanging A and B. This shows that A_0 and B_0 are separated.

(b) Restrict $\mathbf{p}$ to $[0, 1]$, and define A_0 and B_0 correspondingly. The result in (a) still holds. Since $[0, 1]$ is connected, we must have $[0, 1] \supsetneq A_0 \cup B_0$. Choose $t_0 \in [0, 1] \setminus A_0 \cup B_0$. Then $t_0 \neq 0$ and $t_0 \neq 1$, and $\mathbf{p}(t_0) \notin A \cup B$.

(c) Let $S \subset \mathbb{R}^k$. The above result shows that if $\mathbf{p}(t) \in S$ for all $\mathbf{a}, \mathbf{b} \in S$ and $t \in [0, 1]$, then S is connected. This is precisely the condition that S is convex.

42. Exercise 22: Show that $\mathbb{R}^k$ is separable.

Solution: (*ghostofgarborg*)
Assume for contradiction that $\mathbb{R}^k = \bigcup^\infty F_n$ where each F_n is closed and has non-empty interior. Note that $\mathbb{Q}^k$ is countable. Choose $\epsilon > 0$. If $(x_1, \cdots, x_n) \in \mathbb{R}^k$, pick rational r_i such that $|r_i - x_i| < \epsilon/\sqrt{k}$. Then

$$\|(x_1, \cdots, x_k) - (r_1, \cdots, r_k)\| < \left(\sum_1^k \epsilon^2 / k \right)^{\frac{1}{2}} = \epsilon$$

Consequently, any neighborhood around any point of $\mathbb{R}^k$ contains a point of $\mathbb{Q}^k$, and $\overline{\mathbb{Q}^k} = \mathbb{R}^k$, which implies separability.

43. Exercise 23: Every separable space has a countable base.

Solution: (*ghostofgarborg*)
Let B be a countable, dense subset of X. Note that the collection $\mathcal{B} = \{N_{\frac{1}{n}}(x) : x \in B, n \in \mathbb{N}\}$ is countable. We claim that it is a base. Let $x \in U$, U open. Then $N_\epsilon(x) \subset U$ for some ϵ. Pick k such that $\frac{1}{k} < \frac{\epsilon}{2}$. By density, we can find $y \in B$ such that $d(x, y) < \frac{1}{k}$. Then $x \in N_{\frac{1}{k}}(y)$, which is in $\mathcal{B}$, and $N_{\frac{1}{k}}(y) \subset N_\epsilon(x) \subset U$. This finishes the proof.

44. Exercise 24: Let X be a metric space in which every infinite subset has a limit point. Then X is separable.

Solution: (*ghostofgarborg*)
We can assume X is infinite. Pick $\delta > 0$, pick an $x_1 \in X$, and inductively pick x_{n+1} such that $d(x_i, x_{n+1}) > \delta$ for all $1 \leq i < n + 1$ if possible. This process must terminate. Otherwise, $\{x_i\}_{i=1}^\infty$ would have a limit point x, and $N_{\frac{\delta}{2}}(x)$ would contain two distinct points of the sequence, a contradiction.

X can therefore be covered with a finite number of neighborhoods of radius δ around a finite number of points $x_1, \cdots, x_k$. Let $\mathcal{B}$ be the collection of all such neighborhoods for $\delta = \frac{1}{n}$, $n \in \mathbb{N}$. This is a countable collection. Given a neighborhood $N_\epsilon(x)$ of $x \in X$, pick k such that $\frac{1}{k} < \frac{\epsilon}{2}$. Then x is covered by some neighborhood $N \in \mathcal{B}$ of radius $\frac{1}{k}$, and $N \subset N_\epsilon(x)$. This shows that $\mathcal{B}$ is a countable base.

45. Exercise 25: Every compact metric space K has a countable base, and is therefore separable.

Solution: (*ghostofgarborg*)
A finite number of neighborhoods of radius δ cover K. By the reasoning in ex. 24, K has a countable base.

To see that this implies that K is separable, let $\{U_n\}$ be a countable base, and choose $x_n \in U_n$, so that we obtain a countable subset $S = \{x_n\}_{n=1}^\infty$. Any open set in K contains U_n, and therefore an element of S. The S is therefore dense in K.

46. Exercise 26: Let X be a metric space in which every infinite subset has a limit point. X is compact.

Solution: (*ghostofgarborg*)
We know that X is separable by ex. 24, and that it has a countable base by ex. 23. Since any open set is a union of base sets, we can reduce any open cover to a countable subcover. Let $\mathcal{G} = \{G_n\}$ be such a subcover. Assume $\mathcal{G}$ has no finite subcover, so that $F_n = (G_1 \cup \cdots \cup G_n)^c$ is non-empty for all n. Pick $x_n \in F_n$. They constitute an infinite subset, and therefore have a limit point x. This point is in G_k for some k. But then F_k^c is an open set around x that contains only a finite number of the x_n, contradicting x being a limit point. Any open cover of X must therefore have a finite subcover, so X is compact.

47. Exercise 27: Suppose $E \subset \mathbb{R}^k$, E uncountable, and let P be the set of condensation points for E. P is perfect, and at most a countable number of points of E are not in P.

Solution: (*ghostofgarborg*)
Let $\{V_n\}$ be a countable base for $\mathbb{R}^k$. A point x is a condensation point of E if and only if every V_n that contains x contains an uncountable number of points of E. Consequently, P^c is the union of

those V_n for which $V_n \cap E$ is at most countable. $P^c \cap E$ is then at most countable.

48. Exercise 28: Every closed set in a separable metric space is the union of a perfect set and a set which is at most countable.

Solution: (*ghostofgarborg*)
We observe that the proof in ex. 27 goes through for any separable metric space. Assume that E is closed. Since P consists of limit points of E, $P \cap E = P$. We can therefore write $E = P \cup (P^c \cap E)$, which is a union of a perfect set and a set which is at most countable.

49. Exercise 29: Every open set in $\mathbb{R}$ is the union of an at most countable collection of disjoint segments.

Solution: (*ghostofgarborg*)
Let V be the open set in question. We know that the collection $\mathcal{U}$ consisting of all segments of the form $(r - \alpha, r + \beta)$, $r, \alpha \in \mathbb{Q}$, $\alpha, \beta \in \mathbb{Q}^+$, is a countable basis for $\mathbb{R}$.

Let $\mathcal{V}$ be the subcollection of sets of $\mathcal{U}$ that are contained in V. Let I_x be the union of all sets $U \in \mathcal{V}$ such that U intersects a segment in $\mathcal{V}$ that contains x. It is clear that I_x is a segment, $I_x \subset V$, and that if $y \in V$ then either $I_x = I_y$ or they are disjoint.

The collection $\{I_x\}_{x \in V}$ covers V, and since $\mathbb{R}$ is separable, it can be reduced to a countable subcover $\mathcal{I}$. It is then clear that $\mathcal{I}$ has the desired properties.

50. Exercise 30: If $\mathbb{R}^k = \bigcup^\infty F_n$ where each F_n is closed, then at least one F_n has non-empty interior.

Solution: (*ghostofgarborg*)
Assume for contradiction that $\mathbb{R}^k = \bigcup^\infty F_n$ where each F_n is closed and has non-empty interior. Let N_0 be a ball of finite radius around a point $x_1 \in F_1$, so that $\bar{N}_0$ is compact. Assume N_{i-1} is open and does not contain any points of $F_1, \cdots F_{i-1}$. This set must contain a point x_i not in F_i, as it otherwise would belong to the interior of F_i. Furthermore, x_i must be contained in a neighborhood $N_i \subseteq N_{i-1}$ that does not intersect F_i, as x_i otherwise would be a limit point of F_i and therefore belong to F_i. We can choose N_i such that $\bar{N}_i \subseteq N_{i-1}$, and we observe that it does not contain any points of $F_1, \cdots, F_i$.

Now observe that each $\bar{N}_i$ is compact, and that $\bar{N}_{i+1} \subseteq \bar{N}_i$, so that by the corollary to thm. 2.36, $I = \bigcap_i \bar{N}_i$ is non-empty. However, by construction, if $x \in I$, then $x \notin F_i$ for any i. But this implies that $x \notin \bigcup F_i = \mathbb{R}^k$, a contradiction.

3 Numerical Sequences and Series

51. Exercise 1: Show that if $\{s_n\}$ converges, so does $\{|s_n|\}$. Is the converse true?

Solution: (*ghostofgarborg*)
Assume $s_n \to s$. Choose $\epsilon > 0$, and let N be such that $|s_n - s| < \epsilon$ when $n > N$. Then

$$||s_n| - |s|| \leq |s - s_n| < \epsilon$$

whenever $n > N$. This implies convergence of $|s_n|$.

The converse is not true. Let $s_n = (-1)^n$. This sequence does not converge, even though $|s_n|$ does.

52. Exercise 2: Calculate $\lim_{n\to\infty}\big(\sqrt{n^2+n}-n\big)$.

Solution: (*ghostofgarborg*)
Note that $(\sqrt{n^2+n}-n) = n/(\sqrt{n^2+n}+n)$, and and therefore

$$\frac{1}{2+\frac{1}{\sqrt{n}}} = \frac{n}{(n+\sqrt{n})+n} < \sqrt{n^2+n}-n < \frac{n}{n+n} = \frac{1}{2}$$

where $(2+1/\sqrt{n})^{-1} \to \frac{1}{2}$. By theorem 3.19, $\lim_{n\to\infty}\sqrt{n^2+n}-n = \frac{1}{2}$.

Solution: (*Dan "kyp44" Whitman*)
We can calculate this limit directly:

$$\lim_{n\to\infty}(\sqrt{n^2+n}-n) = \lim_{n\to\infty}(\sqrt{n^2+n}-n)\left(\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}\right)$$

$$= \lim_{n\to\infty}\frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}+n} = \lim_{n\to\infty}\frac{1}{\sqrt{1+1/n}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2},$$

noting that clearly $\sqrt{n^2+n}+n > 0$.

53. Exercise 3: If $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2+\sqrt{s_n}} \qquad (n = 1, 2, 3, \ldots),$$

prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, \ldots$.

Solution: (*ghostofgarborg*)
Note that if $s_n < 2$, $s_{n+1} = \sqrt{2+\sqrt{s_n}} < \sqrt{4} = 2$, so the sequence is bounded. Also note that if $s_{n-1} > s_{n-2}$, then $s_n = \sqrt{2+\sqrt{s_{n-1}}} > \sqrt{2+\sqrt{s_{n-2}}} = s_{n-1}$, so the sequence is monotonically increasing. It therefore converges by thm 3.14.

54. Exercise 4: Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0, \qquad s_{2m} = \frac{s_{2m-1}}{2}, \qquad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Solution: (*ghostofgarborg*)
We observe that if $a_n = (2^n-1)/2^n$ and $b_n = (2^{n-1}-1)/2^n$, then $s_{2n+1} = a_n$ and $s_{2n} = b_n$ fulfills both the initial condition and the recursion, so that this is a closed form of the sequence. Note that $a_n \to 1$, $b_n \to \frac{1}{2}$, and that any subsequence of s_n contains either a subsequence of a_n or a subsequence of b_n. Consequently, any convergent subsequence converges to either 1 or $\frac{1}{2}$. We get

$$\limsup_{n\to\infty} s_n = 1 \qquad \liminf_{n\to\infty} s_n = \frac{1}{2}$$

55. Exercise 5: For any two real sequences $\{a_n\}$, $\{b_n\}$, prove that

$$\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

provided that the sum on the right is not of the form $\infty - \infty$.

Solution: (*ghostofgarborg*)
If $\limsup a_n = \infty$ and $\limsup b_n > -\infty$, the result is trivial. If $\limsup a_n = -\infty$ and $\limsup b_n < \infty$, then b_n is bounded above by B and given a number N

$$\#\{n : a_n + b_n > N\} \leq \#\{n : a_n > N - B\}$$

the latter of which is finite. Therefore $\limsup(a_n + b_n) = -\infty = \limsup a_n + \limsup b_n$.

We can therefore assume that both $\limsup a_n$ and $\limsup b_n$ are finite. There is a subsequence such that $a_{n_k} + b_{n_k} \to \limsup(a_n + b_n)$. By choosing this subsequence right, i.e. if necessary passing to another subsequence, we can make sure that a_{n_k} is convergent. Then $b_{n_k} = (a_{n_k} + b_{n_k}) - a_{n_k}$ is convergent, so

$$\limsup_n(a_n+b_n) = \lim_k(a_{n_k}+b_{n_k}) = \lim_k a_{n_k} + \lim_k b_{n_k} \leq \limsup_n a_n + \limsup_n b_n$$

which finishes the proof.

56. Exercise 6: Investigate the behavior (convergence of divergence) of $\sum a_n$ if

(a) $a_n = \sqrt{n+1} - \sqrt{n}$,

(b) $a_n = \dfrac{\sqrt{n+1}-\sqrt{n}}{n}$,

(c) $a_n = \left(\sqrt[n]{n} - 1\right)^n$,

(d) $a_n = \dfrac{1}{1+z^n}$, for complex values of z.

Solution: (*ghostofgarborg*)
(a) Since the series telescopes, $S_n = \sum_{i=0}^n (\sqrt{i+1} - \sqrt{i}) = \sqrt{n+1}$. This shows that it diverges to ∞.

(b)

$$a_n = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} < \frac{1}{n\sqrt{n}}$$

Converges by the comparison test (thm. 3.25).

(c)

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty}(\sqrt[n]{n} - 1) = 1 - 1 = 0$$

Converges by the root test.

(d) When $|z| \leq 1$

$$\left|\frac{1}{1+z^n}\right| \geq \frac{1}{1+|z|^n} \geq \frac{1}{2}$$

violating the necessary condition that $a_n \to 0$. In this case, $\sum a_n$ is divergent. When $|z| > 1$,

$$\left|\frac{1}{1+z^n}\right| \leq \frac{1}{|z|^n - 1} \leq \frac{1}{|z|^n - |z|^{n-1}} = \left(\frac{|z|}{|z|-1}\right)\frac{1}{|z|^n}$$

and $\sum a_n$ converges by comparison with the geometric series $\frac{|z|}{|z|-1}\sum \frac{1}{|z|^n}$.

57. Exercise 7: Prove that the convergence of $\sum a_n$ implies the convergence of

$$\sum \frac{\sqrt{a_n}}{n},$$

if $a_n \geq 0$.

Solution: (*ghostofgarborg*)
Assume $\sum a_n = C$. We know that $\sum \frac{1}{n^2}$ converges. Let D be the number it converges to. Note that $S_n = \sum_{i=0}^{n} \sqrt{a_i}/i$ is monotonic. By the Cauchy-Schwarz inequality

$$\sum_{i=1}^{n} \frac{\sqrt{a_i}}{i} \leq \left(\sum_{i=1}^{n} a_i\right)^{\frac{1}{2}} \left(\sum_{i=0}^{n} \frac{1}{i^2}\right)^{\frac{1}{2}} < \sqrt{C \cdot D}$$

This implies that $\sum \frac{\sqrt{a_n}}{n}$ is bounded, hence converges.

58. Exercise 8: If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

Solution: (*ghostofgarborg*)
We first note that thm 3.42 holds for b_n a monotonously increasing sequence whose limit is 0 as well, since $(-b_n)$ then fulfills the criteria of the theorem, and $\sum a_n b_n = -\sum a_n(-b_n)$.

If $\sum a_n$ converges, the partial sums form a bounded sequence. If b_n is monotonic and bounded it converges to a number B, and we get that

$$\sum a_n b_n = \sum a_n(b_n - B) + B\sum a_n$$

The first sum on the right hand side converges by thm 3.42 and the observation above. The second sum converges because $\sum a_n$ does. Consequently the left hand side converges.

59. Exercise 9: Find the radius of convergence of each of the following power series:

$$\text{(a)} \sum n^3 z^n, \qquad \text{(b)} \sum \frac{2^n}{n!} z^n, \qquad \text{(c)} \sum \frac{2^n}{n^2} z^n, \qquad \text{(d)} \sum \frac{n^3}{3^n} z^n.$$

Solution: (*ghostofgarborg*)
(a)

$$\frac{1}{R} = \lim_{n\to\infty} \frac{(n+1)^3}{n^3} = \lim_{n\to\infty} (1 + \frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3}) = 1$$

So $R = 1$.

(b)

$$\frac{1}{R} = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n\to\infty} \frac{2}{n+1} = 0$$

So $R = \infty$.

(c)

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^2}} = \lim_{n\to\infty} \frac{2}{(\sqrt[n]{n})^2} = 2$$

So $R = \frac{1}{2}$.

(c)

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{\frac{n^3}{3^n}} = \lim_{n\to\infty} \frac{(\sqrt[n]{n})^3}{3} = \frac{1}{3}$$

So $R = 3$.

60. Exercise 10: Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

Solution: (*ghostofgarborg*)
If $|z| \geq 1$ and $a_n \neq 0$ for an infinite number of values of n, then $|a_n z^n| \geq |a_n|$, and does not tend to 0. This makes the series divergent when $|z| \geq 1$.

61. Exercise 11: Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges.

(a) Prove that $\sum\big(a_n/(1+a_n)\big)$ diverges.

(b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum(a_n/s_n)$ diverges.

(c) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum(a_n/s_n^2)$ converges.

(d) What can be said about

$$\sum \frac{a_n}{1+na_n} \quad \text{and} \quad \sum \frac{a_n}{1+n^2 a_n}?$$

Solution: (*ghostofgarborg*)
(a) If $a_n \leq 1$, then $\frac{a_n}{1+a_n} \geq \frac{a_n}{2}$. If $a_n > 1$, then $\frac{a_n}{1+a_n} > \frac{1}{2}$. Therefore

$$\sum \frac{a_n}{1+a_n} \geq \frac{1}{2} \sum \min(a_n, 1)$$

If $\{n : a_n > 1\}$ is infinite, this clearly diverges. Otherwise, there is an N such that $n > N$ implies $a_n \leq 1$, in which case the series diverges by comparison to $\frac{1}{2}\sum a_n$.

(b) Since S_n is monotonically increasing, whenever $j \leq k$

$$\frac{a_{n+j}}{S_{n+j}} \geq \frac{a_{n+j}}{S_{n+k}}$$

Consequently

$$\frac{a_{N+1}}{S_{N+1}} + \cdots + \frac{a_{N+k}}{S_{N+k}} \geq \frac{a_{N+1}}{S_{N+k}} + \cdots + \frac{a + N + k}{S_{N+k}} = \frac{a_{N+1} + \cdots + a_{N+k}}{S_{N+k}} = 1 - \frac{S_N}{S_{N+k}}$$

Since S_n is increasing and unbounded, it is possible to choose k such that S_N/S_{N+k} is arbitrarily close to 0. This shows that $\frac{a_n}{S_n}$ is not Cauchy, and therefore not convergent.

(c) Since S_n is monotonically increasing, we have that $S_n^2 \geq S_{n-1}S_n$, so we can deduce that

$$\frac{a_n}{S_n^2} \leq \frac{S_n - S_{n-1}}{S_{n-1}S_n} = \frac{1}{S_{n-1}} - \frac{1}{S_n}$$

Since $1/S_n$ is bounded

$$\sum_{n=1}^{N} \frac{a_n}{S_n^2} \leq \frac{a_1}{S_1^2} + \sum_{n=2}^{N} \left(\frac{1}{S_{n-1}} - \frac{1}{S_n} \right) \leq \frac{a_1}{S_1^2} + \frac{1}{S_1} - \frac{1}{S_N}$$

so $\sum a_n/S_n^2$ is monotonic and bounded, hence convergent.

(d) Since

$$\frac{a_n}{1+n^2 a_n} \leq \frac{1}{n^2}$$

$\sum a_n/(1+n^2a_n)$ converges. The series $\sum a_n/(1+na_n)$, however, might converge or diverge. Let $a_n = \frac{1}{n}$, and it is clear that it diverges. Let $a_n = 1$ whenever n is a square and $a_n = 2^{-n}$ otherwise. This series clearly diverges, since the terms do not tend to 0 as $n \to \infty$. Then

$$\sum_{n=1}^{N} \frac{a_n}{1+na_n} \leq \sum_{n=1}^{N} \frac{1}{n^2} + \sum_{n=1}^{N} \frac{1}{2^n}$$

and the series therefore converges.

62. Exercise 12: Suppose $a_n > 0$ and $\sum a_n$ converges. Put

$$r_n = \sum_{m=n}^{\infty} a_n.$$

(a) Prove that

$$\frac{a_m}{r_m} + \ldots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if $m < n$, and deduce that $\sum(a_n/r_n)$ diverges.

(b) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right)$$

and deduce that $\sum\left(a_n/\sqrt{r_n}\right)$ converges.

Solution: (*ghostofgarborg*)
(a) Since r_n is monotonically decreasing

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > \frac{a_m}{r_m} + \cdots + \frac{a_{n-1}}{r_m} = \frac{a_m + \cdots + a_{n-1}}{r_m} = \frac{r_m - r_n}{r_m} = 1 - \frac{r_n}{r_m}$$

Since $r_n \to 0$, given any M we can find an $N > M$ such that $1 - \frac{R_N}{R_M}$ is arbitrarily close to 1. This implies that a_n/r_n is not Cauchy, hence not convergent.

(b)

$$\frac{a_n}{\sqrt{r_n}} = \frac{a_n\left(1+\frac{\sqrt{r_{n+1}}}{\sqrt{r_n}}\right)}{\sqrt{r_n}+\sqrt{r_{n+1}}} < \frac{2a_n}{\sqrt{r_n}+\sqrt{r_{n+1}}} = \frac{2(r_n - r_{n+1})}{\sqrt{r_n}+\sqrt{r_{n+1}}} = 2(\sqrt{r_n}-\sqrt{r_{n+1}})$$

Since

$$\sum_{n=1}^{N} 2(\sqrt{r_n}-\sqrt{r_{n+1}}) = 2(\sqrt{r_1}-\sqrt{r_{N+1}}) < 2\sqrt{r_1}$$

the series converges by the comparison test.

63. Exercise 13: Prove that the Cauchy product of two absolutely convergent series converges absolutely.

Solution: (*ghostofgarborg*)
Since $\sum |a_n|$ and $\sum |b_n|$ converge, so does their Cauchy product $\sum C_n$. Let c_n be the Cauchy product of a_n and b_n. Then $|c_n| \le C_n$, and as it is majorized by a convergent series, $\sum |c_n|$ converges . I.e. $\sum c_n$ converges absolutely.

64. Exercise 14: If $\{s_n\}$ is complex sequence, define its arithmetic means σ_n by

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} \qquad (n = 0, 1, 2, \ldots).$$

(a) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.

(b) Construct a sequence $\{s_n\}$ which does not converge, although $\lim \sigma_n = 0$.

(c) Can it happen that $s_n > 0$ for all n and that $\limsup s_n = \infty$, although $\lim \sigma_n = 0$?

(d) Put $a_n = s_n - s_{n-1}$, for $n \ge 1$. Show that

$$s_n - \sigma_n = \frac{1}{n+1}\sum_{k=1}^{n} k a_k.$$

Assume that $\lim(na_n) = 0$ and $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges.

(e) Derive the last conclusion from a weaker hypothesis: Assume $M < \infty$, $|na_n| \le M$ for all n, and $\lim \sigma_n = \sigma$. Prove that $\lim s_n = \sigma$.

Solution: (*ghostofgarborg*)
(a) Choose $\epsilon > 0$ and N such that $|s_n - s_m| < \epsilon/2$ for all $n, m > N$. Then choose M such that

$$\left|\frac{(s_0 - s) + \cdots + (s_{N-1} - s)}{M}\right| < \frac{\epsilon}{2}$$

Then whenever $n > \max(M, N)$

$$\left|\frac{(s_0 - s) + \cdots + (s_n - s)}{n+1}\right| \le \left|\frac{(s_0 - s) + \cdots + (s_{N-1} - s)}{n+1}\right| + \left|\frac{(S_N - s) + \cdots + (S_n - s)}{n+1}\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This shows that $(\sigma_n - s)$ converges to 0, i.e. that $\sigma_n \to s$.

(b) Let $s_n = (-1)^n$. Then s_n does not converge, but $\sigma_n = \frac{1+(-1)^n}{2(n+1)}$ converges to 0.

(c) Yes. Let $a_n = \frac{1}{2^n}$, $b_{2^n} = n$, and $b_n = 0$ when n is not a power of 2. Let $s_n = a_n + b_n$. Then $s_n > 0$, and the series is unbounded, so $\limsup s_n = \infty$. However

$$0 < \sigma_n = \frac{1}{n+1}\sum_{i=0}^{n}\frac{1}{2^i} + \frac{1}{n+1}\sum_{i=0}^{\lfloor \log_2(n) \rfloor} i < \frac{2}{n} + \frac{\log_2(n)^2}{n}$$

where the right hand side tends to 0. This shows that $\sigma_n \to 0$.

(d) Note that

$$\sum_{k=1}^{n} k(s_k - s_{k-1}) \;=\; \sum_{k=1}^{n} ks_k - \sum_{k=0}^{n-1}(k+1)s_k \;=\; ns_n - \sum_{k=0}^{n-1} s_k \;=\; (n+1)s_n - \sum_{k=0}^{n} s_k$$

so that

$$s_n - \sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} ka_k$$

Note that the right hand side is the nth arithmetic mean of the sequence na_n, which by assumption and our result in (a) converges. Therefore $s_n - \sigma_n$ is convergent, and since σ_n is convergent by assumption, $(s_n - \sigma_n) + \sigma_n = s_n$ is convergent.

(e)

$$\begin{aligned}
s_n - \sigma_n &= s_n - \sum_{k=0}^{n}\frac{s_k}{n+1} \\
&= s_n - \sum_{k=0}^{n}\left(\frac{1}{n-m} - \frac{m+1}{(n+1)(n-m)}\right)s_k \\
&= \sum_{k=0}^{m}\left(\frac{(m+1)s_k}{(n+1)(n-m)} - \frac{s_k}{n-m}\right) + \sum_{k=m+1}^{n}\left(\frac{(m+1)s_k}{(n+1)(n-m)} + \frac{s_n - s_k}{n-m}\right) \\
&= \frac{m+1}{n-m}\left(\sum_{k=0}^{n}\frac{s_k}{n+1} - \sum_{k=0}^{m}\frac{s_k}{m+1}\right) + \frac{1}{n-m}\sum_{k=m+1}^{n}(s_n - s_k)
\end{aligned}$$

which is the desired equality. By assumption $|s_n - s_{n-1}| < M/n$, so that

$$|s_n - s_t| = |(s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + \cdots + (s_{i+1} - s_i)| \leq \frac{M}{n} + \cdots + \frac{M}{i+1} \leq \frac{(n-i)M}{i+1}$$

The rest of the argument is covered in sufficient detail in the text.

65. Exercise 15: Definition 3.21 can be extended to the case in which the a_n lie in some fixed $\mathbf{R}^k$. Absolute convergence is defined as convergence of $\sum |\mathbf{a}_n|$. Show that Theorems 3.22, 3.23, 3.25(a), 3.33, 3.34, 3.42, 3.45, 3.47, and 3.55 are true in this more general setting.

Solution: (*Dan "kyp44" Whitman*)
$\boxed{\text{Theorem 3.22}}$ For $a_n \in \mathbb{R}^k$, the series $\sum a_n$ converges if and only if for every $\epsilon > 0$ there is an integer N such that

$$\left|\sum_{k=n}^{m} a_k\right| \leq \epsilon$$

for all $m \geq n \geq N$.

Proof: ($\rightarrow$) Suppose that $\sum a_n$ converges. Then by definition the sequence $\{s_n\}$ of partial sums converges where $s_n = \sum_{k=1}^{n} a_k$. It follows from Theorem 3.11a that $\{s_n\}$ is a Cauchy sequence, noting that this is true in any metric space. So consider any $\epsilon > 0$. Then, since $\{s_n\}$ is Cauchy, there is an integer N such that for every $m \geq N$ and $n \geq N$, $|s_m - s_n| < \epsilon$.

So let $M = N + 1$ and consider any $m \geq n \geq M$. Also let $l = n - 1$, from which it follows immdediately that $n = l + 1$. Clearly then $m \geq l \geq N$ so that $|s_m - s_l| < \epsilon$. We then have

$$\left|\sum_{k=n}^{m} a_k\right| = \left|\sum_{k=l+1}^{m} a_k\right| = \left|\sum_{k=1}^{m} a_k - \sum_{k=1}^{l} a_k\right| = |s_m - s_l| \leq \epsilon$$

as required.

($\leftarrow$) Suppose that for every $\epsilon > 0$ there is an integer N such that

$$\left|\sum_{k=n}^{m} a_k\right| \leq \epsilon$$

for all $m \geq n \geq N$ and consider any $\epsilon > 0$. Then obviously there is an integer N for which

$$\left|\sum_{k=n}^{m} a_k\right| \leq \frac{\epsilon}{2}$$

for every $m \geq n \geq N$. So consider any $m \geq N$ and $n \geq N$. We can assume that $m \geq n$ without loss of generality. If $m = n$ then we have simply $|s_m - s_n| = |s_m - s_m| = |0| = 0 < \epsilon$. If $m > n$ let $l = n + 1$. It then follows that $m \geq l \geq N$ so that

$$\left|\sum_{k=l}^{m} a_k\right| \leq \frac{\epsilon}{2}\,.$$

We then have

$$|s_m - s_n| = \left|\sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k\right| = \left|\sum_{k=n+1}^{m} a_k\right| = \left|\sum_{k=l}^{m} a_k\right| \leq \frac{\epsilon}{2} < \epsilon\,.$$

Thus we have shown that $\{s_n\}$ is a Cauchy sequence. Then by Theorem 3.11c, $\{s_n\}$ converges since we are in $\mathbb{R}^k$. It follows then by definition that $\sum a_n$ converges.

$\boxed{\text{Theorem 3.23}}$ For $a_n \in \mathbb{R}^k$, if $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

Proof: Suppose that $\sum a_n$ converges and consider any $\epsilon > 0$. Then by Theorem 3.22 there is an integer N such that

$$\left|\sum_{k=n}^{m} a_k\right| \leq \frac{\epsilon}{2}$$

for all $m \geq n \geq N$. So consider any $n \geq N$ and let $m = n$. Then clearly $m \geq n \geq N$ is true so that

$$|a_n - 0| = |a_n| = \left|\sum_{k=n}^{n} a_k\right| = \left|\sum_{k=n}^{m} a_k\right| \leq \frac{\epsilon}{2} < \epsilon\,,$$

thereby showing by definition that $\lim_{n\to\infty} a_n = 0$.

$\boxed{\text{Theorem 3.25a}}$ For $a_n \in \mathbb{R}^k$ and $c_n \in \mathbb{R}$, if $|a_n| \leq c_n$ for all $n \geq N_0$ where N_0 is a fixed integer, and if $\sum c_n$ converges, then $\sum a_n$ converges.

Proof: The proof is the same as that in the text since Theorem 3.22 applies to sums in $\mathbb{R}^k$ and the triangle inequality is also true there.

$\boxed{\text{Theorem 3.33}}$ For $a_n \in \mathbb{R}^k$, given $\sum a_n$, put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then

(a) if $\alpha < 1$ then $\sum a_n$ converges
(b) if $\alpha > 1$ then $\sum a_n$ diverges
(c) if $\alpha = 1$, then the test gives no information.

Proof: The proof is again the same as that given in the text since for each part:

(a) The comparison test (Theorem 3.25a) is valid in $\mathbb{R}^k$.
(b) Theorem 3.23 is valid in $\mathbb{R}^k$.
(c) The series given still provide a counter example since they are in $\mathbb{R}^1$

$\boxed{\text{Theorem 3.34}}$ For $a_n \in \mathbb{R}^k$, the series $\sum a_n$

(a) converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$
(b) diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq N_0$, where N_0 is some fixed integer.

Proof: Similar to the root test, the proof is the same as that given in the text since for each part:

(a) The comparison test (Theorem 3.25a) is valid in $\mathbb{R}^k$.
(b) Theorem 3.23 is valid in $\mathbb{R}^k$.

$\boxed{\text{Theorem 3.42}}$ For $a_n \in \mathbb{R}^k$ and $b_n \in \mathbb{R}$ suppose that

(a) the partial sums A_n of $\sum a_n$ form a bounded sequence;
(b) $b_0 \geq b_1 \geq b_2 \geq \cdots$;
(c) $\lim_{n\to\infty} b_n = 0$.

Then $\sum a_n b_n$ converges.

Proof: The proof given in the text still holds since Theorem 3.41 clearly holds for $a_n \in \mathbb{R}^k$ as does the Cauchy criterion (Theorem 3.22).

$\boxed{\text{Theorem 3.45}}$ For $a_n \in \mathbb{R}^k$, if $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Proof: The proof is the same since the Cauchy criterion (Theorem 3.22) and the triangle inequality both hold in $\mathbb{R}^k$.

$\boxed{\text{Theorem 3.47}}$ For $a_n \in \mathbb{R}^k$ and $b_n \in \mathbb{R}^k$, if $\sum a_n = A$ and $\sum b_n = B$, then $\sum(a_n + b_n) = A + B$, and $\sum ca_n = cA$ for any fixed $c \in \mathbb{R}$.

Proof: The proofs are the same as those given in the text since the limit rules used (Theorem 3.3 parts a and b) are trivially shown to hold in $\mathbb{R}^k$.

$\boxed{\text{Theorem 3.55}}$ If $\sum a_n$ is a series of elements in $\mathbb{R}^k$ that converges absolutely, then every rearrangement of $\sum a_n$ converges, and they all converge to the same sum.

Proof: The proof is the same since the Cauchy criterion (Theorem 3.22) holds in $\mathbb{R}^k$.

66. Exercise 16: Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and define $x_2, x_3, \ldots$ by the recursion formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

(a) Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.

(b) Put $\varepsilon = x_n - \sqrt{\alpha}$, and show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

so that, setting $\beta = 2\sqrt{\alpha}$,

$$\varepsilon < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n} \qquad (n = 1, 2, \ldots).$$

(c) this is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\varepsilon_1/\beta < 0.1$ and that therefore $\varepsilon_5 < 4 \cdot 10^{-16}$, $\varepsilon_6 < 4 \cdot 10^{-32}$.

Solution: (*Dan "kyp44" Whitman*)
(a) First we show that $x_n > \sqrt{\alpha}$ for all $n \geq 1$ by induction. The $n = 1$ case is obvious since $x_1 > \sqrt{\alpha}$ by construction. So then assume that $x_n > \sqrt{\alpha}$. We then have

$$\begin{aligned}
x_n &> \sqrt{\alpha} \\
x_n - \sqrt{\alpha} &> 0 \\
(x_n - \sqrt{\alpha})^2 &> 0 && \text{(since both sides} \geq 0) \\
x_n^2 - 2x_n\sqrt{\alpha} + \alpha &> 0 \\
x_n^2 + \alpha &> 2x_n\sqrt{\alpha} \\
\frac{x_n^2 + \alpha}{x_n} &> 2\sqrt{\alpha} && \text{(since } x_n > \sqrt{\alpha} > 0) \\
x_n + \frac{\alpha}{x_n} &> 2\sqrt{\alpha} \\
\frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) &> \sqrt{\alpha} \\
x_{n+1} &> \sqrt{\alpha}
\end{aligned}$$

by definition. Therefore $x_n^2 > \alpha$ for any $n \geq 1$. From this we can easily show that $\{x_n\}$ is decreasing monotonically:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) < \frac{1}{2}\left(x_n + \frac{x_n^2}{x_n}\right) = \frac{1}{2}(x_n + x_n) = x_n\,.$$

Now since $\{x_n\}$ is monotonic and bounded (bounded above by x_1 and bounded below by $\sqrt{\alpha}$), it converges by Theorem 3.14. So suppose that $\lim_{n\to\infty} x_n = x$. Then clearly it must be that $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = x$ also. So we have

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) = \frac{1}{2}\left(\lim_{n\to\infty} x_n + \frac{\alpha}{\lim_{n\to\infty} x_n}\right) = \frac{1}{2}\left(x + \frac{\alpha}{x}\right) = x\,,$$

where we have used Theorem 3.3. Solving this simple equation for x results in $x = \lim_{n\to\infty} x_n = \sqrt{\alpha}$.

(b) We have simply

$$\epsilon_{n+1} = x_{n+1} - \sqrt{\alpha} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \sqrt{\alpha} = \frac{1}{2x_n}(x_n^2 + \alpha) - \sqrt{\alpha} = \frac{1}{2x_n}(x_n^2 + \alpha - 2x_n\sqrt{\alpha})$$

$$\frac{(x_n - \sqrt{\alpha})^2}{2x_n} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$$

since $x_n > \sqrt{\alpha}$. So then, letting $\beta = 2\sqrt{\alpha}$, we claim that

$$\epsilon_{n+1} < \beta \left(\frac{\epsilon_1}{\beta} \right)^{2^n}$$

for all $n \geq 1$. A simple proof by induction shows that this is indeed the case. For $n = 1$ we have

$$\epsilon_{n+1} = \epsilon_2 < \frac{\epsilon_1^2}{\beta} = \beta \frac{\epsilon_1^2}{\beta^2} = \beta \left(\frac{\epsilon_1}{\beta} \right)^2 = \beta \left(\frac{\epsilon_1}{\beta} \right)^{2^1} = \beta \left(\frac{\epsilon_1}{\beta} \right)^{2^n} .$$

Now assume that

$$\epsilon_{n+1} < \beta \left(\frac{\epsilon_1}{\beta} \right)^{2^n} .$$

Then we have

$$\epsilon_{n+2} < \frac{\epsilon_{n+1}^2}{\beta} < \frac{1}{\beta} \left[\beta \left(\frac{\epsilon_1}{\beta} \right)^{2^n} \right]^2 = \frac{1}{\beta} \left[\beta^2 \left(\frac{\epsilon_1}{\beta} \right)^{2^n \cdot 2} \right] = \beta \left(\frac{\epsilon_1}{\beta} \right)^{2^{n+1}} .$$

(c) For $\alpha = 3$ and $x_1 = 2$ we first show that $\epsilon_1/\beta < 1/10$ without using decimal approximations for $\sqrt{\alpha} = \sqrt{3}$, starting with the obvious:

$$\begin{gathered}
100 < 108 = 36 \cdot 3 \\
\sqrt{100} < \sqrt{36}\sqrt{3} \\
10 < 6\sqrt{3} \\
10 - 5\sqrt{3} < \sqrt{3} \\
20 - 10\sqrt{3} < 2\sqrt{3} \\
10(2 - \sqrt{3}) < 2\sqrt{3} \\
\frac{2 - \sqrt{3}}{2\sqrt{3}} < \frac{1}{10} \\
\frac{x_1 - \sqrt{\alpha}}{2\sqrt{\alpha}} < \frac{1}{10} \\
\frac{\epsilon_1}{\beta} < \frac{1}{10} .
\end{gathered}$$

We also have

$$\begin{gathered}
12 < 16 \\
4 \cdot 3 < 16 \\
\sqrt{4}\sqrt{3} < \sqrt{16} \\
2\sqrt{3} < 4 \\
\beta < 4 ,
\end{gathered}$$

from which it follows that

$$\epsilon_5 < \beta \left(\frac{\epsilon_1}{\beta} \right)^{2^4} < 4 \left(\frac{\epsilon_1}{\beta} \right)^{16} < 4 \left(\frac{1}{10} \right)^{16} = 4 \cdot 10^{-16} .$$

Likewise

$$\epsilon_6 < \beta \left(\frac{\epsilon_1}{\beta} \right)^{2^5} < 4 \left(\frac{\epsilon_1}{\beta} \right)^{32} < 4 \left(\frac{1}{10} \right)^{32} = 4 \cdot 10^{-32} .$$

67. Exercise 17: Fix $\alpha > 1$. Take $x_1 > \sqrt{\alpha}$, and define

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}.$$

(a) Prove that $x_1 > x_3 > x_5 > \cdots$.

(b) Prove that $x_2 < x_4 < x_6 < \cdots$.

(c) Prove that $\lim x_n = \sqrt{\alpha}$.

(c) Compare the rapidity of convergence of this process with the one described in Exercise 16.

Solution: (*Dan "kyp44" Whitman*)
(a) First we show that $x_n > \sqrt{\alpha}$ implies that $0 < x_{n+1} < \sqrt{\alpha}$. Since clearly $\sqrt{\alpha} > 1$ (since $\alpha > 1$) we have

$$\begin{aligned}
0 < 1 < \sqrt{\alpha} &< x_n \\
0 < \alpha &< x_n^2 \\
0 < \alpha(\alpha - 1) &< x_n^2(\alpha - 1) && \text{(since } \alpha - 1 > 0\text{)} \\
0 < \alpha^2 - \alpha &< \alpha x_n^2 - x_n^2 \\
0 < x_n^2 + \alpha < \alpha^2 + x_n^2 &< \alpha x_n^2 + \alpha \\
0 < 2\alpha x_n < \alpha^2 + 2\alpha x_n + x_n^2 &< \alpha x_n^2 + 2\alpha x_n + \alpha \\
0 < (\alpha + x_n)^2 &< \alpha(x_n + 1)^2 \\
0 < \alpha + x_n &< \sqrt{\alpha}(x_n + 1) && \text{(since all sides are } \geq 0\text{)} \\
0 < \frac{\alpha + x_n}{x_n + 1} &< \sqrt{\alpha} && \text{(since } x_n + 1 > 0\text{)} \\
0 < x_{n+1} &< \sqrt{\alpha}\,.
\end{aligned}$$

Identical reasoning with the direction reversed shows that $0 < x_n < \sqrt{\alpha}$ implies that $x_{n+1} > \sqrt{\alpha}$. It follows immediately from these that $x_n > \sqrt{\alpha}$ if n is odd and $0 < x_n < \sqrt{\alpha}$ if n is even.

Now for any $n \geq 1$ we have

$$x_{n+2} = \frac{\alpha + x_{n+1}}{1 + x_{n+1}} = \frac{\alpha + \frac{\alpha + x_n}{1 + x_n}}{1 + \frac{\alpha + x_n}{1 + x_n}} = \frac{\alpha(1 + x_n) + \alpha + x_n}{1 + x_n + \alpha + x_n} = \frac{2\alpha + \alpha x_n + x_n}{\alpha + 2x_n + 1} \tag{3}$$

So supposing that n is odd we have

$$\begin{aligned}
\sqrt{\alpha} &< x_n \\
\alpha &< x_n^2 && \text{(since both sides are } \geq 0\text{)} \\
2\alpha &< 2x_n^2 \\
2\alpha + \alpha x_n + x_n &< 2x_n^2 + \alpha x_n + x_n = x_n(2x_n + \alpha + 1) \\
\frac{2\alpha + \alpha x_n + x_n}{\alpha + 2x_n + 1} &< x_n && \text{(since clearly } \alpha + 2x_n + 1 > 0\text{)} \\
x_{n+2} &< x_n\,.
\end{aligned}$$

Thus we have shown that the subsequence of odd indices is monotonically decreasing as required.

(b) An identical derivation as that above with the direction reversed shows that the subsequence of even indices is monotonically increasing.

(c) Consider the subsequence of odd indices, i.e. $\{x_{k_n}\}$ where $k_n = 2n-1$ for $n \geq 1$. We have shown above that this subsequence is bounded (above by x_1 and below by $\sqrt{\alpha}$) and is monotonic. Therefore by Theorem 3.14 it converges. So let $x = \lim_{n\to\infty} x_{k_n}$. Then also clearly $\lim_{n\to\infty} x_{k_{n+1}} = \lim_{n\to\infty} x_{k_n} = x$. So since $k_{n+1} = 2(n+1)-1 = 2n+1 = (2n-1)+2 = k_n+2$ by (3) we have

$$\lim_{n\to\infty} x_{k_{n+1}} = \lim_{n\to\infty} \frac{2\alpha + \alpha x_{k_n} + x_{k_n}}{\alpha + 2x_{k_n} + 1} = \frac{2\alpha + \alpha \lim_{n\to\infty} x_{k_n} + \lim_{n\to\infty} x_{k_n}}{\alpha + 2\lim_{n\to\infty} x_{k_n} + 1} = \frac{2\alpha + \alpha x + x}{\alpha + 2x + 1} = x\,.$$

Solving this for x results in $x = \sqrt{\alpha}$. A similar argument for the subsequence of even indices, i.e. $\{x_{l_n}\}$ where $l_n = 2n$ for $n \geq 1$, shows that it also converges to $\sqrt{\alpha}$.

So consider any $\epsilon > 0$. Then there is an N where $|x_{k_n} - \sqrt{\alpha}| < \epsilon$ for all $n \geq N$. Likewise there is an M where $|x_{l_n} - \sqrt{\alpha}| < \epsilon$ for all $n \geq M$. So let $L = \max(k_N, l_M)$ and consider any $n \geq L$. If n is odd then there is an $m \geq 1$ where $n = 2m-1 = k_m$. We then have

$$n = 2m-1 \geq L \geq k_N = 2N-1$$

$$2m \geq 2N$$

$$m \geq N$$

so that $|x_n - \sqrt{\alpha}| = |x_{k_m} - \sqrt{\alpha}| < \epsilon$. A similar argument shows that $|x_n - \sqrt{\alpha}| < \epsilon$ in the case when n is even as well. Thus we have shown that $\lim_{n\to\infty} x_n = \sqrt{\alpha}$ by definition.

(d)

Lemma 3.17.1: If $s_n \to 0$ for a real sequence $\{s_n\}$ then also

$$\lim_{n\to\infty} (s_n)^n = 0\,.$$

Proof: Consider any $\epsilon > 0$. If $\epsilon \geq 1$ then is an $N \geq 1$ such that $|s_n| < 1$ for any $n \geq N$. So consider any $n \geq N$. We then have

$$|s_n^n - 0| = |s_n^n| = |s_n|^n < 1^n = 1 \leq \epsilon\,.$$

If $\epsilon < 1$ then $\epsilon^n \leq \epsilon$ for $n \geq 1$. Also there is an $N \geq 1$ such that $|s_n| < \epsilon$ for all $n \geq N$. So consider any $n \geq N$, nothing that also $n \geq N \geq 1$ so that

$$|s_n^n - 0| = |s_n^n| = |s_n|^n < \epsilon^n \leq \epsilon\,.$$

So since ϵ was arbitrary we have shown that $(s_n)^n \to 0$. ■

Lemma 3.17.2: For $0 < \xi < 1$ and positive real a

$$\lim_{n\to\infty} \xi^{2^n} a^n = 0\,.$$

Proof: First consider any $n \geq 2$. Then by the Binomial Theorem we have

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots > \binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}\,.$$

Since $\xi < 1$ and $2^n > n(n-1)/2$ it follows that for $n \geq 2$

$$0 < \xi^{2^n} a^n < \xi^{\frac{n(n-1)}{2}} a^n\,. \tag{4}$$

But we have

$$\xi^{\frac{n(n-1)}{2}} a^n = \left(\xi^{\frac{n-1}{2}} a\right)^n\,, \tag{5}$$

and clearly, since $|\xi| < 1$, by Theorem 3.20e

$$\lim_{n\to\infty} \xi^{\frac{n-1}{2}} a = a \lim_{n\to\infty} \xi^{\frac{n-1}{2}} = 0$$

so that by Lemma 3.17.1

$$\lim_{n\to\infty} \left(\xi^{\frac{n-1}{2}} a\right)^n = 0\,.$$

The desired result then follows from (4), (5), and Theorem 3.19. ■

To start the main argument let $\{y_n\}$ denote the sequence from Exercise 3.16 and $\{x_n\}$ denote the sequence from this exercise, setting $y_1 = x_1 > \sqrt{\alpha}$. We also define $\delta_n = y_n - \sqrt{\alpha}$ and $\epsilon_n = |x_n - \sqrt{\alpha}|$, noting that the absolute value for ϵ_n is necessary since it was shown in part a that $x_n < \sqrt{\alpha}$ for even n.

First we show that $\delta_n \to 0$, which is straightforward given that it was shown in Exercise 3.16a that $y_n \to \sqrt{\alpha}$:

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} (y_n - \sqrt{\alpha}) = \lim_{n\to\infty} y_n - \sqrt{\alpha} = \sqrt{\alpha} - \sqrt{\alpha} = 0$$

Next we show that there is an $N \geq 1$ where $\delta_N/\beta < 1$, where again $\beta = 2\sqrt{\alpha}$. This follows immediately from the fact that $\delta_n \to 0$ since this implies that there is an $N \geq 1$ such that $|\delta_n - 0| = |\delta_n| = \delta_n < \beta$ for all $n \geq N$, since $\beta > 0$. In particular $\delta_N < \beta$, from which it immediately follows that $\delta_N/\beta < 1$ since $\beta > 0$.

Now in Exercise 3.16b it was shown that $\delta_{n+1} < \delta_n^2/\beta$ for all $n \geq 1$. Similar to the derivation that follows in Exercise 3.16b we claim that $\delta_n \leq \beta(\delta_N/\beta)^{2^{n-N}}$ for all $n \geq N$, and to the surprise of no one we show this by induction. For $n = N$ we have

$$\delta_n = \delta_N \leq \delta_N = \beta\frac{\delta_N}{\beta} = \beta\left(\frac{\delta_N}{\beta}\right)^1 = \beta\left(\frac{\delta_N}{\beta}\right)^{2^0} = \beta\left(\frac{\delta_N}{\beta}\right)^{2^{N-N}} = \beta\left(\frac{\delta_N}{\beta}\right)^{2^{n-N}}\,.$$

Now assume that $\delta_n \leq \beta(\delta_N/\beta)^{2^{n-N}}$. We then have

$$\delta_{n+1} \leq \frac{\delta_n^2}{\beta} \leq \frac{1}{\beta}\left[\beta\left(\frac{\delta_N}{\beta}\right)^{2^{n-N}}\right]^2 = \frac{\beta^2}{\beta}\left(\frac{\delta_N}{\beta}\right)^{2^{n-N+1}} = \beta\left(\frac{\delta_N}{\beta}\right)^{2^{(n+1)-N}}\,.$$

Turning our attention to $\{x_n\}$, we first claim that $\epsilon_1 > \epsilon_2$. To see this we have

$$\begin{aligned}
\sqrt{\alpha} &< x_1 < 2 + x_1 \\
\sqrt{\alpha} - 1 &< 1 + x_1 \\
(x_1 - \sqrt{\alpha})(\sqrt{\alpha} - 1) &< (1 + x_1)(x_1 - \sqrt{\alpha}) && \text{(since } x_1 - \sqrt{\alpha} > 0\text{)} \\
\sqrt{\alpha} + \sqrt{\alpha}x_1 - \alpha - x_1 &< (1 + x_1)(x_1 - \sqrt{\alpha}) \\
\sqrt{\alpha}(1 + x_1) - (\alpha + x_1) &< (1 + x_1)(x_1 - \sqrt{\alpha}) \\
\sqrt{\alpha} - \frac{\alpha + x_1}{1 + x_1} &< x_1 - \sqrt{\alpha} && \text{(since } 1 + x_1 > 0\text{)} \\
\sqrt{\alpha} - x_2 &< x_1 - \sqrt{\alpha} \\
\epsilon_2 &< \epsilon_1\,.
\end{aligned}$$

We also claim that $\epsilon_{n+1} \geq \epsilon_n\gamma$ for any $n \geq 1$, where we have let

$$\gamma = \frac{\sqrt{\alpha} - 1}{1 + x_1}$$

To see this consider any $n \geq 1$. If n is odd then we have

$$\begin{aligned}
\epsilon_{n+1} &= |x_{n+1} - \sqrt{\alpha}| = \sqrt{\alpha} - x_{n+1} && \text{(since } n+1 \text{ is even)} \\
&= \sqrt{\alpha} - \frac{\alpha + x_n}{1 + x_n} = \frac{\sqrt{\alpha} + \sqrt{\alpha}x_n - \alpha - x_n}{1 + x_n} \\
&= \frac{(\sqrt{\alpha} - x_n)(1 - \sqrt{\alpha})}{1 + x_n} = \frac{(x_n - \sqrt{\alpha})(\sqrt{\alpha} - 1)}{1 + x_n} \\
&= \epsilon_n \frac{\sqrt{\alpha} - 1}{1 + x_n} \geq \epsilon_n \frac{\sqrt{\alpha} - 1}{1 + x_1} = \epsilon_n \gamma \,, && \text{(since } n \text{ is odd)}
\end{aligned}$$

where we have used the fact that $x_n \leq x_1$ for all odd $n \geq 1$ since the odd indexed sequence decreases monotonically as shown in part a. On the other hand if n is even then

$$\begin{aligned}
\epsilon_{n+1} &= |x_{n+1} - \sqrt{\alpha}| = x_{n+1} - \sqrt{\alpha} && \text{(since } n+1 \text{ is odd)} \\
&= \frac{\alpha + x_n}{1 + x_n} - \sqrt{\alpha} = \frac{\alpha + x_n - \sqrt{\alpha} - \sqrt{\alpha}x_n}{1 + x_n} \\
&= \frac{(\sqrt{\alpha} - x_n)(\sqrt{\alpha} - 1)}{1 + x_n} = \epsilon_n \frac{\sqrt{\alpha} - 1}{1 + x_n} && \text{(since } n \text{ is even)} \\
&\geq \epsilon_n \frac{\sqrt{\alpha} - 1}{1 + x_1} = \epsilon_n \gamma \,,
\end{aligned}$$

where we have used the fact that $x_n < \sqrt{\alpha} < x_1$ for even n as shown in part a. So from this we can easily show that $\epsilon_n \geq \epsilon_1 \gamma^{n-1}$ for all $n \geq 1$ by induction. For $n = 1$ we have

$$\epsilon_n = \epsilon_1 \geq \epsilon_1 = \epsilon_1 \cdot 1 = \epsilon_1 \gamma^0 = \epsilon_1 \gamma^{1-1} = \epsilon_1 \gamma^{n-1} \,.$$

Now assume that $\epsilon_n \geq \epsilon_1 \gamma^{n-1}$. Then we have

$$\epsilon_{n+1} \geq \epsilon_n \gamma \geq \epsilon_1 \gamma^{n-1} \gamma = \epsilon_1 \gamma^{n-1+1} = \epsilon_1 \gamma^{(n+1)-1} \,.$$

Since $0 < \delta_N / \beta < 1$ and $\gamma > 0$, Lemma 3.17.2 implies that there is an $M \geq 1$ where

$$\left(\frac{\delta_N}{\beta}\right)^{2^m} \left(\frac{1}{\gamma}\right)^m < \gamma^{N-1} \frac{\epsilon_1}{\beta}$$

for all $m \geq M$, noting that the right hand side is positive. So then let $L = N + M$ and consider an arbitrary $n \geq L$. Letting $m = n - N$ we clearly have $n = m + N$ so that

$$n \geq L$$

$$m + N \geq N + M$$

$$m \geq M \,.$$

Therefore we have

$$\begin{aligned}
\left(\frac{\delta_N}{\beta}\right)^{2^m}\left(\frac{1}{\gamma}\right)^m &< \gamma^{N-1}\frac{\epsilon_1}{\beta} \\
\left(\frac{\delta_N}{\beta}\right)^{2^m} &< \gamma^{m+N-1}\frac{\epsilon_1}{\beta} && \text{(since } \gamma^m > 0\text{)} \\
\beta\left(\frac{\delta_N}{\beta}\right)^{2^m} &< \epsilon_1\gamma^{m+N-1} && \text{(since } \beta > 0\text{)} \\
\beta\left(\frac{\delta_N}{\beta}\right)^{2^{n-N}} &< \epsilon_1\gamma^{n-N+N-1} \\
\beta\left(\frac{\delta_N}{\beta}\right)^{2^{n-N}} &< \epsilon_1\gamma^{n-1}
\end{aligned}$$

But since $n \geq L = N + M \geq N \geq 1$, by what was shown before and transitivity we have

$$\delta_n \leq \beta\left(\frac{\delta_N}{\beta}\right)^{2^{n-N}} < \epsilon_1\gamma^{n-1} \leq \epsilon_n$$

Thus we have shown that for all $n \geq L$

$$\delta_n < \epsilon_n \,.$$

This shows that $\{y_n\}$ converges more rapidly than $\{x_n\}$.

68. Exercise 18: Replace the recursion formula of Exercise 16 by

$$x_{n+1} = \frac{p-1}{p}\,x_n + \frac{\alpha}{p}\,x_n^{-p+1}$$

where p is a fixed positive integer, and describe the behavior of the resulting sequences $\{x_n\}$.

Solution: (*Dan "kyp44" Whitman*)
Lemma 3.18.1: For an integer $p \geq 1$ and real a where $0 < a < 1$

$$p(1-a) \geq 1 - a^p \,.$$

Proof: We have

$$
\begin{aligned}
p &= \sum_{n=0}^{p-1} 1 = \sum_{n=0}^{p-1} 1^n \geq \sum_{n=0}^{p-1} a^n && \text{(equality when } p = 1\text{)} \\
&= \frac{1-a}{1-a} \sum_{n=0}^{p-1} a^n = \frac{1}{1-a} \left(\sum_{n=0}^{p-1} a^n - a \sum_{n=0}^{p-1} a^n \right) && \text{(since } 1 - a \neq 0\text{)} \\
&= \frac{1}{1-a} \left(\sum_{n=0}^{p-1} a^n - \sum_{n=0}^{p-1} a^{n+1} \right) \\
&= \frac{1}{1-a} \left(\sum_{n=0}^{p-1} a^n - \sum_{n=1}^{p} a^n \right) \\
&= \frac{1}{1-a} \left(a^0 + \sum_{n=1}^{p-1} a^n - \sum_{n=1}^{p-1} a^n - a^p \right) \\
&= \frac{1}{1-a} \left(1 - a^p \right) .
\end{aligned}
$$

From this the desired result follows immediately since $1 - a > 0$. ■

Now we assume that for this family of sequences we always have $x_1 > \sqrt[p]{\alpha}$ and we claim that the sequence converges to $\sqrt[p]{\alpha}$ for a given p. So assume that p is a fixed positive integer. First we claim that $x_n > \sqrt[p]{\alpha}$ for all $n \geq 1$. We show this by induction. The $n = 1$ case is by construction so assume that $x_n > \sqrt[p]{\alpha}$. We then have

$$x_n > \sqrt[p]{\alpha} > 0$$

$$1 > \frac{\sqrt[p]{\alpha}}{x_n} > 0\,.$$

So then by Lemma 3.18.1 above we have

$$
\begin{aligned}
p\left(1 - \frac{\sqrt[p]{\alpha}}{x_n}\right) &\geq 1 - \left(\frac{\sqrt[p]{\alpha}}{x_n}\right)^p \\
p - p\frac{\sqrt[p]{\alpha}}{x_n} &\geq 1 - \frac{\alpha}{x_n^p} \\
p + \frac{\alpha}{x_n^p} &\geq 1 + p\frac{\sqrt[p]{\alpha}}{x_n} \\
px_n + \frac{\alpha}{x_n^{p-1}} &\geq x_n + p\sqrt[p]{\alpha} && \text{(since } x_n > 0\text{)} \\
(p-1)x_n + \frac{\alpha}{x_n^{p-1}} &\geq p\sqrt[p]{\alpha} \\
\frac{p-1}{p}x_n + \frac{\alpha}{px_n^{p-1}} &\geq \sqrt[p]{\alpha} \\
x_{n+1} &\geq \sqrt[p]{\alpha}\,.
\end{aligned}
$$

Given this it is easy to show that $\{x_n\}$ decreases monotonically. Since $x_n \geq \sqrt[p]{\alpha}$ for any $n \geq 1$, clearly $x_n^p \geq \alpha$. Therefore we have

$$x_{n+1} = \frac{p-1}{p}x_n + \frac{\alpha}{px_n^{p-1}} \leq \frac{p-1}{p}x_n + \frac{x_n^p}{px_n^{p-1}} = \frac{p-1}{p}x_n + \frac{x_n}{p} = \frac{p-1+1}{p}x_n = \frac{p}{p}x_n = x_n\,.$$

So since $\{x_n\}$ is bounded (above by x_1 and below by $\sqrt[p]{\alpha}$) and monotic, it converges by Theorem 3.14. So let $x = \lim_{n\to\infty} x_n$, and then clearly $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = x$ also. Therefore we have

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \left(\frac{p-1}{p} x_n + \frac{\alpha}{p x_n^{p-1}} \right)$$
$$= \left(\frac{p-1}{p} \lim_{n\to\infty} x_n + \frac{\alpha}{p \lim_{n\to\infty} x_n^{p-1}} \right)$$
$$= \left(\frac{p-1}{p} x + \frac{\alpha}{p x^{p-1}} \right) = x\,.$$

Solving this straightforward equation for x results in $x = \sqrt[p]{\alpha}$.

Thus these sequences provide a means by which the pth root of a real number can be calculated (approximately), and it is worth mentioning that the $p = 2$ case corresponds to the sequence in Exercise 3.16 for the square root.

69. Exercise 19: Associate to each sequence $a = \{\alpha_n\}$, in which α_n is 0 or 2, the real number

$$x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}.$$

Prove that the set of all $x(a)$ is precisely the Cantor set described in Sec. 2.44.

Solution: (*Dan "kyp44" Whitman*)
Let A denote the set of all α sequences as defined in the text and let $x(A) = \{x(\alpha) : \alpha \in A\}$.

For a given sqeuence $\alpha \in A$ let

$$\beta_n = \frac{\alpha_n}{3^n}$$

We first show that $x(\alpha) = \sum \beta_n$ converges. So consider any $\alpha \in A$. If $\{\alpha_n\}$ ends in all zeros (i.e. there is an $N \geq 1$ where $\alpha_n = 0$ for all $n \geq N$) then clearly $\sum \beta_n$ is effectively finite and so therefore converges. If this is not the case then $\{\alpha_n\}$ clearly has a subsequence $\{\alpha_{k_n}\}$ where $\alpha_{k_n} = 2$ for all $n \geq 1$. Then, noting that obviously $\beta_n \geq 0$, we have

$$\limsup_{n\to\infty} \sqrt[n]{|\beta_n|} = \limsup_{n\to\infty} \sqrt[n]{\beta_n} = \limsup_{n\to\infty} \sqrt[n]{\frac{\alpha_n}{3^n}} = \limsup_{n\to\infty} \frac{\sqrt[n]{\alpha_n}}{3} = \lim_{n\to\infty} \frac{\sqrt[n]{\alpha_{k_n}}}{3} = \lim_{n\to\infty} \frac{\sqrt[n]{2}}{3} = \frac{1}{3}\,,$$

where we have invoked Theorem 3.20b. It follows then from the Root Test (Theorem 3.33) that $\sum \beta_n$ converges.

Now, for a sequence $\alpha \in A$, $x(\alpha)$ is the real number whose representation in base three is $0.\alpha_1\alpha_2\alpha_3\ldots$, i.e. α_n is the nth digit after the decimal point. Clearly if $\alpha_n = 0$ for all $n \geq 1$ then $x(\alpha) = 0$. We also note that if $\alpha_n = 2$ for all $n \geq 1$ then we have

$$x(\alpha) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n} = \sum_{n=1}^{\infty} \frac{2}{3^n} = 2\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = 2\left[\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n - 1\right] = 2\left[\frac{1}{1-\frac{1}{3}} - 1\right] = 1\,,$$

analagously to the way in which $0.99999\ldots = 1$. It follows then that $x(A) \subseteq [0,1]$.

First consider any $m \geq 1$ for any $\alpha \in A$. We then have

$$3^m x(\alpha) = 3^m \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n} = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^{n-m}} = \sum_{n=1}^{m-1} \frac{\alpha_n}{3^{n-m}} + \sum_{n-m}^{m} \frac{\alpha_n}{3^{n-m}} + \sum_{n=m+1}^{\infty} \frac{\alpha_n}{3^{n-m}}\,,$$

where we adopt the convention that $\sum_{n=p}^{q} = 0$ for $p > q$, which is needed for the first sum if $m = 1$. Continuing, we have

$$3^m x(\alpha) = \sum_{n=1}^{m-1} 3^{m-n}\alpha_n + \alpha_m + \sum_{n=m+1}^{\infty} \frac{\alpha_n}{3^{n-m}} = 3j_{\alpha,m} + \alpha_m + \sum_{n=m+1}^{\infty} \frac{\alpha_n}{3^{n-m}}\,, \tag{6}$$

where we define

$$j_{\alpha,m} = \sum_{n=1}^{m-1} 3^{m-n-1}\alpha_n$$

noting that for the sum

$$m - 1 \geq n \geq 1$$

$$1 - m \leq -n \leq -1$$

$$1 \leq m - n \leq m - 1$$

$$0 \leq m - n - 1 \leq m - 2$$

$$0 \leq j_{\alpha,m} \leq m - 2$$

so that $j_{\alpha,m}$ is an integer (since the α_n are). Note also that $j_{\alpha,1} = 0$, following the convention. Now we examine the last sum in (6). Letting $l = n - m$ so that $n = l + m$ we let

$$\delta_{\alpha,m} = \sum_{n=m+1}^{\infty} \frac{\alpha_n}{3^{n-m}} = \sum_{l=1}^{\infty} \frac{\alpha_{m+l}}{3^l}\,.$$

This sum of course depends on α but we can find its bounds. Clearly it will be the lowest when $\alpha_{l+m} = 0$ for $l \geq 1$ and the largest when $\alpha_{l+m} = 2$ for $l \geq 1$. We therefore have

$$0 \leq \delta_{\alpha,m} \leq \sum_{l=1}^{\infty} \frac{2}{3^l} = 1\,.$$

Recombining this with (6) we have

$$3^m x(\alpha) = 3j_{\alpha,m} + \alpha_m + \delta_{\alpha,m}\,, \tag{7}$$

where $j_{\alpha,m} \in \mathbb{N}$ (where $0 \in \mathbb{N}$) and $0 \leq \delta_{\alpha,m} \leq 1$.

Moving along, let P denote the Cantor set. Now, by equation (2.24) in the text

$$y \notin P \iff \exists m \in \mathbb{Z}^+ \exists k \in \mathbb{N} \left[y \in \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m} \right) \right],$$

where $\mathbb{Z}^+$ denotes the set of positive integers. Note that the text says that $k \in \mathbb{Z}^+$ but this should really be $k \in \mathbb{N}$ as written above. This is logically equivalent to

$$y \in P \iff \forall m \in \mathbb{Z}^+ \forall k \in \mathbb{N} \left[y \notin \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m} \right) \right]$$

$$y \in P \iff \forall m \in \mathbb{Z}^+ \forall k \in \mathbb{N} \left[\neg \left(y > \frac{3k+1}{3^m} \wedge y < \frac{3k+2}{3^m} \right) \right]$$

$$y \in P \iff \forall m \in \mathbb{Z}^+ \forall k \in \mathbb{N} \left[y \leq \frac{3k+1}{3^m} \vee y \geq \frac{3k+2}{3^m} \right]$$

$$y \in P \iff \forall m \in \mathbb{Z}^+ \forall k \in \mathbb{N} \left[3^m y \leq 3k+1 \vee 3^m y \geq 3k+2 \right] \tag{8}$$

Finally we show that $x(A) = P$.

($\subseteq$) Consider any $\alpha \in A$ and consider any $m \in \mathbb{Z}^+$ and $k \in \mathbb{N}$. If $\alpha_m = 0$ then by (7)

$$3j_{\alpha,m} \leq 3^m x(\alpha) = 3j_{\alpha,m} + \delta_{\alpha,m} \leq 3j_{\alpha,m} + 1\,.$$

So in the sub-case where $j_{\alpha,m} > k$ we have $j_{\alpha,m} \geq k+1$ so that

$$3^m x(\alpha) \geq 3j_{\alpha,m} \geq 3(k+1) = 3k+3 \geq 3k+2\,.$$

In the other sub-case when $j_{\alpha,m} \leq k$ we have

$$3^m x(\alpha) \leq 3j_{\alpha,m} + 1 \leq 3k+1$$

On the other hand if $\alpha_m = 2$ then again by (7)

$$3j_{\alpha,m} + 2 \leq 3^m x(\alpha) = 3j_{\alpha,m} + 2 + \delta_{\alpha,m} \leq 3j_{\alpha,m} + 3$$

So in the sub-case where $j_{\alpha,m} \geq k$ we have

$$3^m x(\alpha) \geq 3j_{\alpha,m} + 2 \geq 3k+2\,,$$

and the other sub-case where $j_{\alpha,m} < k$ we we have $j_{\alpha,m} + 1 \leq k$ so that

$$3^m x(\alpha) \leq 3j_{\alpha,m} + 3 = 3(j_{\alpha,m}+1) \leq 3k \leq 3k+1$$

Thus, since the cases are exhaustive, we have shown that the right side of (8) is true, from which it follows that $x(\alpha) \in P$. Since α was an arbitrary sequence, $x(A) \subseteq P$.

($\supseteq$) Consider any $y \in P$. Then clearly $y \in [0,1]$. So let α be the sequence corresponding to the base three representation of y as described above; thus $y = x(\alpha)$. If α contains only zeros and twos, i.e. $\alpha_n \in \{0,2\}$ for all $n \geq 1$ then clearly $\alpha \in A$ so that $y \in x(A)$. So suppose that there is an $n \geq 1$ where $\alpha_n = 1$. We will show that this is the only digit that is one and that, furthermore, there is an α' such that $\alpha' \in A$ and $x(\alpha') = x(\alpha) = y$.

So let m be the index of the first digit that is a one, i.e. $m = \min\{k \in \mathbb{Z}^+ : \alpha_k = 1\}$. Then since $y \in P$, $m \in \mathbb{Z}^+$, and $j_{\alpha,m} \in \mathbb{N}$ we have $3^m y \leq 3j_{\alpha,m} + 1$ or $3^m y \geq j_{\alpha,m} + 2$ by (8).

If $3^m y \leq 3j_{\alpha,m} + 1$ then since $y = x(\alpha)$ we have by (7)

$$3^m y \leq 3j_{\alpha,m} + 1$$

$$3j_{\alpha,m} + \alpha_m + \delta_{\alpha,m} \leq 3j_{\alpha,m} + 1$$

$$1 + \delta_{\alpha,m} \leq 1$$

$$\delta_{\alpha,m} \leq 0\,.$$

Since also we know that $\delta_{\alpha,m} \geq 0$ it has to be that $\delta_{\alpha,m} = 0$, which implies that every digit after α_m is zero so that α_m is the only digit that is one. But then we can form α' where

$$\alpha'_k = \begin{cases} \alpha_k & \text{if } k < m \\ 0 & \text{if } k = m \\ 2 & \text{if } k > m\,. \end{cases}$$

Clearly $x(\alpha') = x(\alpha) = y$ and $\alpha' \in A$.

If $3^m y \geq 3j_{\alpha,m} + 2$ then we again have by (7)

$$\begin{aligned}3^m y &\geq 3j_{\alpha,m} + 2\\ 3j_{\alpha,m} + \alpha_m + \delta_{\alpha,m} &\geq 3j_{\alpha,m} + 2\\ 1 + \delta_{\alpha,m} &\geq 2\\ \delta_{\alpha,m} &\geq 1\,.\end{aligned}$$

Since we also know that $\delta_{\alpha,m} \leq 1$ it has to be that $\delta_{\alpha,m} = 1$, which means that $\alpha_k = 2$ for $k > m$, i.e. the base three representation of $x(\alpha)$ ends in infinitely many twos after the one. Then we can construct a terminating α' where

$$\alpha'_k = \begin{cases} \alpha_k & \text{if } k < m \\ 2 & \text{if } k = m \\ 0 & \text{if } k > m\,. \end{cases}$$

Again clearly $x(\alpha') = x(\alpha) = y$ and $\alpha' \in A$.

Thus in either case there is an $\alpha' \in A$ where $y = x(\alpha) = x(\alpha')$ so that $y \in x(A)$. Since y was arbitrary this shows that $P \subseteq x(A)$, which completes the proof since it has been shown that $x(A) = P$.

70. Exercise 20: Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

Solution: (*ghostofgarborg*)
Choose $\epsilon > 0$. Since p_n is Cauchy, we can choose an N such that $d(p_m, p_n) < \frac{\epsilon}{2}$ whenever $m, n > N$. Assume $p_{n_i} \to p$. Then there is an M such that whenever $i > M$, $d(p_{n_i}, p) < \frac{\epsilon}{2}$. For $n > N$, there is an $i > M$ such that $n_i > N$. Then

$$d(p_n, p) \leq d(p_n, p_{n_i}) + d(p_{n_i}, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This implies that the sequence converges to p.

71. Exercise 21: Prove the following analogue of Theorem 3.10(b): If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a *complete* metric space X, if $E_n \supset E_{n+1}$, and if

$$\lim_{n\to\infty} \operatorname{diam} E_n = 0,$$

then $\cup_1^\infty E_n$ consists of exactly one point.

Solution: (*ghostofgarborg*)
Pick $x_n \in E_n$ for each n. Since $\{x_k\}_{k=n}^\infty \subset E_n$ and $\lim \operatorname{diam} E_n = 0$, x_n is Cauchy. Since X is complete, x_n converges to a point x. Since x is a limit point for each E_n and each E_n is closed, $x \in E_n$ for all n. Consequently, $x \in \bigcap_{n=1}^\infty E_n$. There cannot be more than one point in this intersection, since that would contradict $\lim \operatorname{diam} E_n = 0$.

72. Exercise 22: Suppose X is a nonempty complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X. Prove Baire's theorem, namely, that $\cup_1^\infty G_n$ is not empty. (In fact, it is dense in X.)

Solution: (*ghostofgarborg*)
Since G_1 is open, we can find a neighborhood E_1 such that $\bar{E}_1 \subset G_1$, and $\operatorname{diam} E_1 < 1$ by letting $E_1 = N_\epsilon(x)$ for a sufficiently small ϵ around $x \in G_1$.

Assume E_n is open, that $\bar{E}_n \subset G_n$, that $\bar{E}_n \subset \bar{E}_{n-1}$ and that $\operatorname{diam} E_n < \frac{1}{n}$. Since G_{n+1} is dense in X and open, $E_n \cap G_{n+1}$ is non-empty and open. We can therefore choose a neighborhood E_{n+1} in the same manner as above such that $\operatorname{diam} E_{n+1} < \frac{1}{n+1}$, $\bar{E}_{n+1} \subset G_{n+1}$ and $\bar{E}_{n+1} \subset \bar{E}_n$. This gives us a sequence of sets with the desired properties.

By ex. 21, $\bigcap \bar{E}_n$ contains a point x. Since $x \in G_n$ for all n, $\bigcap E_n$ is non-empty as well, which is what we wanted to show.

73. Exercise 23: Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\big\{d(p_n, q_n)\big\}$ converges.

Solution: (*ghostofgarborg*)
By applying the triangle inequality, we see that

$$d(p_n, q_n) - d(p_m, q_m) \leq d(p_n, p_m) + d(q_m, q_n)$$

By interchanging the roles of m and n, we in fact get

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n)$$

Let $\epsilon > 0$. Since p_n and q_n are Cauchy, we can find an N such that $m, n > N$ implies that $d(p_n, p_m) < \frac{\epsilon}{2}$ and $d(q_m, q_n) < \frac{\epsilon}{2}$. Then

$$|d(p_n, q_n) - d(p_m, q_m)| \leq \epsilon$$

This shows that $d(p_n, q_n)$ is Cauchy, hence convergent.

74. Exercise 24: Let X be a metric space

(a) Call two Cauchy sequences $\{p_n\}$, $\{q_n\}$ in X *equivalent* if

$$\lim d(p_n, q_n) = 0.$$

Prove that this is an equivalence relation.

(b) Let X^* be the set of all equivalence classes so obtained. If $P \in X^*$, $Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$, define

$$\Delta(P, Q) = \lim_{n \to \infty} d(p_n, q_n);$$

by Exercise 23, this limit exists. Show that the number $\Delta(P, Q)$ is unchanged if $\{p_n\}$, $\{q_n\}$ are replaced by equivalent sequences, and hence that Δ is a distance function in X^*.

(c) Prove that the resulting metric space X^* is complete.

(d) For each $p \in X$, there is a Cauchy sequence all of whose terms are p; let P_p be the element of X^* which contains this sequence. Prove that

$$\Delta(P_p, P_q) = d(p, q)$$

for all $p, q \in X$. In other words, the mapping φ defined by $\varphi(p) = P_p$ is an isometry of X into X^*.

(e) Prove that $\varphi(X)$ is dense in X^*, and that $\varphi(X) = X^*$ if X is complete. By (d), we may identify X and $\varphi(X)$ and thus regard X as embedded in the complete metric space X^*. We call X^* the *completion* of X.

Solution: (*ghostofgarborg*)
(a) Let $\equiv$ be the relation. Assume $p_n \equiv q_n$ and $q_n \equiv r_n$. We know that $d(p_n, p_n) \to 0$ whenever p is Cauchy, so that $p_n \equiv p_n$. By the symmetry of d, it is clear that $q_n \equiv p_n$. By the triangle inequality,

$$d(p_n, r_n) \le d(p_n, q_n) + d(q_n, r_n)$$

which shows that $d(p_n, r_n) \to 0$ and $p_n \equiv r_n$. This shows that $\equiv$ is an equivalence relation.

(b) Let $p_n \equiv p_n'$. It suffices to show that $\lim d(p_n, q_n) = \lim d(p_n', q_n)$.

$$d(p_n, q_n) \le d(p_n, p_n') + d(p_n', q_n)$$

Consequently, since $d(p_n, p_n') \to 0$, $\lim d(p_n, q_n) \le \lim d(p_n', q_n)$. By interchanging p_n and p_n', we also get that $\lim d(p_n', q_n) \le \lim d(p_n, q_n)$, which proves equality. Δ is therefore well-defined. We now need to show that it is a metric. It is clear that Δ is positive, and that $\Delta(P, Q) = 0$ iff $P = Q$. The symmetry of Δ follows from the symmetry of d. It remains to be shown that Δ satisfies the triangle inequality.

$$\Delta(P, R) = \lim_{n\to\infty} d(p_n, r_n) \le \lim_{n\to\infty} d(p_n, q_n) + \lim_{n\to\infty} d(q_n, r_n) = \Delta(P, Q) + \Delta(Q, R)$$

(c) We first prove a lemma: Let r_k be a sequence, and define the family of equivalence classes $Q_k = [q_k]$ by letting q_k be constant sequences given by $q_{k,n} = r_k$ for all n. If r_n is Cauchy, then Q_k converges to the equivalence class $R = [r]$. To see this, choose an $\epsilon > 0$ and pick N such that $d(r_k, r_\ell) < \epsilon$ whenever $k, \ell > N$. Then

$$\lim_{n\to\infty} d(q_{k,n}, r_n) = \lim_{n\to\infty} d(r_k, r_n) < \epsilon$$

Consequently

$$\lim_{k\to\infty} \Delta(Q_k, R) = \lim_{k\to\infty} \lim_{n\to\infty} d(q_{k,n}, r_n) = 0$$

as desired.

Now for the main result, let $P_k = [p_k]$ be a Cauchy sequence in X^*. For each k, choose a number $N(k)$ such that $d(p_{k,m}, p_{k,n}) < \frac{1}{2^k}$ whenever $m, n > N(k)$, and let $r_k = p_{k,N(k)}$. We now want to show that r is Cauchy: Choose an $\epsilon > 0$. First remember that P_i is Cauchy, so that $\Delta(P_m, P_n) = \lim_k d(p_{m,k}, p_{n,k}) < \frac{\epsilon}{6}$ when the indices are sufficiently large. The convergence of that limit means that $d(p_{m,s}, p_{n,s}) < \frac{\epsilon}{3}$ whenever s is sufficiently large. Also note that $\frac{1}{2^m} < \frac{\epsilon}{3}$ and $\frac{1}{2^m} < \frac{\epsilon}{3}$ whenever m and n are sufficiently large. Choose m, n sufficiently large to fulfill these criteria simultaneously. Then

$$\begin{aligned} d(r_m, r_n) &= d(p_{m,N(m)}, p_{n,N(n)}) \\ &\le d(p_{m,N(m)}, p_{m,s}) + d(p_{m,s}, p_{n,s}) + d(p_{n,s}, p_{n,N(n)}) \\ &\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

So r is Cauchy as desired.

This allows us to reach our desired conclusion: Define Q_k in terms of r as we did above. Now note that $\Delta(P_k, Q_k) < \frac{1}{2^k}$, so that $\Delta(P_k, Q_k) \to 0$. Since Q converges to R, so does P. Consequently, any Cauchy sequence in X^* converges. This makes X^* complete.

(d)

$$\Delta(P_p, P_q) = \lim_n d(p, q) = d(p, q)$$

(e) Let $R = [r]$ be any element of X^*. We have shown that Q_k as defined above is a sequence in X that converges to R. This implies that X is dense in X^*.

75. Exercise 25: Let X be the metric space whose points are the rational numbers, with the metric $d(x, y) = |x - y|$. What is the completion of this space?

> **Solution:** (*Dan "kyp44" Whitman*)
> The completion of $\mathbb{Q}$ in the manner described in Exercise 3.24 results in the real numbers. In fact this is one way to define the real numbers.

4 Continuity

76. Exercise 1: Suppose f is a real function defined on $\mathbf{R}$ which satisfies

$$\lim_{h \to 0} \big[f(x+h) - f(x-h)\big] = 0$$

for every $x \in \mathbf{R}$. Does this imply that f is continuous?

> **Solution:** (*ghostofgarborg*)
> No. As an example, take the function
>
> $$f(x) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases}$$
>
> which is discontinuous at 0, although $\lim_{h \to 0}[f(x+h) - f(x-h)] = 0$ everywhere.

77. Exercise 2: If f is a continuous mapping of a metric space X into a metric space Y, prove that

$$f(\overline{E}) \subset \overline{f(E)}$$

for every set $E \subset X$. Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.

> **Solution:** (*ghostofgarborg*)
>
> $$f^{-1}(\overline{f(E)}) = f^{-1}\left(\bigcap_{\substack{V \supseteq f(E) \\ V \text{ closed}}} V\right) = \bigcap_{\substack{V \supseteq f(E) \\ V \text{ closed}}} f^{-1}(V) \supseteq \bigcap_{\substack{W \supseteq E \\ W \text{ closed}}} W = \bar{E}$$
>
> Consequently, $f(\bar{E}) \subseteq \overline{f(E)}$. To see that the inclusion can be proper, let $X = (0, 1)$ seen as a subspace of $\mathbb{R}$, and f the inclusion into $\mathbb{R}$. If we let $E = X$, then
>
> $$f(\bar{E}) = f((0,1)) = (0,1) \subsetneq \overline{(0,1)} = [0,1]$$

78. Exercise 3: Let f be a continuous real function on a metric space X. Let $Z(f)$ (the *zero set* of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

> **Solution:** (*ghostofgarborg*)
> Note that $\{0\}$ is closed, so that $Z(f) = f^{-1}(\{0\})$ is closed by the continuity of f.

79. Exercise 4: Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$.

Solution: (*ghostofgarborg*)
Let $U \subset f(X)$ be open. Then $f^{-1}(U)$ is open, and since E is dense, it contains points of E. Therefore U contains points of $f(E)$. This makes $f(E)$ dense in $f(X)$.

Let $p \in X$, and p_n a sequence in E that converges to p. By theorem 4.6 and the equality of f and g on E

$$f(p) = \lim_{n\to\infty} f(p_n) = \lim_{n\to\infty} g(p_n) = g(p)$$

so that f and g agree on all of X.

80. Exercise 5: If f is a real continuous function defined on a closed set $E \subset \mathbf{R}$, prove that there exist continuous real functions g on $\mathbf{R}$ such that $g(x) = f(x)$ for all $x \in E$. Show that the result becomes false if the word "closed" is omitted. Extend the result to vector-valued functions.

Solution: (*analambanomenos*)
By Exercise 2.29, the open complement of E is a countable collection of disjoint open intervals $\cup_i(a_i, b_i)$. If $b_i = \infty$ define g_i on $[a_i, \infty)$ to take the constant value $f(a_i)$. Similarly, if $a_i = -\infty$, define g_i on $(-\infty, b_i]$ to take the constant value $f(b_i)$. Otherwise, on $[a_i, b_i]$, let g_i be the linear function

$$g_i(x) = \frac{f(b_i) - f(a_i)}{b_i - a_i}x + \frac{f(a_i)b_i - f(b_i)a_i}{b_i - a_i}.$$

Note that $g_i(a_i) = f(a_i)$, $g_i(b_i) = f(b_i)$, and the values of g_i lie between $f(a_i)$ and $f(b_i)$ on (a_i, b_i).

Let $g(x) = f(x)$ for $x \in E$ and $g(x) = g_i(x)$ for $x \in (a_i, b_i)$. It is clear that g is continuous at any $x \in (a_i, b_i)$, so suppose $x \in E$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ such that $\big|f(x) - f(y)\big| < \varepsilon$ for $y \in (x - \delta, x + \delta) \cap E$. If $x - \delta \notin E$, then $x \in (a_i, b_i)$ for some i, and we can replace $x - \delta$ with $x - \delta_1 = b_i \in E$. Similarly, if $x + \delta \notin E$, we can replace $x + \delta$ with some $x + \delta_2 = a_j \in E$. Hence, if any of the open intervals (a_i, b_i) intersect $(x - \delta_1, x + \delta_2)$, then both a_i and b_i must be in $(x - \delta_1, x + \delta_2)$. By the construction of the g_i, we must have $\big|g(x) - g(y)\big| < \varepsilon$ for $y \in (x - \delta_1, x + \delta_2)$.

If $\delta_1 = 0$, so that $x = b_i$ for some i, then by the linearity of g_i we can increase δ_1 by an amount small enough so that $\big|g(x) - g(y)\big| = \big|g_i(x) - g_i(y)\big| < \varepsilon$ for $y \in (x - \delta_1, x)$. Similarly, if $\delta_2 = 0$ so that $x = a_j$ for some j, we can increase δ_2 by an amount small enough so that $\big|g(x) - g(y)\big| < \varepsilon$ for $y \in (x, x + \delta_2)$. Hence both $\delta_1 > 0$ and $\delta_2 > 0$, so we can conclude that g is continuous at x.

Let $f(x) = x^{-1}$ for $x \in (0, 1)$. There can be no extension of f to all of $\mathbf{R}$ since $\lim_{x\to 0+} f(x) = \infty$.

If $\mathbf{f}(x) = \big(f_1(x), \ldots, f_k(x)\big)$ is a continuous map from a closed set E in $\mathbf{R}$ into $\mathbf{R}^k$, then each of the component functions f_n are continuous functions on E by Theorem 4.10(a). Extend each of the f_n to a continuous function g_n defined on all of $\mathbf{R}$. Then, also by Theorem 4.10(a), the vector-valued function $\mathbf{g}(x) = \big(g_1(x), \ldots, g_k(x)\big)$ is a continuous extension of $\mathbf{f}$ to all of $\mathbf{R}$.

81. Exercise 6: If f is defined on E, the *graph* of f is the set of points $\big(x, f(x)\big)$, for $x \in E$. Suppose E is compact, and prove that f is continuous on E if and only if its graph is compact.

Solution: (*ghostofgarborg*)
Assume $f : X \to Y$ is continuous. Then the map $\Gamma(x) = (x, f(x))$ is also continuous by thm. 4.10a. Since E is compact, so is $\Gamma(E)$, which is the graph of F.

If $\Gamma(E)$ is compact, let V be a closed subset of Y. The set $V' = (X \times V) \cap \Gamma(E)$ is closed in $\Gamma(E)$, hence compact. The projection $\pi : X \times Y \to X$ is continuous, so $f^{-1}(V) = \pi(V')$ is compact, hence closed (since X is a metric space and therefore Hausdorff). This makes f continuous.

82. Exercise 7: If $E \subset x$ and if f is a function defined on X, the *restriction* of f to E is the function g whose domain of definition is E, such that $g(p) = f(p)$ for $p \in E$. Define f and g on $\mathbf{R}^2$ by: $f(0,0) = g(0,0) = 0$,

$$f(x,y) = \frac{xy^2}{x^2+y^4}, \quad g(x,y) = \frac{xy^2}{x^2+y^6}$$

if $(x,y) \neq (0,0)$. Prove that f is bounded on $\mathbf{R}^2$, that g is unbounded in every neighborhood of $(0,0)$, and that f is not continuous at $(0,0)$; nevertheless, the restrictions of both f and g to every straight line in $\mathbf{R}^2$ are continuous!

Solution: (*analambanomenos*)
Note that both f and g are equal to 0 on the x and y axes.

For $k \in \mathbf{R}$, $f(ky^2, y) = k/(k^2+1)$ is constant for $y \neq 0$. Since the value of f along the parabola (ky^2, y) drops from $k/(k^2+1)$ to 0 at $y = 0$, f is not continuous at $(0,0)$. These parabolas sweep out $\mathbf{R}^2$ except for the x axis, and the values of f along these parabolas reach a maximum value of $1/2$ for $k = 1$, so the values of f lie in $[0, 1/2]$.

For $y \neq 0$, $g(ky^3, y) = k/\big((k^3+1)y\big) \to \infty$ as $y \to 0$, so g is unbounded in every neighborhood of $(0,0)$.

Since f and g are continuous away from the origin, their restrictions to any line which doesn't intersect the origin is also continuous. And since

$$f(x,kx) = \frac{k^2x}{1+k^4x^2} \qquad g(x,kx) = \frac{k^2x}{1+k^6x^4}$$

restrictions of f and g to lines which go through the origin are also continuous.

83. Exercise 8: Let f be a real uniformly continuous function on the bounded set E in $\mathbf{R}$. Prove that f is bounded on E. Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

Solution: (*analambanomenos*)
The quickest solution uses Exercise 13 below. Note that the closure $\bar{E}$ is also bounded, since if $E \subset [m, M]$ and $x > M$, then x has a neighborhood which doesn't intersect E so $x \notin \bar{E}$, similarly $y \notin \bar{E}$ if $y < m$, so that $\bar{E} \subset [m, M]$ also. Hence $\bar{E}$ is compact. By Exercise 13, f can be extended to a continuous function $\bar{f}$ on $\bar{E}$ whose range is also compact. Hence $\bar{f}$ is bounded on $\bar{E}$, so f is bounded on E.

The identity function $f(x) = x$ on $E = \mathbf{R}$ (which is not bounded) is uniformly continuous but not bounded on E.

84. Exercise 9: Show that the requirement in the definition of uniform continuity can be rephrased as follows, in terms of diameters of sets: To every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\operatorname{diam} f(E) < \varepsilon$ for all $E \subset X$ with $\operatorname{diam} E < \delta$.

Solution: (*analambanomenos*)
Suppose f is a uniformly continuous function from the metric space X to the metric space Y. Let

$\varepsilon > 0$. Then there is a $\delta > 0$ such that $d_Y\big(f(p), f(q)\big) < \varepsilon$ if $d_X(p,q) < \delta$. Let $E \subset X$ with $\operatorname{diam} E < \delta$. If $p, q \in E$, then $d_X(p,q) \le \operatorname{diam} E < \delta$, so $d_Y\big(f(p), f(q)\big) < \varepsilon$. Hence $\operatorname{diam} f(E) < \varepsilon$.

Conversely, let f be a function from the metric space X to the metric space Y with the diameter property. Let $\varepsilon > 0$ and $\delta > 0$ such that $\operatorname{diam} f(E) < \varepsilon$ for all $E \subset X$ with $\operatorname{diam} E < \delta$. Let $p, q \in X$ such that $d_X(p,q) < \delta$. Letting $E = \{p, q\}$ we have $\operatorname{diam} E < \delta$, so $\operatorname{diam} f(E) = d_Y\big(f(p), f(q)\big) < \varepsilon$. Hence f is uniformly continuous.

85. Exercise 10: Complete the details of the following alternative proof of Theorem 4.19: If f is not uniformly continuous, then for some $\varepsilon > 0$ there are sequences $\{p_n\}$, $\{q_n\}$ in X such that $d_X(p_n, q_n) \to 0$ but $d_Y\big(f(p_n), f(q_n)\big) > \varepsilon$. Use Theorem 2.37 to obtain a contradiction.

Solution: (*analambanomenos*)
By Theorem 2.37, $\{p_n\}$ has a subsequence which converges to $p \in X$. Replace $\{p_n\}$ with this subsequence and replace $\{q_n\}$ with the corresponding subsequence. Similarly, $\{q_n\}$ has a subsequence which converges to $q \in X$. Again replace $\{q_n\}$ with this subsequence and replace $\{p_n\}$ with the corresponding subsequence. Since $d_X(p_n, q_n)$ converges to 0, we must have $p = q$. Hence by continuity, $f(p_n)$ and $f(q_n)$ must both converge to $f(p) = f(q)$, contradicting the assumption that $d_Y\big(f(p_n), f(q_n)\big) > \varepsilon$.

86. Exercise 11: Suppose f is a uniformly continuous mapping of a metric space X into a metric space Y and prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X. Use this result to give an alternative proof of the theorem stated in Exercise 13.

Solution: (*analambanomenos*)
Let $\varepsilon > 0$ and let $\delta > 0$ such that $d_Y\big(f(x), f(y)\big) < \varepsilon$ if $d_X(x,y) < \delta$. Let $\{x_n\}$ be a Cauchy sequence in X. Then there is an integer N such that $d_X(x_n, x_m) < \delta$ if $n \ge N$ and $m \ge N$. Hence $d_Y\big(f(x_n), f(x_m)\big) < \varepsilon$ if $n \ge N$ and $m \ge N$, so $\{f(x_n)\}$ is a Cauchy sequence in Y.

Let E be a dense subset of a metric space X, and let f be a uniformly continuous mapping of X into a complete metric space Y. We can extend f to a function on all of X as follows. Let $x \in X$ and $\{p_n\}$ be a sequence in E which converges to x. Then $\{f(p_n)\}$ is a Cauchy sequence in Y which converges to $y \in Y$ since Y is complete. This sets up a well-defined function g from X to Y since if $\{p'_n\}$ is another sequence in X converging to x, then $\{f(p_n)\}$ converges to $y' \in Y$. Since f is uniformly continuous on D, for $\varepsilon > 0$ there is $\delta > 0$ such that if $p, p' \in E$ with $d_X(p, p') < \delta$ then $d_Y\big(f(p), f(p')\big) < \epsilon/3$. There is an integer N such that $d_X(p_n, x) < \delta/2$ and $d_X(p'_n, x) < \delta/2$ for all $n \ge N$. Then for such n we have

$$d_X(p_n, p'_n) \le d_X(p_n, x) + d_X(x, p'_n) < \delta.$$

Also, there is an integer $M \ge N$ such that for all $n > M$ we have $d_Y\big(y, f(p_n)\big) < \varepsilon/3$ and $d_Y\big(y', f(p'_n)\big) < \varepsilon/3$. Hence, for large enough n we have

$$d_Y(y, y') \le d_Y\big(y, f(p_n)\big) + d_Y\big(f(p_n), f(p'_n)\big) + d_Y\big(f(p'_n), y'\big) < \varepsilon.$$

Considering the constant sequence $\{x\}$ for $x \in E$, we see that g extends f to all of X. To see that g is uniformly continuous, let $\varepsilon > 0$. Since f is uniformly continuous on E, there is $\delta > 0$ such that if $p, p' \in E$ with $d_X(p, p') < \delta$ then $d_Y\big(f(p), f(p')\big) < \epsilon/3$. Let $x, x' \in X$ such that $d_X(x, x') < \delta/3$, and let $\{p_n\}$ and $\{p'_n\}$ be sequences in E which converge to x and x', respectively. Let n be a large enough integer so that $d_X(p_n, x) < \delta/3$ and $d_X(p'_n, x') < \delta/3$. Then

$$d_X(p_n, p'_n) \le d(p_n, x) + d(x, x') + d(x', p'_n) < \delta$$

so that $d_Y\big(f(p_n), f(p_n')\big) < \epsilon/3$. Also, if n is large enough we have $d_Y\big(g(x), f(p_n)\big) < \varepsilon/3$ and $d_Y\big(g(x'), f(p_n')\big) < \varepsilon/3$. Hence, for large enough n we have

$$d_Y\big(g(x), g(x')\big) \leq d_Y\big(g(x), f(p_n')\big) + d_Y\big(f(p_n), f(p_n')\big) + d_Y\big(f(p_n'), g(x')\big) < \varepsilon.$$

Hence g is uniformly continuous on X.

87. Exercise 12: Suppose f is a uniformly continuous mapping of a metric space X into a metric space Y, and suppose g is a uniformly continuous mapping of Y into a metric space Z. Then show that $g \circ f$ is a uniformly continuous mapping of X into Z.

Solution: (*analambanomenos*)
Let $\varepsilon > 0$. Then there is $\eta > 0$ such that $d_Z\big(g(x), g(y)\big) < \varepsilon$ if $d_Y(x, y) < \eta$. There is also $\delta > 0$ such that $d_Y\big(f(x), f(y)\big) < \eta$ if $d_X(p, q) < \delta$. Hence if $d_X(p, q) < \delta$, then $d_Z\big(g(f(p)), g(f(q))\big) < \varepsilon$, so $g \circ f$ is uniformly continuous.

88. Exercise 13: Let E be a dense subset of a metric space X, and let f be a uniformly continuous *real* function defined on E. Prove that f has a continuous extension from E to X. Could the range space $\mathbf{R}$ be replaced by $\mathbf{R}^k$? By any compact metric space? By any complete metric space? By any metric space?

Solution: (*analambanomenos*)
Following the hint, for $p \in X$ let $V_n(p)$ be the set of $q \in E$ such that $d(p, q) < 1/n$. By Exercise 9, there is an integer N such that $\operatorname{diam} f\big(V_N(p)\big) < 1$ so the closure F_n of $f\big(V_n(p)\big)$ for $n \geq N$ is a compact subset of $\mathbf{R}$. Since $F_N \supset F_{N+1} \supset \cdots$, the intersection $\cap_N^\infty F_n$ is nonempty by Theorem 2.36. This intersection consists of a single point. For suppose $x, y \in \cap F_n$ and $|x - y| = \varepsilon > 0$. Let $n \geq N$ be large enough so that if $q_1, q_2 \in V_n(p)$ then $\big|f(q_1) - f(q_2)\big| < \varepsilon/3$. Since both x and y are in F_n, they have open neighborhoods of radius $\varepsilon/3$ which intersect $f\big(V_n(p)\big)$, let $q_1, q_2 \in V_n(p)$ such that $f(q_1)$ and $f(q_2)$ are in these neighborhoods of x and y, respectively. Then

$$\varepsilon = |x - y| \leq \big|x - f(q_1)\big| + \big|f(q_1) - f(q_2)\big| + \big|f(q_2) - y\big| < \varepsilon,$$

which is a contradiction.

Let $g(p)$ be this single point in $\cap F_n$. If $p \in E$, then $f(p) \in F_n$ for all n, so $g(p) = f(p)$, hence g is an extension of f to all of X. To show that g is continuous, let $\varepsilon > 0$. Since f is uniformly continuous on E, there is $\delta > 0$ so that $\big|f(q_1) - f(q_2)\big| < \varepsilon/3$ if $q_1, q_2 \in E$ and $d(q_1, q_2) < \delta$. Let $p_1, p_2 \in X$ such that $d(p_1, p_2) < \delta/3$, and let n be a large enough integer so that $1/n < \delta/3$ and large enough so that, if $q_1 \in V_n(p_1)$ and $q_2 \in V_n(p_2)$, then $\big|g(p_1) - f(q_1)\big| < \varepsilon/3$ and $\big|g(p_2) - f(q_2)\big| < \varepsilon/3$. Then

$$d(q_1, q_2) \leq d(q_1, p_1) + d(p_1, p_2) + d(p_2, q_2) < \delta$$

so that

$$\big|g(p_1) - g(p_2)\big| \leq \big|g(p_1) - f(q_1)\big| + \big|f(q_1) - f(q_2)\big| + \big|f(q_2) - g(p_2)\big| < \varepsilon.$$

Hence g is uniformly continuous on X.

The only topological property of $\mathbf{R}$ that was used was that closed and bounded sets are compact, so the assertion can probably also be extended to $\mathbf{R}^k$. Also, the F_n will also be compact if the target space is compact, so the assertion is probably also true for that case. For a complete metric space, the assertion was proved in the answer to Exercise 11. The assertion is probably not true for a general metric space, but I can't come up with an example.

89. Exercise 14: Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

Solution: (*Matt "frito" Lundy*)
Because both $h(x) = x$ and f are continuous mappings on I, $g(x) = f(x) - x$ is also continuous on I. We are searching for an x such that $g(x) = 0$. $g(0) \in [0, 1]$ and $g(1) \in [-1, 0]$, and if either $g(0) = 0$ or $g(1) = 0$, we are done, so assume that both $g(0) \neq$ and $g(1) \neq 0$. Then $g(0) \in (0, 1]$ and $g(1) \in [-1, 0)$, and because g is continuous on I, with I connected, the intermediate value theorem applies, and there exists an $x \in (0, 1)$ such that $f(x) = 0$.

90. Exercise 15: Call a mapping of X into Y *open* if $f(V)$ is an open set in Y whenever V is an open set in X. Prove that every continuous open mapping f of $\mathbf{R}$ into $\mathbf{R}$ is monotonic.

Solution: (*analambanomenos*)
By Theorem 4.16, there is an x and y in any closed interval $[a, b]$ such that $f(x)$ is the minimum value of f on $[a, b]$ and $f(y)$ is the maximum value. If $a < x < y < b$, then $f\big((a, b)\big)$ is the closed interval $\big[f(x), f(y)\big]$, contradicting the openness of f. Similarly, if $a = x < y < b$ or $a < x < y = b$ the image of (a, b) is a half-closed interval. Hence f must attain its maximum or minimum values at the endpoints of any closed interval. Suppose $f(a)$ is the minimum value and $f(b)$ is the maximum value of f on $[a, b]$, and let $a < y \leq b$. Then $f(y)$ would be the maximum value of f on $[a, y]$, so $f(x) \leq f(y)$ for $a \leq x < y$, that is, f is monotonically increasing on $[a, b]$.

Hence f is monotonic on any closed interval, and if f is monotonically increasing on one closed interval, then it must also be monotonically increasing on any larger interval. Hence f is monotonic on all of $\mathbf{R}$.

91. Exercise 16: Let $[x]$ denote the largest integer contained in x and let $(x) = x - [x]$ denote the fractional part of x. What discontinuities do the functions $[x]$ and (x) have?

Solution: (*analambanomenos*)
Let $f(x) = [x]$ and $g(x) = (x)$. Then f and g, being constant or linear functions on $(n, n + 1)$ for each integer n, are continuous except at the integers. If n is an integer, then $f(n-) = n - 1$ and $f(n+) = f(n) = n$, and $g(n-) = 1$ and $g(n+) = g(n) = 0$. These are all simple discontinuities.

92. Exercise 17: Let f be a real function defined on (a, b). Prove that the set of points at which f has a simple discontinuity is at most countable.

Solution: (*analambanomenos*)
Following the hint, let E_1 be the set of points in (a, b) on which $f(x-) < f(x+)$. With each $x \in E_1$ associate a triple (p, q, r) of rational numbers such that $f(x-) < p < f(x+)$, $f(y) < p$ for $a < q < y < x$, and $f(y) > p$ for $x < y < r < b$. Each triple is associated with at most one point of E_1. Otherwise there would be x_1 and x_2 in E_1 such that $f(y) < p$ for $q < y < x_1$ and $q < y < x_2$, and $f(y) > p$ for $x_1 < y < r$ and $x_2 < y < r$, which is a set of contradictory conditions on $f(y)$ for y between x_1 and x_2. Hence E_1 can be mapped in a one-to-one fashion to a subset of the set of all rational triples, which is countable, and so E_1 is at most countable. Similarly, the set E_2 of points in (a, b) on which $f(x-) > f(x+)$ is also at most countable.

Let F_1 be the set of points in (a, b) on which $f(x-) = f(x+) < f(x)$. With each $x \in F_1$ associate a triple (p, q, r) of rational numbers such that $f(y) < p < f(x)$ for $a < q < y < x$ and $x < y < r < b$. Each triple is associated with at most one point of F_1. Otherwise there would be x_1 and x_2 in F_1

such that $f(y) < p < f(x_1)$ for $q < y < x_1$ and $x_1 < y < r$, and $f(y) < p < f(x_2)$ for $q < y < x_2$ and $x_2 < y < r$, which is a set of contradictory conditions on $f(x_1)$ and $f(x_2)$. Hence, as with E_1, F_1 is at most countable. Similarly, the set F_2 of points in (a, b) on which $f(x-) = f(x+) > f(x)$ is also at most countable.

Since the set of points in (a, b) at which f has a simple discontinuity is $E_1 \cup E_2 \cup F_1 \cup F_2$, the set of such points is at most countable.

93. Exercise 18: Every rational x can be written in the form $x = m/n$, where $n > 0$ and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on $\mathbf{R}$ by

$$f(x) = \begin{cases} 0 & x \text{ irrational} \\ \dfrac{1}{n} & x = \dfrac{m}{n}. \end{cases}$$

Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point.

Solution: (*Matt "frito" Lundy*)
Because every point $x \in \mathbf{R}$ is a limit point of $\mathbf{R}$, we can show that for $r \in \mathbf{Q}$, we have $\lim_{x\to r} f(x) = 0 \neq f(r)$ and for $y \in \mathbf{R}, y \notin \mathbf{Q}$, we have $\lim_{x\to y} f(x) = 0 = f(y)$.

Fix $x \in \mathbf{R}$. Let $\{x_n\}$ be a sequence in $\mathbf{R}$ such that $x_n \neq x$ and $x_n \to x$ as $n \to \infty$.

If $\{x_n\}$ has only finitely many $x_n \in \mathbf{Q}$, then there exists some $N \in \mathbf{N}$ such that $n \geq N$ implies that $f(x_n) = 0 < \varepsilon$, because each x_n is irrational after some finite number of terms. If $\{x_n\}$ has infinitely many $x_n \in \mathbf{Q}$, let $\{x_{n_l}\}$ be a subsequence of $\{x_n\}$ consisting of all the rational x_n. x_{n_l} must converge to x because x_n converges to x.

Each $x_{n_l} = \frac{p_{n_l}}{q_{n_l}}$ where $p_{n_l} \in \mathbf{Z}$, $q_{n_l} \in \mathbf{N}$ and $\gcd(p_{n_l}, q_{n_l}) = 1$. We claim that $q_{n_l} \to +\infty$, as $n_l \to +\infty$, so that

$$\lim_{n_l\to\infty} f(x_{n_l}) = \lim_{n_l\to\infty} \frac{1}{q_{n_l}} = 0.$$

Suppose that $\lim_{n_l\to\infty} q_{n_l} \neq +\infty$, then there exists some $M \in \mathbf{N}$ such that $q_{n_l} < M$ for all n_l. Then we have:

$$\begin{aligned} |x - x_{n_l}| &= \left| x - \frac{p_{n_l}}{q_{n_l}} \right| && (9) \\ &= \left| \frac{q_{n_l}x - p_{n_l}}{q_{n_l}} \right| && (10) \\ &> \frac{|q_{n_l}x - p_{n_l}|}{M}. && (11) \end{aligned}$$

If $\{p_{n_l}\}$ is unbounded, then, $\{q_{n_l}\}$ bounded means that (11) is unbounded. If $\{p_{n_l}\}$ is bounded, then there are finitely many values of p_{n_l} and q_{n_l}. $x \neq x_n$ means that $q_{n_l}x - p_{n_l} \neq 0$ for any n_l, so let $\eta = \min(|q_{n_l}x - p_{n_l}|) > 0$ then:

$$\begin{aligned} |x - x_{n_l}| &> \frac{|q_{n_l}x - p_{n_l}|}{M} && (12) \\ &> \frac{\eta}{M}. && (13) \end{aligned}$$

So, $\lim_{n_l \to \infty} q_{n_l} \neq +\infty$ leads to x_{n_l} not converging to x, a contradiction. So $q_{n_l} \to \infty$, and for any $p \in \mathbf{R}$, $\lim_{p \to x} f(x) = 0$, and thus $f(x)$ is continuous at every irrational p, and discontinuous at every rational p.

Solution: (*analambanomenos*)
Let p be any real number, let $\varepsilon > 0$, and let N be an integer large enough so that $1/N < \varepsilon$. Since in any interval around p there are only a finite number of rational numbers m/n such that $n < N$, there is a $\delta > 0$ small enough so that if $0 < |x-p| < \delta$ then $\bigl|f(x)\bigr| < 1/N < \varepsilon$. Hence $\lim_{x \to p} f(x) = 0$ for all real p. Since $f(p) = 0$ for irrational p, f is continuous at such p. And since $f(p) > 0$ for rational p, f has a simple discontinuity at such p.

94. Exercise 19: Suppose f is a real function with domain $\mathbf{R}$ which has the intermediate value property: If $f(a) < c < f(b)$, then $f(x) = c$ for some x between a and b. Suppose also, for every rational r, that the set of all x with $f(x) = r$ is closed. Prove that f is continuous.

Solution: (*analambanomenos*)
Suppose f is not continuous at x_0. Then there is a sequence of real numbers y_n converging to x_0 such that $f(y_n)$ doesn't converge to $f(x_0)$, that is, there is a $\delta > 0$ such that $(f(x_0) - \delta, f(x_0) + \delta)$ contains only a finite number of the $f(y_n)$. Hence there is an infinite subsequence x_n of the y_n such that either all the $f(x_n)$ are greater than $f(x_0) + \delta$ or less than $f(x_0) - \delta$. Suppose the first case is true (the second case can be handled by considering the function $-f$, which also satisifies the hypotheses). Following the hint, let r be a rational number such that $f(x_n) > r > f(x_0)$ for all n. Then by the intermediate value property satisfied by f, for each n there is a number t_n such that t_n lies between x_0 and x_n and $f(t_n) = r$. Since t_n converges to x_0, x_0 is in the closure of the set of all x such that $f(x) = r$. This set is closed by hypothesis, so we must have $f(x_0) = r$, contradicting the assumption that $f(x_0) < r$.

95. Exercise 20: If E is a nonempty subset of a metric space X, define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{x \in E} d(x, z).$$

(a) Prove that $\rho_E(x) = 0$ if and only if $x \in \bar{E}$.

(b) Prove that ρ_E is a uniformly continuous function on X, by showing that

$$\bigl|\rho_E(x) - \rho_E(y)\bigr| \leq d(x, y)$$

for all $x \in X$, $y \in X$.

Solution: (*analambanomenos*)
(a): The condition $\rho_E(x) = 0$ is equivalent to having every neighborhood of x of radius $\delta > 0$ having some element z of E, that is, $x \in \bar{E}$.

(b): Following the hint, if $z \in E$, then $\rho_E(x) \leq d(x, y) + d(y, z)$ by the triangle inequality, so that $\rho_E(x) \leq d(x, y) + \rho_E(y)$. Symmetrically, $\rho_E(y) \leq d(x, y) + \rho_E(x)$, so that $\bigl|\rho_E(x) - \rho_E(y)\bigr| \leq d(x, y)$ for all $x \in X$, $y \in X$.

96. Exercise 21: Suppose K and F are disjoint sets in a metric space X, K is compact, F is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K$, $q \in F$. Show that the conclusion may fail for two disjoint closed sets if neither is compact.

Solution: (*analambanomenos*)
Since ρ_F is a continous function on the compact set K, by Theorem 4.16 it must attain its minimum value on K at some $x \in K$. This minimum value δ must be nonzero, since if $\rho_F(x) = 0$ then by Exercise 20(a) $x \in \bar{F} = F$ which contradicts the disjointness of K and F.

An example where this fails for two planar closed sets is

$$\begin{aligned} F_1 &= \{(x, y) \in \mathbf{R}^2 | x < 0, y \geq 1/x^2\} \\ F_2 &= \{(x, y) \in \mathbf{R}^2 | x > 0, y \geq 1/x^2\} \end{aligned}$$

A one-dimensional example is

$$\begin{aligned} F_1 &= \text{set of positive integers} \\ F_2 &= \cup_{n=0}^{\infty} \left[n + 2^{-(n+2)}, n + 1 - 2^{-(n+2)}\right] \\ &= [\tfrac{1}{4}, \tfrac{3}{4}] \cup [1\tfrac{1}{8}, 1\tfrac{7}{8}] \cup [2\tfrac{1}{16}, 2\tfrac{15}{16}] \cup \cdots \end{aligned}$$

97. Exercise 22: Let A and B be disjoint nonempty closed sets in a metric space X and define

$$f(p) = \frac{\rho_A(p)}{\rho_A(p) + \rho_B(p)} \quad (p \in X).$$

Show that f is a continuous function on X whose range lies in $[0, 1]$, that $f(p) = 0$ precisely on A and $f(p) = 1$ precisely on B. This establishes a converse of Exercise 3: Every closed set $A \subset X$ is $Z(f)$ for some continuous real f on X. Setting

$$V = f^{-1}\big([0, 1/2)\big), \quad W = f^{-1}\big((1/2, 1]\big),$$

show that V and W are open and disjoint, and that $A \subset V$, $B \subset W$.

Solution: (*analambanomenos*)
If $\rho_A(p)+\rho_B(p) = 0$, then $\rho_A(p) = \rho_B(p) = 0$. By Exercise 20(a) this implies that $p \in \bar{A}\cap\bar{B} = A\cap B$. But A and B are disjoint, hence $\rho_A(p) + \rho_B(p) > 0$. Hence, since ρ_A and ρ_B are continuous on X, $f(p)$ is also continuous on X by Theorem 4.9. Also, $f(p) = 0$ if and only if $\rho_A(p) = 0$, that is, $p \in \bar{A} = A$, and $f(p) = 1$ if and only if $\rho_B(p) = 0$, that is, $p \in \bar{B} = B$.

Since $[0, 1/2)$ and $(1/2, 1]$ are disjoint open sets in $[0, 1]$ and f is continuous on X, V and W are disjoint open sets of X by Theorem 4.8.

98. Exercise 23: A real-valued function f defined in (a, b) is said to be *convex* if

$$f\big(\lambda x + (1 - \lambda x)y\big) \leq \lambda f(x) + (1 - \lambda)f(y)$$

whenever $a < x < b$, $a < y < b$, $0 < \lambda < 1$. Prove that every convex function is continuous. Prove that every increasing convex function of a convex function is convex. If f is convex in (a, b), and if $a < s < t < u < b$, show that

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

Solution: (*analambanomenos*)
Letting $a < s < t < u < b$, then $t = \lambda u + (1+\lambda)s$ for $\lambda = (t-s)/(u-s)$. Hence

$$\begin{aligned} f(t) &\le \lambda f(u) + (1-\lambda)f(s) \\ f(t) - f(s) &\le \lambda\big(f(u) - f(s)\big) \\ \frac{f(t)-f(s)}{t-s} &\le \frac{f(u)-f(s)}{u-s} \end{aligned}$$

Also, $t = \lambda' s + (1-\lambda')u$ for $\lambda' = (u-t)/(u-s)$. Hence

$$\begin{aligned} f(t) &\le \lambda' f(s) + (1-\lambda')f(u) \\ \lambda'\big(f(u) - f(s)\big) &\le f(u) - f(t) \\ \frac{f(u)-f(s)}{u-s} &\le \frac{f(u)-f(t)}{u-t} \end{aligned}$$

The first inequality can be rewritten $f(t) - f(s) = K(t-s)$ for $s < t < u$, which shows that f is continuous from the right at s. Similarly, the second inequality can be rewritten $f(u) - f(t) = K'(u-t)$ for $s < t < u$, which shows that f is continuous from the left at u. Since s and u were arbitrary elements of (a,b), we see that f is continuous on (a,b).

Let f be a convex function defined in (a,b) and let g be an increasing convex function defined on the range of f. Then for $a < x < b$, $a < y < b$, $0 < \lambda < 0$, we have

$$\begin{aligned} g\Big(f\big(\lambda x + (1-\lambda)y\big)\Big) &\le g\big(\lambda f(x) + (1-\lambda)f(y)\big) \\ &\le \lambda g\big(f(x)\big) + (1-\lambda)g\big(f(y)\big). \end{aligned}$$

Hence $g \circ f$ is convex on (a,b).

99. Exercise 24: Assume that f is a continuous real function defined in (a,b) such that

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}$$

for all $x, y \in (a,b)$. Prove that f is convex.

Solution: (*analambanomenos*)
Fix $x, y \in (a,b)$. We can show by induction that $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$ for all rational λ between 0 and 1 with denominator a power of 2. This is true for $\lambda = 1/2$ by assumption. Assume that the inequality holds for all $\lambda = j/2^m$ for $m < n$ and $1 < j < 2^m$, and let $\lambda = j/2^n$. If j is even, then the inequality holds by the induction hypothesis, so assume $j = 2k+1$ is odd. Let

$$x_1 = \left(\frac{k}{2^{n-1}}\right)x + \left(1 - \frac{k}{2^{n-1}}\right)y \quad \text{and} \quad y_1 = \left(\frac{k+1}{2^{n-1}}\right)x + \left(1 - \frac{k+1}{2^{n-1}}\right)y.$$

Then $\lambda x + (1-\lambda)y = (x_1 + y_1)/2$, so we have

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\le \frac{f(x_1)+f(y_1)}{2} \\ &\le \frac{1}{2}\left(\left(\frac{k}{2^{n-1}}\right)f(x) + \left(1 - \frac{k}{2^{n-1}}\right)f(y)\right) + \frac{1}{2}\left(\left(\frac{k+1}{2^{n-1}}\right)f(x) + \left(1 - \frac{k+1}{2^{n-1}}\right)f(y)\right) \\ &= \lambda f(x) + (1-\lambda)f(y) \end{aligned}$$

Both sides of the inequality $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$ are continuous for $\lambda \in (0,1)$, and the inequality holds for a dense subset of $(0,1)$. Hence it holds for all $\lambda \in (0,1)$.

100. Exercise 25: If $A \subset \mathbf{R}^k$ and $B \subset \mathbf{R}^k$, define $A+B$ to be the set of all sums $\mathbf{x}+\mathbf{y}$ with $\mathbf{x} \in A$, $\mathbf{y} \in B$.

   (a) If K is compact and C is closed in $\mathbf{R}^k$, prove that $K+C$ is closed.

   (b) Let α be an irrational real number. Let C_1 be the set of all integers, let C_2 be the set of all $n\alpha$ with $n \in C_1$. Show that C_1 and C_2 are closed subsets of $\mathbf{R}$ whose sum $C_1 + C_2$ is *not* closed, by showing that $C_1 + C_2$ is a countable dense subset of $\mathbf{R}$.

Solution: (*analambanomenos*)
(a): Following the hint, take $\mathbf{z} \notin K+C$ and put $F = \{\mathbf{z}\} - C$. K and F are disjoint since if there were a $\mathbf{y} \in F \cap K$ then there would be $\mathbf{x} \in C$ such that $\mathbf{y} = \mathbf{z} - \mathbf{x}$, but then $\mathbf{z} = \mathbf{y} + \mathbf{x}$ contradicts $\mathbf{z} \notin K + C$. F is closed since it is the inverse image of C by the continuous map $\mathbf{x} \mapsto \mathbf{z} - \mathbf{x}$. Hence by Exercise 21 there is a $\delta > 0$ such that $|\mathbf{x} - \mathbf{y}| > \delta$ if $\mathbf{x} \in K$ and $\mathbf{y} \in F$. If $\mathbf{x} + \mathbf{w} \in K + C$, then $\big|(\mathbf{x}+\mathbf{w}) - \mathbf{z}\big| = \big|\mathbf{x} - (\mathbf{z} - \mathbf{w})\big| > \delta$, so that the open neighborhood of $\mathbf{z}$ of radius δ is disjoint from $K + C$. Hence the complement of $K+C$ is open.

(b): Each element of $C_1 + C_2$ can be associated with a unique pair of integers, since if $n_1 + n_2\alpha = m_1 + m_2\alpha$ for $n_2 \ne m_2$, then this equation can be rewritten to express α as a rational number. Since the collection of such integer pairs is countable, $C_1 + C_2$ is countable.

Consider the fractional parts of the integer multiples of α, the set of $p\alpha - [p\alpha]$ for p an integer, which are all in $C_1 + C_2$. These are all distinct by the argument above. Let n be any positive integer. By the pigeonhole principle, there are at least two such fractional parts in one of the subintervals $(0, 1/n), (1/n, 2/n), \ldots, ((n-1)/n, 1)$ of $(0,1)$. That is, there are integers p, q such that $0 < (p-q)\alpha - ([p\alpha] - [q\alpha]) < 1/n$, so that $C_1 + C_2 \cap (0, 1/n)$ is nonempty.

Let x be any real number and let $\epsilon > 0$. Let n be a large enough positive integer such that $1/n < \epsilon$, and let $y \in C_1 + C_2 \cap (0, 1/n)$. Then some multiple of y lies in $[x, x + 1/n)$, so that some element of $C_1 + C_2$ is within ϵ of x. Hence the closure of $C_1 + C_2$ is $\mathbf{R}$, and since it is a proper subset of $\mathbf{R}$, it is not closed.

101. Exercise 26: Suppose X, Y, Z are metric spaces, and Y is compact. Let f map X into Y, let g be a continuous one-to-one mapping of Y into Z, and put $h(x) = g(f(x))$ for $x \in X$. Prove that f is uniformly continuous if h is uniformly continuous. Prove also that f is continuous if h is continuous. Show that the compactness of Y cannot be omitted from the hypotheses, even when X and Z are compact.

Solution: (*analambanomenos*)
Since g is one-to-one, it defines a function g^{-1} from $g(Y)$ to Y, which is continuous by Theorem 4.17. If h is continuous, then $f = g^{-1} \circ h$ is continuous by Theorem 4.7, and if h is uniformly continuous, then f is uniformly continuous by Exercise 12.

Let $X = Z = [0,2]$ and $Y = [0,1) \cup [2,3]$. Let $f(x) = x$ for $0 \le x < 1$ and $f(x) = x+1$ for $1 \le x \le 2$. Let $g(x) = x$ for $0 \le x < 1$ and $g(x) = x - 1$ for $2 \le x \le 3$. Then X and Z are compact, Y is not compact, g is continuous and one-to-one, $h(x) = g(f(x)) = x$ for $0 \le x \le 2$ is uniformly continous, but f is discontinuous at $x = 1$.

5 Differentiation

102. Exercise 1: Let f be defined for all real x, and suppose that $\big|f(x)-f(y)\big| \le (x-y)^2$ for all real x and y. Prove that f is constant.

Solution: (*Matt "Frito" Lundy*)
For any x, we have:

$$\begin{aligned} \left|\frac{f(t)-f(x)}{t-x}\right| &\le \left|\frac{(t-x)^2}{t-x}\right| \\ &= |t-x| \to 0 \end{aligned}$$

as $t \to x$. Because $f'(x) = 0$ for all x, $f(x)$ is a constant by Theorem 5.11(b).

103. Exercise 2: Suppose $f'(x) > 0$ in (a,b). Prove that f is strictly increasing in (a,b), and let g be its inverse function. Prove that g is differentiable, and that

$$g'\big(f(x)\big) = \frac{1}{f'(x)} \qquad (a < x < b).$$

Solution: (*Matt "Frito" Lundy*)
If f were not strictly increasing in (a,b), there would exist x, y with both $a < x < y < b$ and $f(x) \ge f(y)$. By "the" mean value theorem, there would exist a $z \in (x,y)$ such that $f'(z) = \frac{f(y)-f(x)}{y-x} \le 0$, which contradicts $f'(x) > 0$ in (a,b), so f must be strictly increasing.

Fix $\varepsilon > 0$ and $x \in (a,b)$. $f'(x) > 0$ means there exists a $\eta > 0$ and a $\delta_1 > 0$ such that $0 < |x-t| < \delta_1$ implies

$$\left|\frac{f(t)-f(x)}{t-x}\right| > \eta$$

So:

$$\frac{1}{\left|\frac{f(t)-f(x)}{t-x}\right|} < \frac{1}{\eta}$$

From the definition of $f'(x)$, we also have for $\eta f'(x)\varepsilon > 0$, there exists a $\delta_2 > 0$ such that $0 < |x-t| < \delta_2$ implies

$$\left|f'(x) - \frac{f(t)-f(x)}{t-x}\right| < \eta f'(x)\varepsilon$$

Let $y = f(x)$ and $\delta = \min\{\delta_1, \delta_2\}$. Then for any $u \in (f(x-\delta), f(x+\delta))$, let $g(u) = t$ so $t \in (x-\delta, x+\delta)$ and:

$$\begin{aligned} \left|\frac{g(u)-g(y)}{u-y} - \frac{1}{f'(x)}\right| &= \left|\frac{f'(x) - \frac{f(t)-f(x)}{t-x}}{f'(x)\left[\frac{f(t)-f(x)}{t-x}\right]}\right| \\ &< \frac{\eta f'(x)\varepsilon}{\eta f'(x)} \\ &= \varepsilon \end{aligned}$$

Which shows that
$$g'(f(x)) = \frac{1}{f'(x)}.$$

104. Exercise 3: Suppose g is a real function on $\mathbf{R}$, with bounded derivative (say $|g'| \le M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is one-to-one if ε is small enough. (A set of admissible values of ε can be determined which depends only on M.)

Solution: (*analambanomenos*)
The derivative of f is $f' = 1 + \varepsilon g'$, which is positive if $\varepsilon < 1/M$. In that case, by Exercise 2, f is strictly increasing, hence one-to-one.

105. Exercise 4: If
$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$
where $C_0, \ldots, C_n$ are real constants, prove that the equation
$$C_0 + C_1 x + \ldots + C_{n-1}x^{n-1} + C_n x^n = 0$$
has at least one real root between 0 and 1.

Solution: (*Matt "Frito" Lundy*)
Let:
$$f(x) = \sum_{i=0}^{n} \frac{C_i x^{i+1}}{i+1}.$$
Then we are looking for an $x \in (0,1)$ such that $f'(x) = 0$. But $f(x)$ is continuous in $[0,1]$, differentiable in $(0,1)$, $f(0) = 0$, and $f(1) = 0$. So by "the" mean value theorem, there exists an $x \in (0,1)$ such that $f'(x) = 0$, as desired.

106. Exercise 5: Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

Solution: (*Matt "Frito" Lundy*)
Fix $\varepsilon > 0$. $f'(x) \to 0$ as $x \to +\infty$ means that there exists an $M \in \mathbf{R}$ such that $x \ge M$ implies $|f'(x)| < \varepsilon$. Now for any $a, b \in \mathbf{R}$ such that $M < a < b$, "the" mean value theorem implies that there exists an $x \in (a,b)$ such that
$$\frac{f(b) - f(a)}{b-a} = f'(x).$$
But $x \in (a,b)$ implies that $x > M$ so:
$$\left|\frac{f(b) - f(a)}{b-a}\right| = |f'(x)| < \varepsilon.$$
Taking $b = a+1$ means if $a > M$, then $|g(a)| < \epsilon$, which shows that $g(x) \to 0$ as $x \to +\infty$.

107. Exercise 6: Suppose f is continuous for $x \ge 0$, $f'(x)$ exists for $x > 0$, $f(0) = 0$, and f' is monotonically increasing. Put
$$g(x) = \frac{f(x)}{x} \quad (x > 0)$$

and prove that g is monotonically increasing.

Solution: (*analambanomenos*)
By Theorem 5.10, for $x > 0$ we have $f(x) = f(x) - f(0) = (x - 0)f'(y)$ for some $y \in (0, x)$, so that $f(x) \le xf'(x)$ since f' is monotonically increasing. Hence

$$g'(x) = \frac{xf'(x) - f(x)}{x^2} \ge 0 \quad (x > 0),$$

and so g is monotonically increasing by Theorem 5.11(a).

108. Exercise 7: Suppose $f'(x)$, $g'(x)$ exist, $g'(x) \ne 0$, and $f(x) = f(x) = 0$. Prove that

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

Solution: (*Matt "Frito" Lundy*)
We have:

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \frac{t - x}{g(t) - g(x)} = \frac{f'(x)}{g'(x)}$$

109. Exercise 8: Suppose f' is continuous on $[a, b]$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon$$

whenever $0 < |t - x| < \delta$, $a \le x \le b$, $a \le t \le b$. Does this hold for vector-value functions too?

Solution: (*Matt "Frito" Lundy*)
Because f' is continuous on the compact set $[a, b]$, f' is uniformly continuous (Theorem 4.19). So there exists $\delta > 0$ such that for any x, y where $|x - y| < \delta$ we have $|f'(x) - f'(y)| < \varepsilon$. But by "the" mean value theorem, for any $x, t \in [a, b]$, there exists a $y \in (a, b)$ such that $|x - y| < |x - t|$ and

$$f'(y) = \frac{f(t) - f(x)}{t - x}.$$

So if $|x - t| < \delta$ we have:

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| = |f'(y) - f'(x)| < \varepsilon$$

because $|x - y| < |x - t| < \delta$.

Solution: (*analambanomenos*)
Let $\mathbf{f}$ be a vector-valued function such that $\mathbf{f}'$ is continuous on $[a, b]$ and $\varepsilon > 0$. Define the vector-valued function $\mathbf{g}$ on the rectangle $[a, b] \times [a, b]$ as follows:

$$\mathbf{g}(x, t) = \begin{cases} \dfrac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{f}'(x) & x \ne t \\ \mathbf{0} & x = t \end{cases}$$

Then $\mathbf{g}$ is continuous on its compact domain, and so is uniformly continuous by Theorem 4.19. Hence there is $\delta > 0$ such that $|\mathbf{g}(x,t)| = \left|\mathbf{g}(x,t) - \mathbf{g}(x,x)\right| < \varepsilon$ if $\left|(x,t) - (x,x)\right| = |t - x| < \delta$.

110. Exercise 9: Let f be a continuous real function on $\mathbf{R}$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(3) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists?

Solution: (*Matt "Frito" Lundy*)
By definition
$$f'(0) = \lim_{t\to 0} \frac{f(t) - f(0)}{t}.$$
Because f is continuous on $\mathbf{R}$ and differentiable on $\mathbf{R} \setminus 0$, "the" mean value theorem says for any $t \in \mathbf{R} \setminus 0$, there exists an $x_t \in (0,t)$ (or $(t,0)$ if t is negative) such that
$$\frac{f(t) - f(0)}{t} = f'(x_t).$$
So we have:
$$f'(0) = \lim_{t\to 0} \frac{f(t) - f(0)}{t} = \lim_{t\to 0} f'(x_t) = 3.$$
Because $x_t \to 0$ as $t \to 0$. So f' exists at 0, and is 3.

111. Exercise 10: Suppose f and g are complex differentiable functions on $(0,1)$, $f(x) \to 0$, $g(x) \to 0$, $f'(x) \to A$, $g'(x) \to B$ as $x \to 0$, where A and B are complex numbers, $B \neq 0$. Prove that
$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{A}{B}.$$
Compare with Example 5.18.

Solution: (*analambanomenos*)
Let $f(x) = u(x) + iv(x)$ where u, v are real-valued and differentiable on $(0,1)$, and let $A = a + ib$. Following the hint, since
$$\frac{u'(x)}{1} \to a \quad \text{and} \quad \frac{v'(x)}{1} \to b \quad \text{as } x \to 0,$$
we have by Theorem 5.13
$$\frac{u(x)}{x} \to a \quad \text{and} \quad \frac{v(x)}{x} \to b \quad \text{as } x \to 0$$
so that
$$\frac{f(x)}{x} \to A \quad \text{as } x \to 0.$$
Similarly, by breaking g into its real and imaginary parts, we get
$$\frac{g(x)}{x} \to B \quad \text{and so} \quad \frac{x}{g(x)} \to \frac{1}{B} \quad \text{as } x \to 0.$$
Hence
$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \left(\left(\frac{f(x)}{x} - A \right) \frac{x}{g(x)} + A \frac{x}{g(x)} \right) = \frac{A}{B}.$$
In Example 5.18 it was shown that the derivative of the denominator g satisfies $\left|g'(x)\right| \geq (2/x) - 1$. Since the limit of $g'(x)$ as $x \to 0$ does not exist, we can't apply the above result to this case.

112. Exercise 11: Suppose f is defined in a neighborhood of x, and suppose $f''(x)$ exists. Show that

$$\lim_{h\to 0}\frac{f(x+h)-f(x-h)-2f(x)}{h^2}=f''(x).$$

Show by an example that the limit may exist even if $f''(x)$ does not.

Solution: (*analambanomenos*)
Let $g(h)=f(x+h)+f(x-h)-2f(x)$. Then g is differentiable in a neighborhood of 0, and $g(0)=0$. Applying Theorem 5.13, we get

$$\begin{aligned}\lim_{h\to 0}\frac{g(h)}{h^2}&=\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}\\&=\lim_{h\to 0}\frac{f'(x+h)-f(x)}{2h}-\lim_{k\to 0}\frac{f(x+k)-f(x)}{-2k}\qquad\text{(where } k=-h)\\&=\frac{f''(x)}{2}+\frac{f''(x)}{2}=f''(x)\end{aligned}$$

Letting $f(x)=x^2\sin(1/x)$, $f(0)=0$, then f is continuous and differentiable at $x=0$, and since f is an odd function, $f(h)-f(-h)-2f(0)=0$ for all $h\neq 0$ so that the limit above exists for $x=0$. However f' is not continuous at 0, so $f''(0)$ does not exist.

113. Exercise 12: If $f(x)=|x|^3$, compute $f'(x)$, $f''(x)$ for all real x, and show that $f^{(3)}(0)$ does not exist.

Solution: (*analambanomenos*)
For $x>0$, $f(x)=x^3$, $f'(x)=3x^2$, $f''(x)=6x$, $f^{(3)}(x)=6$, and for $x<0$, $f(x)=-x^3$, $f'(x)=-3x^2$, $f''(x)=-6x$, $f^{(3)}(x)=-6$.

$$\begin{aligned}f'(0)&=\lim_{t\to 0}\frac{f(t)}{t}=\lim_{t\to 0}\big(\operatorname{sign}(t)t^2\big)=0\\f''(0)&=\lim_{t\to 0}\frac{f'(t)}{t}=\lim_{t\to 0}\big(\operatorname{sign}(t)3t\big)=0\end{aligned}$$

$f^{(3)}(0)$ does not exist for otherwise $f^{(3)}(x)$ would have a simple discontinuity at $x=0$, which cannot happen by the Corollary to Theorem 5.12.

114. Exercise 13: Suppose a and c are real numbers, $c>0$, and f is defined on $[-1,1]$ by

$$f(x)=\begin{cases}x^a\sin\big(|x|^{-c}\big) & \text{(if } x\neq 0)\\ 0 & \text{(if } x=0)\end{cases}$$

Prove the following statements:

(a) f is continuous if and only if $a>0$.

(b) $f'(0)$ exists if and only if $a>1$.

(c) f' is bounded if and only if $a\geq 1+c$.

(d) f' is continuous if and only if $a>1+c$.

(e) $f''(0)$ exists if and only if $a>2+c$.

(f) f'' is bounded if and only if $a\geq 2+2c$.

(g) f'' is continuous if and only if $a > 2 + 2c$.

Solution: (*analambanomenos*)
First note that $\sin\bigl(|x|^{-c}\bigr)$ fluctuates between -1 and 1, and each neighborhood of 0 has an infinite number of elements from each of the sets $f^{-1}(0)$, $f^{-1}(1)$ and $f^{-1}(-1)$.

(a) f is continuous at 0 if and only if $\lim_{x\to 0} f(x) = 0$ which only happens if $\lim_{x\to 0}|x|^a = 0$, that is $a > 0$.

(b) $f'(0) = \lim_{t\to 0} f(t)/t = \lim_{t\to 0} t^{a-1}\sin\bigl(|t|^{-c}\bigr)$ exists if and only if $\lim_{x\to 0}|x|^{a-1}$ exists, that is $a > 1$.

(c) For $x > 0$,

$$f'(x) = ax^{a-1}\sin(x^{-c}) - cx^{a-c-1}\cos(x^{-c})$$

which is bounded on $(0, 1]$ if and only if $a - c - 1 > 0$, that is $a > 1 + c$. By symmetry, the same is true on $[-1, 0)$.

(d) f' is continuous at 0 if and only if $\lim_{x\to 0} f'(x) = 0$ which only happens if $\lim_{x\to 0}|x|^{a-c-1} = 0$, that is $a > 1 + c$.

(e) $\lim_{t\to 0+} f'(t)/t = \lim_{t\to 0+}\bigl(at^{a-2}\sin(t^{-c}) - ct^{a-c-2}\cos(t^{-c})\bigr)$ exists and is equal to 0 if and only if $\lim_{x\to 0}|x|^{a-c-2}$ exists, that is $a > 2 + c$. We have the same result when taking the limit from the left.

(f) For $x > 0$,

$$f''(x) = \bigl(a(a-1)x^{a-2} - c^2x^{a-2c-2}\bigr)\sin(x^{-c}) - \bigl(acx^{a-c-1} + c(a-c-1)x^{a-c-2}\bigr)\cos(x^{-c})$$

which is bounded on $(0, 1]$ if and only if $a - 2c - 2 > 0$, that is $a > 2 + 2c$. By symmetry, the same is true on $[-1, 0)$.

(g) f'' is continuous at 0 if and only if $\lim_{x\to 0} f''(x) = 0$ which only happens if $\lim_{x\to 0}|x|^{a-2c-2} = 0$, that is $a > 2 + 2c$.

115. Exercise 14: Let f be a differentiable real function defined in (a, b). Prove that f is convex if and only if f' is monotonically increasing. Assume next that $f''(x)$ exists for every $x \in (a, b)$, and prove that f is convex if and only if $f''(x) \geq 0$ for all $x \in (a, b)$.

Solution: (*analambanomenos*)
If $f''(x)$ exists for every $x \in (a, b)$, then $f''(x) \geq 0$ for all $x \in (a, b)$ if and only if f' is monotonically increasing, so the second part of the Exercise follows from the first part.

Suppose that f is convex and let $a < x < y < b$. By Exercise 4.23

$$\frac{f(t) - f(x)}{t - x} \leq \frac{f(u) - f(y)}{u - y}$$

for t and u close to x and y, respectively. Taking limits as $t \to x$ and $u \to y$, we get $f'(x) \leq f'(y)$.

Conversely, suppose is f' is monotonically increasing, let $a < x < y < b$, and let $z = \lambda x + (1 - \lambda)y$

for $0 < \lambda < 1$. By Theorem 5.10 there are points w_1, w_2 such that $x < w_1 < z < w_2 < y$ such that

$$\begin{aligned}\frac{f(z)-f(x)}{z-x} &= f'(w_1) \le f'(w_2) = \frac{f(y)-f(z)}{y-z} \\ \frac{f(z)-f(x)}{(\lambda-1)x+(1-\lambda)y} &\le \frac{f(y)-f(z)}{\lambda(y-x)} \\ f(z)(y-x) &\le \big(\lambda(y-x)\big)f(x) + \big((1-\lambda)(y-x)\big)f(y) \\ f\big(\lambda x+(1-\lambda)y\big) &\le \lambda f(x) + (1-\lambda)f(y)\end{aligned}$$

which shows that f is convex on (a,b). (The algebra above is much easier if you just take $\lambda = 1/2$, then apply Exercise 4.24.)

116. Exercise 15: Suppose $a \in \mathbf{R}$, f is twice-differentiable real function on (a,∞), and M_0, M_1, M_2 are the least upper bounds of $\big|f(x)\big|$, $\big|f'(x)\big|$, $\big|f''(x)\big|$, respectively, on (a,∞). Prove that $M_1^2 \le 4M_0M_2$.

To show that $M_1^2 = 4M_0M_2$ can actually happen, take $a = -1$, define

$$f(x) = \begin{cases} 2x^2-1 & (-1 < x < 0) \\ \dfrac{x^2-1}{x^2+1} & (0 \le x < \infty) \end{cases}$$

and show that $M_0 = 1$, $M_1 = 4$, $M_2 = 4$.

Does $M_1^2 \le 4M_0M_2$ hold for vector-valued functions too?

Solution: (*analambanomenos*)
Let $g(x) = A/x + Bx$ for $x > 0$ where A and B are positive real numbers. Then $g'(x) = -A/x^2 + B$ and $g''(x) = 2A/x^3 > 0$. Since $g'(x) = 0$ for $x = \sqrt{A/B}$, g has the minimum value $g\big(\sqrt{A/B}\big) = 2\sqrt{AB}$.

Following the hint, by Theorem 5.15, for $h > 0$ there is $\xi \in (x, x+2h)$ such that

$$\begin{aligned} f(x+2h) &= f(x) + 2hf'(x) + 2h^2f''(\xi) \\ f'(x) &= \frac{f(x+2h)-f(x)}{2h} - hf''(\xi)\end{aligned}$$

Hence $\big|f'(x)\big| \le M_0/h + M_2h \le 2\sqrt{M_0M_2}$ by the previous result, or $M_1^2 \le 4M_0M_2$.

Letting $f(x)$ be the example above, we get

$$f'(x) = \begin{cases} 4x & (-1 < x < 0) \\ \dfrac{4x}{(x^2+1)^2} & (0 < x < \infty) \end{cases} \qquad f''(x) = \begin{cases} 4 & (-1 < x < 0) \\ \dfrac{4(1-3x^2)}{(x^2+1)^3} & (0 < x < \infty) \end{cases}$$

$$f^{(3)}(x) = \begin{cases} 0 & (-1 < x < 0) \\ \dfrac{48x(x^2-1)}{(x^2+1)^4} & (0 < x < \infty) \end{cases}$$

For $x < 0$, f' is negative, so $f(x)$ decreases from 1 to -1, and for $x > 0$, f' is positive, so $f(x)$ increases monotonically from -1 to a limit of 1. Hence $M_0 = 1$.

For $x < 0$, f' increases linearly from -4 to 0. For $x > 0$, $f''(x) = 0$ has a single solution at $x = \sqrt{3}/3$, so $f'(x)$ has a maximum value of $3\sqrt{3}/4$. Hence $M_1 = 4$.

For $x > 0$, $f^{(3)}(x)$ has the single solution $x = 1$, so $f''(x)$ decreases from 4 to a minimum value of $f''(1) = -1$, then increases monotonically to a limit of 0. Hence $M_2 = 4$.

Hence for this example, $M_1^2 = 4M_0M_1 = 16$.

Solution: (*Dan "kyp44" Whitman*)
Regarding whether this theorem holds for vector-valued functions, the following proof comes from Roger Cooke's solution manual here `https://minds.wisconsin.edu/handle/1793/67009` and is reproduced here for completeness.

Consider a vector-valued function f on (a, ∞). As before let

$$\begin{aligned} M_0 &= \sup_{a<x<\infty} |f(x)| \\ M_1 &= \sup_{a<x<\infty} |f'(x)| \\ M_2 &= \sup_{a<x<\infty} |f''(x)| . \end{aligned}$$

We then must show that $M_1^2 \le 4M_0M_2$.

Clearly if $M_0 = 0$ then $f(x) = 0$ (zero vector) for all $x \in (a, \infty)$. It then follows that $f'(x) = f''(x) = 0$ so that $M_0 = M_1 = M_2 = 0$, resulting in the inequality clearly being true. It is also cleary satisfied if $M_1 = 0$ since the right side is always greater than or equal to zero.

If $M_2 = 0$ then $f''(x) = 0$ for all $x \in (a, \infty)$. It then must be that f' is constant so that $f(x)$ is linear in x, i.e. if $f(x) = (f_1(x), \ldots, f_n(x))$ then every $f_k(x)$ is linear or constant. But since f is bounded (since M_1 exists) it has to be that f is constant. From this it follows that $M_1 = 0$ so that the inequality is again always satisfied.

So then suppose that $M_0 > 0$, $M_1 > 0$, and $M_2 > 0$. Then there is an $\alpha \in \mathbb{R}$ where $0 < \alpha < M_1$. Then α is *not* an upper bound of $|f'(x)|$ so that there is an $x_0 \in (a, \infty)$ where $|f'(x_0)| > \alpha$. Let

$$u = \frac{1}{|f'(x_0)|} f'(x_0)$$

so that $|u| = 1$. Consider then the real-valued function

$$\phi(x) = u \cdot f(x)$$

and let

$$N_0 = \sup_{a<x<\infty} |\phi(x)| ,$$

which exists since for all $x \in (a, \infty)$

$$|\phi(x)| \le |u||f(x)| = |f(x)| \le M_0 .$$

Thus $N_0 \le M_0$ since M_0 is an upper bound of $|\phi(x)|$. Similarly letting

$$\begin{aligned} N_1 &= \sup_{a<x<\infty} |\phi'(x)| \\ N_2 &= \sup_{a<x<\infty} |\phi''(x)| \end{aligned}$$

results in $N_1 \le M_1$ and $N_2 \le M_2$.

Also, however,

$$N_1 \geq |\phi'(x_0)| = |u \cdot f'(x_0)| = \left| \frac{f'(x_0)}{|f'(x_0)|} \cdot f'(x_0) \right| = \frac{|f'(x_0)|^2}{|f'(x_0)|} = |f'(x_0)| > \alpha \,.$$

Then since ϕ is real-valued it follows from the solution above that

$$N_1^2 \leq 4N_0N_2$$

so that

$$\alpha^2 < N_1^2 \leq 4N_0N_2 \leq 4M_0M_2$$

since $0 < \alpha < N_1$, $N_0 \leq M_0$, and $N_2 \leq M_2$. Since this is true for *every* α less than M_1 it has to be that

$$M_1^2 \leq 4M_0M_2$$

so that the result also holds for vector-valued functions.

117. Exercise 16: Suppose f is twice-differentiable on $(0, \infty)$, f'' is bounded on $(0, \infty)$, and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$.

Solution: (*analambanomenos*)
Following the hint, let f'' be bounded by M on $(0, \infty)$. Then by Exercise 15,

$$\sup_{x>a} \big|f'(x)\big| \leq 4M \sup_{x>a} \big|f(x)\big|$$

which tends to 0 as $a \to \infty$.

118. Exercise 17: Suppose f is a real, three-times differentiable function on $[-1, 1]$, such that

$$f(-1) = 0, \quad f(0) = 0, \quad f(1) = 1, \quad f'(0) = 0.$$

Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1, 1)$.

Solution: (*analambanomenos*)
Following the hint, applying Theorem 5.15 with $\alpha = 0$ and $\beta = \pm 1$, we get

$$\begin{aligned} f(1) = 1 &= \frac{f''(0)}{2} + \frac{f^{(3)}(s)}{6} \\ f(-1) = 0 &= \frac{f''(0)}{2} - \frac{f^{(3)}(t)}{6} \end{aligned}$$

for some $s \in (0, 1)$ and $t \in (-1, 0)$. Subtracting the second equation from the first, we get $6 = f^{(3)}(s) + f^{(3)}(t)$, so at least one of the two terms is ≥ 3.

119. Exercise 18: Suppose f is a real function on $[a, b]$, n is a positive integer, and $f^{(n-1)}$ exists for every $t \in [a, b]$. Let α, β, and P be as in Taylor's theorem (Theorem 5.15). Define

$$Q(t) = \frac{f(t) - f(\beta)}{t - \beta}$$

for $t \in [a, b]$, $t \neq \beta$, differentiate

$$f(t) - f(\beta) = (t - \beta)Q(t)$$

$n-1$ times at $t=\alpha$, and derive the following version of Taylor's theorem:

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n.$$

Solution: (*analambanomenos*)
We have

$$\begin{aligned} f'(t) &= Q(t) + (t-\beta)Q'(t) \\ f''(t) &= 2Q'(t) + (t-\beta)Q''(t) \\ f^{(3)}(t) &= 3Q''(t) + (t-\beta)Q^{(3)}(t) \end{aligned}$$

and so forth, which can be rewritten

$$\begin{aligned} Q(t) &= f'(t) + Q'(t)(\beta-t) \\ Q'(t) &= \frac{1}{2}\big(f''(t) + Q''(t)(\beta-t)\big) \\ Q''(t) &= \frac{1}{3}\big(f^{(3)}(t) + Q^{(3)}(t)(\beta-t)\big). \end{aligned}$$

Hence

$$\begin{aligned} f(\beta) &= f(\alpha) + Q(\alpha)(\beta-\alpha) \\ &= f(\alpha) + f'(\alpha)(\beta-\alpha) + Q'(\alpha)(\beta-\alpha)^2 \\ &= f(\alpha) + f'(\alpha)(\beta-\alpha) + \frac{1}{2}f''(\alpha)(\beta-\alpha)^2 + \frac{1}{2}Q''(\alpha)(\beta-\alpha)^3 \\ &= f(\alpha) + f'(\alpha)(\beta-\alpha) + \frac{1}{2}f''(\alpha)(\beta-\alpha)^2 + \frac{1}{3!}f^{(3)}(\beta-\alpha)^3 + \frac{1}{3!}Q^{(3)}(\alpha)(\beta-\alpha)^4 \end{aligned}$$

which easily leads to the desired formula.

120. Exercise 19: Suppose f is defined in $(-1,1)$ and $f'(0)$ exists. Suppose $-1<\alpha_n<\beta_n<1$, $\alpha_n\to 0$, and $\beta_n\to 0$ as $n\to\infty$. Define the difference quotients

$$D_n = \frac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}.$$

Prove the following statements:

(a) If $\alpha_n<0<\beta_n$, then $\lim D_n = f'(0)$.

(b) If $0<\alpha_n<\beta_n$ and $\big\{\beta_n/(\beta_n-\alpha_n)\big\}$ is bounded, then $\lim D_n = f'(0)$.

(c) If f' is continuous in $(-1,1)$, then $\lim D_n = f'(0)$.

Give an example in which α_n, β_n tend to 0 in such a way that $\lim D_n$ exists but is different from $f'(0)$.

Solution: (*analambanomenos*)
For (a) and (b), we need to find an algebraic expression that relates D_n to the difference quotients found in the definition of $f'(0)$, $\big(f(\alpha_n)-f(0)\big)/\alpha_n$ and $\big(f(\beta_n)-f(0)\big)/\beta_n$ in such a way that we can safely let $n\to\infty$. To simplify the algebra, replace f with $F=f-f(0)$ so that $F(0)=0$. We

start with

$$\begin{aligned}\frac{F(\beta_n)}{\beta_n}-\frac{F(\alpha_n)}{\alpha_n} &= \frac{\alpha_n F(\beta_n)-\beta_n F(\alpha_n)}{\alpha_n\beta_n}\\ &= \frac{\alpha_n F(\beta_n)-\alpha_n F(\alpha_n)+\alpha_n F(\alpha_n)-\beta_n F(\alpha_n)}{\alpha_n\beta_n}\\ &= \frac{F(\beta_n)-F(\alpha_n)}{\beta_n}+\frac{(\alpha_n-\beta_n)F(\alpha_n)}{\alpha_n\beta_n}.\end{aligned}$$

To get D_n, we need to multiply by $\beta_n/(\beta_n-\alpha_n)$, and this gives us what we need:

$$\begin{aligned}\left(\frac{F(\beta_n)}{\beta_n}-\frac{F(\alpha_n)}{\alpha_n}\right)\frac{\beta_n}{\beta_n-\alpha_n} &= \left(\frac{F(\beta_n)-F(\alpha_n)}{\beta_n}+\frac{(\alpha_n-\beta_n)F(\alpha_n)}{\alpha_n\beta_n}\right)\frac{\beta_n}{\beta_n-\alpha_n}\\ &= \frac{F(\beta_n)-F(\alpha_n)}{\beta_n-\alpha_n}-\frac{F(\alpha_n)}{\alpha_n}.\end{aligned}$$

Rearranging and substituting back $f-f(0)$ for F, we get

$$D_n=\frac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}=\left(\frac{f(\beta_n)-f(0)}{\beta_n}-\frac{f(\alpha_n)-f(0)}{\alpha_n}\right)\frac{\beta_n}{\beta_n-\alpha_n}+\frac{f(\alpha_n)-f(0)}{\alpha_n}.$$

For (b), the $\beta_n/(\beta_n-\alpha_n)$ factor is assumed to be bounded, and for part (a) it is bounded by 1. In either case, we can pass to a subsequence where the factor converges to a finite limit, so letting $n\to\infty$ we get $\lim D_n=f'(0)$.

For part (c), we can apply Theorem 5.10 to get $D_n=f'(\gamma_n)$ for some γ_n between α_n and β_n. Since $\lim\gamma_n=0$ and f' is continuous, we get $\lim D_n=f'(0)$.

Let f be the function in Example 5.6(b), $f(x)=x^2\sin(1/x)$ for $x\neq 0$ and $f(0)=0$. It was shown that $f'(0)=0$, although f' is not continuous at 0. Let

$$\alpha_n=\frac{2}{(4n-1)\pi}\qquad \beta_n=\frac{2}{(4n-3)\pi}.$$

Then $\sin(1/\alpha_n)=-1$ and $\sin(1/\beta_n)=1$, so that $f(\alpha_n)=-\alpha_n^2$ and $f(\beta_n)=\beta_n^2$. Hence

$$D_n=\frac{\beta_n^2+\alpha_n^2}{\beta_n-\alpha_n}=\cdots=\frac{2}{\pi}\left(\frac{16n^2-16n+5}{16n^2-16n+3}\right)$$

which tends to $2/\pi$ as $n\to\infty$.

121. Exercise 20: Formulate and prove an inequality which follows from Taylor's theorem and which remains valid for vector-valued functions.

Solution: (*analambanomenos*, with fixes suggested by *Dan "kyp44" Whitman*
As in Theorem 5.15, let f be a real function on $[a,b]$ and n a positive integer such that $f^{(n-1)}$ is continuous on $[a,b]$ and $f^{(n)}(t)$ exists for $t\in(a,b)$. Let α and β be distinct points of $[a,b]$ and define

$$P_{f,\alpha}(t)=\sum_{k=0}^{n-1}\frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k.$$

Then by Theorem 5.15 there is a point x between α and β such that

$$\left|f(\beta) - P_{f,\alpha}(\beta)\right| \le \frac{(\beta-\alpha)^n}{n!}\left|f^{(n)}(x)\right|.$$

We can extend this result to vector-valued functions just as Theorem 5.19 extended the Mean-Value theorem. Let $\mathbf{f}$ be a continuous mapping of $[a,b]$ into $\mathbf{R}^k$ such that $\mathbf{f}^{(n-1)}$ is continuous on $[a,b]$ and $\mathbf{f}^{(n)}$ exists for $t \in (a,b)$. Let α and β be distinct points in $[a,b]$ and define

$$\mathbf{P}_{\mathbf{f},\alpha}(t) = \sum_{k=0}^{n-1} \frac{\mathbf{f}^{(k)}(\alpha)}{k!}(t-\alpha)^k.$$

Put $\mathbf{z} = \mathbf{f}(\beta) - \mathbf{P}_{\mathbf{f},\alpha}(\beta)$, and let $\varphi(t) = \mathbf{z}\cdot\mathbf{f}(t)$. Then by Theorem 5.15 there is a point x between α and β such that

$$\left|\varphi(\beta) - P_{\varphi,\alpha}(\beta)\right| = \frac{(\beta-\alpha)^n}{n!}\left|\varphi^{(n)}(x)\right| = \frac{(\beta-\alpha)^n}{n!}\left|\mathbf{z}\cdot\mathbf{f}^{(n)}(x)\right|.$$

We also have

$$\left|\varphi(\beta) - P_{\varphi,\alpha}(\beta)\right| = \left|\mathbf{z}\cdot\mathbf{f}(\beta) - \mathbf{z}\cdot\mathbf{P}_{\mathbf{f},\alpha}(\beta)\right| = \left|\mathbf{z}\cdot\mathbf{z}\right| = |\mathbf{z}|^2.$$

By the Schwartz inequality, we have

$$|\mathbf{z}|^2 = \frac{(\beta-\alpha)^n}{n!}\left|\mathbf{z}\cdot\mathbf{f}^{(n)}(x)\right| \le \frac{(\beta-\alpha)^n}{n!}|\mathbf{z}|\left|\mathbf{f}^{(n)}(x)\right|$$

so that

$$\left|\mathbf{f}(\beta) - \mathbf{P}_{\mathbf{f},\alpha}(\beta)\right| = |\mathbf{z}| \le \frac{(\beta-\alpha)^n}{n!}\left|\mathbf{f}^{(n)}(x)\right|.$$

122. Exercise 21: Let E be a closed subset of $\mathbf{R}$. We saw in Exercise 4.22 that there is a real continuous function f on $\mathbf{R}$ whose zero set is E. Is it possible, for each closed set E, to find such an f which is differentiable on $\mathbf{R}$, or one which is n times differentiable, or even one which has derivatives for all orders on $\mathbf{R}$?

Solution: (*analambanomenos*)
I'm going to show this for the infinitely differentiable case (and even that won't be complete). Cauchy's function $F(x) = e^{-1/x^2}$ for $x \ne 0$, $F(0) = 0$, is the classic counterexample of a non-constant, infinitely differentiable real function such that $f^{(n)}(0) = 0$ for all orders n (you can easily find a proof of this, or you can try it yourself). We can use this to define a function $\tilde{F}(x) = F(x)$ for $x > 0$, $\tilde{F}(x) = 0$ for $x \le 0$, which is infinitely differentiable everywhere on $\mathbf{R}$ and whose zero set is $(-\infty, 0]$. Finally, for $a < b$ let $F_{a,b}(x) = \tilde{F}(x-a)\tilde{F}(b-x)$, an infinitely differentiable function on $\mathbf{R}$ whose zero set is $(-\infty, a] \cup [b, \infty)$.

The complement of a closed set E of $\mathbf{R}$ is an open set consisting of a (possibly infinite) collection of open intervals (a_n, b_n). The function $f(x) = \sum F_{a_n,b_n}(x)$ is well-defined since at most one of the terms in the sum is non-zero for any given x, infinitely differentiable everywhere on $\mathbf{R}$, and has zero set E.

123. Exercise 22: Suppose f is a real function on $(-\infty, \infty)$. Call x a *fixed point* of f if $f(x) = x$.

(a) If f is differentiable and $f'(t) \ne 1$ for every real t, prove that f has at most one fixed point.

(b) Show that the function f defined by $f(t) = t + (1+e^t)^{-1}$ has no fixed point, although $0 < f'(t) < 1$

for all real t.

(c) However, if there is a constant $A < 1$ such that $|f'(t)| \le A$ for all real t, prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real number and $x_{n+1} = f(x_n)$ for $n = 1, 2, 3, \ldots$.

(d) Show that the process described in (c) can be visualized by the zig-zag path

$$(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to (x_3, x_4) \to \cdots .$$

Solution: (*analambanomenos*)
(a) Suppose $f(a) = a$ and $f(b) = b$ for $a < b$. By Theorem 5.10, there is a point t, $a < t < b$, such that $f'(t) = \big(f(b) - f(a)\big)/(b - a) = 1$, contradicting $f'(t) \ne 1$ for all real t.

(b) If $t = f(t) = t + (1 + e^t)^{-1}$, then $(1 + e^t)^{-1} = 0$, which is impossible. We have

$$f(t) = 1 - \frac{e^t}{(1 + e^t)^2} \qquad f''(t) = \frac{e^t(e^t - 1)}{(1 + e^t)^3}.$$

Since $f''(t)$ has a single zero, at $t = 0$, $f'(t)$ decreases from $\lim_{t\to-\infty} f'(t) = 1$ to $f'(0) = 3/4$, then increases to $\lim_{t\to\infty} f'(t) = 1$, so that the range of f' is $[3/4, 1)$.

(c) Since $x_{n+1} = f(x_n)$ and $x_n = f(x_{n-1})$, by Theorem 5.10 there is a point t between x_{n-1} and x_n such that

$$(x_{n+1} - x_n) = (x_n - x_{n-1})f'(t).$$

Hence

$$|x_{n+1} - x_n| < A|x_n - x_{n-1}| < A^2|x_{n-1} - x_{n-2}| < \cdots < A^{n-1}|x_2 - x_1|.$$

So for m and n greater than N, we have

$$\begin{aligned}
|x_n - x_m| &\le |x_n - x_{n+1}| + \cdots + |x_{m-1} - x_m| \\
&\le (A^{n-1} + \cdots + A^{m-2})|x_2 - x_1| \\
&\le |x_2 - x_1| \sum_{k=N}^{\infty} A^k \\
&\le \frac{|x_2 - x_1|}{1 - A} A^N
\end{aligned}$$

Since $A < 1$, this last term goes to 0 as $N \to \infty$, so $\{x_n\}$ is a Cauchy sequence, converging to x. Since f is differentiable, it is continuous, hence

$$f(x) = \lim_n f(x_n) = \lim_n x_{n+1} = x$$

that is, x is a fixed point of f.

(d) This zig-zag path is described in $\mathbf{R}^2$ with respect to the graph of f in that space. It starts with the point $\big(x_1, x_2 = f(x_1)\big)$ on the graph of f, goes horizonatally until it meets the diagonal $y = x$ at (x_2, x_2) then goes vertically until it hits the graph of f again at $\big(x_2, x_3 = f(x_2)\big)$, and so forth. It will zig-zag or spiral to the point on the graph of f corresponding to a fixed point of f, where it crosses the diagonal $y = x$.

124. Exercise 23: The function f defined by $f(x) = (x^3 + 1)/3$ has three fixed points, say α, β, γ. where

$$-2 < \alpha < -1, \qquad 0 < \beta < 1, \qquad 1 < \gamma < 2.$$

For arbitrarily chosen x_1, define $\{x_n\}$ by setting $x_{n+1} = f(x_n)$.

(a) If $x_1 < \alpha$, prove that $x_n \to -\infty$ as $n \to \infty$.

(b) If $\alpha < x_1 < \gamma$, prove that $x_n \to \beta$ as $n \to \infty$.

(c) If $\gamma < x_1$, prove that $x_n \to +\infty$ as $n \to \infty$.

Thus β can be located by this method, but α and γ cannot.

Solution: (*analambanomenos*)
The fixed points of f are the zeros of $g(x) = f(x) - x$, of which there are at most three. Since $g(-2) = -1/3$, $g(-1) = 1$, $g(0) = 1/3$, $g(1) = -1/3$, and $g(2) = 1$, there are three fixed points of f, lying in the intervals $(-2, -1)$, $(0, 1)$, and $(1, 2)$, as asserted.

Note that $\Delta x_n = x_n - x_{n-1} = f(x_{n-1}) - x_{n-1} = g(x_{n-1})$. For $x < \alpha$, $g(x) < 0$, and g decreases monotonically to $-\infty$ as $x \to -\infty$. Hence if $x_1 < \alpha$, then $g(x_{n-1}) = \Delta x_n < 0$, so $x_n < x_{n-1}$ and $\Delta x_{n+1} = g(x_n) < \Delta x_n$. Hence $x_n \to -\infty$ as $n \to \infty$.

Similarly, for $x > \gamma$, $g(x) > 0$, and g increases monotonically to $+\infty$ as $x \to +\infty$. Hence if $x_1 > \gamma$, then $g(x_{n-1}) = \Delta x_n > 0$, so $x_n > x_{n-1}$ and $\Delta x_{n+1} = g(x_n) > \Delta x_n$. Hence $x_n \to +\infty$ as $n \to \infty$.

For $x \in (\alpha, \beta)$ we have $g(x) > 0$, and for $x \in (\beta, \gamma)$ we have $g(x) < 0$. I want to compare g with the linear function $\beta - x$, which also has a zero at $x = \beta$ and has a slope of -1. Since $g'(\beta) = \beta^2 - 1 > -1$, we have $g(x) < \beta - x$ for $x \in (\alpha, \beta)$ and $g(x) > \beta - x$ for $x \in (\beta, \gamma)$.

Putting this together, we get that if $x_n \in (\alpha, \beta)$, then $0 < \Delta x_{n+1} < \beta - x_n$ so that $x_{n+1} \in (x_n, \beta)$. Similarly, if $x_n \in (\beta, \gamma)$, then $\beta - x < \Delta x_{n+1} < 0$ so that $x_{n+1} \in (\beta, x_n)$.

That is, if $x_1 \in (\alpha, \beta)$, then $\{x_n\}$ is a monotonically increasing sequence with upper bound β, and if $x_1 \in (\beta, \gamma)$, then $\{x_n\}$ is a monotonically decreasing sequence with lower bound β. In either case they must have a limit β'. This must be a zero of g (and so a fixed point of f) since otherwise $g(\beta') \neq 0$, so for x_n close to β', the value of $|\Delta x_n|$ would be larger than $|\beta' - x_n|$ which would contradict β' being a limit point of the monotonic sequence $\{x_n\}$. Hence $\beta' = \beta$.

125. Exercise 24: The process described in part (c) of Exercise 22 can of course also be applied to functions that map $(0, \infty)$ to $(0, \infty)$.

Fix some $\alpha > 1$, and put

$$f(x) = \frac{1}{2}\left(x + \frac{\alpha}{x}\right), \qquad g(x) = \frac{\alpha + x}{1 + x}.$$

Both f and g have $\sqrt{\alpha}$ as their only fixed point in $(0, \infty)$. Try to explain, on the basis of properties of f and g, why the convergence in Exercise 3.16 is so much more rapid than it is in Exercise 3.17. Do the same when $0 < \alpha < 1$.

Solution: (*analambanomenos*)
The distance between the successive elements of the sequence $x_n = f(x_{n-1})$ is given by the function $F(x) = f(x) - x$, while the distance between the successive elements of the sequence $y_n = g(y_{n-1})$ is given by the function $G(y) = g(y) - y$. We have

$$F(x) = -\frac{1}{2}\left(x - \frac{\alpha}{x}\right)$$
$$F\big(\sqrt{\alpha}\big) = 0, \quad F'\big(\sqrt{\alpha}\big) = -1$$
$$G(y) = \frac{\alpha - y^2}{1 + \alpha}$$
$$G\big(\sqrt{\alpha}\big) = 0, \quad G'\big(\sqrt{\alpha}\big) = -\frac{2}{1 + \sqrt{\alpha}}.$$

By Taylor's theorem we have near $\sqrt{\alpha}$

$$F(x) = -(x-\sqrt{\alpha}) + K_1(x-\sqrt{\alpha})^2$$
$$G(y) = -\frac{2}{1+\sqrt{\alpha}}(y-\sqrt{\alpha}) + K_2(y-\sqrt{\alpha})^2$$

Now $\sqrt{\alpha}-x$ is the distance from x to $\sqrt{\alpha}$, so we see that as x_n approaches $\sqrt{\alpha}$ the differences between the successive elements of the first sequence become very close to the distance to $\sqrt{\alpha}$, but not so much in the case of the second sequence, which has the additional factor of $2/(1+\sqrt{\alpha})$. In other words, the difference between the successive steps of the x_n and the distance to $\sqrt{\alpha}$ is quadratic in $x-\sqrt{\alpha}$, while in the case of the second sequece the difference is only linear in $y-\sqrt{\alpha}$.

126. Exercise 25: Suppose f is twice differentiable on $[a,b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f'(x) \leq M$ for all $x \in [a,b]$. Let ξ be the unique point in (a,b) at which $f(\xi) = 0$.

(a) Choose $x_1 \in (\xi, b)$, and define $\{x_n\}$ by $x_{n+1} = x_n - f(x_n)/f'(x_n)$. Interpret this geometrically, in terms of a tangent to the graph of f.

(b) Prove that $x_{n+1} < x_n$ and that $\lim_{x\to\infty} x_n = \xi$.

(c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n-\xi)^2$$

for some $t_n \in (\xi, x_n)$.

(d) if $A = M/2\delta$, deduce that

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}\big(A(x_1-\xi)\big)^{2^n}.$$

Compare with Exercises 3.16 and 3.18.

(e) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

How does $g'(x)$ behave for x near ξ?

(f) Put $f(x) = x^{1/3}$ on $(-\infty,\infty)$ and try Newton's method. What happens?

Solution: (*analambanomenos*)
(a) The tangent to the graph of f at the point $\big(x_n, f(x_n)\big)$ has the equation

$$y - f(x_n) = f'(x_n)(x-x_n).$$

Letting $y=0$ and solving for x, we see that this line intersects the x-axis at the point $(x_{n+1}, 0)$.

(b) Note that if $x_n > \xi$ then $f(x_n) > 0$ so that $x_{n+1} = x_n - f(x_n)/f'(x_n) < x_n$. By Theorem 5.10, there is a $t \in (\xi, x_n)$ such that

$$f(x_n) = f(x_n) - f(\xi) = f'(t)(x_n-\xi) \quad \text{or} \quad \xi = x_n - \frac{f(x_n)}{f'(t)} \leq x_n - \frac{f(x_n)}{f'(x_n)} = x_{n+1}$$

since $f'' \geq 0$ implies $f'(t) \leq f'(x_n)$. If for some m $x_m = \xi$ then $x_{m+1} = x_{m+2} = \cdots = \xi$ and so the infinite sequence converges to ξ. Otherwise, it is a monotonically decreasing sequence with lower bound ξ and so by Theorem 3.14 converges to a limit $y \geq \xi$.

To see that $y = \xi$, let g be the continuous function defined in part (e), $g(x) = x - f(x)/f'(x)$. Assume that $y > \xi$. Then $f(y) > 0$, so $g(y) < y$. Let $\varepsilon > 0$ be small enough so that $g(y) + \varepsilon < y$. Then there is a $\delta > 0$ such that $\left|g(x) - g(y)\right| < \varepsilon$ if $|x - y| < \delta$. Let N be large enough so that $|x_N - y| < \delta$. Then $g(x_N) < g(y) + \varepsilon < y$, but this contradicts $g(x_N) = x_{N+1} \geq y$. Hence $y = \xi$.

(c) Letting $\alpha = x_n$ and $\beta = \xi$ in Theorem 5.15, and using $f(x_n) = f'(x_n)(x_n - x_{n+1})$, there is some $t_n \in (\xi, x_n)$ such that

$$\begin{aligned}
f(\xi) &= f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(\xi - x_n)^2 \\
0 &= f'(x_n)(x_n - x_{n+1}) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(\xi - x_n)^2 \\
0 &= f'(x_n)(\xi - x_{n+1}) + \frac{f''(t_n)}{2}(\xi - x_n)^2 \\
x_{n+1} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2
\end{aligned}$$

(d) Part (c) gives us

$$\begin{aligned}
(x_{n+1} - \xi) &\leq A(x_n - \xi)^2 \\
&\leq A \cdot A^2(x_{n-1} - \xi)^{2^2} \\
&\leq A \cdot A^2 \cdot A^{2^2}(x_{n-2} - \xi)^{2^3} \\
&\leq \cdots \\
&\leq A^{1+2+2^2+\cdots+2^{n-1}}(x_1 - \xi)^{2^n} = A^{2^n-1}(x_1 - \xi)^{2^n} = \frac{1}{A}\big(A(x_1 - \xi)\big)^{2^n}.
\end{aligned}$$

The algorithm described in Exercises 3.16 and 3.18 is Newton's method applied to the functions $x^2 - \alpha$ and $x^p - \alpha$, respectively.

(e) We have $g(\xi) = \xi$ if and only if $f(\xi) = 0$. Since

$$g'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2} \leq f(x)\frac{M}{\delta^2},$$

$g'(x)$ tends to 0 as x approaches ξ.

(f) For $f(x) = x^{1/3}$, note that

$$x_{n+1} = x_1 - \frac{x_n^{1/3}}{(1/3)x_n^{-2/3}} = -2x_n$$

so that $x_n = (-2)^{n-1}x_1$. That is, $\{x_n\}$ is an alternating sequence such that $|x_n| \to \infty$ as $x \to \infty$.

127. Exercise 26: Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $\left|f'(x)\right| \leq A\left|f(x)\right|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Solution: (*analambanomenos*)
I'm going to show this for vector-valued functions $\mathbf{f} = (f_1, \ldots, f_k)$ mapping $[a, b]$ into $\mathbf{R}^k$ such that $\mathbf{f}(a) = \mathbf{0}$ and such that $\left|\mathbf{f}'(x)\right| \leq A|\mathbf{f}(x)|$, since this is needed for Exercise 28.

If $A = 0$, then $\mathbf{f}' = 0$, so $\mathbf{f} = \mathbf{f}(a) = \mathbf{0}$ by Theorem 5.11(b). So assume that $A > 0$. Following the hint, let $\delta > 0$ be small enough so that $\delta < 1/A$ and $x_0 = a + \delta \le b$. For $i = 1, \ldots, k$, let $M_{i,0} = \sup\left|f_i(x)\right|$, $M_{i,1} = \sup\left|f_i'(x)\right|$ for $a \le x \le x_0$. By Theorem 5.19, for any $x \in (a, x_0)$ and for every $i = 1, \ldots, k$ there is a $t_i \in (a, x)$ such that

$$\left|f_i(x)\right| = \left|f_i(x) - f_i(a)\right| = \left|f_i'(t)\right|(x - a) \le M_{i,1}(x - a) \le AM_{i,0}\delta.$$

Since $A\delta < 1$, this can only happen if $M_{i,0} = 0$, so $f_i(x) = 0$ in $[a, a + \delta]$ for each $i = 1, \ldots, k$, that is, $\mathbf{f} = \mathbf{0}$ in $[a, a + \delta]$. Repeating this argument with $a + \delta$ replacing a, we get $\mathbf{f}(x) = \mathbf{0}$ in $[a, a + 2\delta]$. After about $(b - a)/\delta$ steps, we've shown that $\mathbf{f}(x) = \mathbf{0}$ in $[a, b]$.

128. Exercise 27: Let ϕ be a real function defined on a rectangle R in the plane, given by $a \le x \le b$, $\alpha \le y \le \beta$. A *solution* of the initial-value problem

$$y' = \phi(x, y), \quad y(a) = c \quad (a \le c \le b)$$

is, by definition, a differentiable function f on $[a, b]$ such that $f(a) = c$, $\alpha \le f(x) \le \beta$, and

$$f'(x) = \phi\big(x, f(x)\big) \quad (a \le x \le b).$$

Prove that such a problem has at most one solution if there is a constant A such that

$$\left|\phi(x, y_2) - \phi(x, y_1)\right| \le A|y_2 - y_1|$$

whenever $(x, y_1) \in R$ and $(x, y_2) \in R$.

Solution: (*analambanomenos*)
Following the hint, let f_1, f_2 be two solutions of the initial value problem and let $f = f_1 - f_2$. Then f is differentiable on $[a, b]$, $f(a) = f_1(a) - f_2(a) = 0$, and

$$\left|f'(x)\right| = \left|f_1'(x) - f_2'(x)\right| = \left|\phi(x, f_1(x)) - \phi(x, f_2(x))\right| \le A\left|f_1(x) - f_2(x)\right| = A\left|f(x)\right|.$$

Applying Exercise 27, we get $f(x) = 0$ for all $x \in [a, b]$.

129. Exercise 28: Formulate and prove an analogous uniqueness theorem for systems of differential equations of the form

$$y_j'\phi_j(x, y_1, \ldots, y_k), \quad y_j(a) = c_j \quad (j = 1, \ldots, k).$$

Note that this can be rewritten in the form

$$\mathbf{y}' = \boldsymbol{\phi}(x, \mathbf{y}), \quad \mathbf{y}(a) = \mathbf{c}$$

where $\mathbf{y} = (y_1, \ldots, y_k)$ ranges over a k-cell, $\boldsymbol{\phi}$ is the mapping of a $(k+1)$-cell into the Euclidean k-space whose components are the functions $\phi_1, \ldots, \phi_k$, and $\mathbf{c}$ is the vector $(c_1, \ldots, c_k)$.

Solution: (*analambanomenos*)
If there is a constant A such that

$$\left|\boldsymbol{\phi}(x, \mathbf{y}_2) - \boldsymbol{\phi}(x, \mathbf{y}_1)\right| \le A|\mathbf{y}_2 - \mathbf{y}_1|,$$

then $\mathbf{y}' = \boldsymbol{\phi}(x, \mathbf{y})$, $\mathbf{y}(a) = \mathbf{c}$ has at most one solution. For suppose $\mathbf{f}_1$ and $\mathbf{f}_2$ are two such solutions, and let $\mathbf{f} = \mathbf{f}_1 - \mathbf{f}_2$. Then $\mathbf{f}$ is differentiable on $[a, b]$, $\mathbf{f}(a) = \mathbf{0}$, and

$$\left|\mathbf{f}'(x)\right| = \left|\boldsymbol{\phi}(x, \mathbf{f}_1(x)) - \boldsymbol{\phi}(x, \mathbf{f}_2(x))\right| \le A|\mathbf{f}_1 - \mathbf{f}_2| = A|\mathbf{f}|.$$

Hence by the vector-valued version of Exercise 26 shown above, $\mathbf{f}(x) = \mathbf{0}$ for all $x \in [a, b]$.

130. Exercise 29: Specialize Exercise 28 by considering the system

$$y_j' = y_{j+1} \quad (j = 1, \ldots, k-1),$$
$$y_k' = f(x) - \sum_{j=1}^{k} g_j(x) y_j,$$

where $f, g_1, \ldots, g_k$ are continuous real functions on $[a, b]$, and derive a uniqueness theorem for solutions of the equation

$$y^{(k)} + g_k(x) y^{(k-1)} + \cdots + g_2(x) y' + g_1(x) y = f(x),$$

subject to the initial conditions

$$y(a) = c_1, \quad y'(a) = c_2, \quad \ldots, \quad y^{(k-1)}(a) = c_k.$$

Solution: (*analambanomenos*)
Using the notation of Exercise 28, $\boldsymbol{\phi}(x, \mathbf{y}) = \big(\phi_1(x, \mathbf{y}), \ldots, \phi_k(x, \mathbf{y})\big)$ where

$$\phi_j(x, \mathbf{y}) = \begin{cases} y_{j+1} & j = 1, \ldots, k-1 \\ f(x) + \sum_{i=1}^{k} g_i(x) y_i = f(x) + \mathbf{g}(x) \cdot \mathbf{y} & j = k \end{cases}$$

where $\mathbf{g}(x) = \big(g_1(x), \ldots, g_k(x)\big)$. Then

$$\begin{aligned} \big|\boldsymbol{\phi}(x, \mathbf{y}_2) - \boldsymbol{\phi}(x, \mathbf{y}_2)\big|^2 &= \sum_{j=1}^{k} \big|\phi_j(x, \mathbf{y}_2) - \phi_j(x, \mathbf{y}_1)\big|^2 \\ &= \sum_{j=1}^{k-1} |y_{2,j+1} - y_{1,j+1}|^2 + \big|\mathbf{g}(x) \cdot (\mathbf{y}_2 - \mathbf{y}_1)\big|^2 \\ &\le |\mathbf{y}_2 - \mathbf{y}_1|^2 + |\mathbf{g}(x)|^2 |\mathbf{y}_2 - \mathbf{y}_1|^2 \\ &= \big(1 + |\mathbf{g}(x)|^2\big) |\mathbf{y}_2 - \mathbf{y}_1|^2 \end{aligned}$$

$$\big|\boldsymbol{\phi}(x, \mathbf{y}_2) - \boldsymbol{\phi}(x, \mathbf{y}_2)\big| \le \sqrt{\big(1 + |\mathbf{g}(x)|^2\big)}\ |\mathbf{y}_2 - \mathbf{y}_1|$$

Since the $g_i(x)$ are continuous on $[a, b]$, they are bounded, so that there is a constant A such that $\sqrt{\big(1 + |\mathbf{g}(x)|^2\big)} \le A$ for all $x \in [a, b]$. Hence by Exercise 28, the equation has at most one solution.

6 The Riemann-Stieltjes Integral

131. Exercise 1: Suppose α increase on $[a, b]$, $a \le x_0 \le b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \ne x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f\, d\alpha = 0$.

Solution: (*Matt "Frito" Lundy*)
(*Note: I should probably consider the cases where $x \pm d \notin [a, b]$ in the solutions to 1 and 2 below*)
First note that for any partition P of $[a, b]$, $L(P, f, \alpha) = 0$, so we have

$$\underline{\int_a^b} f\, d\alpha = 0$$

and $0 \leq U(P, f, \alpha)$.

Let $\varepsilon > 0$ be given. α is continuous at x_0 means there exists a $\delta > 0$ such that $|x - x_0| < \delta$ and $a \leq x \leq b$ implies $|\alpha(x) - \alpha(x_0)| < \varepsilon$.

Let $P = \{a, x_0 - \delta, x_0 + \delta, b\}$. Then we have

$$U(P, f, \alpha) = \alpha(x_0 + \delta) - \alpha(x_0 - \delta) < 2\varepsilon$$

and

$$0 \leq U(P, f, \alpha) < 2\varepsilon.$$

Because ε was arbitrary, we have

$$\overline{\int_a^b} f \; d\alpha = 0$$

so that $f \in \mathscr{R}(\alpha)$ and

$$\int_a^b f \; d\alpha = 0.$$

132. Exercise 2: Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x)\, dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Solution: (*Matt "Frito" Lundy*)
Suppose there exists an $x \in [a, b]$ such that $f(x) > 0$. f is continuous at on $[a, b]$ means there exists a δ such that $|t - x| < \delta$ and $a \leq t \leq b$ implies $f(t) > 0$. Let $P = \{a, x - \delta, x + \delta, b\}$, then $L(P, f) > 0$, and for any partition P

$$\int_a^b f(x) \; dx \geq L(P, x) > 0,$$

a contradiction.

Solution: (*Dan "kyp44" Whitman*)
Define the real function F on $[a, b]$ by

$$F(x) = \int_a^x f(t) dt\,.$$

Then, since $f \in \mathscr{R}$, F is continuous by Theorem 6.20. Moreover since f is continuous on $[a, b]$, F is differentiable with

$$F'(x) = f(x)$$

for any $x \in [a, b]$. Since $F'(x) = f(x) \geq 0$ for all $x \in [a, b]$, F is monotonically increasing by Theorem 5.11a. But since we also we have

$$F(a) = \int_a^a f(t) dt = 0$$
$$F(b) = \int_a^b f(t) dt = 0$$

so that it must be that $F(x) = 0$ for all $x \in [a, b]$, since otherwise F would not be monotonic. From this it follows that $f(x) = F'(x) = 0$ for all $x \in [a, b]$ also since F is constant.

133. Exercise 3: Define three functions β_1, β_2, β_3 as follows: $\beta_j(x) = 0$ if $x < 0$, $\beta_j(x) = 1$ if $x > 0$ for $j = 1, 2, 3$, and $\beta_1(0) = 0$, $\beta_2(0) = 1$, $\beta_3(0) = \frac{1}{2}$. Let f be a bounded function on $[-1, 1]$.

(a) Prove that $f \in \mathscr{R}(\beta_1)$ if and only if $f(0+) = f(0)$ and that then $\int f\, d\beta_1 = f(0)$.

(b) State and prove a similar result for β_2.

(c) Prove that $f \in \mathscr{R}(\beta_3)$ if and only if f is continuous at 0.

(d) If f is continuous at 0 prove that $\int f\, d\beta_1 = \int f\, d\beta_2 = \int f\, d\beta_3 = f(0)$.

Solution: (*Matt "Frito" Lundy*)
(a) First suppose that $f(0+) = f(0)$ and let $\varepsilon > 0$ be given. Then there exists a $\delta^* > 0$ such that $0 < x < \delta^*$ implies $|f(x) - f(0)| < \epsilon$. Let $\delta = \min\{1, \delta^*/2\}$ and form a partition $P = \{-1, 0, \delta, 1\}$. Then we have

$$U(P, f, \beta_1) - L(P, f, \beta_1) = f(s) - f(t)$$

for some $s, t \in [0, \delta]$. But

$$f(s) - f(t) \leq |f(s) - f(0)| + |f(0) - f(t)|$$

so we have

$$U(P, f, \beta_1) - L(P, f, \beta_1) \leq |f(s) - f(0)| + |f(0) - f(t)| < 2\varepsilon,$$

which shows that $f \in \mathscr{R}(\beta_1)$.

Now suppose that $f \in \mathscr{R}(\beta_1)$ and let $\varepsilon > 0$ be given. Then there exists a partition P for which

$$U(P, f, \beta_1) - L(P, f, \beta_1) < \varepsilon.$$

Let P^* be a refinement of P that includes 0. Then we have:

$$U(P^*, f, \beta_1) - L(P^*, f, \beta_1) < \varepsilon.$$

But if $[0, \delta]$ is the subinterval of P^* that contains 0, then:

$$U(P^*, f, \beta_1) - L(P^*, f, \beta_1) = f(s) - f(t)$$

for some $s, t \in [0, \delta]$ where $f(s) - f(t) \geq |f(x_1) - f(x_2)|$ for any $x_1, x_2 \in [0, \delta]$. So we have for any $0 < x < \delta$

$$|f(0) - f(x)| \leq f(s) - f(t) < \varepsilon,$$

which means that $f(0+) = f(0)$.

As above, for any partition P that contains 0 we have:

$$U(P, f, \beta_1) = M_l \quad L(P, f, \beta_1) = m_l$$

in the interval $[0, x_l]$ of P. Because f is right-continuous at 0, both M_l and m_l converge to $f(0)$ as $x_l \to 0$, so

$$\int f\ d\beta_1 = f(0).$$

(b) The statement is: $f \in \mathscr{R}(\beta_2)$ if and only if $f(0-) = f(0)$ and then

$$\int f\ d\beta_2 = f(0).$$

The proof is similar to part (a).

(c) Suppose that f is continuous at 0 and let ε be given. Then there exists a $\delta^* > 0$ such that $|x| < 0$ implies $|f(x) - f(0)| < \varepsilon$. Let $\delta = \min\{1, \delta^*/2\}$, and $P = \{-1, -\delta, \delta, 1\}$. Then

$$U(P, f, \beta_3) - L(P, f, \beta_3) = f(s) - f(t)$$

where $s, t \in [-\delta, \delta]$. But

$$f(s) - f(t) \leq |f(s) - f(0)| + |f(0) - f(t)|$$

so

$$U(P, f, \beta_3) - L(P, f, \beta_3) < 2\varepsilon$$

and $f \in \mathscr{R}(\beta_3)$.

Now suppose that $f \in \mathscr{R}(\beta_3)$ and let $\varepsilon > 0$ be given. There exists a partition P such that

$$U(P, f, \beta_3) - L(P, f, \beta_3) < \varepsilon.$$

Let P^* be a refinement of P that contains 0 so that the partitions around 0 are $[x_{l-1}, 0]$ and $[0, x_l]$, let $\delta = \min\{|x_l|, |x_{l-1}|\}$ and let $P^{\#}$ be a refinement of P^* that contains $\pm\delta$. Then

$$U(P^{\#}, f, \beta_3) - L(P^{\#}, f, \beta_3) = \frac{1}{2}\left[f(s) - f(t) + f(q) - f(r)\right]$$

where $s, t \in [0, \delta]$, $q, r \in [-\delta, 0]$

$$f(s) - f(t) \geq |f(0) - f(x)| \quad \text{for any} \quad x \in [0, \delta]$$

$$f(q) - f(q) \geq |f(0) - f(x)| \quad \text{for any} \quad x \in [-\delta, 0].$$

So for any $x \in [-\delta, \delta]$ we have

$$|f(0) - f(x)| < \epsilon$$

which shows that f is continuous at 0.

(d) The result follows from parts (a) - (c) and the fact that if f is continuous at 0, then $f(0-) = f(0) = f(0+)$.

134. Exercise 4: If $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x, prove that $f \notin \mathscr{R}$ on $[a, b]$ for any $a < b$.

Solution: (*Matt "Frito" Lundy*)
For any partition P, we have

$$U(P, f) = b - a > 0$$

$$L(P, f) = 0,$$

so $f \notin \mathscr{R}$.

135. Exercise 5: Suppose f is a bounded real function on $[a, b]$, and $f^2 \in \mathscr{R}$ on $[a, b]$. Does it follow that $f \in \mathscr{R}$? Does the answer change if we assume that $f^3 \in \mathscr{R}$?

Solution: (*Matt "Frito" Lundy*)
To answer the question "Does $f^2 \in \mathscr{R}$ imply that $f \in \mathscr{R}$?" consider the function

$$f(x) = \begin{cases} 1 & x \text{ rational} \\ -1 & x \text{ irrational.} \end{cases}$$

Then $f^2 = 1$ and $f^2 \in \mathscr{R}$, but $f \notin \mathscr{R}$.

To answer the question "Does $f^3 \in \mathscr{R}$ imply that $f \in \mathscr{R}$?" use the fact that f is bounded on $[a, b]$ implies that f^3 is bounded on $[a, b]$, so there exists $m, M \in \mathbf{R}$ such that $m \leq f^3 \leq M$ for all $x \in [a, b]$. Let $\phi(x) = x^{1/3}$, then ϕ is continuous on $[m, M]$, and $f = \phi(f^3)$, so $f \in \mathscr{R}$ on $[a, b]$ by theorem 6.11.

Notice that $\phi(x) = x^{1/2}$ does not work for the first question because ϕ is not continuous on $[-1, 1]$.

136. Exercise 6: Let P be the Cantor set. Let f be a bounded real function on $[0, 1]$ which is continuous at every point outside P. Prove that $f \in \mathscr{R}$ on $[0, 1]$.

Solution: (*analambanomenos*)
Following the hint, recall that $P = \cup E_n$ where E_n is a set of 2^n disjoint close intervals of length 3^{-n} obtained by removing the middle thirds of the intervals in E_{n-1}. The total length of the intervals of E_n is $(2/3)^n$ and so $\to 0$ as $n \to \infty$. We can replace the closed intervals of E_n, $[a_{i,n}, b_{i,n}]$ with slightly larger open intervals $(a_{i,n} - \delta/2^n, b_{i,n} + \delta/2^n)$, with total length $(2/3)^n + \delta$, so that we can cover P with a set of disjoint open intervals with total length as small as possible.

Now proceeding as in the proof of Theorem 6.10, let $M = \sup\left|f(x)\right|$, and cover P with a collection of open intervals (u_j, v_j) with total length less than ε. The complement K of the union of the open intervals in $[a, b]$ is compact. Then f is uniformly continuous on K and there exists $\delta > 0$ such that $\left|f(s) - f(t)\right| < \varepsilon$ if $s, t \in K$, $|s - t| < \delta$.

Form a partition $Q = \{x_0, \ldots, x_n\}$ of $[a, b]$ such that each u_j and v_j occurs in Q, no point of any segment (u_j, v_j) occurs in Q, and if x_{i-1} is not one of the u_j, then $\Delta x_i < \delta$.

Note that $M_i - m_i \leq 2M$ for every i, and that $M_i - m_i \leq \varepsilon$ if x_{i-1} is not one of the u_j. Hence

$$\begin{aligned} U(Q, f) - L(Q, f) &= \sum_i (M_i - m_i)\Delta x_i \\ &= \sum_{x_{i-1} \in \{u_j\}} (M_i - m_i)\Delta x_i + \sum_{x_{i-1} \notin \{u_j\}} (M_i - m_i)\Delta x_i \\ &\leq 2M\varepsilon + \varepsilon(b - a) = (2M + b - a)\varepsilon. \end{aligned}$$

Hence $f \in \mathscr{R}$ by Theorem 6.6.

137. Exercise 7: Suppose f is a real function on $(0, 1]$ and $f \in \mathscr{R}$ on $[c, 1]$ for every $c > 0$. Define

$$\int_0^1 f(x)\,dx = \lim_{c \to 0} \int_c^1 f(x)\,dx$$

if this limit exists (and is finite).

(a) If $f \in \mathscr{R}$ on $[0, 1]$, show that this definition of the integral agrees with the old one.

(b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f.

Solution: (*analambanomenos*)
(a) By Theorems 6.12(c) and 6.20,

$$\lim_{c \to 0} \int_c^1 f(x)\,dx = \int_0^1 f(x)\,dx - \lim_{c \to 0} \int_0^c f(x)\,dx = \int_0^1 f(x)\,dx.$$

(b) Let

$$f(x) = \begin{cases} 0 & x = 0 \\ -\dfrac{2^{2n-1}}{2n-1} & 2^{-(2n-1)} < x \le 2^{-(2n-2)},\ n = 1, 2, \dots \\ \dfrac{2^{2n}}{2n} & 2^{-2n} < x \le 2^{-(2n-1)},\ n = 1, 2, \dots \end{cases}$$

Then for any positive integer N,

$$\int_{2^{-N}}^{1} f(x)\,dx = \sum_{n=1}^{N} \frac{(-1)^n}{n}$$

which converges as $N \to \infty$ by Theorem 3.43. However,

$$\int_{2^{-N}}^{1} \big|f(x)\big|\,dx = \sum_{n=1}^{N} \frac{1}{n}$$

fails to converge as $N \to \infty$ by Theorem 3.28.

138. Exercise 8: Suppose $f \in \mathscr{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Define

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left *converges.* If it also converges after f has been replaced by $|f|$, it is said to converge *absolutely.*

Assume that $f(x) \ge 0$ and that f decreases monotonically on $[1, \infty)$. Prove that $\int_1^\infty f(x)\,dx$ converges if and only if $\sum_{n=1}^\infty f(n)$ converges.

Solution: (*analambanomenos*)
For any positive integer n define $g_1(x) = f(n)$ and $g_2(x) = f(n+1)$ for $x \in (n, n+1]$. Since f decreases monotonically, $g_1(x) \ge f(x) \ge g_2(x)$ for all $x \in [1, \infty)$. And since

$$\int_n^{n+1} g_1(x)\,dx = f(n) \qquad \int_n^{n+1} g_2(x)\,dx = f(n+1)$$

we get by Theorem 6.12(b), for any positive integer N,

$$\sum_1^N f(n) = \int_1^N g_1(x)\,dx \le \int_1^N f(x)\,dx \le \int_1^N g_2(x)\,dx = \sum_2^{N+1} f(n).$$

Hence $\int_1^N f(x)\,dx$ converges as $N \to \infty$ if and only if $\sum_1^N f(n)$ converges. In that case, if $A = \sum_1^\infty f(n)$, then $\int_1^\infty f(x)\,dx$ lies between $A - f(1)$ and A.

139. Exercise 9: Show that integration by parts can sometimes be applied to the "improper" integrals defined in Exercises 7 and 8. (State appropriate hypotheses, formulate a theorem, and prove it.) For instance, show that

$$\int_0^\infty \frac{\cos x}{1+x}\,dx = \int_0^\infty \frac{\sin x}{(1+x)^2}\,dx.$$

Show that one of these integrals converges *absolutely,* but that the other does not.

Solution: (*analambanomenos*)
If F and G are functions which are differentiable on $[c,1]$ for all $c>0$ and such that $F'=f\in\mathscr{R}$ and $G'=g\in\mathscr{R}$ on $[c,1]$ for all $c>0$, then by Theorem 6.22 we have

$$\int_c^1 F(x)g(x)\,dx = F(1)G(1)-F(c)G(c)-\int_c^1 f(x)G(x)\,dx.$$

Suppose that two of the limits

$$\lim_{c\to 0}\int_c^1 F(x)g(x)\,dx=\int_0^1 F(x)g(x)\,dx,\ \ \lim_{c\to 0}\int_c^1 f(x)G(x)\,dx=\int_0^1 f(x)G(x)\,dx,\ \ \lim_{c\to 0}F(c)G(c)$$

exist and are finite. Then the third limit exists and is finite, and we have

$$\int_0^1 F(x)g(x)\,dx = F(1)G(1)-\lim_{c\to 0}F(c)G(c)-\int_0^1 F(x)g(x)\,dx.$$

Similarly, if F and G are functions which are differentiable on $[a,b]$ for all $b>a$ and such that $F'=f\in\mathscr{R}$ and $G'=g\in\mathscr{R}$ on $[a,b]$ for all $b>a$, then by Theorem 6.22 we have

$$\int_a^b F(x)g(x)\,dx = F(b)G(b)-F(a)G(a)-\int_a^b f(x)G(x)\,dx.$$

Suppose that two of the limits

$$\lim_{b\to\infty}\int_a^b F(x)g(x)\,dx=\int_a^\infty F(x)g(x)\,dx,\ \ \lim_{b\to\infty}\int_a^b f(x)G(x)\,dx=\int_a^\infty f(x)G(x)\,dx,\ \ \lim_{b\to\infty}F(b)G(b)$$

exist and are finite. Then the third limit exists and is finite, and we have

$$\int_a^\infty F(x)g(x)\,dx = \lim_{b\to\infty}F(b)G(b)-F(a)G(a)-\int_a^\infty F(x)g(x)\,dx.$$

For the example, let

$$F(x)=\frac{1}{1+x},\quad F'(x)=f(x)=-\frac{1}{(1+x)^2},\quad G(x)=\sin x,\quad G'(x)=g(x)=\cos x.$$

The functions F and G are differentiable on $[0,b)$, for all $b>0$, and $f\in\mathscr{R}$, $g\in\mathscr{R}$ on $[0,b]$ for all $b>0$. Also

$$\lim_{b\to\infty}F(b)G(b)=\lim_{b\to\infty}\frac{\sin b}{1+b}=0$$

and

$$\lim_{b\to\infty}\left|\int_0^b\frac{\sin x}{(1+x)^2}\,dx\right|\le\lim_{b\to\infty}\int_0^b\frac{|\sin x|}{(1+x)^2}\,dx\le\lim_{b\to\infty}\int_0^b\frac{1}{(1+x)^2}\,dx$$

which converges by Exercise 8 since $\sum_0^\infty 1/(1+n)^2$ converges. Hence we can apply the results of the first part of this exercise and conclude that

$$\int_0^\infty\frac{\cos x}{1+x}\,dx=\lim_{b\to\infty}\frac{\sin b}{1+b}-\frac{\sin 0}{1+0}+\int_0^\infty\frac{\sin x}{(1+x)^2}\,dx=\int_0^\infty\frac{\sin x}{(1+x)^2}\,dx.$$

We've seen above that $\sin x/(1+x)^2$ converges absolutely on $[0,\infty)$. To show that $\cos x/(1+x)$ diverges absolutely,

$$\begin{aligned}\int_0^\infty \frac{|\cos x|}{1+x}\,dx &= \sum_{k=0}^\infty \int_{2\pi k}^{2\pi(k+1)} \frac{|\cos x|}{1+x}\,dx \\ &\geq \sum_{k=0}^\infty \frac{1}{2\pi(k+1)+1}\int_{2\pi k}^{2\pi(k+1)} |\cos x|\,dx \\ &\geq \sum_{k=0}^\infty \frac{1}{2\pi(k+1)+2\pi}\int_0^{2\pi} |\cos x|\,dx \\ &= \frac{2}{\pi}\sum_{k=0}^\infty \frac{1}{k+2}\end{aligned}$$

a sum which diverges.

140. Exercise 10: Let p and q be positive real numbers such that

$$\frac{1}{p}+\frac{1}{q}=1.$$

Prove the following statements.

(a) If $u \geq 0$ and $v \geq 0$, then

$$uv \leq \frac{u^p}{p}+\frac{v^q}{q}.$$

Equality holds if and only if $u^p = v^q$.

(b) If $f \in \mathscr{R}(\alpha)$, $g \in \mathscr{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and

$$\int_a^b f^p\,d\alpha = 1 = \int_a^b g^q\,d\alpha,$$

then

$$\int_a^b fg\,d\alpha \leq 1.$$

(c) If f and g are complex functions in $\mathscr{R}(\alpha)$, then

$$\left|\int_a^b fg\,d\alpha\right| \leq \left(\int_a^b |f|^p\,d\alpha\right)^{1/p}\left(\int_a^b |g|^q\,d\alpha\right)^{1/q}.$$

This is *Hölder's inequality.* When $p=q=2$ it is usually called the Schwarz inequality.

(d) Show that Hölder's inequality is also true for the "improper" integrals defined in Exercises 7 and 8.

Solution: (*analambanomenos*)
Note that since q and $p = q/(q-1)$ are positive, then $q-1$ is positive.

(a) Fix u and let $f(v) = (u^p/p) + (v^q/q) - uv$. Then $f'(v) = v^{q-1} - u$ and $f''(v) = (q-1)v^{q-2}$ is non-negative for non-negative v, so the critical point $v = u^{1/(q-1)}$ is a minimum. Hence

$$\begin{aligned}\frac{u^p}{p} + \frac{v^q}{q} - uv &\geq \frac{u^p}{p} + \frac{u^{q/(q-1)}}{q} - u^{1+1/(q-1)} \\ &= \left(\frac{1}{p} + \frac{1}{q} - 1\right) u^p \\ &= 0\end{aligned}$$

so that $uv \leq (u^p/p) + (v^q/q)$. Equality holds at the critical value $v = u^{1/(q-1)}$, or $v^q = u^{q/(q-1)} = u^p$.

(b) From part (a) we have

$$\begin{aligned}\int_a^b fg\,d\alpha &\leq \int_a^b \left(\frac{f^p}{p} + \frac{g^q}{q}\right) d\alpha \\ &= \frac{1}{p}\int_a^b f^p\,d\alpha + \frac{1}{q}\int_a^b g^q\,d\alpha \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1.\end{aligned}$$

(c) Define

$$A = \left(\int_a^b |f|^p\,d\alpha\right)^{1/p} \qquad B = \left(\int_a^b |g|^q\,d\alpha\right)^{1/q}.$$

Then

$$\begin{aligned}\int_a^b \left|\frac{f}{A}\right|^p d\alpha &= \frac{1}{A^p}\int_a^b |f|^p\,d\alpha = 1 \\ \int_a^b \left|\frac{g}{B}\right|^q d\alpha &= \frac{1}{B^q}\int_a^b |g|^q\,d\alpha = 1.\end{aligned}$$

Applying part (b), we get

$$\int_a^b \left|\frac{f}{A}\right| \cdot \left|\frac{g}{B}\right| d\alpha \leq 1,$$

or

$$\left|\int_a^b fg\,d\alpha\right| \leq \int_a^b |fg|\,d\alpha \leq AB = \left(\int_a^b |f|^p\,d\alpha\right)^{1/p} \left(\int_a^b |g|^q\,d\alpha\right)^{1/q}.$$

(d) Let $f \in \mathscr{R}$, $g \in \mathscr{R}$ on $[c, 1]$ for all $c > 0$ such that the improper integrals $\int_0^1 |f|^p\,dx$ and $\int_0^1 |g|^q\,dx$ exist. Then for all $c > 0$ we have

$$\left|\int_c^1 fg\,dx\right| \leq \left(\int_c^1 |f|^p\,dx\right)^{1/p} \left(\int_c^1 |g|^q\,dx\right)^{1/q}.$$

Since the right side increases monotonically as $c \to 0$, we can take the limit of both sides to get the desired result.

Similarly, let $f \in \mathscr{R}$, $g \in \mathscr{R}$ on $[a, b]$ for all $b > a$ such that the improper integrals $\int_a^\infty |f|^p\,dx$ and $\int_a^\infty |g|^q\,dx$ exist. Then for all $b > a$ we have

$$\left|\int_a^b fg\,dx\right| \leq \left(\int_a^b |f|^p\,dx\right)^{1/p} \left(\int_a^b |g|^q\,dx\right)^{1/q}.$$

Since the right side increases monotonically as $b \to \infty$, we can take the limit of both sides to get the desired result.

141. Exercise 11: Let α be a fixed increasing function on $[a, b]$. For $u \in \mathscr{R}(\alpha)$, define

$$||u||_2 = \left(\int_a^b |u|^2 \, d\alpha\right)^{1/2}.$$

Suppose $f, g, h \in \mathscr{R}(\alpha)$, and prove the triangle inequality

$$||f - h||_2 \leq ||f - g||_2 + ||g - h||_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

Solution: (*analambanomenos*)
As in the proof of Theorem 1.37(f), we show that

$$\begin{aligned}
||u + v||_2^2 &= \int_a^b |u + v|^2 \, d\alpha \\
&\leq \int_a^b \big(|u|^2 + 2|uv| + |v|^2\big) \, d\alpha \\
&= ||u||_2^2 + 2\int_a^b |uv| \, d\alpha + ||v||_2^2 \\
&\leq ||u||_2^2 + 2||u||_2||v||_2 + ||v||_2^2 \\
&= \big(||u||_2 + ||v||_2\big)^2
\end{aligned}$$

so that $||u + v||_2 \leq ||u||_2 + ||v||_2$. The result follows by letting $u = f - g$ and $v = g - h$.

142. Exercise 12: With the notations of Exercise 11, suppose $f \in \mathscr{R}(\alpha)$ and $\varepsilon > 0$. Prove that there exists a continuous function g on $[a, b]$ such that $||f - g||_2 < \varepsilon$.

Solution: (*analambanomenos*)
Since $f \in \mathscr{R}(\alpha)$, There are bounds $m \leq f(x) \leq M$ for the values of f in $[a, b]$. Let $\varepsilon > 0$ and let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$ such that $U(f, P, \alpha) - L(f, P, \alpha) \leq \varepsilon^2/(M - m)$. Following the hint, for $t \in [x_{i-1}, x_i]$ define

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i).$$

Then g is linear on $[x_{i-1}, x_i]$ and the definitions of g on $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ both define $g(x_i) = f(x_i)$, and so g is a continuous function on $[a, b]$. Also, $m_i \leq g(t) \leq M_i$ on each $[x_{i-1}, x_i]$. Hence

$$\begin{aligned}
||f - g||_2^2 &= \int_a^b |f - g|^2 \, d\alpha \\
&= \sum_{i=0}^n \int_{x_{i-1}}^{x_i} |f - g|^2 \, d\alpha \\
&\leq \sum_{i=0}^n (M_i - m_i)^2 \Delta\alpha_i \\
&\leq (M - m) \sum_{i=0}^n (M_i - m_i) \Delta\alpha_i \\
&= (M - m)\big(U(f, P, \alpha) - L(f, P, \alpha)\big) \\
&\leq \varepsilon^2
\end{aligned}$$

143. Exercise 13: Define

$$f(x) = \int_x^{x+1} \sin t^2 \, dt.$$

(a) Prove that $|f(x)| < 1/x$ if $x > 0$.

(b) Prove that

$$2xf(x) = \cos x^2 - \cos(x+1)^2 + r(x)$$

where $|r(x)| < c/x$ and c is a constant.

(c) Find the upper and lower limits of $xf(x)$, as $x \to \infty$.

(d) Does $\int_0^\infty \sin t^2 \, dt$ converge?

Solution: (*analambanomenos*)
(a) Following the hint (substituting u for t^2, integration by parts using $\sin u$ and $1/(2\sqrt{u})$, replacing $|\cos u|$ with 1), we get

$$\begin{aligned}
f(x) &= \int_{x^2}^{(x+1)^2} \frac{\sin u}{2\sqrt{u}} \, du \\
&= -\frac{\cos(x+1)^2}{2(x+1)} + \frac{\cos x^2}{2x} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} \, du \\
|f(x)| &\le \frac{|\cos(x+1)^2|}{2(x+1)} + \frac{|\cos x^2|}{2x} + \int_{x^2}^{(x+1)^2} \frac{|\cos u|}{4u^{3/2}} \, du \\
&< \frac{1}{2(x+1)} + \frac{1}{2x} + \int_{x^2}^{(x+1)^2} \frac{1}{4u^{3/2}} \, du \\
&= \frac{1}{2(x+1)} + \frac{1}{2x} + \left(\frac{1}{2x} - \frac{1}{2(x+1)} \right) \\
&= \frac{1}{x}
\end{aligned}$$

(The strict inequality is due to the fact that $|\cos u|$ will not be constantly equal to its maximum possible value 1 throughout the interval of integration.)

(b) From the results in part (a), we get

$$\begin{aligned}
2xf(x) &< \cos x^2 - \frac{x\cos(x+1)^2}{x+1} + \frac{1}{x+1} \\
r(x) = 2xf(x) - \cos x^2 + \cos(x+1)^2 &< \frac{\cos(x+1)^2}{x+1} + \frac{1}{x+1} \\
|r(x)| &< \frac{2}{x+1} < \frac{2}{x}
\end{aligned}$$

(c) (partial) From part (b), we get that as $x \to \infty$,

$$xf(x) \approx \frac{\cos x^2 - \cos(x+1)^2}{2}.$$

Hence $xf(x)$ lies (more or less) between $(-1-1)/2 = -1$ and $(1+1)/2 = 1$.

(d) Note that $\sin t^2$ is positive between $\sqrt{n\pi}$ and $\sqrt{(n+1)\pi}$ for any even integer n and negative for any odd integer n. Hence $\int_0^\infty \sin t^2 \, dt$ can be reduced to an alternating series, whose terms satisfy

$$\left| \int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} \sin t^2 \, dt \right| < \sqrt{n\pi} - \sqrt{(n+1)\pi} = \frac{\sqrt{\pi}}{\sqrt{n} + \sqrt{n+1}}$$

which goes to 0 as $n \to \infty$. Hence, by Theorem 3.43, the integral converges.

Solution: (*Dan "kyp44" Whitman*)
(c) From before we have

$$xf(x) = \frac{1}{2}\cos(x^2) - \frac{1}{2}\cos[(x+1)^2] + \frac{1}{2}r(x)$$

and since $r(x) \to 0$ as $x \to \infty$ it follows that $\limsup xf(x)$ and $\liminf xf(x)$ are the same as those for

$$h(x) = \frac{1}{2}\cos(x^2) - \frac{1}{2}\cos[(x+1)^2]\,.$$

Now we have by the cosine addition formula

$$\begin{aligned} h(x) = \frac{\cos(x^2) - \cos[(x+1)^2]}{2} &= \frac{1}{2}\left[-2\sin\left(\frac{x^2+(x+1)^2}{2}\right)\sin\left(\frac{x^2-(x+1)^2}{2}\right)\right] \\ = -\sin\left(\frac{x^2+x^2+2x+1}{2}\right)\sin\left(\frac{x^2-x^2-2x-1}{2}\right) &= -\sin\left(\frac{2x^2+2x+1}{2}\right)\sin\left(\frac{-2x-1}{2}\right) \\ = \sin\left(x^2+x+\frac{1}{2}\right)\sin\left(x+\frac{1}{2}\right)\,. \end{aligned}$$

If we let $s = x + \frac{1}{2}$ then this can be expressed as

$$h(x) = \sin(s)\sin\left(s^2 + \frac{1}{4}\right)\,.$$

Now, clearly $-1 \le h(x) \le 1$ but we claim that the upper limit of $h(x)$, and therefore $xf(x)$, is 1. To see this first choose any $\varepsilon \in \mathbb{R}^+$ and let

$$N = \max\left(1, \left\lceil \frac{2-\varepsilon}{8\varepsilon} \right\rceil + 1\right)\,.$$

Consider then any $n \ge N$ so that $n \in \mathbb{Z}^+$ and

$$n \ge N \ge \left\lceil \frac{2-\varepsilon}{8\varepsilon} \right\rceil + 1 > \left\lceil \frac{2-\varepsilon}{8\varepsilon} \right\rceil \ge \frac{2-\varepsilon}{8\varepsilon}\,.$$

Now let

$$p_n = \left(2n + \frac{1}{2}\right)\pi$$

and consider then the interval $[a_n, b_n]$ where

$$\begin{aligned} a_n &= p_n - \varepsilon \\ b_n &= p_n + \varepsilon\,. \end{aligned}$$

Since the function $s^2 + \frac{1}{4}$ is strictly increasing and continuous for $s > 0$, $[a_n, b_n]$ maps bijectively onto $[a_n^2 + \frac{1}{4}, b_n^2 + \frac{1}{4}]$, and the length of this latter interval is

$$\begin{aligned}
\left(b_n^2 + \frac{1}{4}\right) - \left(a_n^2 + \frac{1}{4}\right) &= (p_n + \varepsilon)^2 + \frac{1}{4} - (p_n - \varepsilon)^2 - \frac{1}{4} \\
&= p_n^2 + 2p_n\varepsilon + \varepsilon^2 - p_n^2 + 2p_n\varepsilon - \varepsilon^2 = 4p_n\varepsilon \\
&= 4\left(2n + \frac{1}{2}\right)\pi\varepsilon = (8n+2)\pi\varepsilon \\
&> \left(8\frac{2-\varepsilon}{8\varepsilon} + 2\right)\pi\varepsilon = (2 - \varepsilon + 2\varepsilon)\pi \\
&= 2\pi + \pi\varepsilon > 2\pi\,.
\end{aligned}$$

It then follows that there is a $t_n \in [a_n, b_n]$ where $\sin(t_n^2 + \frac{1}{4}) = 1$. Now suppose that $t_n \le p_n$. Consider then any $s < p_n$ and the function $g_n(u) = (u - p_n) + 1$ on the interval $[s, p_n]$. Then since $\cos(u) \le 1$ for all u it follows by Theorem 6.12b that

$$\begin{gathered}
\int_s^{p_n} \cos u\, du \le \int_s^{p_n} 1\, du \\
\sin p_n - \sin s \le g_n(p_n) - g_n(s) \\
1 - \sin s \le 1 - g_n(s) \\
g_n(s) \le \sin s
\end{gathered}$$

since $g_n'(u) = 1$. Thus for all $s \le p_n$ we have $g_n(s) \le \sin s$. So since g_n is clearly strictly increasing and $a_n \le t_n$ we have

$$1 - \varepsilon = g_n(a_n) \le g_n(t_n) \le \sin t_n\,.$$

An analagous argument shows that $\sin t_n \ge 1 - \varepsilon$ when $t_n \ge p_n$ as well. Thus if we let $x_n = t_n - \frac{1}{2}$ then we have

$$h(x_n) = \sin t_n \sin\left(t_n^2 + \frac{1}{4}\right) = \sin t_n \ge 1 - \varepsilon\,.$$

Since this is true for any $n \ge N$ and it is easily verified than for such n none of the intervals $[a_n, b_n]$ overlap we can form a sequence $\{x_n\}$ where $h(x_n) \ge 1-\varepsilon$ for all $n \ge N$ (x_n for $n < N$ are irrelevent). Also $x_n \to \infty$ since clearly $a_n \to \infty$ and $x_n \ge a_n - \frac{1}{2}$.

But since ε was arbitrary our sequence can get as close to 1 as we want so that clearly the upper limit must be 1. A similar argument shows that the lower limit is -1.

(d) Let

$$a_n = \int_{\sqrt{\pi n}}^{\sqrt{\pi(n+1)}} \sin t^2\, dt = \int_{\pi n}^{\pi(n+1)} \frac{\sin u}{2\sqrt{u}}\, du$$

so that

$$\int_0^\infty \sin t^2\, dt = \sum_{n=0}^{\infty} \int_{\sqrt{\pi n}}^{\sqrt{\pi(n+1)}} \sin t^2\, dt = \sum_{n=0}^{\infty} a_n\,.$$

Thus it suffices to show that $\sum a_n$ converges. To this end let us first note that

$$a_n = \int_{\pi n}^{\pi(n+1)} \frac{\sin u}{2\sqrt{u}}\, du = \int_{\pi n}^{\pi(n+1)} (-1)^n \frac{|\sin u|}{2\sqrt{u}}\, du = (-1)^n \int_{\pi n}^{\pi(n+1)} \frac{|\sin u|}{2\sqrt{u}}\, du$$

since $\sin u$ is always positive for even n and negative for odd n with $u \in [\pi n, \pi(n+1)]$. We therefore have

$$|a_n| = \int_{\pi n}^{\pi(n+1)} \frac{|\sin u|}{2\sqrt{u}}\, du$$

Setting $v = u - \pi$ we have

$$|a_{n+1}| = \int_{\pi(n+1)}^{\pi(n+2)} \frac{|\sin u|}{2\sqrt{u}}\,du = \int_{\pi n}^{\pi(n+1)} \frac{|\sin(v+\pi)|}{2\sqrt{v+\pi}}\,dv = \int_{\pi n}^{\pi(n+1)} \frac{|\sin v|}{2\sqrt{v+\pi}}\,dv$$
$$= \int_{\pi n}^{\pi(n+1)} \frac{|\sin u|}{2\sqrt{u+\pi}}\,du$$

since $|\sin u|$ has a period of π and v is just a dummy variable inside the integrand. Then since

$$\frac{|\sin u|}{2\sqrt{u}} \geq \frac{|\sin u|}{2\sqrt{u+\pi}}$$

for all $u \in [\pi n, \pi(n+1)]$ and the intervals are the same it follows from Theorem 6.12b that $|a_n| \geq |a_{n+1}|$. Since this is true for any $n \geq 0$ clearly the sequence $\{|a_n|\}$ is monotonically decreasing.

We also have

$$\begin{aligned} 0 \leq |a_n| &= \int_{\pi n}^{\pi(n+1)} \frac{|\sin u|}{2\sqrt{u}}\,du \\ &\leq \int_{\pi n}^{\pi(n+1)} \frac{1}{2\sqrt{u}}\,du = (\sqrt{u})\Big|_{\pi n}^{\pi(n+1)} \\ &= \sqrt{\pi(n+1)} - \sqrt{\pi n} = \sqrt{\pi}(\sqrt{n+1} - \sqrt{n}) \\ &= \frac{\sqrt{\pi}}{\sqrt{n+1} + \sqrt{n}}\,. \end{aligned}$$

Clearly the right side goes to zero as $n \to \infty$ so that necessarily $|a_n| \to 0$, from which it immediately follows that $a_n \to 0$.

Since $\{a_n\}$ is alternating, $\sum a_n$ then converges by Theorem 3.43 (actually a slightly modified version of Theorem 3.43 in which $a_n \geq 0$ for even n and $a_b \leq 0$ for odd n, which is trivial to prove).

144. Exercise 14: Deal similarly with

$$f(x) = \int_x^{x+1} \sin e^t\,dt.$$

Show that

$$e^x\big|f(x)\big| < 2$$

and that

$$e^x f(x) = \cos e^x - e^{-1}\cos e^{x+1} + r(x),$$

where $\big|r(x)\big| < Ce^{-x}$, for some constant C.

Solution: (*analambanomenos*)
Substituting u for e^t and integrating by parts, we get

$$\begin{aligned} f(x) &= \int_x^{x+1} \sin e^t\,dt \\ &= \int_{e^x}^{e^{x+1}} \frac{\sin u}{u}\,du \\ &= \frac{\cos e^x}{e^x} - \frac{\cos e^{x+1}}{e^{x+1}} - \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2}\,du \end{aligned}$$

Hence, substituting -1 for $\cos u$ in the integral, we get

$$\begin{aligned} e^x f(x) &< \cos e^x - \frac{\cos e^{x+1}}{e} - e^x \int_{e^x}^{e^{x+1}} u^{-2}\, du \\ &= (1+\cos e^x) - e^{-1}(1+\cos e^{x+1}), \end{aligned}$$

so that $e^x\big|f(x)\big| < 2(1+1/e)$, which isn't quite 2 (more work would need to be done to get this below 2). Doing another integration by parts, we get

$$\begin{aligned} r(x) &= -e^x \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2}\, du \\ &= e^x\left(\frac{\sin e^x}{e^{2x}} - \frac{\sin e^{x+1}}{e^{2x+2}} - 2\int_{e^x}^{e^{x+1}} \frac{\sin u}{u^3}\, du\right) \\ &< \frac{\sin e^x}{e^x} - \frac{\sin e^{x+1}}{e^{x+2}} + 2e^x \int_{e^x}^{e^{x+1}} u^{-3}\, du \\ &= \frac{1+\sin e^x}{e^x} - \frac{1+\sin e^{x+1}}{e^{x+2}}, \end{aligned}$$

so that $\big|r(x)\big| < 2(1+1/e^2)e^{-x}$.

Solution: (*Dan "kyp44" Whitman*)
Define $\varphi(u) = \ln u$, noting that φ is strictly increasing and continuous for $u > 0$. Also let $h(t) = \sin(e^t)$, $\alpha(t) = t$ so that $\alpha'(t) = 1$, $A = e^x$ (where $x > 0$) so that $a = \varphi(A) = x$, $B = e^{x+1}$ so that $b = \varphi(B) = x+1$. Then put $\beta(u) = \alpha(\varphi(u)) = \varphi(u) = \ln u$ so that $\beta'(u) = \varphi'(u) = 1/u$ and $g(u) = h(\varphi(u)) = \sin u$. We then have by Theorem 6.19 and Theorem 6.17

$$f(x) = \int_x^{x+1} \sin(e^t)\, dt = \int_a^b h(t)\alpha'(t)\, dt = \int_a^b h\, d\alpha = \int_A^B g\, d\beta = \int_A^B g(u)\beta'(u)\, du = \int_{e^x}^{e^{x+1}} \frac{\sin u}{u}\, du$$

Now let $F(u) = 1/u$ so that $F'(u) = -1/u^2$ and $G(u) = -\cos u$ so that $G'(u) = \sin u$. Then by Theorem 6.22 we have

$$\begin{aligned} f(x) = \int_{e^x}^{e^{x+1}} \frac{\sin u}{u}\, du &= \int_{e^x}^{e^{x+1}} F(u)G'(u)\, du \\ &= F(e^{x+1})G(e^{x+1}) - F(e^x)G(e^x) - \int_{e^x}^{e^{x+1}} F'(u)G(u)\, du \\ &= -\cos(e^{x+1})e^{-(x+1)} + \cos(e^x)e^{-x} - \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2}\, du\,. \end{aligned}$$

It then follows that

$$
\begin{aligned}
|f(x)| &\le |\cos(e^{x+1})e^{-(x+1)}| + |\cos(e^x)e^{-x}| + \left| \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2}\, du \right| \\
&\le |\cos(e^{x+1})|e^{-(x+1)} + |\cos(e^x)|e^{-x} + \int_{e^x}^{e^{x+1}} \frac{|\cos u|}{u^2}\, du \\
&\le e^{-(x+1)} + e^{-x} + \int_{e^x}^{e^{x+1}} \frac{|\cos u|}{u^2}\, du \\
&< e^{-(x+1)} + e^{-x} + \int_{e^x}^{e^{x+1}} \frac{1}{u^2}\, du \\
&= e^{-(x+1)} + e^{-x} + \left(\frac{-1}{u} \right) \Bigg|_{e^x}^{e^{x+1}} \\
&= e^{-(x+1)} + e^{-x} - e^{-(x+1)} + e^{-x} = 2e^{-x}
\end{aligned}
$$

so that $e^x|f(x)| < 2$ since $e^x > 0$.

Now let

$$
r(x) = -e^x \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2}\, du
$$

so that from the above we have

$$
e^x f(x) = \cos(e^x) - e^{-1}\cos(e^{x+1}) + r(x)
$$

Now let $F(u) = 1/u^2$ so that $F'(u) = -2/u^3$ and $G(u) = \sin u$ so that $G'(u) = \cos u$. Then by Theorem 6.22 we have

$$
\begin{aligned}
r(x) &= -e^x \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2}\, du = -e^x \int_{e^x}^{e^{x+1}} F(u)G'(u)\, du \\
&= -e^x \left(F(e^{x+1})G(e^{x+1}) - F(e^x)G(e^x) - \int_{e^x}^{e^{x+1}} F'(u)G(u)\, du \right) \\
&= -e^x e^{-2(x+1)} \sin(e^{x+1}) + e^x e^{-2x} \sin(e^x) + e^x \int_{e^x}^{e^{x+1}} \frac{-2\sin u}{u^3}\, du \\
&= e^{-x}\sin(e^x) - e^{-x-2}\sin(e^{x+1}) - 2e^x \int_{e^x}^{e^{x+1}} \frac{\sin u}{u^3}\, du\,.
\end{aligned}
$$

It then follows that

$$\begin{aligned}
|r(x)| &\le |e^{-x}\sin(e^x)| + |e^{-x-2}\sin(e^{x+1})| + \left|2e^x \int_{e^x}^{e^{x+1}} \frac{\sin u}{u^3}\,du\right| \\
&= e^{-x}|\sin(e^x)| + e^{-x-2}|\sin(e^{x+1})| + 2e^x \left|\int_{e^x}^{e^{x+1}} \frac{\sin u}{u^3}\,du\right| \\
&\le e^{-x} + e^{-x-2} + 2e^x \int_{e^x}^{e^{x+1}} \frac{|\sin u|}{u^3}\,du \\
&< e^{-x} + e^{-x-2} + 2e^x \int_{e^x}^{e^{x+1}} \frac{1}{u^3}\,du \\
&= e^{-x} + e^{-x-2} + 2e^x \left(\frac{-1}{2u^2}\right)\Bigg|_{e^x}^{e^{x+1}} = e^{-x} + e^{-x-2} - e^x(e^{-2(x+1)} - e^{-2x}) \\
&= e^{-x} + e^{-x-2} - e^{-x-2} + e^{-x} = 2e^{-x}\,.
\end{aligned}$$

145. Exercise 15: Suppose f is a real, continuously differentiable function on $[a,b]$, $f(a) = f(b) = 0$, and

$$\int_a^b f^2(x)\,dx = 1.$$

Prove that

$$\int_a^b xf(x)f'(x)\,dx = -\frac{1}{2}$$

and that

$$\int_a^b f'^2(x)\,dx \cdot \int_a^b x^2f^2(x)\,dx > \frac{1}{4}.$$

Solution: (*analambanomenos*)
Applying integration by parts, applied to the functions f^2 and 1, we get

$$1 = \int_a^b f^2(x)\,dx = bf^2(b) - af^2(a) - \int_a^b 2xf(x)f'(x)\,dx = -2\int_a^b xf(x)f'(x)\,dx$$

which yields the first assertion. In the solution to Exercise 10 I showed that

$$\left(\int_a^b F^2\,dx\right)^{1/2}\left(\int_a^b G^2\,dx\right)^{1/2} \ge \int_a^b |FG|\,dx$$

with equality only if F^2 is a multiple of G^2. Letting $F(x) = f'(x)$ and $G(x) = xf(x)$, then

$$\left(\int_a^b f'^2(x)\,dx\right)^{1/2}\left(\int_a^b x^2f^2(x)\,dx\right)^{1/2} \ge \int_a^b \big|xf(x)f'(x)\big|\,dx \ge \left|\int_a^b xf(x)f'(x)\,dx\right| = \frac{1}{2}.$$

Squaring both sides almost yields the second assertion, with $\ge$ instead of $>$. To get the strict inequality, note that the equality only holds if $f'^2(x)$ is a multiple of $x^2f^2(x)$. Since $\int_a^b f^2(x)\,dx = 1$, the continuous function $f(x)$ cannot have the constant value 0. Since $f(a) = f(b) = 0$, there are points c, d such that $a \le c < d \le b$, $f(c) = f(d) = 0$ and $f(x) \ne 0$ for $c < x < d$. Then by Theorem 5.10 there is a point x between c and d such that $f'(x) = 0$. If equality were to hold above, then we would have $f'^2(x) = 0 = x^2f^2(x) \ne 0$, which is impossible.

146. Exercise 16: For $1 < s < \infty$, define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{(Riemann's zeta function).}$$

Prove that

$$\text{(a)} \quad \zeta(s) = s \int_1^{\infty} \frac{[x]}{x^{s+1}}\, dx$$

$$\text{(b)} \quad \zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{x-[x]}{x^{s+1}}\, dx,$$

where $[x]$ denotes the greatest integer $\leq x$.

Prove that the integral in (b) converges for all $s > 0$.

Solution: (*analambanomenos*)
(a) We have

$$\begin{aligned}
s \int_1^N \frac{[x]}{x^{s+1}}\, dx &= s \sum_{n=1}^{N-1} n \int_n^{n+1} x^{-s-1}\, dx \\
&= \sum_{n=1}^{N-1} n\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) \\
&= 1\left(\frac{1}{1^s}\right) + \left(\sum_{n=2}^{N-1} (n-(n-1))\frac{1}{n^s}\right) - \frac{N-1}{N^s} \\
&= \left(\sum_{n=1}^{N} \frac{1}{n^s}\right) - \frac{1}{N^s} - \frac{N-1}{N^s} \\
&= \left(\sum_{n=1}^{N} \frac{1}{n^s}\right) - \frac{1}{N^{s-1}}
\end{aligned}$$

Taking the limit as $N \to \infty$, we get the desired result.

(b) From the work done in part (a), we get

$$\begin{aligned}
\frac{s}{s-1} - s\int_1^N \frac{x-[x]}{x^{s+1}}\, dx &= \frac{s}{s-1} - s\int_1^N x^{-s}\, dx + s\int_1^N \frac{[x]}{x^{s+1}}\, dx \\
&= \frac{s}{s-1} + \frac{s}{s-1}\left(\frac{1}{N^{s-1}} - 1\right) + \left(\sum_{n=1}^{N} \frac{1}{n^s}\right) - \frac{1}{N^{s-1}} \\
&= \frac{1}{s-1}\left(\frac{1}{N^{s-1}}\right) + \left(\sum_{n=1}^{N} \frac{1}{n^s}\right)
\end{aligned}$$

Taking the limit as $N \to \infty$, we get the desired result.

Note that

$$0 < \int_1^b \frac{x-[x]}{x^{s+1}}\, dx < \int_1^b \frac{1}{x^{s+1}}\, dx,$$

and the integral on the right converges as $b \to \infty$ for all $s > 0$ by the integral test of Exercise 8. Hence the integral in part (b) also converges as $b \to \infty$ for all $s > 0$.

147. Exercise 17: Suppose α increases monotonically on $[a,b]$, g is continuous, and that $g(x) = G'(x)$ for $a \le x \le b$. Prove that

$$\int_a^b \alpha(x)g(x)\,dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G\,d\alpha.$$

Solution: (*analambanomenos*)
Following the hint, let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. We may assume that g is real-valued. By the mean-value theorem, Theorem 5.10, each interval (x_{i-1}, x_i) contains t_i such that $G(x_i) - G(x_i) = g(t_i)\Delta x_i$. Hence we have

$$\begin{aligned}
&\sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i \\
&\quad = \alpha(x_1)G(x_1) - \alpha(x_1)G(x_0) + \alpha(x_2)G(x_2) - \alpha(x_2)G(x_1) + \cdots + \alpha(x_n)G(x_n) - \alpha(x_n)G(x_{n-1}) \\
&\quad = -G(x_0)\alpha(x_0) - G(x_0)\big(\alpha(x_1) - \alpha(x_0)\big) - \cdots - G(x_{n-1})\big(\alpha(x_n) - \alpha(x_{n-1})\big) + G(x_n)\alpha(x_n) \\
&\quad = G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1} G(x_{i-1})\Delta\alpha_i.
\end{aligned}$$

By Theorem 6.10, $\alpha g \in \mathscr{R}$, and by Theorem 6.8, $G \in \mathscr{R}(\alpha)$. Hence as the partition becomes finer, the sum on the right tends to $\int \alpha g\,dx$, and the sum on the left tends to $\int G\,d\alpha$.

148. Exercise 18: Let $\gamma_1, \gamma_2, \gamma_3$ be curves in the complex plane, defined on $[0, 2\pi]$ by

$$\gamma_1(t) = e^{it}, \qquad \gamma_2(t) = e^{2it}, \qquad \gamma_3(t) = e^{2\pi it\sin(1/t)}.$$

Show that these three curves have the same range, that γ_1 and γ_2 are rectifiable, that the length of γ_1 is 2π, that the length of γ_2 is 4π, and that γ_3 is not rectifiable.

Solution: (*analambanomenos*)
Since for any $x \in \mathbf{R}$ we have $|e^{ix}| = |\cos x + i\sin x| = \cos^2 x + \sin^2 x = 1$, the images of all three curves lie on the unit circle S^1. And if $z = \cos x + i\sin x$ is an arbitrary point of S^1, $0 \le x < 2\pi$, then $z = \gamma_1(x) = \gamma_2(x/2)$, so the images of γ_1 and γ_2 are all of S^1. Also, since

$$2\pi(\pi)\sin\left(\frac{1}{\pi}\right) \approx 6.18 \qquad 2\pi\left(\frac{2}{3\pi}\right)\sin\left(\frac{3\pi}{2}\right) \approx -1.33,$$

by the intermediate-value theorem (Theorem 4.23) if $-1.33 \le y \le 6.18$, there is a $0 \le t \le 2\pi$ such that $2\pi t\sin(1/t) = y$, so that $\gamma_3(t) = e^{iy} = z$. Hence the image of γ_3 is also all of S^1.

Since $\big|\gamma_1(t)\big| = |ie^{it}| = 1$ and $\big|\gamma_2(t)\big| = |2ie^{2it}| = 2$, by Theorem 6.27 we have

$$\Lambda(\gamma_1) = \int_0^{2\pi} 1\,dt = 2\pi \qquad \Lambda(\gamma_2) = \int_0^{2\pi} 2\,dt = 4\pi.$$

For γ_3 we have

$$\begin{aligned}
\gamma_3'(t) &= 2\pi i\big(\sin(t^{-1}) - t^{-1}\cos(t^{-1})\big)\gamma_3(t) \\
\big|\gamma_3'(t)\big| &= 2\pi\big|\sin(t^{-1}) - t^{-1}\cos(t^{-1})\big|
\end{aligned}$$

Let $a_n = (2n+1)\pi$, $b_n = (2n+1/2)\pi$ for any integer $n \geq 1$. Then $[a_n^{-1}, b_n^{-1}]$ is a subinterval of $[0, 2\pi]$ on which $\sin(1/t) \geq 0$ and $\cos(1/t) \leq 0$. Hence the length of γ_3 on this subinterval is

$$\begin{aligned}
\int_{a_n^{-1}}^{b_n^{-1}} \left|\gamma_3'(t)\right| dt &= 2\pi \int_{a_n^{-1}}^{b_n^{-1}} \sin(t^{-1}) - t^{-1}\cos(t^{-1})\, dt \\
&= 2\pi \int_{a_n}^{b_n} -u^{-2}\sin(u) + u^{-1}\cos(u)\, du \\
&= 2\pi\bigl(b_n^{-1}\sin(b_n) - a_n^{-1}\sin(a_n)\bigr) \\
&= \frac{1}{n+(1/4)}
\end{aligned}$$

Since $\Lambda(\gamma_3)$ must be larger than the divergent sum of such terms, γ_3 is not rectifiable.

149. Exercise 19: Let γ_1 be a curve in $\mathbf{R}^k$, defined on $[a,b]$, let ϕ be a continuous one-to-one mapping of $[c,d]$ onto $[a,b]$ such that $\phi(c) = a$, and define $\gamma_2(s) = \gamma_1\bigl(\phi(s)\bigr)$. Prove that γ_2 is an arc, a closed curve, or a rectifiable curve if and only if the same is true of γ_1. Prove that γ_2 and γ_1 have the same length.

Solution: (*analambanomenos*)
Suppose γ_1 is an arc. Since the composition of one-to-one mappings is clearly one-to-one, γ_2 is also an arc.

Note that ϕ is strictly monotonically increasing. For if not, there are points $c < x_1 < x_2 < d$ such that $\phi(c) = a < \phi(x_2) < \phi(x_1)$. by the intermediate-value theorem (Theorem 4.23) there is a point x_3, $c < x_3 < x_1$ such that $\phi(x_3) = \phi(x_2)$, contradicting the fact that ϕ is one-to-one. Hence $\phi(d) = b$, so if γ_1 is a closed curve, then $\gamma_2(d) = \gamma_1\bigl(\phi(d)\bigr) = \gamma_1(b) = \gamma_1(a) = \gamma_1\bigl(\phi(c)\bigr) = \gamma_2(c)$, so that γ_2 is also closed.

Suppose that γ_1 is rectifiable. Let $P = \{x_0 = a < x_2 < \cdots < x_n = c\}$ be a partition of $[a,b]$. Then since ϕ is strictly monotonically increasing, $\phi^{-1}(P)$ is a partition of $[c,d]$, and we have

$$\begin{aligned}
\Lambda\bigl(\phi^{-1}(P), \gamma_2\bigr) &= \sum_{i=1}^{n} \Bigl|\gamma_2\bigl(\phi^{-1}(x_i)\bigr) - \gamma_2\bigl(\phi^{-1}(x_{i-1})\bigr)\Bigr| \\
&= \sum_{i=1}^{n} \bigl|\gamma_1(x_i) - \gamma_1(x_{i-1})\bigr| \\
&= \Lambda(P, \gamma_1).
\end{aligned}$$

Hence γ_2 is rectifiable if and only if γ_1 is rectifiable, in which case they have the same length.

7 Sequences and Series of Functions

150. Exercise 1: Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Solution: (*analambanomenos*)
Let $\{f_n\}$ be a uniformly convergent sequence of bounded functions on a set E. For each n, there is a number M_n such that $\bigl|f_n(x)\bigr| < M_n$ for all $x \in E$. By Theorem 7.8, there is an integer N such that $\bigl|f_n(x) - f_N(x)\bigr| < 1$ if $n \geq N$ for all $x \in E$. Let $M = \max\{M_1, \ldots, M_N\}$. Then for $n \leq N$ and

$x \in E$ we have $|f_n(x)| < M+1$, and for $n \geq N$ and $x \in E$ we have

$$|f_n(x)| \leq |f_n(x) - f_N(x)| + |f_N(x)| < M+1.$$

That is, the f_n are uniformly bounded by $M+1$ in E.

151. Exercise 2: If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.

Solution: (*analambanomenos*)
Let $\varepsilon > 0$. By Theorem 7.8 there is an integer N such that for all $n, m \geq N$ and $x \in E$,

$$|f_n(x) - f_m(x)| < \varepsilon/2 \qquad |g_n(x) - g_m(x)| < \varepsilon/2.$$

Hence for all $n, m \geq N$ and $x \in E$,

$$|(f_n + g_n)(x) - (f_m + g_m)(x)| \leq |f_n(x) - f_m(x)| + |g_n(x) - g_m(x)| < \varepsilon.$$

Hence, also by Theorem 7.8, $\{f_n + g_n\}$ converges uniformly on E.

If $\{f_n\}$ and $\{g_n\}$ are uniformly convergent sequences of bounded functions, then by Exercise 1 they are uniformly bounded. That is, there is a number M such that $|f_n(x)| < M$ and $|g_n(x)| < M$ for all n and for all $x \in E$. By Theorem 7.8 there is an integer N such that for all $n \geq N$ and for all $x \in E$ we have

$$|f_n(x) - f_m(x)| < \varepsilon/M \qquad |g_n(x) - g_m(x)| < \varepsilon/M.$$

Hence for all $n, m \geq N$ and $x \in E$,

$$\begin{aligned} |f_n(x)g_n(x) - f_m(x)g_m(x)| &\leq |f_n(x)g_n(x) - f_m(x)g_n(x)| + |f_m(x)g_n(x) - f_m(x)g_m(x)| \\ &\leq M|f_n(x) - f_m(x)| + M|g_n(x) - g_m(x)| \\ &\leq \varepsilon \end{aligned}$$

Hence, also by Theorem 7.8, $\{f_n g_n\}$ converges uniformly on E.

152. Exercise 3: Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).

Solution: (*analambanomenos*)
Let $f_n(x) = x^{-1} + n^{-1}$ on $(0,1)$. Then f_n converges uniformly to x^{-1} on $(0,1)$, and

$$f_n^2(x) = \frac{1}{x^2} + \frac{2}{nx} + \frac{1}{n^2}$$

converges to x^{-2} on $(0,1)$, but not uniformly, since the difference $f_n^2(x) - x^{-2} = 2/(nx) + 1/n^2$ is arbitrarily large as $x \to 0$.

153. Exercise 4: Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2 x}.$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly?

On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

Solution: (*analambanomenos*)
Let $f_n(x)$ be the n^{th} term of the series. Since $f_n(0) = 1$ for all n, the series diverges for $x = 0$. If $x > 0$, then

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x} < \frac{1}{x}\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$$

by Theorem 3.28, so the series converges if $x > 0$. The convergence is absolute since $f_n(x) > 0$ for $x > 0$, and since $f_n(x) \to \infty$ as $x \to 0+$, $f(x)$ is not bounded on $(0, \infty)$. If $a > 0$, then since $f_n(x) \le 1/(an^2)$ for $a \le x$ and all n, the series converges uniformly to $f(x)$ on $[a, \infty)$ by Theorem 7.10. Since the partial sums are continuous, $f(x)$ is continuous on $[a, \infty)$ by Theorem 7.12, so it is continuous on all of $(0, \infty)$. The series does not converge uniformly on $(0, \infty)$ since the difference between $f(x)$ and a partial sum is a series $\sum_N^\infty f_n(x) > 1/(1 + N^2x)$, which is arbitrarily large as $x \to 0+$.

The case $x < 0$ is more complicated. First note that $f_n(x)$ is not even defined for $x = -n^{-2}$, so we are limited to considering the intervals $(-\infty, -1)$ and $\big(-n^{-2}, -(n+1)^{-2}\big)$ for $n = 1, 2, \dots$ For $-n^{-2} < x < -(n+1)^{-2}$ and $m > n+1$ we have

$$\frac{1}{1-\big(m/(n+1)\big)^2} < f_m(x) < \frac{1}{1-(m/n)^2} < 0$$

so that

$$\big|f_m(x)\big| < \frac{1}{\big(m/(n+1)\big)^2 - 1}.$$

Similarly, for $x < -1$ and $m > 1$, we get

$$\big|f_m(x)\big| < \frac{1}{m^2-1},$$

which is the same inequality, with $n = 0$. The series

$$\sum_{m=n+2}^{\infty} \frac{1}{\big(m/(n+1)\big)^2 - 1}$$

converges, which can be shown either by the integral test or the "limit comparison test," which was not covered in Rudin's text. Hence the series converges uniformly and absolutely for $-n^{-2} < x < (n+1)^{-2}$, or $x < -1$ by Theorem 7.10. Since the partial sums are continuous, $f(x)$ is continuous on this interval by Theorem 7.12. It is not bounded, since $f_n(x) \to -\infty$ as $x \to -n^{-2}$, while the rest of the series (everything except the n^{th} term) sums to a finite value.

154. Exercise 5: Let

$$f_n(x) = \begin{cases} 0 & \left(x < \dfrac{1}{n+1}\right), \\ \sin^2 \dfrac{\pi}{x} & \left(\dfrac{1}{n+1} \le x \le \dfrac{1}{n}\right), \\ 0 & \left(\dfrac{1}{n} < x\right). \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x, does not imply uniform convergence.

Solution: (*analambanomenos*)
For $x \leq 0$, we have $f_n(x) = 0$ for all n, and for $x > 0$ and n large enough so that $n^{-1} < x$, we have $f_n(x) = 0$. Hence $f_n(x)$ converges to the constant function $f(x) = 0$. Since for all n there is an x such that $f_n(x) = 1$ (viz. $x = 2/(2n+1)$), for any $0 < \varepsilon < 1$ there is no N such that $\big|f_n(x) - f(x)\big| = \big|f_n(x)\big| < \varepsilon$ for all x and all integers $n > N$, so $\{f_n(x)\}$ does not converge uniformly to $f(x)$.

Let

$$F_N(x) = \sum_{n=1}^{N} f_n(x) = \begin{cases} 0 & \left(x < \dfrac{1}{N+1}\right), \\ \sin^2 \dfrac{\pi}{x} & \left(\dfrac{1}{n+1} \leq x\right). \end{cases}$$

Then $F_N(x)$ converges to

$$F(x) = \begin{cases} 0 & (x \leq 0), \\ \sin^2 \dfrac{\pi}{x} & (0 < x). \end{cases}$$

Since for all N there is a positive number $x < (N+1)^{-1}$ such that $\big|F(x) - F_N(x)\big| = 1$, the convergence is not uniform.

Solution: (*Dan "kyp44" Whitman*)

Lemma 7.5.1: For a sequence of functions $\{f_n\}$ on a set E, if $\sum f_n$ converges uniformly then $f_n \to 0$ uniformly.

Proof: Suppose that $\sum f_n$ converges uniformly and let

$$F_n(x) = \sum_{k=1}^{n} f_k(x)$$

be the partial sums of $\sum f_n$. Consider any $\varepsilon \in \mathbb{R}^+$. Then since $\sum f_n$ converges uniformly it follows from Theorem 7.8 that there is an $N \in \mathbb{Z}^+$ where

$$|F_n(x) - F_m(x)| \leq \varepsilon$$

for every $n \geq N$, $m \geq N$, and $x \in E$. So consider any $n \geq N+1$ and $x \in E$. Then clearly we have

$$|f_n(x) - 0| = |f_n(x)| = \left|\sum_{k=n}^{n} f_k(x)\right| = \left|\sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{n-1} f_k(x)\right| = |F_n(x) - F_{n-1}(x)| \leq \varepsilon$$

since $n \geq N$ and $n-1 \geq N$. Since n and x were arbitrary we've shown by definition that $f_n \to 0$ uniformly. ■

Main Problem

Claim 1: The sequence $\{f_n\}$ converges pointwise to the continuous function $f = 0$.

Proof: Consider any $x \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}^+$.

Case: $x > 0$. Then let

$$N = \left\lceil \frac{1}{x} \right\rceil + 1\,,$$

noting that $N \in \mathbb{Z}^+$ since $1/x > 0$. Consider any $n \geq N$ so that we have

$$\begin{aligned} n \geq N = \left\lceil \frac{1}{x} \right\rceil + 1 > \left\lceil \frac{1}{x} \right\rceil &\geq \frac{1}{x} \\ nx &> 1 && \text{(since } x > 0\text{)} \\ x &> \frac{1}{n}\,. && \text{(since } n \geq N \geq 1 > 0\text{)} \end{aligned}$$

Then by the defintion of f_n we have $f_n(x) = 0$.

Case: $x \leq 0$. Then let $N = 1$ and consider any $n \geq N$. Then

$$\begin{aligned} 0 &< 1 \\ 0 &< \frac{1}{n+1} && \text{(since } n+1 > n \geq N = 1 > 0\text{)} \\ x \leq 0 &< \frac{1}{n+1}\,. \end{aligned}$$

Hence by the definition of f_n it follows that $f_n(x) = 0$.

Thus in all cases there is an $N \in \mathbb{Z}^+$ where $f_n(x) = 0$ for all $n \geq N$. Clearly then for any such n we have

$$|f_n(x) - f(x)| = |0 - 0| = |0| = 0 < \varepsilon$$

so that $f_n \to f = 0$ since x was arbitrary. ■

Claim 2: $\{f_n\}$ does *not* converge uniformly.

Proof: Let $\varepsilon = 1/2$ and consider any $N \in \mathbb{Z}^+$. Let $n = N$ and $m = N + 1$ so that clearly $n \geq N$ and $m \geq N$. Then let

$$x = \frac{2}{2n+1}$$

We then have that

$$\begin{aligned} 0 &\leq 1 \\ 2n &\leq 2n+1 \\ \frac{2n}{2n+1} &\leq 1 && \text{(since } 2n+1 > 0\text{)} \\ \frac{2}{2n+1} &\leq \frac{1}{n} && \text{(since } n > 0\text{)} \\ x &\leq \frac{1}{n}\,. \end{aligned}$$

Similarly we have

$$\begin{aligned} 1 &< 2 \\ 2n+1 &< 2n+2 \\ 2n+1 &< 2(n+1) \\ \frac{2n+1}{n+1} &< 2 && \text{(since } 2n+2 > 0\text{)} \\ \frac{1}{n+1} &< \frac{2}{2n+1} && \text{(since } 2n+1 > 0\text{)} \\ \frac{1}{n+1} &< x \\ \frac{1}{m} &< x\,. \end{aligned}$$

Thus by definition $f_m(x) = 0$ whereas $x \in [1/(n+1), 1/n]$ so that

$$f_n(x) = \sin^2 \frac{\pi}{x} = \sin^2 \frac{\pi(2n+1)}{2} = (\pm 1)^2 = 1$$

since $2n+1$ is odd. Hence we have

$$|f_n(x) - f_m(x)| = |1 - 0| = |1| = 1 > \frac{1}{2} = \varepsilon$$

Thus we have succeeded in showing that

$$\begin{aligned} &\exists \varepsilon \in \mathbb{R}^+ \forall N \in \mathbb{Z}^+ \exists n \geq N \exists m \geq N \exists x \in \mathbb{R} (|f_n(x) - f_m(x)| > \varepsilon) \\ &\equiv \neg \forall \varepsilon \in \mathbb{R}^+ \exists N \in \mathbb{Z}^+ \forall n \geq N \forall m \geq N \forall x \in \mathbb{R} (|f_n(x) - f_m(x)| \leq \varepsilon) \,, \end{aligned}$$

Which shows that $\{f_n\}$ does not converge uniformly by Theorem 7.8. ■

Claim 3: $\sum f_n$ converges absolutely (pointwise) on the entirety of $\mathbb{R}$.

Proof: Consider any $x \in \mathbb{R}$.

Case: $x > 1$. Consder any $n \in \mathbb{Z}^+$. Then

$$\begin{aligned} 1 &< x \\ \frac{1}{x} &< 1 \leq n && \text{(since } x > 1 > 0\text{)} \\ 1 &< nx && \text{(since } x > 1 > 0\text{)} \\ \frac{1}{n} &< x && \text{(since } n \geq 1 > 0\text{)} \end{aligned}$$

so that by definition $f_n(x) = 0$. Since n was arbitary this clearly shows that $\sum |f_n(x)| = \sum |0| = 0$.

Case: $x \leq 0$. Again consider any $n \in \mathbb{Z}^+$. Then

$$\begin{aligned} 0 &< 1 \\ 0 &< \frac{1}{n+1} && \text{(since } n+1 > 0\text{)} \\ x \leq 0 &< \frac{1}{n+1} \end{aligned}$$

so that by definition we again have $f_n(x) = 0$. Likewise since n was arbitary it shows that $\sum |f_n(x)| = \sum |0| = 0$.

Case: $0 < x \leq 1$. Then let

$$N = \left\lfloor \frac{1}{x} \right\rfloor ,$$

noting that $N \in \mathbb{Z}^+$ since $1/x \geq 1$ since $x \leq 1$ and $x > 0$. Then $N \leq 1/x < N+1$ so that

$$\begin{aligned} N &\leq \frac{1}{x} \\ Nx &\leq 1 && \text{(since } x > 0\text{)} \\ x &\leq \frac{1}{N} && \text{(since } N > 0\text{)} \end{aligned}$$

and

$$\begin{aligned}
\frac{1}{x} &< N+1 \\
1 &< x(N+1) && \text{(since } x > 0\text{)} \\
\frac{1}{N+1} &< x && \text{(since } N+1 > 0\text{)}\,.
\end{aligned}$$

Hence $x \in [1/(N+1), 1/N]$ so that by definition

$$f_N(x) = \sin^2 \frac{\pi}{x}\,,$$

which is clearly just a positive real number.

Now consider any $n \in \mathbb{Z}^+$ where $n \neq N$. If $n < N$ then

$$\begin{aligned}
n+1 &\leq N \\
\frac{n+1}{N} &\leq 1 && \text{(since } N > 0\text{)} \\
\frac{1}{N} &\leq \frac{1}{n+1} && \text{(since } N > 0\text{)} \\
x \leq \frac{1}{N} &\leq \frac{1}{n+1}\,.
\end{aligned}$$

Then if $x < 1/(n+1)$ by definition $f_n(x) = 0$. If we have $x = 1/(n+1)$ then

$$f_n(x) = \sin^2 \frac{\pi}{x} = \sin^2 \pi(n+1) = 0$$

since $n+1$ is an integer. On the other hand if $n > N$ then we have

$$\begin{aligned}
N+1 &\leq n \\
\frac{N+1}{n} &\leq 1 && \text{(since } n > 0\text{)} \\
\frac{1}{n} &\leq \frac{1}{N+1} && \text{(since } N+1 > 0\text{)} \\
\frac{1}{n} &\leq \frac{1}{N+1} < x
\end{aligned}$$

so that by again definition $f_n(x) = 0$.

Putting this all together we have

$$\begin{aligned}
\sum_{n=1}^{\infty} |f_n(x)| &= \sum_{n=1}^{N-1} |f_n(x)| + \sum_{n=N}^{N} |f_n(x)| + \sum_{n=N+1}^{\infty} |f_n(x)| \\
&= \sum_{n=1}^{N-1} |0| + |f_N(x)| + \sum_{n=N+1}^{\infty} |0| \\
&= 0 + \left|\sin^2 \frac{\pi}{x}\right| + 0 \\
&= \sin^2 \frac{\pi}{x}\,.
\end{aligned}$$

Thus in all cases clearly $\sum f_n(x)$ converges absolutely. ■

Claim 4: $\sum f_n$ does *not* converge uniformly.

Proof: This follows immediately from the contrapositive of Lemma 7.5.1 since we have shown that $f_n \to 0$ but not uniformly. ■

155. Exercise 6: Prove that the series
$$\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$$
converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

Solution: (*analambanomenos*)
Since
$$\lim_{n\to\infty}\frac{x^2+n}{n^2}=0,$$
the alternating series converges for all x by Theorem 3.43. It doesn't converge absolutely for any x since
$$\sum_{n=1}^{\infty}\frac{x^2+n}{n^2}\geq\sum_{n=1}^{\infty}\frac{1}{n}$$
which diverges. The partial sums are
$$f_m(x)=\sum_{n=1}^{m}(-1)^n\frac{x^2+n}{n^2}=x^2\sum_{n=1}^{m}\frac{(-1)^n}{n^2}+\sum_{n=1}^{m}\frac{(-1)^n}{n}=A_mx^2+B_m\to Ax^2+B.$$
Let $\varepsilon>0$ and let N be large enough so that $|A-A_n|<\varepsilon$ and $|B-B_n|<\varepsilon$ for all $n>N$. Let $[a,b]$ be a bounded interval, and let $c=\max\bigl(|a|,|b|\bigr)$. Then for $n>N$ and $x\in[a,b]$ we have $\bigl|f(x)-f_n(x)\bigr|<c^2\varepsilon+\varepsilon$, so that $\{f_n\}$ converges uniformly to $f(x)$ on $[a,b]$.

156. Exercise 7: For $n=1,2,3,\ldots$, x real, put
$$f_n(x)=\frac{x}{1+nx^2}.$$
Show that $\{f_n\}$ converges uniformly to a function f, and that the equation
$$f'(x)=\lim_{n\to\infty}f_n'(x)$$
is correct if $x\neq 0$, but false if $x=0$.

Solution: (*analambanomenos*)
Calculating the minimum and maximum values of f_n using elementary calculus, we get that
$$\bigl|f_n(x)\bigr|\leq\frac{1}{2\sqrt{n}},$$
so that $f_n(x)$ converges uniformly to the constant function $f(x)=0$ on $\mathbf{R}$. For $x\neq 0$,
$$f_n'(x)=\frac{1-nx^2}{(1+nx^2)^2}$$
converges to $f'(x)=0$, but $f_n'(0)=1$ does not.

157. Exercise 8: If
$$I(x)=\begin{cases}0 & (x\leq 0),\\ 1 & (x>0),\end{cases}$$
if $\{x_n\}$ is sequence of distinct points of (a,b), and if $\sum|c_n|$ converges, prove that the series
$$f(x)=\sum_{n=1}^{\infty}c_nI(x-x_n)\qquad(a\leq x\leq b)$$

converges uniformly, and that f is continuous for every $x \neq x_n$.

Solution: (*analambanomenos*)
Let $f_n(x)$ be the partial sum $\sum_{k=1}^n c_k I(x - x_k)$. Then

$$\left|f_n(x) - f_m(x)\right| \leq \sum_{k=m+1}^{n} \left|c_k I(x - x_k)\right| \leq \sum_{k=m+1}^{n} |c_k|.$$

Since $\sum |c_k|$ converges, by Theorem 3.22 for any $\varepsilon > 0$ there is an integer N such that for $m \geq N$ and $n \geq N$ we have

$$\left|f_n(x) - f_m(x)\right| \leq \sum_{k=m+1}^{n} |c_k| < \varepsilon.$$

Hence by Theorem 7.8 the series converges uniformly.

Let $x \in (a, b)$ such that $x \neq x_n$ for any n. Then the partial sum f_n is constant in any neighborhood of x not containing any of $x_1, \ldots, x_n$. Hence by Theorem 7.11,

$$\lim_{t \to x} f(x) = \lim_{n \to \infty} f_n(x) = f(x),$$

that is, f is continuous at x.

158. Exercise 9: Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that

$$\lim_{n \to \infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$. Is the converse of this true?

Solution: (*analambanomenos*)
The problem didn't state it explicitly, but in order to apply the Chapter's Theorems let's assume that E is a set in a metric space.

Let $\varepsilon > 0$. Since $\{f_n\}$ converges uniformly to f, there is an integer N_1 such that if $n \geq N_1$ then for any $y \in E$ we have

$$\left|f_n(y) - f(y)\right| < \frac{\varepsilon}{2}.$$

Since f is continuous by Theorem 7.12, there is an integer N_2 such that if $n \geq N_2$ then

$$\left|f(x_n) - f(x)\right| < \frac{\varepsilon}{2}.$$

Hence, if $n \geq \max(N_1, N_2)$, we have

$$\left|f_n(x_n) - f(x)\right| \leq \left|f_n(x_n) - f(x_n)\right| + \left|f(x_n) - f(x)\right| < \varepsilon.$$

The converse is not true. For example, let $E = [0, 1)$ and let $f_n(x) = x^n$. Then this sequence converges to the constant function $f(x) = 0$, but not uniformly. If $\{x_n\}$ is a sequence of points of E converging to a point x in E, then it is contained in a closed subinterval $[0, a]$ where $a < 1$. Since f_n converges uniformly on this subinterval, $f_n(x_n)$ converges to $f(x)$ by the first part of this problem. Hence this property is insufficient for determining that the convergence is uniform.

159. Exercise 10: Letting (x) denote the fractional part of the real number x, consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2} \qquad (x \text{ real}).$$

Find all discontinuities of f, and show that they form a countable dense set. Show that f is nevertheless Riemann-integrable on every bounded interval.

Solution: (*analambanomenos*)
The discontinuities of the n^{th} term, $f_n(x) = (nx)/n^2$, occur at the rational numbers m/n for any integer m. Let $y = p/q$ be a rational number where p and q have no common divisors. Then f_n is discontinuous at y if and only if n is a multiple of q. Let $g_q = \sum_m (mqx)/(mq)^2$, which is discontinuous at y, and let $h_q = f - g_q$. Then the partial sums of h_q are all continuous at y, hence by Theorem 7.11, h_q is also continuous at y, and so $f = g_q + h_q$ is discontinuous at y. If the real number z is irrational, then the partial sums of f are all continuous at z, hence by again applying Theorem 7.11 we have f is continuous at z. Therefore the discontinuities of f are the rational numbers, a countable dense subset of the real numbers.

The partial sums of f converge uniformly to f and are all Riemann-integrable on bounded intervals. Hence by Theorem 7.16, f is also Riemann-integrable on bounded intervals.

160. Exercise 11: Suppose that $\{f_n\}$, $\{g_n\}$ are defined on E, and

(a) $\sum f_n$ has uniformly bounded partial sums;

(b) $g_n \to 0$ uniformly on E;

(c) $g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots$ for every $x \in E$.

Prove that $\sum f_n g_n$ converges uniformly on E.

Solution: (*analambanomenos*)
Theorem 3.42 gives us pointwise convergence of the series. Uniform convergence follows by repeating the proof of that Theorem in this context, as follows.

Let $F_n(x)$ be the partial sums of $\sum f_n(x)$. Choose M such that $\left|F_n(x)\right| < M$ for all n and all $x \in E$. Given $\varepsilon > 0$ there is an integer N such that $g_N(x) \leq \varepsilon/(2M)$ for all $x \in E$. For $N \leq p \leq q$ we have by Theorem 3.41

$$\begin{aligned}
\left|\sum_{n=p}^{q} f_n(x)g(x)\right| &= \left|\sum_{n=p}^{q-1} F_n(x)\big(g_n(x) - g_{n+1}(x)\big) + F_q(x)g_q(x) - F_{p-1}(x)g_p(x)\right| \\
&\leq M\left|\sum_{n=p}^{q-1}\big(g_n(x) - g_{n+1}(x)\big) + g_q(x) + g_p\right| \\
&= 2Mg_p(x) \leq 2Mg_N(x) \leq \varepsilon.
\end{aligned}$$

Uniform convergence follows from Theorem 7.8 (the Cauchy condition for uniform convergence).

161. Exercise 12: Suppose $\{f_n\}$, $\{g_n\}$ are defined on $(0, \infty)$, are Riemann-integrable on $[t, T]$ whenever $0 < t < T < \infty$, $|f_n| \leq g$, $f_n \to f$ uniformly on every compact subset of $(0, \infty)$, and

$$\int_0^{\infty} g(x)\,dx < \infty.$$

Prove that

$$\lim_{n\to\infty}\int_0^\infty f_n(x)\,dx = \int_0^\infty f(x)\,dx.$$

Solution: (*analambanomenos*)
Fix $t > 0$ and let $k(T) = \int_t^T \big|f(x)\big|\,dx$. Then $k(T)$ is a monotonically increasing function which is bounded by $\int_t^\infty g(x)\,dx$. Hence by Theorem 3.14 (that theorem refers to sequences, but can be easily extended to this case), $k(T)$ converges as $T \to \infty$, and since $\big|\int_t^T f(x)\,dx\big| \le k(T)$, $\int_t^\infty f(x)\,dx$ converges. Similarly, $\int_t^\infty f_n(x)\,dx$ also converges.

For $t > 0$ define the functions

$$h_n(t) = \int_t^\infty f_n(x)\,dx \qquad h(t) = \int_t^\infty f(x)\,dx.$$

The problem is to show that

$$\lim_{n\to\infty}\left(\lim_{t\to 0} h_n(t)\right) = \lim_{t\to 0} h(t).$$

By Theorem 7.11, this follows if we can show that $h_n(t)$ converges uniformly to $h(t)$ on $(0,\infty)$.

Let $\varepsilon > 0$. Since $\int_0^\infty g(x)\,dx$ converges, there are numbers $0 < T_1 < T_2 < \infty$ such that

$$\int_0^{T_1} g(x)\,dx < \frac{\varepsilon}{2} \qquad \int_{T_2}^\infty g(x)\,dx < \frac{\varepsilon}{2}.$$

Since f_n converges to f uniformly on $[T_1, T_2]$, there is a positive integer N such that for $n > N$ and $T_1 \le x \le T_2$ we have

$$\big|f_n(x) - f(x)\big| < \frac{\varepsilon}{T_2 - T_1}.$$

Hence, for $0 < t < T_1$ and $n > N$, we have

$$\begin{aligned}
\big|h_n(t) - h(t)\big| &\le \int_t^{T_1} \big|f_n(x) - f(x)\big|\,dx + \int_{T_1}^{T_2} \big|f_n(x) - f(x)\big|\,dx + \int_{T_2}^\infty \big|f_n(x) - f(x)\big|\,dx \\
&\le 2\int_0^{T_1} g(x)\,dx + \frac{\varepsilon}{T_2 - T_1}(T_2 - T_1) + 2\int_{T_2}^\infty g(x)\,dx \\
&< 3\varepsilon.
\end{aligned}$$

Similar arguments yielding similar results can be made for $t \ge T_1$. This shows that $h_n(t)$ converges uniformly to $h(t)$ on $(0,\infty)$.

162. Exercise 13: Assume that $\{f_n\}$ is a sequence of monotonically increasing functions on $\mathbf{R}$ such that $0 \le f_n(x) \le 1$ for all x and all n.

(a) Prove that there is a function f and a sequence $\{n_k\}$ such that for every $x \in \mathbf{R}$

$$f(x) = \lim_{k\to\infty} f_{n_k}(x).$$

(b) If, moreover, f is continuous, prove that $f_{n_k} \to f$ uniformly on compact sets.

Solution: (*analambanomenos, fixed after kyp44 found errors in the original solution*)
(a) Following the steps of the hint: (i) Show that some subsequence $\{f_{n_i}\}$ converges at all rational points r, say, to $f(r)$.

Let $\{r_m\}$ be an enumeration of the rational points of $\mathbf{R}$. Since $\{f_n(r_1)\} \subset [0,1]$, a compact set, there is a subsequence $\{f_{1,n_j}(r_1)\}$ which converges to $f(r_1)$, by Theorem 2.41. Similarly, the $\{f_{1,n_j}(r_2)\}$ has a subsequence $\{f_{2,n_j}(r_2)\}$ which converges to $f(r_2)$. Continuing in this fashion, we get a sequence of subsequences $\{f_{m,n_j}\}$ such that each $\{f_{m,n_j}\}$ is a subsequence of $\{f_{m-1,n_j}\}$ and $f_{m,n_j}(r_m)$ converges to $f(r_m)$. Letting $f_{n_i} = f_{i,n_i}$, we get a subsequence of $\{f_n\}$ which converges on all rational points r of $\mathbf{R}$ to $f(r)$.

(ii) Define $f(x)$, for any $x \in \mathbf{R}$, to be $\sup f(r)$, the sup being taken over all rational points $r \le x$.

First we need to show that this new definition of $f(r)$ agrees with the previous one; let $f^*(r)$ be the new definition of $f(r)$. It is clear from the definition that $f^*(r) \ge f(r)$. Suppose $f^*(r) > f(r)$, that is, there is a rational point $q < r$ and a real number a such that $f(q) > a > f(r)$. Since $f(q) = \lim f_{n_i}(q)$, and $f(r) = \lim f_{n_i}(r)$, there is an integer N such that for all $i \ge N$ we have $f_{n_i}(q) > a > f_{n_i}(r)$, but this contradicts the fact that these functions are monotonically increasing.

It is clear from the definition that f is monotonically increasing.

(iii) Show that $f_{n_i}(x) \to f(x)$ at every x at which f is continuous.

Suppose f is continuous at x and let $\varepsilon > 0$. Then there are rational numbers $r_1 < x < r_2$ such that

$$f(x) - \frac{\varepsilon}{2} < f(r_1) \le f(x) \le f(r_2) < f(x) + \frac{\varepsilon}{2}.$$

Since $f_{n_i}(r_1) \to f(r_1)$ and $f_{n_i}(r_2) \to f(r_2)$, there is an integer N such that for all $i > N$ we have

$$\left|f_{n_i}(r_1) - f(r_1)\right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left|f_{n_i}(r_2) - f(r_2)\right| < \frac{\varepsilon}{2}.$$

And since f_{n_i} is monotonically increasing, we have for all $i > N$

$$f(x) - \varepsilon < f(r_1) - \frac{\varepsilon}{2} < f_{n_i}(r_1) \le f_{n_i}(x) \le f_{n_i}(r_2) < f(r_2) + \frac{\varepsilon}{2} < f(x) + \varepsilon.$$

That is, $\left|f(x) - f_{n_i}(x)\right| < \varepsilon$, so $f_{n_i}(x) \to f(x)$ as $i \to \infty$.

(iv) Show that a subsequence of $\{f_{n_i}\}$ converges at every point of discontinuity of f since there are at most countably many such points.

Since f is monotonically increasing, f has at most countably many simple points of discontinuity by Theorems 4.29 and 4.30. Let $\{x_m\}$ be an enumeration of the points of discontinuity of f. Repeating the argument of step (i), there is a subsequence $\{f_{1,n_j}(x_1)\}$ of $\{f_{n_i}(x_1)$ which converges to a value a between 0 and 1. We can continue this way for all the x_m, getting a sequence of subsequences $\{f_{m,n_j}\}$ of $\{f_{n_i}\}$ such that each $\{f_{m,n_j}\}$ is a subsequence of $\{f_{m-1,n_j}\}$ for which $f_{m,n_j}(x_m)$ converges. Letting $f_{n_k} = f_{k,n_k}$, we get a subsequence of $\{f_n\}$ which converges at all the points $\{x_m\}$. We can redefine $f(x_i)$ at these points to be those values.

Since we have already shown that $\{f_{n_k}\}$ converges at the points where f is continuous, we have $f_{n_k}(x)$ converging to $f(x)$ for all real values x.

(b) Let K be a compact subset of $\mathbf{R}$ and let $\varepsilon > 0$. It was shown in step (iii) of part (a) that, since f is everywhere continuous, $f(x) = \lim f_{n_k}(x)$ for all points x. Also, we are using the initial definition of f, so that is it monotonically increasing. For each $x \in K$ there is a $\delta_x > 0$ such that for all $y \in [x - \delta_x, x + \delta_x]$ we have

$$\left|f(y) - f(x)\right| < \frac{\varepsilon}{4}.$$

Also, there is an integer N_x such that for all $k > N_x$

$$\left|f_{n_k}(x-\delta_x)\right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left|f_{n_k}(x+\delta_x)\right| < \frac{\varepsilon}{4}.$$

Since f_{n_k} is monotonically increasing, we must have, for all $y \in (x-\delta_x, x+\delta_x)$ and all $k > N_x$,

$$f(x) - \frac{\varepsilon}{2} < f(x-\delta_x) - \frac{\varepsilon}{4} < f_{n_k}(x-\delta_x) \leq f_{n_k}(y) \leq f_{n_k}(x+\delta_x) + \frac{\varepsilon}{4} < f(x) + \frac{\varepsilon}{2},$$

that is, $\left|f_{n_k}(y) - f(x)\right| < \varepsilon/2$. Hence, for $y \in (x-\delta_x, x+\delta_x)$ and $k > N_x$,

$$\left|f_{n_k}(y) - f(y)\right| \leq \left|f_{n_k}(y) - f(x)\right| + \left|f(x) - f(y)\right| < \varepsilon.$$

Since the open neighborhoods $(x-\delta_x, x+\delta_x)$ cover the compact set K, a finite subset of these neighborhoods, centered at $x_1, \ldots, x_M$, also covers K. Letting $N = \max(N_{x_1}, \ldots, N_{x_M})$, we have for all $y \in K$ and all $k > N$ that $\left|f_{n_k}(y) - f(y)\right| \leq \varepsilon$, that is, $\{f_{n_k}\}$ converges uniformly to f on K.

Solution: (*Dan "kyp44" Whitman*)
(a) Since $\{f_n\}$ is uniformly bounded (by 1) clearly it is also pointwise bounded (by the function $g(x) = 1$). Since $\mathbb{Q}$ is countable, it then follows from Theorem 7.23 that there is a subsequence $\{f_{n_k}\}$ such that the sequence $\{f_{n_k}(x)\}$ converges for every $x \in \mathbb{Q}$. Now for any $x \in \mathbb{R}$ define the set

$$R_x = \{p \in \mathbb{Q} \mid p \leq x\}.$$

Then also define a function f on $\mathbb{R}$ as follows:

$$f(x) = \begin{cases} \lim_{k\to\infty} f_{n_k}(x) & x \in \mathbb{Q} \\ \sup_{p \in R_x} f(p) & x \notin \mathbb{Q}. \end{cases}$$

It was already shown that the limit exists in the $x \in \mathbb{Q}$ case. In the $x \notin \mathbb{Q}$ case we claim that the supremum also exists so that f is defined for all $x \in \mathbb{R}$. To see this note that $f_{n_k}(p) \leq 1$ for any $p \in \mathbb{Q}$ and $k \in \mathbb{Z}^+$ so that by Theorem 3.19 $f(p) \leq 1$ also. Hence for any $x \notin \mathbb{Q}$, 1 is an upper bound of the set $\{f(p) \mid p \in R_x\}$ so that $f(x) = \sup_{p \in R_x} f(p)$ exists since $\mathbb{R}$ has the least upper bound property.

First we show that f is monotonically increasing on all of $\mathbb{R}$. So consider any $x \in \mathbb{R}$ and $y \in \mathbb{R}$ and suppose that $x \leq y$.

Case: $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$. Then since each f_n is monotonically increasing on $\mathbb{R}$ it follows that $f_{n_k}(x) \leq f_{n_k}(y)$ for all $k \in \mathbb{Z}^+$. Then from Theorem 3.19 again we have that

$$f(x) = \lim_{k\to\infty} f_{n_k}(x) \leq \lim_{k\to\infty} f_{n_k}(y) = f(y).$$

Case: $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$. Then since $x \leq y$ and $x \in \mathbb{Q}$ clearly by definition $x \in R_y$. Then since $f(y) = \sup_{p \in R_y} f(p)$ is an upper bound of $\{f(p) \mid p \in R_y\}$ it has to be that $f(x) \leq f(y)$.

Case: $x \notin \mathbb{Q}$ and $y \in \mathbb{Q}$. Consider any $p \in R_x$. Then clearly $p \leq x \leq y$ and $p \in \mathbb{Q}$ so that by the first case above $f(p) \leq f(y)$. Since p was arbitrary clearly $f(y)$ is an upper bound of $\{f(p) \mid p \in R_x\}$. Thus since $f(x)$ is the *least* upper bound of that set it follows that $f(x) \leq f(y)$.

Case: $x \notin \mathbb{Q}$ and $y \notin \mathbb{Q}$. Consider again any $p \in R_x$. Then $p \leq x \leq y$ so that also $p \in R_y$. Hence $R_x \subseteq R_y$, from which it immediately follows that

$$f(x) = \sup_{p \in R_x} f(p) \leq \sup_{p \in R_y} f(p) = f(y).$$

Thus in all cases $f(x) \le f(y)$ so that f is monotonically increasing as claimed.

Now we show that if f is continuous at $x \in \mathbb{R}$ then $f_{n_k}(x) \to f(x)$. So suppose that $x \in \mathbb{R}$ and f is continuous at x and choose any $\varepsilon \in \mathbb{R}^+$. Then by the definition of continuity there is a $\delta \in \mathbb{R}^+$ such that $|f(y) - f(x)| < \varepsilon/6$ when $|y - x| < \delta$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ there is a $p \in \mathbb{Q}$ where $p \in N_{\frac{\delta}{2}}(x - \delta/2)$ and a $q \in \mathbb{Q}$ where $q \in N_{\frac{\delta}{2}}(x + \delta/2)$. It is then trivial to show that $p < x < q$, $|p - x| < \delta$, and $|q - x| < \delta$. It then follows that

$$|f(p) - f(x)| < \frac{\varepsilon}{6}$$

and

$$|f(q) - f(x)| < \frac{\varepsilon}{6}.$$

Now since $p \in \mathbb{Q}$ we have that $f_{n_k}(p) \to f(y)$. Hence there is an $N_1 \in \mathbb{Z}^+$ where $|f_{n_k}(p) - f(p)| < \varepsilon/6$ for all $k \ge N_1$. Likewise there is an $N_2 \in \mathbb{Z}^+$ where $|f_{n_k}(q) - f(q)| < \varepsilon/6$ for all $k \ge N_2$. So consider any $k \ge \max(N_1, N_2)$ so that the preceding inequalities hold.

Now since f_{n_k} is monotonically increasing and $p < x < q$ it follows that

$$\begin{gathered} f_{n_k}(p) \le f_{n_k}(x) \le f_{n_k}(q) \\ 0 \le f_{n_k}(x) - f_{n_k}(p) \le f_{n_k}(q) - f_{n_k}(p) \\ 0 \le |f_{n_k}(x) - f_{n_k}(p)| \le |f_{n_k}(q) - f_{n_k}(p)|. \end{gathered}$$

We therefore have that

$$\begin{aligned} |f_{n_k}(x) - f(x)| &\le |f_{n_k}(x) - f_{n_k}(p)| + |f_{n_k}(p) - f(x)| \\ &\le |f_{n_k}(q) - f_{n_k}(p)| + |f_{n_k}(p) - f(x)| \\ &\le |f_{n_k}(q) - f(q)| + |f(q) - f_{n_k}(p)| + |f_{n_k}(p) - f(p)| + |f(p) - f(x)| \\ &< \frac{\varepsilon}{6} + |f(q) - f_{n_k}(p)| + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} \\ &= |f(q) - f_{n_k}(p)| + \frac{\varepsilon}{2} \\ &\le |f(q) - f(x)| + |f(x) - f_{n_k}(p)| + \frac{\varepsilon}{2} \\ &\le |f(q) - f(x)| + |f(x) - f(p)| + |f(p) - f_{n_k}(p)| + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

Thereby showing that $f_{n_k}(x) \to f(x)$ since ε was arbitrary.

So we have shown that f_{n_k} converges to f at rational points and points where f is continuous, but what about points where f is discontinuous (and the point is irrational)? Since we have shown that f is monotonically increasing it follows from Theorem 4.30 that f has at most countable discontinuities. Then, since $\{f_{n_k}\}$ is pointwise bounded, it follows again from Theorem 7.23 that there is a sub-sub-sequence $\{f_{n_{k_l}}\}$ that converges to f at those points of discontinuity. And since all subsequences of a convergent sequence also converge to the same point (by the remark after Definition 3.5) it must be that $f_{n_{k_l}} \to f$ also at rational points and points at which f is continuous. Thus $f_{n_{k_l}} \to f$ at every point in $\mathbb{R}$. ■

(b) Now suppose that f as defined above is continuous on all of $\mathbb{R}$. Then we have shown above that $f_{n_k} \to f$ pointwise on $\mathbb{R}$. Consider any compact $K \subseteq \mathbb{R}$. Then K is bounded so that clearly there is an interval $[a,b]$ such that $K \subseteq [a,b]$. Consider any $\varepsilon \in \mathbb{R}^+$. Since f is continuous on $[a,b]$, by Theorem 4.19 f is uniformly continuous on $[a,b]$. Hence there is a $\delta \in \mathbb{R}^+$ where $|f(x) - f(y) < \varepsilon/5$ for any x and y in $[a,b]$ where $|x-y| < \delta$.

So let
$$M = \left\lceil \frac{2(b-a)}{\delta} \right\rceil$$
and
$$\delta_0 = \frac{b-a}{M}\,.$$
and $t_i = a + i\delta_0$ for $0 \le i \le M$. We then clearly have that
$$a = t_0 < t_1 < \ldots < t_M = b\,.$$
We also clearly have that $t_i - t_{i-1} = \delta_0$ so that
$$\begin{aligned}
\frac{2(b-a)}{\delta} &\le \left\lceil \frac{2(b-a)}{\delta} \right\rceil = M \\
\frac{2(b-a)}{M\delta} &\le 1 && \text{(since } M > 0\text{)} \\
\frac{b-a}{M} &\le \frac{\delta}{2} && \text{(since } \delta/2 > 0\text{)} \\
\delta_0 &\le \frac{\delta}{2} < \delta \\
t_i - t_{i-1} &< \delta \\
|t_i - t_{i-1}| &< \delta
\end{aligned}$$
from which it then follows that $|f(t_i) - f(t_{i-1})| < \varepsilon/5$ for all $1 \le i \le M$.

Then, for any $0 \le i \le M$, since $f_{n_k}(t_i) \to f(t_i)$ there is an N_i where $|f_{n_k}(t_i) - f(t_i)| < \varepsilon/5$ for all $k \ge N_i$. So let $N = \max_{0 \le i \le M} N_i$ and consider any $k \ge N$ and any $x \in [a,b]$. Then clearly there is an $1 \le i \le M$ where $x \in [t_{i-1}, t_i]$. Since $k \ge N \ge N_i$ and $k \ge N \ge N_{i-1}$ it follows that
$$|f_{n_k}(t_i) - f(t_i)| \le \frac{\varepsilon}{5}$$
and
$$|f_{n_k}(t_{i-1}) - f(t_{i-1})| \le \frac{\varepsilon}{5}\,.$$
Since $t_{i-1} \le x \le t_i$ and f and f_{n_k} are both monotonic it follows that
$$\begin{gathered}
f(t_{i-1}) \le f(x) \le f(t_i) \\
0 \le f(x) - f(t_{i-1}) \le f(t_i) - f(t_{i-1}) \\
0 \le |f(x) - f(t_{i-1})| \le |f(t_i) - f(t_{i-1})| < \frac{\varepsilon}{5}
\end{gathered}$$
and
$$\begin{gathered}
f_{n_k}(t_{i-1}) \le f_{n_k}(x) \le f_{n_k}(t_i) \\
0 \le f_{n_k}(x) - f_{n_k}(t_{i-1}) \le f_{n_k}(t_i) - f_{n_k}(t_{i-1}) \\
0 \le |f_{n_k}(x) - f_{n_k}(t_{i-1})| \le |f_{n_k}(t_i) - f_{n_k}(t_{i-1})|\,.
\end{gathered}$$

We then have that

$$\begin{aligned}|f_{n_k}(x) - f_{n_k}(t_{i-1})| &\le |f_{n_k}(t_i) - f_{n_k}(t_{i-1})| \\ &\le |f_{n_k}(t_i) - f(t_i)| + |f(t_i) - f_{n_k}(t_{i-1})| \\ &\le |f_{n_k}(t_i) - f(t_i)| + |f(t_i) - f(t_{i-1})| + |f(t_{i-1}) - f_{n_k}(t_{i-1})| \\ &< \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \frac{3\varepsilon}{5}.\end{aligned}$$

Lastly, we have

$$\begin{aligned}|f_{n_k}(x) - f(x)| &\le |f_{n_k}(x) - f_{n_k}(t_{i-1})| + |f_{n_k}(t_{i-1}) - f(x)| \\ &\le |f_{n_k}(x) - f_{n_k}(t_{i-1})| + |f_{n_k}(t_{i-1}) - f(t_{i-1})| + |f(t_{i-1}) - f(x)| \\ &< \frac{3\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon\end{aligned}$$

Since x and k were arbitrary this shows that $f_{n_k} \to f$ uniformly on $[a, b]$. It is then trivial to show that the convergence is uniform on K since $K \subseteq [a, b]$. ■

163. Exercise 14: Let f be a continuous real function on $\mathbf{R}$ with the following properties: $0 \le f(t) \le 1$, $f(t + 2) = f(t)$ for every t, and

$$f(t) = \begin{cases} 0 & \left(0 \le t \le \frac{1}{3}\right) \\ 1 & \left(\frac{2}{3} \le t \le 1\right). \end{cases}$$

Put $\Phi(t) = \big(x(t), y(t)\big)$, where

$$x(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1}t), \qquad y(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n}t).$$

Prove that Φ is *continuous* and that Φ maps $I = [0, 1]$ *onto* the unit square $I^2 \subset \mathbf{R}^2$. In fact, show that Φ maps the Cantor set onto I^2.

Solution: (*analambanomenos*)
Since for each n we have $\left|2^{-n} f(3^{2n-1}t)\right| \le 2^{-n}$ and $\left|2^{-n} f(3^{2n}t)\right| \le 2^{-n}$, the series defining $x(t)$ and $y(t)$ converge uniformly since $\sum 2^{-n}$ converges, by Theorem 7.10. And since the partial sums are continuous functions, $x(t)$ and $y(t)$ are continuous functions, by Theorem 7.12. Hence Φ is continuous.

Following the hint, let $(x_0, y_0) \in I^2$, and let

$$x_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1}, \qquad y_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n}$$

be the binary expansions of x_0 and y_0, where each a_i is 0 or 1. Let

$$t_0 = \sum_{i=1}^{\infty} 3^{-i-1}(2a_i).$$

By Exercise 3.19, the set of all such t_0 is precisely the Cantor set.

We have for $k = 1, 2, 3, \ldots,$

$$\begin{aligned} 3^k t_0 &= \sum_{i=1}^{\infty} 3^{k-i-1}(2a_i) \\ &= 2\sum_{n=0}^{k-2} 3^n a_{k-1-n} + \frac{2}{3}\,a_k + \frac{2}{3}\sum_{n=1}^{\infty} 3^{-n} a_{k+n} \end{aligned}$$

Note that the last term lies between 0 and

$$\frac{2}{3}\sum_{n=1}^{\infty} 3^{-n} = \frac{2}{3}\left(\frac{1/3}{1-(1/3)}\right) = \frac{1}{3}.$$

Also, the first term is an even integer, so that, since $f(t+2) = f(t)$, we have

$$f(3^k t_0) = f\left(\frac{2}{3}\,a_k + \frac{2}{3}\sum_{n=1}^{\infty} 3^{-n} a_{k+n}\right).$$

If $a_k = 0$, then the expression in the argument of f lies between 0 and $\frac{1}{3}$, so that $f(3^k t_0) = 0$. And if $a_k = 1$, then the expression in the argument of f lies between $\frac{2}{3}$ and 1, so that $f(3^k t_0) = 1$. In either case, we have $f(3^k t_0) = a_k$. Hence $\Phi(t_0) = \big(x(t_0), y(t_0)\big) = (x_0, y_0)$.

164. Exercise 15: Suppose f is a real continuous function on $\mathbf{R}$, $f_n(t) = f(nt)$ for $n = 1, 2, 3, \ldots,$ and $\{f_n\}$ is equicontinuous on $[0, 1]$. What conclusion can you draw about f?

Solution: (*analambanomenos*)
The function f must be constant on $[0, \infty]$. Let $0 \le x < y$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ such that for all $n = 1, 2, 3, \ldots,$ we have $\big|f_n(z) - f_n(w)\big| < \varepsilon$ if $0 \le w, z \le 1$ and $|w - z| < \delta$. For large enough N, we have

$$\left|\frac{y}{N} - \frac{x}{N}\right| = \frac{|y-x|}{N} < \delta, \qquad 0 \le \frac{x}{N} < \frac{y}{N} \le 1.$$

Hence

$$\big|f(y) - f(x)\big| = \left|f_N\left(\frac{y}{N}\right) - f_N\left(\frac{x}{N}\right)\right| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows that $f(x) = f(y)$.

By the way, it is easy to extend this argument to show that if $\{f_n\}$ were equicontinuous on $[-1, 1]$, then f would be constant on all of $\mathbf{R}$.

165. Exercise 16: Suppose $\{f_n\}$ is an equicontinuous sequence of functions on a compact set K, and $\{f_n\}$ converges pointwise on K. Prove that $\{f_n\}$ converges uniformly on K.

Solution: (*analambanomenos*)
Let $\varepsilon > 0$ and let $\delta > 0$ such that if $d(x, y) < \delta$ then $\big|f_n(x) - f_n(y)\big| < \varepsilon/3$ for all n. Let $x \in K$ and let N_x be an integer large enough so that if $m > N_x$ and $n > N_x$, then $\big|f_m(x) - f_n(x)\big| < \varepsilon$. Then for all $m > N_x$ and $n > N_x$ and all $y \in K$ such that $d(x, y) < \delta$, we have

$$\big|f_n(y) - f_m(y)\big| \le \big|f_n(y) - f_n(x)\big| + \big|f_n(x) - f_m(x)\big| + \big|f_m(x) - f_m(y)\big| < \varepsilon.$$

Since K is compact, there are a finite number of points $x_1, \ldots, x_M$ in K such that the neighborhoods of radius δ centered at the x_i cover K. So if we let $N = \max(N_{x_1}, \ldots, N_{x_M})$, then for $m > N$ and $n > N$ and for all $y \in K$, we have $\left|f_m(y) - f_n(y)\right| < \varepsilon$. Hence by Theorem 7.8, $\{f_n\}$ converges uniformly on K.

Solution: (*Dan "kyp44" Whitman*)

Lemma 7.16.1: If a sequence of complex functions $\{f_n\}$ is equicontinuous on a metric space X and $f_n \to f$ pointwise then f is also equicontinuous with $\{f_n\}$ on X.

Proof: Consider any $\varepsilon \in \mathbb{R}^+$. Then, since $\{f_n\}$ is equicontinuous, there is a $\delta \in \mathbb{R}^+$ where

$$|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$$

for all $n \in \mathbb{Z}^+$, $x \in X$, and $y \in X$ where $d(x,y) < \delta$. Now consider any $g \in \{f_n \,|\, n \in \mathbb{Z}^+\} \cup \{f\}$ and any x and y in X where $d(x,y) < \delta$.

Case: $g \in \{f_n \,|\, n \in \mathbb{Z}^+\}$. Then $g = f_n$ for some $n \in \mathbb{Z}^+$ and we clearly have

$$|g(x) - g(y)| = |f_n(x) - f_n(y)| < \frac{\varepsilon}{3} < \varepsilon$$

since $d(x,y) < \delta$.

Case: $g = f$. Then since $f_n \to f$ there is an N_1 where $|f_n(x) - f(x)| < \varepsilon/3$ for all $n \geq N_1$ and similarly an N_2 where $|f_n(y) - f(y)| < \varepsilon/3$ for all $n \geq N_2$. So, letting $N = \max(N_1, N_2)$ we have

$$\begin{aligned} |g(x) - g(y)| = |f(x) - f(y)| &\leq |f(x) - f_N(x)| + |f_N(x) - f(y)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

since $N \geq N_1$, $d(x,y) < \delta$, and $N \geq N_2$.

Thus since g, x, and y were arbitrary we've shown that $\{f_n \,|\, n \in \mathbb{Z}^+\} \cup \{f\}$ is equicontinuous. ■

Main Problem: Consider any $\varepsilon \in \mathbb{R}^+$. Since f_n converges pointwise, let f be the function to which it converges. Then since $\{f_n\}$ is equicontinuous, by Lemma 7.16.1 f is also equicontinuous with $\{f_n\}$ so that there is $\delta \in \mathbb{R}^+$ where

$$|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$$

and

$$|f(x) - f(y)| < \frac{\varepsilon}{3}$$

for any $n \in \mathbb{Z}^+$ and x and y in K where $d(x,y) < \delta$.

Now clearly $\{N_\delta(x) \,|\, x \in K\}$ is an open cover of K and since K is compact there are finite $x_1, \ldots, x_m$ where $\{N_\delta(x_i) \,|\, 1 \leq i \leq m\}$ covers K. Since $f_n \to f$ pointwise there is associated with each x_i an N_i such that $|f_n(x_i) - f(x_i)| < \varepsilon/3$ for all $n \geq N_i$. So let $N = \max_{1 \leq i \leq m} N_i$ and consider any $n \geq N$ and any $x \in K$. Then there is an $1 \leq i \leq m$ where $x \in N_\delta(x_i)$. Thus $d(x, x_i) < \delta$ and $n \geq N \geq N_i$ so that

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f(x)| \\ &\leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f(x_i)| + |f(x_i) - f(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\,. \end{aligned}$$

Hence since n and x were arbitrary, by definition $f_n \to f$ uniformly. ■

166. Exercise 17: Define the notions of uniform convergence and equicontinuity for mappings into any metric space. Show that Theorems 7.9 and 7.12 are valid for mappings into any metric space, that Theorems 7.8 and 7.11 are valid for mappings into any complete metric space, and that Theorems 7.10, 7.16, 7.17, 7.24, and 7.25 hold for vector-valued functions, that is, for mappings into any $\mathbf{R}^k$.

Solution: (*analambanomenos*)
The extensions of the definitions, and the statements and proofs of Theorems 7.8 through 7.11, are trivial, so I will simply copy them from the text, italicizing the changes.

Definition 7.7 We say that a sequence of *mappings* $\{f_n\}$, $n = 1, 2, 3, \ldots$, *into a metric space* F converges uniformly on E to a function f if for every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies

$$d\big(f_n(x), f(x)\big) \leq \varepsilon \tag{13}$$

for all $x \in E$.

Definition 7.22 A family $\mathscr{F}$ of *mappings* f defined on a set E in a metric space X *with values in a metric space* Y is said to be equicontinuous on E if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y\big(f(x), f(y)\big) < \varepsilon$$

whenever $d_X(x, y) \leq \delta$, $x \in E$, $y \in E$, and $f \in \mathscr{F}$.

Theorem 7.8 The sequence of *mappings* $\{f_n\}$ *into a complete metric space* F, defined on E, converges uniformly on E if and only if for every $\varepsilon > 0$ there exists an integer N such that $m \geq N$, $n \geq N$, $x \in E$ implies $d\big(f_n(x), f_m(x)\big) \leq \varepsilon$.

Proof: Suppose $\{f_n\}$ converges uniformly on E, and let f be the limit function. There there is an integer N such that $n \geq E$ implies $d\big(f_n(x), f(x)\big) \leq \varepsilon/2$ so that

$$d\big(f_n(x), f_m(x)\big) \leq d\big(f_n(x), f(x)\big) + d\big(f(x), f_m(x)\big) \leq \varepsilon$$

if $n \geq N$, $m \geq M$, $x \in E$.

Conversely, suppose the Cauchy condition holds. *Since F is a complete metric space*, the sequence $\{f_n(x)\}$ converges, for every x, to a limit which we may call $f(x)$. Thus the sequence $\{f_n\}$ converges on E, to f. We have to prove that the convergence is uniform.

Let $\varepsilon > 0$ be given, and choose N such that (13) holds. Fix n, and let $m \to \infty$ in (13). Since $f_m(x) \to f(x)$ as $m \to \infty$, this gives $d\big(f_n(x), f(x)\big) \leq \varepsilon$ for every $n \geq N$ and every $x \in E$, which completes the proof.

Theorem 7.9 Suppose $\{f_n\}$ *is a sequence of mappings from E into a metric space F such that* $\lim f_n(x) = f(x)$ for $x \in E$. Put

$$M_n = \sup_{x \in E} d\big(f_n(x), f(x)\big).$$

Then $f_n \to f$ uniformly on E if and only if $M_n \to 0$ as $n \to \infty$.

Proof: Since this is an immediate consequence of Definition 7.7, we omit the details of the proof.

Theorem 7.10 Suppose $\{f_n\}$ is a sequence *of vector-valued functions with values in* $\mathbf{R}^k$ defined on E, and suppose

$$\big|\big|f_n(x)\big|\big| \leq M_n \qquad (x \in E,\ n = 1, 2, 3, \ldots).$$

Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

Proof: If $\sum M_n$ converges, then, for arbitrary $\varepsilon > 0$,

$$\left\| \sum_{i=n}^{m} f_i(x) \right\| \leq \sum_{i=n}^{m} M_i \leq \varepsilon \qquad (x \in E),$$

provided m and n are large enough. Uniform convergence now follows from Theorem 7.8.

Theorem 7.11 Suppose $\{f_n\}$, *a sequence of mappings with values in a complete metric space* F, converges uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that

$$\lim_{t \to x} f_n(t) = A_n \qquad (n = 1, 2, 3, \ldots).$$

Then $\{A_n\}$ converges and

$$\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n. \tag{16}$$

Proof: Let $\varepsilon > 0$ be given. By the uniform convergence of $\{f_n\}$, there exists N such that $n \geq N$, $m \geq N$, $t \in E$ imply

$$d\big(f_n(t), f_m(t)\big) \leq \varepsilon. \tag{18}$$

Letting $t \to x$ in (18), we obtain

$$d(A_n, A_m) \leq \varepsilon$$

for $n \geq N$, $m \geq M$, so that $\{A_n\}$ is a Cauchy sequence and therefore converges, say to A.

Next,

$$d\big(f(t), A\big) \leq d\big(f(t), f_n(t)\big) + d\big(f_n(t), A_n\big) + d\big(A_n, A\big). \tag{19}$$

We first choose n such that

$$d\big(f(t), f_n(t)\big) \leq \frac{\varepsilon}{3} \tag{20}$$

for all $t \in E$ (this is possible by the uniform convergence), and such that

$$d\big(A_n, A\big) \leq \frac{\varepsilon}{3}. \tag{21}$$

Then, for this n we choose a neighborhood V of x such that

$$d\big(f_n(t) - A_n\big) \leq \frac{\varepsilon}{3}, \tag{22}$$

if $t \in V \cap E$, $t \neq x$.

Substituting the inequalities (20) to (22) into (19), we see that $d\big(f(t), A\big) \leq \varepsilon$, provided $t \in V \cap E$, $t \neq x$. This is equivalent to (16).

Theorem 7.12 If $\{f_n\}$ is a sequence of continuous *mappings on* E *into a metric space* F, and if $f_n \to f$ uniformly on E, then f is continuous on E.

Proof: (In the text, Theorem 7.12 is an immediately corollary of Theorem 7.11. Here, that would require that F be complete, which is not necessarily the case. So the following proof is entirely new.)

By Theorem 4.8, to show that $f$is continuous, it suffices to show that $f^{-1}(V)$ is open for any open set $V \subset F$. If $x \in f^{-1}(V)$, we need to find a neighborhood N_x of x which is contained in $f^{-1}(V)$, for then $f^{-1}(V)$ would be the union of the open neighborhoods N_x, $x \in f^{-1}(V)$, and so be open.

Let $\varepsilon > 0$ be small enough so that the neighborhood M_y of $y = f(x)$ of radius ε is contained in V. Let n be large enough so that $d\big(f_n(z), f(z)\big) \leq \varepsilon/3$ for all $z \in E$, which is possible by the assumption of uniform convergence. Let $\delta > 0$ such that $d\big(f_n(z), f_n(x)\big) < \varepsilon/3$ for all z in the neighborhood N_x of x of radius δ, which is possible since f_n is continuous. Then, for all $z \in N_x$,

$$d\big(f(z), f(x)\big) \leq d\big(f(z), f_n(z)\big) + d\big(f_n(z), f_n(x)\big) + d\big(f_n(x), f(x)\big) \leq \varepsilon.$$

That is, $f(z) \in M_y \subset V$, or $z \in f^{-1}(V)$, so that $N_x \subset f^{-1}(V)$.

The remaining Theorems, for mappings into $\mathbf{R}^k$, are largely corollaries of their scalar counterparts in the text.

Theorem 7.16 Let α be monotonically increasing on $[a,b]$. Suppose $\mathbf{f}_n = (f_{n1}, \ldots, f_{nk}) \in \mathscr{R}(\alpha)$ on $[a,b]$, for $n = 1, 2, 3, \ldots$, and suppose $\mathbf{f}_n \to \mathbf{f} = (f_1, \ldots, f_k)$ uniformly on $[a,b]$. Then $\mathbf{f} \in \mathscr{R}(\alpha)$ on $[a,b]$, and

$$\int_a^b \mathbf{f}\, d\alpha = \lim_{n\to\infty} \int_a^b \mathbf{f}_n\, d\alpha.$$

Theorem 7.17 Suppose $\big\{\mathbf{f}_n = (f_{n1}, \ldots, f_{nk})\big\}$ is a sequence of vector-valued functions, differentiable on $[a,b]$ and such that $\big\{\mathbf{f}(x_0)\big\}$ converges for some point x_0 on $[a,b]$. If $\{\mathbf{f}_n'\}$ converges uniformly on $[a,b]$, to a function $\mathbf{f} = (f_1, \ldots, f_k)$, and

$$\mathbf{f}'(x) = \lim_{n\to\infty} f_n'(x) \qquad (a \leq x \leq b).$$

Proofs: If $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbf{R}^k$, then $|x_i| \leq ||\mathbf{x}|| \leq \sum_i^k |x_i|$. Hence $\mathbf{f}_n$ convergences uniformly if and only if the component functions $f_{n1}, \ldots, f_{nk}$ converge uniformly. And since the integers or derivatives, and integrability or differentiability, of vector-valued functions are defined by component, the vector-valued versions of the Theorems 7.16 and 7.17 are immediate corollaries of the scalar versions in the text.

Theorem 7.24 If K is a compact metric space, if $\mathbf{f}_n = (f_{n1}, \ldots, f_{nk}) \in \mathscr{C}(K)$ for $n = 1, 2, 3, \ldots$, and if $\{\mathbf{f}_n\}$ converges uniformly on K, then $\{\mathbf{f}_n\}$ is equicontinuous on K.

Proof: If $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbf{R}^k$, then $|x_i| \leq ||\mathbf{x}|| \leq \sum_i |x_i|$. Hence $\{\mathbf{f}_n\}$ is equicontinuous if and only if each of the $\{f_{ni}\}$, $i = 1, \ldots, k$ are equicontinuous. Hence the vector-valued version of Theorem 7.24 is an immediate corollary of the scalar version in the text.

Theorem 7.25 If K is compact, if $\mathbf{f}_n = (f_{n1}, \ldots, f_{nk})$ is pointwise bounded and equicontinuous on K, then

(a) $\{\mathbf{f}_n\}$ is uniformly bounded on K,

(b) $\{\mathbf{f}_n\}$ contains a uniformly convergent subsequence.

Proof: Since each of the sequences $\{f_{ni}\}$, $i = 1, \ldots, k$, is pointwise bounded and equicontinuous on K, by the scalar version of Theorem 7.25 in the text each of them is uniformly bounded on K, so $\{\mathbf{f}_n\}$ is uniformly bounded on K. Also, $\{f_{n1}\}$ contains a uniformly convergent subsequence, $\{f_{n_j 1}\}$. Since the subsequence $\{f_{n_j 2}\}$ is also pointwise bounded and equicontinuous on K, it also has a uniformly convergent subsequence such that the corresponding subsequence of $\{f_{n_j 1}\}$ also converges uniformly. Continuing in this manner, after k steps we have a subsequence $\{\mathbf{f}_{n_j}\}$ whose component functions all converge uniformly, so $\{\mathbf{f}_{n_j}\}$ also converges uniformly.

167. Exercise 18: Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann-integrable on $[a, b]$, and put

$$F_n(x) = \int_a^x f_n(t)\, dt \qquad (a \le x \le b).$$

Prove that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a, b]$.

Solution: (*analambanomenos*)
This follows from Theorem 7.25 if we show that $\{F_n\}$ is pointwise-bounded and equicontinuous on $[a, b]$. Let $|f_n| \le K$ on $[a, b]$. Then for all n

$$\big|F_n(x)\big| \le \int_a^x \big|f_n(t)\big|\, dt \le K(x - a),$$

so $\{F_n\}$ is pointwise-bounded on $[a, b]$. And if $\varepsilon > 0$ and $x < y$ are points of $[a, b]$ such that $y - x \le \varepsilon/K$, then

$$\big|F_n(x) - F_n(y)\big| \le \int_x^y \big|f_n(t)\big|\, dt \le K(y - x) \le \varepsilon,$$

showing that $\{F_n\}$ is equicontinuous on $[a, b]$.

168. Exercise 19: Let K be a compact metric space, let S be a subset of $\mathscr{C}(K)$. Prove that S is compact (with respect to the metric defined in Section 7.14) if and only if S is uniformly closed, pointwise bounded, and equicontinuous. (If S is not equicontinuous, then S contains a sequence which has no equicontinuous subsequence, hence has no subsequence that converges uniformly on K.)

Solution: (*analambanomenos*)
Suppose S is uniformly closed, pointwise bounded, and equicontinuous, and let $\{f_n\}$ be a sequence of elements of S. By Theorem 7.25, $\{f_n\}$ has a subsequence converging uniformly to $f \in \mathscr{C}(K)$, and since f is uniformly closed, we have $f \in S$. Hence by Exercise 2.26, S is compact.

Conversely, suppose that S is compact. By Theorem 2.34 it is closed with respect to the supremum norm, that is, S is uniformly closed. Let $x \in K$ and define the complex-valued function F_x on $\mathscr{C}(K)$ by $F_x(f) = f(x)$. F_x is continuous since if $g \to f$, then

$$\big|F_x(f) - F_x(g)\big| = \big|f(x) - g(x)\big| \le ||f - g|| \to 0.$$

Hence $F_x(S)$ is bounded by Theorem 4.15, that is, S is pointwise bounded. Since S is compact, by Theorem 2.37 every infinite subset of S has a limit point in S, that is, it has a subsequence which converges uniformly on K. If S were not equicontinuous, then there would exist an $\varepsilon > 0$ such that for all positive integers there exists $x_n, y_n \in K$ and $f_n \in S$ such that

$$d(x_n, y_n) < 1/n \text{ but } \big|f_n(x_n) - f_n(y_n)\big| \ge \varepsilon.$$

Hence $\{f_n\}$ would have no equicontinuous subsequence, so by Theorem 7.24 it would have no subsequence which converges uniformly on K, which we have seen contradicts the compactness of S.

169. Exercise 20: If f is continuous on $[0, 1]$ and if

$$\int_0^1 f(x)x^n\, dx = 0 \qquad (n = 0, 1, 2, \ldots),$$

prove that $f(x) = 0$ on $[0, 1]$.

Solution: (*analambanomenos*)
Following the hint, note that by the linearity of Riemann-integration, $\int_0^1 f(x)P(x)\,dx = 0$ for all polynomials P. By Theorem 7.26 there exists a sequence of polynomials P_n which converge to f uniformly on $[0,1]$. Since $[0,1]$ is compact, f and each P_n are bounded on $[0,1]$, so $f(x)P_n(x)$ converges uniformly to $f^2(x)$ by Exercise 7.2. By Theorem 7.16, $0 = \int_0^1 f(x)P_n(x)\,dx$ must converge to $\int_0^1 f^2(x)\,dx$, so this integral equals 0. Hence $f(x) = 0$ on $[0,1]$ by Exercise 6.2.

170. Exercise 21: Let K be the unit circle in the complex plane and let $\mathscr{A}$ be the algebra of all functions of the form

$$f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta} \qquad (\theta \text{ real}).$$

Then $\mathscr{A}$ separates points on K and $\mathscr{A}$ vanishes at no point of K, but nevertheless there are continuous functions on K which are not in the uniform closure of $\mathscr{A}$.

Solution: (*analambanomenos*)
Since $e^{in\theta}e^{im\theta} = e^{i(m+n)\theta}$, it is clear that $\mathscr{A}$ is an algebra, and since it contains the identity function $f(e^{i\theta}) = e^{i\theta}$, $\mathscr{A}$ separates points on K and vanishes at no point of K. Following the hint, for $f = \sum_0^N c_n e^{in\theta} \in \mathscr{A}$ we have

$$\begin{aligned}
\int_0^{2\pi} f(e^{i\theta})e^{i\theta}\,d\theta &= \sum_{n=0}^{N} c_n \int_0^{2\pi} e^{i(n+1)\theta}\,d\theta \\
&= \sum_{n=0}^{N} c_n \int_0^{2\pi} \cos\big((n+1)\theta\big)\,d\theta + \sum_{n=0}^{N} ic_n \int_0^{2\pi} \sin\big((n+1)\theta\big)\,d\theta \\
&= 0.
\end{aligned}$$

Hence if g is in the uniform closure of $\mathscr{A}$, we have by the same reasoning used in Exercise 7.20 that $\int_0^{2\pi} g(e^{i\theta})e^{i\theta}\,d\theta = 0$. However, $e^{-i\theta}$ is a continuous function on K such that $\int_0^{2\pi} e^{-i\theta}e^{i\theta}\,d\theta = 2\pi$, so that $e^{-i\theta}$ is not in the uniform closure of $\mathscr{A}$.

Solution: (*Dan "kyp44" Whitman*)

Note that for $z \in K$ we have $\mathscr{A}$ as being the set of functions of the form

$$f(z) = \sum_{n=0}^{N} c_n z^n\,,$$

i.e. the set of polynomials in z. It is trivial to show that $\mathscr{A}$ is an algebra so we omit the proof here. It is also easy to see that $\mathscr{A}$ separates points. To see this consider two distinct x_1 and x_2 in K. Then since clearly the identity function $f(z) = z$ is in $\mathscr{A}$ we have $f(x_1) = x_1 \neq x_2 = f(x_2)$. Similarly the identity function shows that $\mathscr{A}$ vanishes at no point of K since $z \neq 0$ for every $z \in K$.

For any $f \in \mathscr{A}$ we have that

$$\begin{aligned}
\int_0^{2\pi} f(e^{i\theta})e^{i\theta}d\theta &= \int_0^{2\pi} \left(\sum_{n=0}^{N} c_n e^{in\theta}\right) e^{i\theta}d\theta = \int_0^{2\pi} \sum_{n=0}^{N} c_n e^{i\theta(n+1)}d\theta \\
&= \sum_{n=0}^{N} c_n \int_0^{2\pi} e^{i\theta(n+1)}d\theta = \sum_{n=0}^{N} c_n \left(\frac{e^{i\theta(n+1)}}{i(n+1)}\right)\Bigg|_0^{2\pi} \\
&= \sum_{n=0}^{N} c_n \left(\frac{e^{i\theta(n+1)2\pi}}{i(n+1)} - \frac{e^0}{i(n+1)}\right) = \sum_{n=0}^{N} c_n \left(\frac{1}{i(n+1)} - \frac{1}{i(n+1)}\right) \\
&= \sum_{n=0}^{N} c_n \cdot 0 = 0\,.
\end{aligned}$$

Now suppose that the uniform closure includes all continuous functions on K and define f on K by

$$f(z) = \frac{1}{z}$$

Since clearly this is continuous on K (noting that it is defined on K since $0 \notin K$) there must be a sequence $\{f_n\}$ in $\mathscr{A}$ where $f_n \to f$ uniformly. Let $g(z) = z$ be the identity function on K. Now also consider the function $h(\theta) = e^{i\theta}$, which clearly maps $[0, 2\pi]$ onto K. Then define

$$\begin{aligned}
\dot{f}_n &= f_n \circ h \\
\dot{f} &= f \circ h \\
\dot{g} &= g \circ h
\end{aligned}$$

that all clearly map $[0, 2\pi]$ to $\mathbb{C}$. Since h is clearly continuous as are f_n, f, and g it follows from Theorem 4.7 that the above functions are also all continuous. Then since $[0, 2\pi]$ is compact and they are continuous, they are all bounded as well. It is also trivial to show that $\dot{f}_n \to \dot{f}$ uniformly. Then by Exercise 7.2 the sequence $\{\dot{f}_n\dot{g}\}$ converges uniformly to $\dot{f}\dot{g}$. Then since clearly each $\{\dot{f}_n\dot{g}\} \in \mathscr{R}$ since they are continuous it follows from Theorem 7.16 that

$$\int_0^{2\pi} \dot{f}(\theta)\dot{g}(\theta)d\theta = \lim_{n\to\infty} \int_0^{2\pi} \dot{f}_n(\theta)\dot{g}(\theta)d\theta\,. \tag{14}$$

However, we have that

$$\lim_{n\to\infty} \int_0^{2\pi} \dot{f}_n(\theta)\dot{g}(\theta)d\theta = \lim_{n\to\infty} \int_0^{2\pi} f_n(e^{i\theta})e^{i\theta}d\theta = \lim_{n\to\infty} 0 = 0$$

by what was shown above since $f_n \in \mathscr{A}$. We also have that

$$\int_0^{2\pi} \dot{f}(\theta)\dot{g}(\theta)d\theta = \int_0^{2\pi} f(e^{i\theta})e^{i\theta}d\theta = \int_0^{2\pi} \frac{e^{i\theta}}{e^{i\theta}}d\theta = \int_0^{2\pi} 1d\theta = 2\pi - 0 = 2\pi$$

so that the equality (14) does not hold. Since this is a contradiction it must be that the uniform closure of $\mathscr{A}$ does *not* include all continuous functions on K. ■

Note that $\mathscr{A}$ is not self-adjoint. For example consider again the identity function $f(z) = z$. Since K is the unit circle, for $z = e^{i\theta}$, we have

$$\bar{f}(z) = \overline{f(z)} = \bar{z} = \overline{e^{i\theta}} = e^{-i\theta} = z^{-1}\,,$$

which is not a polynomial and so not in $\mathscr{A}$.

171. Exercise 22: Assume $f \in \mathscr{R}(\alpha)$ on $[a,b]$, and prove that there are polynomials P_n such that

$$\lim_{n\to\infty} \int_a^b |f - P_n|^2 \, d\alpha = 0.$$

Solution: (*analambanomenos*)
Recall the notation of Exercise 6.11: $||u||_2 = \left(\int_a^b |u^2| \, d\alpha\right)^{1/2}$. We want to show that, if $\varepsilon > 0$, there is a polynomial P such that $||f - P||_2 < \varepsilon$. By Exercise 6.12, there is a continuous function g on $[a,b]$ such that $||f - g||_2 < \varepsilon/2$. By Theorem 7.26, there is a polynomial P such that $\sup\big|g(x) - P(x)\big| < \varepsilon/\big(2\sqrt{\alpha(b) - \alpha(a)}\big)$. Then

$$||g - P||_2^2 = \int_a^b \big|g(x) - P(x)\big|^2 \, d\alpha < \frac{\varepsilon^2}{4\big(\alpha(b) - \alpha(a)\big)}\big(\alpha(b) - \alpha(a)\big) = \frac{\varepsilon^2}{4}.$$

Hence, by Exercise 6.11, $||f - P||_2 \le ||f - g||_2 + ||g - P||_2 < \varepsilon$.

Solution: (*Dan "kyp44" Whitman*)

Consider any $n \in \mathbb{Z}^+$. Since $f \in \mathscr{R}(\alpha)$ there is by Exercise 6.12 a continuous function g_n such that

$$\|f - g_n\|_2 < \frac{1}{2\sqrt{n}}\,,$$

where

$$\|u\|_2 = \left(\int_a^b |u|^2 d\alpha\right)^{1/2}$$

is as defined in Exercise 6.11. Since g_n is continuous on $[a,b]$ there is a sequence of polynomials $\{S_{n,m}\}$ such that $S_{n,m} \to g_n$ uniformly as $m \to \infty$ by Theorem 7.26. Hence by definition there is an $N \in \mathbb{Z}^+$ where

$$|S_{n,m}(x) - g_n(x)| \le \frac{1}{2\sqrt{n[\alpha(b) - \alpha(a)]}}$$

for all $x \in [a,b]$ and $m \ge N$. In particular the above is true for $S_{n,N}$ so define the polynomial $P_n = S_{n,N}$. Hence we have

$$|P_n(x) - g_n(x)| \le \frac{1}{2\sqrt{n[\alpha(b) - \alpha(a)]}}$$
$$|P_n(x) - g_n(x)|^2 \le \frac{1}{4n[\alpha(b) - \alpha(a)]}$$

for all $x \in [a,b]$ so that by Theorem 6.12b we have

$$\int_a^b |P_n - g_n|^2 d\alpha \le \int_a^b \frac{1}{4n[\alpha(b) - \alpha(a)]} d\alpha = \frac{1}{4n}$$
$$\left(\int_a^b |P_n - g_n|^2 d\alpha\right)^{1/2} \le \left(\frac{1}{4n}\right)^{1/2}$$
$$\|P_n - g_n\|_2 \le \frac{1}{2\sqrt{n}}\,.$$

Then by Exercise 6.11 we have

$$\|f - P_n\|_2 \leq \|f - g_n\|_2 + \|g_n - P_n\|_2 < \frac{1}{2\sqrt{n}} + \frac{1}{2\sqrt{n}} = \frac{1}{\sqrt{n}}$$
$$\left(\int_a^b |f - P_n|^2 d\alpha\right)^{1/2} < \frac{1}{\sqrt{n}}$$
$$\int_a^b |f - P_n|^2 d\alpha < \frac{1}{n}\,.$$

Thus we have a sequence of polynomials $\{P_n\}$ such that the above is true. So consider any $\varepsilon \in \mathbb{R}^+$ and let

$$N = \left\lceil \frac{1}{\varepsilon} \right\rceil\,.$$

Now choose any $n \geq N$ so that clearly

$$\frac{1}{n} \leq \frac{1}{N} \leq \varepsilon\,.$$

We then have that

$$\left|\int_a^b |f - P_n|^2 d\alpha - 0\right| = \left|\int_a^b |f - P_n|^2 d\alpha\right| = \int_a^b |f - P_n|^2 d\alpha < \frac{1}{n} \leq \varepsilon\,,$$

which shows that

$$\lim_{n\to\infty} \int_a^b |f - P_n|^2 d\alpha = 0$$

by definition as desired. ■

172. Exercise 23: Put $P_0 = 0$, and define, for $n = 0, 1, 2, \ldots,$

$$P_{n+1}(x) = P_n + \frac{x^2 - P_n^2(x)}{2}.$$

Prove that

$$\lim_{n\to\infty} P_n(x) = |x|,$$

uniformly on $[-1, 1]$.

Solution: (*analambanomenos*)
From the definition, it is easy to see that if P_n is an even function, then so is P_{n+1}. Hence by induction, since P_0 is trivially an even function, all the P_n are even functions. Since $|x|$ is also even, it suffices to show that the P_n converge uniformly to x on $[0, 1]$. In what follows, it is always understood that the variable x lies in $[0, 1]$.

Following the hint, note that

$$(*) \qquad \big(x - P_n(x)\big)\left(1 - \frac{x + P_n(x)}{2}\right) = x - P_n(x) - \frac{x^2 - P_n(x)}{2} = x - P_{n+1}(x).$$

I want to show that $0 \leq P_n(x) \leq P_{n+1}(x) \leq x$, by induction. Since $P_0(x) = 0$ and $P_1(x) = x^2/2$, this is true for the case $n = 0$. Suppose it is true for the case n. Then the factors

$$x - P_n(x) \quad \text{and} \quad 1 - \left(\frac{x + P_n(x)}{2}\right)$$

in $(*)$ are nonnegative, so $x - P_{n+1}(x) \geq 0$ or $P_{n+1} \leq x$. Also, the terms

$$P_n(x) \quad \text{and} \quad \frac{x^2 - P_n^2(x)}{2}$$

in the definition of P_{n+1} are nonnegative, so $P_{n+1}(x) \geq 0$. Hence the factor

$$1 - \frac{x + P_n(x)}{2}$$

in $(*)$ lies between 0 and 1, so that $x - P_{n+1}(x) \leq x - P_n(x)$, or $P_n(x) \leq P_{n+1}(x)$. Putting this all together, $\{P_n\}$ is a monotonically increasing sequence of polynomials on $[0,1]$ which lie between 0 and x.

By $(*)$ we have

$$\begin{aligned}
x - P_n(x) &= \left(1 - \frac{x + P_{n-1}(x)}{2}\right)\big(x - P_{n-1}(x)\big) \\
&= \left(1 - \frac{x + P_{n-1}(x)}{2}\right)\left(1 - \frac{x + P_{n-2}(x)}{2}\right)\big(x - P_{n-2}(x)\big) \\
&= \cdots \\
&= x \prod_{i=0}^{n-1} \left(1 - \frac{x + P_i(x)}{2}\right) \\
&\leq x\left(1 - \frac{x}{2}\right)^n.
\end{aligned}$$

By elementary calculus, the function $x(1 - x/2)^n$ has a maximum value at $x = 2/(n+1)$ of

$$\frac{2}{n+1}\left(\frac{n}{n+1}\right) \to 0 \text{ as } n \to \infty.$$

Hence $P_n(x)$ increases monotonically and uniformly to x as $n \to \infty$.

173. Exercise 24: Let X be a metric space, with metric d. Fix a point $a \in X$. Assign to each $p \in X$ the function f_p defined by

$$f_p(x) = d(x,p) - d(x,a) \qquad (x \in X).$$

Prove that $\left|f_p(x)\right| \leq d(a,p)$ for all $x \in X$, and that therefore $f_p \in \mathscr{C}(X)$. Prove that

$$||f_p - f_q|| = d(p,q)$$

for all $p, q \in X$.

If $\Phi(p) = f_p$ it follows that Φ is an *isometry* (a distance-preserving mapping) of X onto $\Phi(X)$ in $\mathscr{C}(X)$.

Let Y be the closure of $\Phi(X)$ in $\mathscr{C}(X)$. Show that Y is complete. *Conclusion: X is isometric to a dense subset of a complete metric space Y.*

Solution: (*analambanomenos*)
Note that the triangle inequality gives us, for all $x \in X$, $y \in X$, and $z \in X$,

$$\begin{aligned}
d(x,z) - d(x,y) &\leq d(y,z) \\
d(x,y) - d(x,z) &\leq d(z,y) = d(y,z),
\end{aligned}$$

so that

$$\left|d(x,z) - d(x,y)\right| \le d(y,z).$$

Hence, for all $x \in X$,

$$\left|f_p(x)\right| = \left|d(x,p) - d(x,a)\right| \le d(a,p).$$

To show that f_p is continuous, let $\varepsilon > 0$ and let $x \in X$ and $y \in X$ such that $d(x,y) < \delta = \varepsilon/2$. Then

$$\begin{aligned}\left|f_p(x) - f_p(y)\right| &= \left|d(x,p) - d(x,a) - d(y,p) + d(y,a)\right| \\ &\le \left|d(x,p) - d(y,p)\right| + \left|d(y,a) - d(x,a)\right| \\ &\le d(x,y) + d(x,y) \\ &\le \varepsilon.\end{aligned}$$

(This shows that f_p is uniformly continuous.)

Note that if $p \in E$ and $q \in E$, then for all $x \in E$ we have

$$f_p(x) - f_q(x) = d(x,p) - d(x,a) - d(x,p) + d(x,a) = d(x,p) - d(x,q)$$

so that

$$\begin{aligned}||f_p - f_q|| &= \sup_{x\in E}\left|f_p(x) - f_q(x)\right| \\ &= \sup_{x\in E}\left|d(x,p) - d(x,q)\right| \\ &\le d(p,q).\end{aligned}$$

And since $f_p(q) - f_q(q) = d(p,q)$, we have

$$||f_p - f_q|| = d(p,q)$$

for all $p, q \in X$.

By Theorem 7.15, $\mathscr{C}(X)$ with the uniform convergence metric is complete. Since it is clear that any closed subset of a complete metric space is also complete (if a sequence of elements in the closed subset satisfies the Cauchy condition, then it must converge to an element of the complete metric space, which must be an element of the closed subset since it is closed), we see that the closure Y if $\Phi(X)$ is complete.

174. Exercise 25: Suppose ϕ is a continuous bounded real function in the strip defined by $0 \le x \le 1$, $-\infty < y < \infty$. Prove that the initial-value problem

$$y' = \phi(x,y), \qquad y(0) = c$$

has a solution.

Solution: (*analambanomenos*)
Following the hint, let n be a positive integer. For $i = 0, \ldots, n$, put $x_i = i/n$. Let f_n be a continuous, piecewise-linear function on $[0,1]$ such that $f_n(0) = c$ and has slope $f_n'(t) = \phi\big(x_i, f_n(x_i)\big)$ if $x_i < t < x_{i+1}$.

Let $|\phi|$ be bounded by M, so that $|f_n'| \le M$. Note that

$$\int_{x_i}^{x_{i+1}} f_n'(t)\,dt = f_n(x_{i+1}) - f_n(x_i),$$

so that for $0 \le x \le 1$, $f_n(x) - c$ is the sum of integrals of f_n' over intervals where it is defined. Hence

$$\big|f_n(x)\big| \le |c| + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \big|f_n'(t)\big|\,dt \le |c| + M = M_1$$

so that $\{f_n\}$ is uniformly bounded on $[0,1]$. Also, since the continuous, piecewise-linear functions f_n have slopes lying between $-M$ and M on their linear parts, if $\varepsilon > 0$ then for $0 \le x \le 1$, $0 \le y \le 1$, $|x-y| \le \varepsilon/M$, we have $\big|f_n(x) - f_n(y)\big| \le \varepsilon$. That is, $\{f_n\}$ is equicontinuous on $[0,1]$. Hence by Theorem 7.25, there is a subsequence $\{f_{n_k}\}$ which converges uniformly to a continuous function f on $[0,1]$.

By Theorem 4.19, ϕ is uniformly continuous on the compact rectangle R given by $0 \le x \le 1$, $|y| \le M_1$. That is, if $\varepsilon > 0$, there is a $\delta > 0$ such that if the distance between the points (x_1, y_1) and (x_2, y_2) in R is less than δ, then $\big|\phi(x_1,y_1) - \phi(x_2,y_2)\big| < \varepsilon$. Since there is a K such that for all $k \ge K$ and all $t \in [0,1]$, we have $\big|f_{n_k}(t) - f(t)\big| < \delta$, we have

$$\big|\phi\big(t, f_{n_k}(t)\big) - \phi\big(t, f(t)\big)\big| < \varepsilon.$$

That is $\phi\big(t, f_{n_k}(t)\big)$ converges uniformly to $\phi\big(t, f(t)\big)$ as $k \to \infty$. Hence, if we let

$$\Delta_n(t) = \begin{cases} \phi\big(x_i, f_n(x_i)\big) - \phi\big(t, f_n(t)\big) & x_i < t < t_{i+1}, \quad i = 0, \dots, n-1, \\ 0 & t = x_i, \quad i = 0, \dots, n, \end{cases}$$

then, since $\phi\big(x_i, f_{n_k}(x_i)\big)$ converges to $\phi\big(x_i, f(x_i)\big)$, and $\phi\big(t, f_{n_k}(t)\big)$ converges to $\phi\big(t, f(t)\big)$, and $\phi\big(t, f(t)\big)$ is uniformly continuous on $[0,1]$, and the distance between the x_i and the t in the definition of Δ_n is less than $1/n$, it's not hard to see that $\Delta_{n_k}(x)$ will converge uniformly to 0 as $k \to \infty$.

Here are the gory details. Let $\varepsilon > 0$. There is a $\delta > 0$ such that if $|t_1 - t_2| < \delta$ we have

$$\big|\phi\big(t_1, f(t_1)\big) - \phi\big(t_2, f(t_2)\big)\big| < \varepsilon/3.$$

There is a K such that for $k > K$ we have $1/n_k < \delta$ and such that for all $t \in [0,1]$,

$$\big|\phi\big(t, f_{n_k}(t)\big) - \phi\big(t, f(t)\big)\big| < \varepsilon/3.$$

Then, for $k > K$ and for t such that $x_i < t < x_{i+1}$

$$\begin{aligned} \big|\Delta_{n_k}(t)\big| &= \big|\phi\big(x_i, f_{n_k}(x_i)\big) - \phi\big(t, f_{n_k}(t)\big| \\ &\le \big|\phi\big(x_i, f_{n_k}(x_i)\big) - \phi\big(x_i, f(x_i)\big)\big| + \big|\phi\big(x_i, f(x_i)\big) - \phi\big(t, f(t)\big)\big| + \big|\phi\big(t, f(t)\big) - \phi\big(t, f_{n_k}(t)\big)\big| \\ &\le \varepsilon. \end{aligned}$$

By the definition of f_n, $\Delta_n(t) = f_n'(t) - \phi\big(t, f_n(t)\big)$ for $x_i < t < t_{i+1}$, so

$$(*) \qquad f_{n_k}(x) = c + \int_0^x \phi\big(t, f_{n_k}(t)\big) + \Delta_{n_k}(t)\,dt.$$

Since $f_{n_k}(x)$ converges uniformly to $f(x)$ on $[0,1]$, and $\phi\big(t, f_{n_k}(t)\big)$ converges uniformly to $\phi\big(t, f(t)\big)$ on $[0,1]$, and $\Delta_{n_k}(t)$ converges uniformly to 0 on $[0,1]$, letting $k \to \infty$ in $(*)$, by Theorem 7.16 we have

$$f(x) = c + \int_0^x \phi\big(t, f(t)\big)\,dt.$$

Hence $f(0) = c$, and by Theorem 6.20, $f'(x) = \phi\big(x, f(x)\big)$ for $0 \le x \le 1$.

175. Exercise 26: Prove an analogous existence theorem for the initial-value problem

$$\mathbf{y}' = \mathbf{\Phi}(x, \mathbf{y}), \qquad \mathbf{y}(0) = \mathbf{c},$$

where now $\mathbf{c} \in \mathbf{R}^k$, $\mathbf{y} \in \mathbf{R}^k$, and $\mathbf{\Phi}$ is a continuous bounded mapping of the part of $\mathbf{R}^{k+1}$ defined by $0 \le x \le 1$, $\mathbf{y} \in \mathbf{R}^k$ into $\mathbf{R}^k$.

Solution: (*analambanomenos*)
Repeating the argument of the solution to Exercise 7.25, making changes where necessary, let n be a positive integer. For $i = 0, \dots, n$, put $x_i = i/n$. Let $\mathbf{f}_n$ be a continuous, piecewise-linear vector-valued function on $[0, 1]$ into $\mathbf{R}^k$ such that $\mathbf{f}_n(0) = \mathbf{c}$ and has slope $\mathbf{f}_n'(t) = \mathbf{\Phi}\big(x_i, \mathbf{f}_n(x_i)\big)$ if $x_i < t < x_{i+1}$.

Let $||\mathbf{\Phi}||$ be bounded by M, so that $||\mathbf{f}_n'|| \le M$. Note that

$$\int_{x_i}^{x_{i+1}} \mathbf{f}_n'(t)\,dt = \mathbf{f}_n(x_{i+1}) - \mathbf{f}_n(x_i),$$

so that for $0 \le x \le 1$, $\mathbf{f}_n(x) - \mathbf{c}$ is the sum of integrals of $\mathbf{f}_n'$ over intervals where it is defined. Hence

$$\big|\big|\mathbf{f}_n(x)\big|\big| \le ||\mathbf{c}|| + \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \big|\big|\mathbf{f}_n'(t)\big|\big|\,dt \le ||\mathbf{c}|| + M = M_1$$

so that $\{\mathbf{f}_n\}$ is uniformly bounded on $[0, 1]$. Let $\mathbf{f}_n = (f_{n1}, \dots, f_{nk})$. Then the continuous, piecewise-linear f_{nm} have slopes lying between $-M$ and M on their linear parts, so if $\varepsilon > 0$ then for $0 \le x \le 1$, $0 \le y \le 1$, $|x - y| \le \varepsilon/Mk$, we have $\big|f_{nm}(x) - f_{nm}(y)\big| \le \varepsilon/k$, so $\big|\big|\mathbf{f}_n(x)\big|\big| \le \varepsilon$. That is, $\{\mathbf{f}_n\}$ is equicontinuous on $[0, 1]$, using the extended definition of "equicontinuous" given in Exercise 17. Hence by the extended version Theorem 7.25 also given in Exercise 25, there is a subsequence $\{\mathbf{f}_{n_j}\}$ which converges uniformly to a continuous function $\mathbf{f}$ on $[0, 1]$.

By Theorem 4.19, $\mathbf{\Phi}$ is uniformly continuous on the compact parallelpiped R given by $0 \le x \le 1$, $||y|| \le M_1$. That is, if $\varepsilon > 0$, there is a $\delta > 0$ such that if the distance between the points $(x_1, \mathbf{y}_1)$ and $(x_2, \mathbf{y}_2)$ in R is less than δ, then $\big|\big|\mathbf{\Phi}(x_1, \mathbf{y}_1) - \mathbf{\Phi}(x_2, \mathbf{y}_2)\big|\big| < \varepsilon$. Since there is a K such that for all $k \ge K$ and all $t \in [0, 1]$, we have $\big|\big|\mathbf{f}_{n_k}(t) - \mathbf{f}(t)\big|\big| < \delta$, we have

$$\big|\big|\mathbf{\Phi}\big(t, \mathbf{f}_{n_k}(t)\big) - \mathbf{\Phi}\big(t, \mathbf{f}(t)\big)\big|\big| < \varepsilon.$$

That is, $\mathbf{\Phi}\big(t, \mathbf{f}_{n_k}(t)\big)$ converges uniformly to $\mathbf{\Phi}\big(t, \mathbf{f}(t)\big)$ as $k \to \infty$. Hence, if we let

$$\mathbf{\Delta}_n(t) = \begin{cases} \mathbf{\Phi}\big(x_i, \mathbf{f}_n(x_i)\big) - \mathbf{\Phi}\big(t, \mathbf{f}_n(t)\big) & x_i < t < t_{i+1}, \quad i = 0, \dots, n-1, \\ \mathbf{0} & t = x_i, \quad i = 0, \dots, n, \end{cases}$$

then, since $\mathbf{\Phi}\big(x_i, \mathbf{f}_{n_k}(x_i)\big)$ converges to $\mathbf{\Phi}\big(x_i, \mathbf{f}(x_i)\big)$, and $\mathbf{\Phi}\big(t, \mathbf{f}_{n_k}(t)\big)$ converges to $\mathbf{\Phi}\big(t, \mathbf{f}(t)\big)$, and $\mathbf{\Phi}\big(t, \mathbf{f}(t)\big)$ is uniformly continuous on $[0, 1]$, and the distance between the x_i and the t in the definition of $\mathbf{\Delta}_n$ is less than $1/n$, it's not hard to see that $\mathbf{\Delta}_{n_k}(x)$ will converge uniformly to $\mathbf{0}$ as $k \to \infty$.

Here are the gory details. Let $\varepsilon > 0$. There is a $\delta > 0$ such that if $|t_1 - t_2| < \delta$ we have

$$\big|\big|\mathbf{\Phi}\big(t_1, \mathbf{f}(t_1)\big) - \mathbf{\Phi}\big(t_2, f(t_2)\big)\big|\big| < \varepsilon/3.$$

There is a K such that for $k > K$ we have $1/n_k < \delta$ and such that for all $t \in [0, 1]$,

$$\big|\big|\mathbf{\Phi}\big(t, \mathbf{f}_{n_k}(t)\big) - \mathbf{\Phi}\big(t, \mathbf{f}(t)\big)\big|\big| < \varepsilon/3.$$

Then, for $k > K$ and for t such that $x_i < t < x_{i+1}$

$$\begin{aligned}\big|\big|\mathbf{\Delta}_{n_k}(t)\big|\big| &= \big|\big|\mathbf{\Phi}\big(x_i, \mathbf{f}_{n_k}(x_i)\big) - \mathbf{\Phi}\big(t, \mathbf{f}_{n_k}(t)\big|\big| \\ &\le \big|\big|\mathbf{\Phi}\big(x_i, \mathbf{f}_{n_k}(x_i)\big) - \mathbf{\Phi}\big(x_i, \mathbf{f}(x_i)\big)\big|\big| + \big|\big|\mathbf{\Phi}\big(x_i, \mathbf{f}(x_i)\big) - \mathbf{\Phi}\big(t, \mathbf{f}(t)\big)\big|\big| + \\ &\quad \big|\big|\mathbf{\Phi}\big(t, \mathbf{f}(t)\big) - \mathbf{\Phi}\big(t, \mathbf{f}_{n_k}(t)\big)\big|\big| \\ &\le \varepsilon.\end{aligned}$$

By the definition of $\mathbf{f}_n$, $\mathbf{\Delta}_n(t) = \mathbf{f}'_n(t) - \mathbf{\Phi}\big(t, \mathbf{f}_n(t)\big)$ for $x_i < t < t_{i+1}$, so

$$(*) \qquad \mathbf{f}_{n_k}(x) = \mathbf{c} + \int_0^x \mathbf{\Phi}\big(t, \mathbf{f}_{n_k}(t)\big) + \mathbf{\Delta}_{n_k}(t)\, dt.$$

Since $\mathbf{f}_{n_k}(x)$ converges uniformly to $\mathbf{f}(x)$ on $[0,1]$, and $\mathbf{\Phi}\big(t, \mathbf{f}_{n_k}(t)\big)$ converges uniformly to $\mathbf{\Phi}\big(t, \mathbf{f}(t)\big)$ on $[0,1]$, and $\mathbf{\Delta}_{n_k}(t)$ converges uniformly to $\mathbf{0}$ on $[0,1]$, letting $k \to \infty$ in $(*)$, by the extended version of Theorem 7.16 given in Exercise 17 we have

$$\mathbf{f}(x) = \mathbf{c} + \int_0^x \mathbf{\Phi}\big(t, \mathbf{f}(t)\big)\, dt.$$

Hence $\mathbf{f}(0) = \mathbf{c}$, and by Theorem 6.20, $\mathbf{f}'(x) = \mathbf{\Phi}\big(x, \mathbf{f}(x)\big)$ for $0 \le x \le 1$.

8 Some Special Functions

176. Exercise 1: Define

$$f(x) = \begin{cases} e^{1/x^2} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Prove that f has derivatives of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \ldots$.

Solution: (*Matt "Frito" Lundy*)
First note that according to the limit definition of the derivative:

$$\begin{aligned} f'(0) &= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \to 0} \frac{e^{-1/x^2}}{x} \\ &= 0 \end{aligned}$$

by theorem 8.6(f). This establishes a base case and we proceed with induction by noticing that for $x \neq 0$, $n \in \mathbf{N}$, we have:

$$f^{(n)}(x) = e^{-1/x^2} r_n(x)$$

where $r_n(x)$ is some rational function of x. Then, according to the limit definition of the derivative and using the inductive hypothesis that $f^{(n)}(0) = 0$ we have:

$$\begin{aligned} f^{(n+1)}(0) &= \lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{e^{-1/x^2} r_n(x)}{x} \\ &= 0 \end{aligned}$$

again using theorem 8.6(f). This establishes that $f(x)$ is infinitely differentiable at $x = 0$ and that $f^{(n)}(0) = 0$.

The point here is that even though $f(x)$ is infinitely differentiable at $x = 0$, it does not have a taylor series expansion about $x = 0$.

177. Exercise 2: Let a_{ij} be the number in the ith row and jth column of the array

$$\begin{matrix} -1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & -1 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{2} & -1 & 0 & \cdots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

so that

$$a_{ij} = \begin{cases} 0 & (i < j), \\ -1 & (i = j), \\ 2^{j-i} & (i > j). \end{cases}$$

Prove that

$$\sum_i \sum_j a_{ij} = -2, \qquad \sum_j \sum_i a_{ij} = 0.$$

Solution: (*Matt "Frito" Lundy*)
Along any row, the positive entries form a finite geometric sequence, and the sum of all the entries along a row is

$$\sum_{j=1}^{\infty} a_{ij} = -\left(\frac{1}{2}\right)^{i-1},$$

and so we have

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = -\sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^{i-1} = -2.$$

Whereas along any column, the positive entries form an infinite geometric sequence, and the sum of all the entries along a column is

$$\sum_{i=1}^{\infty} a_{ij} = -1 + \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n} = -1 + \left(\frac{1}{1-\frac{1}{2}} - 1\right) = 0,$$

so that

$$\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij} = 0.$$

Note that this double sequence $\{a_{ij}\}$ fails to meet the criteria of theorem 8.3 because if

$$\sum_{j=1}^{\infty} |a_{ij}| = b_i$$

then $\sum b_i$ diverges.

178. Exercise 3: Prove that

$$\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$$

if $a_{ij} \geq 0$ for all i and j (the case $+\infty = +\infty$ may occur).

Solution: (*analambanomenos*)
Let $\sum_j a_{ij} = b_i \in [0, \infty]$. If $\sum b_i$ diverges, then the conclusion follows from Theorem 8.3, so suppose $\sum b_i$ converges. Let $\sum_i a_{ij} = c_j$, we want to show that $\sum c_j = \sum b_i = \infty$. But if $\sum c_j$ converges, then we can define the transposed double sequence $\tilde{a}_{ij} = a_{ji}$, and let $\sum_j \tilde{a}_{ij} = \tilde{b}_i = c_i$. Since $\sum \tilde{b}_i = \sum c_j$ converges, we can apply Theorem 8.3 to conclude that $\sum \tilde{c}_i$ converges, where $\tilde{c}_i = \sum_j \tilde{a}_{ij} = b_i$. But this contradicts the assumption that $\sum b_i$ diverges.

179. Exercise 4: Prove the following limit relations:

(a) $\displaystyle\lim_{x\to 0} \frac{b^x - 1}{x} = \log(b) \quad (b > 0).$

(b) $\displaystyle\lim_{x\to 0} \frac{\log(1+x)}{x} = 1.$

(c) $\displaystyle\lim_{x\to 0} (1+x)^{1/x} = e.$

(d) $\displaystyle\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$

Solution: (*analambanomenos*)
(a) By L'Hospital's Rule, Theorem 5.13,

$$\lim_{x\to 0} \frac{b^x - 1}{x} = \lim_{x\to 0} \frac{(\log b)b^x}{1} = \log b.$$

(b) By L'Hospital's Rule,

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{x\to 0} \frac{1/(1+x)}{1} = 1.$$

Since the exponential function is continuous, if $g(x)$ is a continuous function, and $f(x) = \log\big(g(x)\big)$, and $\lim_{x\to a} f(x) = A$, then $\lim_{x\to a} g(x) = e^A$.
(c) Let $f(x) = \log(1+x)^{1/x} = \big(\log(1+x)\big)/x$. Then, from part (b),

$$\lim_{x\to 0} (1+x)^{1/x} = e^{\lim_{x\to 0} f(x)} = e^1 = e.$$

(d) Let $f(x) = \log\big(1 + (x/n)\big)^n = n\log\big(1 + (x/n)\big)$. By L'Hospital's Rule

$$\lim_{n\to\infty} f(x) = \lim_{n\to\infty} \frac{\log\left(1+\dfrac{x}{n}\right)}{1/n} = \lim_{n\to\infty} \frac{-\dfrac{x}{n^2}\left(\dfrac{1}{1+x/n}\right)}{-1/n^2} = \lim_{n\to\infty} \frac{x}{1+x/n} = x.$$

Hence $\lim_{n\to\infty}\big(1 + (x/n)\big)^n = e^x$.

180. Exercise 5: Find the following limits:

(a) $\displaystyle\lim_{x\to 0} \frac{e - (1+x)^{1/x}}{x}$.

(b) $\displaystyle\lim_{n\to\infty} \frac{n}{\log n}(n^{1/n} - 1)$.

(c) $\displaystyle\lim_{x\to 0} \frac{\tan x - x}{x(1-\cos x)}$.

(d) $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{\tan x - x}$.

Solution: (*analambanomenos*)
(a) Note from Exercise 3(c) that the limit of the numerator is 0, so by L'Hospital's Rule,

$$\begin{aligned}
\lim_{x\to 0} \frac{e - (1+x)^{1/x}}{x} &= -\lim_{x\to 0} \frac{d}{dx}\left(\frac{\log(1+x)}{x}\right)(1+x)^{1/x} \\
&= -e \lim_{x\to 0} \frac{x - (1+x)\log(1+x)}{x^2} && \text{(Exercise 3(c))} \\
&= -e \lim_{x\to 0} \frac{-\log(1+x)}{2x} && \text{(L'Hospital's Rule)} \\
&= -e \lim_{x\to 0} \frac{-1/(1+x)}{2} && \text{(L'Hospital's Rule)} \\
&= \frac{e}{2}.
\end{aligned}$$

(b) Applying L'Hospital's Rule to $\log(n^{1/n}) = \log n/n$, we get $\lim_{n\to\infty} \log n/n = \lim_{n\to\infty} 1/n = 0$, so that $\lim_{x\to\infty} n^{1/n} = e^0 = 1$. Hence we can apply L'Hospital's Rule to get

$$\lim_{n\to\infty} \frac{n^{1/n} - 1}{\log n/n} = \lim_{n\to\infty} \frac{n^{1/n}\dfrac{d}{dn}\left(\dfrac{\log n}{n}\right)}{\dfrac{d}{dn}\left(\dfrac{\log n}{n}\right)} = \lim_{n\to\infty} n^{1/n} = 1.$$

(c) Note that the derivatives of the numerator and denominator are

$$\begin{aligned}
f(x) &= \tan x - x \\
f'(x) &= \cos^{-2} x - 1 \\
f''(x) &= 2\cos^{-3} x \sin x \\
f'''(x) &= 6\cos^{-4} x \sin^2 x + 2\cos^{-2}(x) \\
g(x) &= x - x\cos x \\
g'(x) &= 1 - \cos x + x\sin x \\
g''(x) &= 2\sin x + x\cos x \\
g'''(x) &= 3\cos x - x\sin x
\end{aligned}$$

so that $f(0) = f'(0) = f''(0) = g(0) = g'(0) = g''(0) = 0$, while $f'''(0) = 2$ and $g'''(0) = 3$. Hence applying L'Hospital's Rule three times, we have

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{f''(x)}{g''(x)} = \lim_{x\to 0}\frac{f'''(x)}{g'''(x)} = \frac{2}{3}.$$

(d) Note that the derivatives of the numerator and denominator are

$$\begin{aligned}
f(x) &= x - \sin x \\
f'(x) &= 1 - \cos x \\
f''(x) &= \sin x \\
f'''(x) &= \cos x \\
g(x) &= \tan x - x \\
g'(x) &= \cos^{-2} x - 1 \\
g''(x) &= 2\cos^{-3} x \sin x \\
g'''(x) &= 6\cos^{-4} x \sin^2 x + 2\cos^{-2} x
\end{aligned}$$

so that $f(0) = f'(0) = f''(0) = g(0) = g'(0) = g''(0) = 0$, while $f'''(0) = 1$ and $g'''(0) = 2$. Hence applying L'Hospital's Rule three times, we have

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{f''(x)}{g''(x)} = \lim_{x\to 0}\frac{f'''(x)}{g'''(x)} = \frac{1}{2}.$$

181. Exercise 6: Suppose $f(x)f(y) = f(x+y)$ for all real x and y.

(a) Assuming that f is differentiable and not zero, prove that $f(x) = e^{cx}$ where c is a constant.

(b) Prove the same thing, assuming only that f is continuous.

Solution: (*analambanomenos*)
(a) By assumption, there is a real x such that $f(x) \neq 0$. Hence $f(x) = f(x+0) = f(x)f(0)$, showing that $f(0) = 1$. Since f is differentiable, we have for all real x

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{f(x)f(h) - f(x)}{h} = f(x)\lim_{h\to 0}\frac{f(h) - f(0)}{h} = f(x)f'(0).$$

So, letting $c = f'(0)$, we have $f'(x) = cf'(x)$ for all real x. Since this is also true for the function

$g(x) = e^{cx}$, we have

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{cfg - cfg}{g^2} = 0,$$

so that $f = Kg$ for some constant K. Since $f(0) = 1 = g(0)$, we have $K = 1$, or $f(x) = g(x) = e^{cx}$.

(b) For all rational numbers $p = m/n$, we have $f(p) = f(1)^{m/n}$. Let $c = \log f(1)$, and let $g(x) = e^{cx}$. Since we also have $g(p) = g(1)^{m/n}$ and $g(1) = e^{\log f(1)} = f(1)$, we have $f(x) = g(x)$ on all rational numbers. Hence, since the continuous functions f and g are equal on a dense subset of the real numbers, they are equal for all real numbers.

182. Exercise 7: If $0 < x < \pi/2$ prove that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1.$$

Solution: (*analambanomenos*)
Note that by L'Hospital's Rule,

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

If $f(x) = \sin x/x$, then

$$f'(x) = \frac{x\cos x - \sin x}{x^2}.$$

Letting $g(x) = x\cos x - \sin x$, the numerator in the expression above for $f'(x)$, we have $g'(x) = -x\sin x \le 0$ for $0 \le x \le \pi/2$, so that $g(x)$ decreases in this interval from $g(0) = 0$ to $g(\pi/2) = -1$. Hence $f'(x) < 0$ for $0 < x < \pi/2$, so that it decreases steadily in this interval, and its values lie between $\lim_{x\to 0} f(x) = 1$ and $f(\pi/2) = 2/\pi$.

183. Exercise 8: For $n = 0, 1, 2, \ldots$, and x real, prove that $|\sin nx| \le n|\sin x|$.

Solution: (*analambanomenos*)
Since the functions on both sides of the inequality are periodic functions with period π, and since they are both even functions, we only have to show this for $x \in [0, \pi/2]$.

Let $f(x) = n\sin x - \sin nx$, $f(0) = 0$, $f'(x) = n(\cos x - \cos nx)$. Since $\cos x$ is monotonically decreasing on $[0, \pi/2]$, we have $f'(x) \ge 0$ for $x \in \big[0, \pi/(2n)\big]$, hence $f(x) \ge 0$ in this interval, that is, $n\sin x \ge \sin nx$.

For $x \in [\pi/(2n), \pi/2]$, since $\sin x$ is monotonically increasing, we have $n\sin x \ge n\sin\big(\pi/(2n)\big) \ge \sin\big(n\pi/(2n)\big) = 1$, while $|\sin nx| \le 1$. Hence $n\sin x \ge |\sin nx|$ on all of $[0, \pi/2]$.

Solution: (*Dan "kyp44" Whitman*)
We show this by induction. So for $n = 0$ we clearly have

$$|\sin nx| = |\sin 0| = 0 \le 0 = 0 \cdot |\sin x| = n\,|\sin x|\ .$$

Now assume that $|\sin nx| \le n\,|\sin x|$. Then

$$\begin{aligned}|\sin(n+1)x| &= |\sin(nx+x)| = |\sin nx\cos x + \cos nx \sin x| \\ &\le |\sin nx \cos x| + |\cos nx \sin x| = |\sin nx|\,|\cos x| + |\cos nx|\,|\sin x| \\ &\le n\,|\sin x|\,|\cos x| + |\cos nx|\,|\sin x| \le n\,|\sin x| \cdot 1 + 1 \cdot |\sin x| \\ &= (n+1)\,|\sin x|\ ,\end{aligned}$$

where we have used the angle addition formula, which is trivial to derive. This completes the inductive proof. ■

184. Exercise 9: (a) Put $s_N = 1 + (1/2) + \cdots + (1/N)$. Prove that

$$\lim_{N\to\infty} (s_N - \log N)$$

exists.

(b) Roughly how large must m be so that $N = 10^m$ satisfies $s_N > 100$.

Solution: (*analambanomenos*)
(a) Since the minimum and maximum values of $1/x$ in the interval $[n, n+1]$ are $1/(n+1)$ and $1/n$, respectively, we have $1/(n+1) \le \int_n^{n+1} (1/x)\,dx = \log(n+1) - \log(1) \le 1/n$. Hence,

$$\big(s_{N+1} - \log(N+1)\big) - (s_N - \log N) = \frac{1}{N} - \int_N^{N+1} \frac{1}{x}\,dx \ge 0,$$

so that the sequence $s_N - \log N$ is nondecreasing. Also, since $\log(1) = 0$, we have

$$\begin{aligned}\frac{1}{2} + \cdots + \frac{1}{N} &\le \sum_{n=0}^{N-1} \big(\log(n+1) - \log(n)\big) = \log N \\ -\log N &\le -\frac{1}{2} - \cdots - \frac{1}{N} \\ s_N - \log N &\le 1.\end{aligned}$$

so that the sequence is also bounded above. Hence by Theorem 3.14 the limit of the sequence exists.

(b) From part (a), $s_N \approx \gamma + \log N$. Using the estimate $\gamma \approx 0.577$ and solving for $100 \approx \gamma + \log 10^m$, we get the approximate solution $m = 43$.

185. Exercise 10: Prove that $\sum 1/p$ diverges, the sum extends over all primes.

Solution: (*analambanomenos*)
Following the hint, let $p_1, \ldots, p_k$ be distinct primes. Then the product of the convergent geometric series of p_k^{-1} yields

$$\prod_{j=1}^{k} \left(1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \cdots\right) = \sum \frac{1}{p_1^{n_1} \cdots p_k^{n_k}}$$

where the sum is over all ordered k-tuples $(n_1, \ldots, n_k)$ of non-negative integers.

Hence if N is a positive integer, and $p_1, \ldots, p_k$ are the prime numbers that divide at least one integer

$\leq N$, we have, using Theorem 3.26,

$$\sum_{n=1}^{N} \frac{1}{n} \leq \prod_{j=1}^{k} \left(1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \cdots\right) = \prod_{j=1}^{k} \left(1 - \frac{1}{p_j}\right)^{-1}.$$

Let $f(x) = 2x + \log(1-x)$ for $x \in [0, 1/2]$. Then $f'(x) = (1-2x)/(1-x) \geq 0$ in this interval. Since $f(0) = 0$, we have $f(x) \geq 0$ for $0 \leq x \leq 1/2$, or

$$\begin{aligned} 0 &\leq 2x + \log(1-x) \\ \log(1-x)^{-1} &\leq 2x \\ (1-x)^{-1} &\leq e^{2x}. \end{aligned}$$

Applying this to the above, we get

$$\begin{aligned} \sum_{n=1}^{N} \frac{1}{n} &\leq \prod_{j=1}^{k} \left(1 - \frac{1}{p_j}\right)^{-1} \\ &\leq \prod_{j=1} e^{2/p_j} \\ &= \exp\left(2\sum_{j=1}^{k} \frac{1}{p_j}\right). \end{aligned}$$

Since the harmonic series diverges, so must the series in the argument of exp above.

186. Exercise 11: Suppose $f \in \mathscr{R}$ on $[0, A]$ for all $A < \infty$, and $f(x) \to 1$ as $x \to +\infty$. Prove that

$$\lim_{t \to 0} \int_0^\infty e^{-tx} f(x)\, dx = 1 \qquad (t > 0).$$

Solution: (*analambanomenos*)
Since $|e^{-tx}| \leq 1$ for $x \geq 0$ and $t > 0$, we have, for $A > 0$ and $t > 0$

$$(*) \qquad \lim_{t\to 0} \left| t \int_0^A e^{-tx} f(x)\, dx \right| \leq \lim_{t \to 0} t \int_0^A \big|f(x)\big|\, dx = 0.$$

Let $\varepsilon > 0$ and let A be large enough so that $1 - \varepsilon < f(x) < 1 + \varepsilon$ for $x \geq A$. For any constant K we have, for $t > 0$,

$$t \int_A^B K e^{-tx}\, dx = K(e^{-tA} - e^{-tB}) \to K e^{-tA} \text{ as } B \to \infty.$$

Hence

$$(1-\varepsilon)e^{-tA} = t\int_A^\infty (1-\varepsilon) e^{-tx}\, dx \leq t \int_A^\infty e^{-tx} f(x)\, dx \leq t \int_A^\infty (1+\varepsilon) e^{-tx}\, dx = (1+\varepsilon)e^{-tA}.$$

Letting $t \to 0+$, we get

$$(1-\varepsilon) \leq \lim_{t \to 0} t \int_A^\infty e^{-tx} f(x)\, dx \leq (1+\varepsilon).$$

Combining this with $(*)$, we get

$$(1-\varepsilon) \leq \lim_{t\to 0} t \int_0^\infty e^{-tx} f(x)\,dx \leq (1+\varepsilon).$$

Since ε was arbitrary, we finally get

$$\lim_{t\to 0} t \int_0^\infty e^{-tx} f(x)\,dx = 1.$$

187. Exercise 12: Suppose $0 < \delta < \pi$, $f(x) = 1$ if $|x| \leq \delta$, $f(x) = 0$ if $\delta < |x| \leq \pi$, and $f(x+2\pi) = f(x)$ for all x.

(a) Compute the Fourier coefficients of f.

(b) Conclude that

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2}.$$

(c) Deduce from Parseval's theorem that

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi - \delta}{2}.$$

(d) Let $\delta \to 0$ and prove that

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}.$$

(e) Put $\delta = \pi/2$ in (c). What do you get?

Solution: (*analambanomenos*)
(a) For $n \neq 0$

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\delta}^{\delta} e^{-inx}\,dx \\ &= \frac{1}{2\pi} \cdot \frac{1}{in}(e^{in\delta} - e^{-in\delta}) \\ &= \frac{\sin n\delta}{\pi n} \\ c_0 &= \frac{1}{2\pi} \int_{-\delta}^{\delta} 1\,dx \\ &= \frac{\delta}{\pi} \end{aligned}$$

(b) By Theorem 8.14, the Fourier series for $f(x)$ converges for $x = 0$ to $f(0) = 1$, hence (noting that

$(\sin x)/x$ is an even function)

$$
\begin{aligned}
1 &= \frac{\delta}{\pi} + \sum_{|n|\neq 0} \frac{\sin n\delta}{\pi n} \\
1 - \frac{\delta}{\pi} &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\delta}{n} \\
\frac{\pi - \delta}{2} &= \sum_{n=1}^{\infty} \frac{\sin n\delta}{n}
\end{aligned}
$$

(c) We have by Parseval's theorem, using $c_{-n} = c_n$ for $n \neq 0$,

$$
\begin{aligned}
\frac{1}{2\pi} \int_{-\delta}^{\delta} \left|f(x)\right|^2 dx &= \sum_{-\infty}^{\infty} |c_n|^2 \\
\frac{1}{2\pi} \int_{-\delta}^{\delta} 1\, dx &= \frac{\delta^2}{\pi^2} + 2\sum_{n=1}^{\infty} \frac{\sin^2 n\delta}{\pi^2 n^2} \\
\frac{\delta}{\pi} - \frac{\delta^2}{\pi^2} &= \frac{2\delta}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin^2 n\delta}{n^2\delta} \\
\frac{\pi^2}{2\delta} \cdot \frac{\delta(\pi - \delta)}{\pi^2} &= \sum_{n=1}^{\infty} \frac{\sin^2 n\delta}{n^2\delta} \\
\frac{\pi - \delta}{2} &= \sum_{n=1}^{\infty} \frac{\sin^2 n\delta}{n^2\delta}
\end{aligned}
$$

(d) First note that by L'Hospital's Rule,

$$
\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{x} = 1,
$$

so that the integral

$$
\int_0^A \left(\frac{\sin x}{x}\right)^2 dx
$$

is well-defined. Also, for $\delta > 0$,

$$
\begin{aligned}
\lim_{A\to\infty} \int_{\delta}^{A} \left|\frac{\sin x}{x}\right|^2 dx &\leq \lim_{A\to\infty} \int_{\delta}^{A} \frac{1}{x^2}\, dx \\
&= \lim_{A\to\infty} \left(\frac{1}{\delta} - \frac{1}{A}\right) \\
&= \frac{1}{\delta}
\end{aligned}
$$

so that the improper integral converges. That is, if $\varepsilon > 0$, there is an $A > 0$ large enough so that

$$
\left| \int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx - \int_0^{A} \left(\frac{\sin x}{x}\right)^2 dx \right| \leq \frac{\varepsilon}{4}.
$$

Let $\delta = A/M$ for some large integer M, and let $P = \{0, \delta, \ldots, M\delta = A\}$ be a partition of $[0, A]$. Then

$$
\sum_{n=1}^{M} \frac{\sin^2(n\delta)}{n^2\delta^2} \cdot \delta = \sum_{n=1}^{M} \frac{\sin^2(n\delta)}{n^2\delta}
$$

is a Riemann sum of the finite integral. Hence there is an M large enough so that

$$\left| \int_0^A \left(\frac{\sin x}{x} \right)^2 dx - \sum_{n=1}^{M} \frac{\sin^2(n\delta)}{n^2\delta} \right| \leq \frac{\varepsilon}{4}.$$

From part (c), we can make M large enough so that

$$\left| \sum_{n=1}^{M} \frac{\sin^2(n\delta)}{n^2\delta} - \frac{\pi - \delta}{2} \right| \leq \frac{\varepsilon}{4}$$

and we can make M large enough so that $\delta/2 \leq \varepsilon/4$, so that

$$\left| \frac{\pi - \delta}{2} - \frac{\pi}{2} \right| \leq \frac{\varepsilon}{4}.$$

Putting together these four inequalities, we get that, for all $\varepsilon > 0$,

$$\left| \int_0^\infty \left(\frac{\sin x}{x} \right)^2 dx - \frac{\pi}{2} \right| \leq \varepsilon,$$

so that the equality follows.

(e) Since $\sin^2(n\pi/2) = 1$ if n is odd, and 0 otherwise, we get from part (c) that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi}{2} \cdot \frac{\pi - (\pi/2)}{2} = \frac{\pi^2}{8}.$$

188. Exercise 13: Put $f(x) = x$ if $0 \leq x < 2\pi$, and apply Parseval's theorem to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution: (*analambanomenos*)

The Fourier coefficients for f are (letting $n \neq 0$ and using $e^{i2\pi n} = 1$)

$$\begin{aligned}
c_0 &= \frac{1}{2\pi}\int_0^{2\pi} x\,dx \\
&= \frac{1}{2\pi}\left(\frac{(2\pi)^2 - 0}{2}\right) \\
&= \pi \\
c_n &= \frac{1}{2\pi}\int_0^{2\pi} xe^{-inx}\,dx \\
&= \frac{1}{2\pi}\left(\frac{2\pi}{-in} - 0\right) - \frac{1}{2\pi}\cdot\frac{1}{-in}\int_0^{2\pi} e^{-inx}\,dx \\
&= -\frac{1}{in} + \frac{1}{2\pi in}\cdot\frac{1-1}{-in} \\
&= \frac{i}{n}.
\end{aligned}$$

Hence by Parseval's theorem

$$\begin{aligned}
\sum_{-\infty}^{\infty} |c_n|^2 &= \frac{1}{2\pi}\int_0^{2\pi} x^2\,dx \\
\pi^2 + 2\sum_1^{\infty}\frac{1}{n^2} &= \frac{4\pi^2}{3} \\
\sum_1^{\infty}\frac{1}{n^2} &= \frac{1}{2}\left(\frac{4\pi^2}{3} - \pi^2\right) = \frac{\pi^2}{6}
\end{aligned}$$

189. Exercise 14: If $f(x) = \big(\pi - |x|\big)^2$ on $[-\pi, \pi]$, prove that

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx$$

and deduce that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

Solution: (*analambanomenos*)
The maximum local rate of change of f will occur at the multiples of 2π, where $\lim |f'| = 2\pi$. Hence f satisfies the condition of Theorem 8.14 with constant $M = 2\pi$, so we can conclude that the Fourier series of f converges to $f(x)$ for all x.

Here are the Fourier coefficients of f (where $n \neq 0$, and using $e^{in\pi} = e^{-in\pi}$ for all integers n):

$$\begin{aligned}
c_0 &= \frac{1}{2\pi}\int_{-\pi}^{0} (\pi + x)^2\, dx + \frac{1}{2\pi}\int_0^{\pi} (\pi - x)^2\, dx \\
&= \frac{1}{2\pi}\left(\frac{\pi^3}{3} + \frac{\pi^3}{3}\right) \\
&= \frac{\pi^2}{3} \\
c_n &= \frac{1}{2\pi}\int_{-\pi}^{0} (\pi^2 + 2\pi x + x^2)e^{-inx}\, dx + \frac{1}{2\pi}\int_0^{\pi} (\pi^2 - 2\pi x + x^2)e^{-inx}\, dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi^2 + x^2)e^{-inx}\, dx + \int_{-\pi}^{0} xe^{-inx}\, dx - \int_0^{\pi} xe^{-inx}\, dx
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi^2 + x^2)e^{-inx}\, dx &= \frac{1}{2\pi}\cdot\frac{i}{n}(2\pi^2 e^{-in\pi} - 2\pi^2 e^{in\pi}) - \frac{1}{2\pi}\cdot\frac{2i}{n}\int_{-\pi}^{\pi} xe^{-inx}\, dx \\
&= 0 - \frac{i}{\pi n}\cdot\frac{i}{n}(\pi e^{-in\pi} + \pi e^{in\pi}) + \frac{i}{\pi n}\cdot\frac{i}{n}\int_{-\pi}^{\pi} e^{-inx}\, dx \\
&= \frac{1}{n^2}(e^{-in\pi} + e^{in\pi}) - \frac{i}{\pi n^2}(e^{-in\pi} - e^{in\pi}) \\
&= \frac{1}{n^2}(e^{-in\pi} + e^{in\pi})
\end{aligned}$$

$$\begin{aligned}
\int_{-\pi}^{0} xe^{-inx}\, dx &= \frac{i}{n}(0e^0 + \pi e^{in\pi}) - \frac{i}{n}\int_{-\pi}^{0} e^{-inx}\, dx \\
&= \frac{i\pi}{n}e^{in\pi} - \frac{i}{n}\cdot\frac{i}{n}(e^0 - e^{in\pi}) \\
&= \frac{i\pi}{n}e^{in\pi} + \frac{1}{n^2}(1 - e^{in\pi})
\end{aligned}$$

$$\begin{aligned}
\int_0^{\pi} xe^{-inx}\, dx &= \frac{i}{n}(\pi e^{-in\pi} - 0e^0) - \frac{i}{n}\int_0^{\pi} e^{-inx}\, dx \\
&= \frac{i\pi}{n}e^{-in\pi} - \frac{i}{n}\cdot\frac{i}{n}(e^{-in\pi} - e^0) \\
&= \frac{i\pi}{n}e^{-in\pi} + \frac{1}{n^2}(e^{-in\pi} - 1)
\end{aligned}$$

$$c_n = \frac{1}{n^2}(e^{-in\pi} + e^{in\pi}) + \frac{i\pi}{n}e^{in\pi} + \frac{1}{n^2}(1 - e^{in\pi}) - \frac{i\pi}{n}e^{-in\pi} - \frac{1}{n^2}(e^{-in\pi} - 1) = \frac{2}{n^2}$$

Hence

$$\begin{aligned} f(x) &= \frac{\pi^2}{3} + \sum_{|n|\neq 0} \frac{2}{n^2} e^{inx} \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2}(e^{inx} + e^{-inx}) \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx \end{aligned}$$

Letting $n = 0$, we get

$$\begin{aligned} \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} &= \pi^2 \\ \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{6} \end{aligned}$$

And by Parseval's theorem, we get

$$\begin{aligned} \frac{\pi^4}{9} + 2\sum_{n=1}^{\infty} \frac{4}{n^4} &= \frac{1}{2\pi}\int_{-\pi}^{0} (\pi + x)^4 \, dx + \frac{1}{2\pi}\int_{0}^{\pi} (\pi - x)^4 \, dx \\ \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty} \frac{1}{n^4} &= \frac{1}{2\pi} \cdot \frac{1}{5}(\pi^5 + \pi^5) \\ \sum_{n=1}^{\infty} \frac{1}{n^4} &= \frac{1}{8}\left(\frac{\pi^4}{5} - \frac{\pi^4}{9}\right) \\ \sum_{n=1}^{\infty} \frac{1}{n^4} &= \frac{\pi^4}{90} \end{aligned}$$

190. Exercise 15: With D_n as defined in (77), put

$$K_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} D_n(x).$$

Prove that

$$K_N(x) = \frac{1}{N+1} \cdot \frac{1 - \cos(N+1)x}{1 - \cos x}$$

and that

(a) $K_N \geq 0$,

(b) $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} K_N(x)\, dx = 1$,

(c) $K_N(x) \leq \dfrac{1}{N+1} \cdot \dfrac{2}{1 - \cos\delta}$ if $0 < \delta \leq |x| \leq \pi$.

If $s_N = s_N(f; x)$ is the Nth partial sum ofthe Fourier series of f, consider the arithmetic means

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_N}{N+1}.$$

Prove that

$$\sigma_N(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)\,dt,$$

and hence prove Fejér's theorem: *If f is continuous, with period 2π, then $\sigma_N(f;x) \to f(x)$ uniformly on $[-\pi,\pi]$.*

Solution: (*analambanomenos*)
We want to show that

$$\sum_{n=0}^{N} D_n(x) = \frac{1-\cos(N+1)x}{1-\cos x}.$$

I'll show this by induction. It is true for the case $N=0$ since both sides equal 1, so assume that

$$\sum_{n=0}^{N-1} D_n(x) = \frac{1-\cos Nx}{1-\cos x}.$$

Then we have

$$\begin{aligned}
(1-\cos x)\sum_{n=0}^{N} D_n(x) &= (1-\cos x)\sum_{n=0}^{N-1} D_n(x) + (1-\cos x)D_N(x) \\
&= (1-\cos Nx) + \frac{(1-\cos x)\sin(N+1/2)x}{\sin(x/2)} \\
&= (1-\cos Nx) + \frac{2\sin^2(x/2)\sin(N+1/2)x}{\sin(x/2)} \\
&= (1-\cos Nx) + 2\sin(x/2)\sin(N+1/2)x \\
&= (1-\cos Nx) + \big(\cos Nx - \cos(N+1)x\big) \\
&= 1-\cos(N+1)x
\end{aligned}$$

Dividing both sides by $(1-\cos x)$ gives the desired result.

(a) Since $\cos x \le 1$ for all x, we have $1-\cos(N+1)x \ge 0$ and $1-\cos x \ge 0$, hence $K_N(x) \ge 0$.

(b) For $n \ne 0$,

$$\int_{-\pi}^{\pi} e^{inx}\,dx = \frac{1}{in}(e^{in\pi} - e^{-in\pi}) = 0$$

since $e^{in\pi} = e^{-in\pi}$ for all all integers n. Hence

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^0\,dx = 1,$$

and so

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)\,dx = \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x)\,dx = \frac{1}{N+1}(N+1) = 1.$$

(c) Since $1-\cos x$ monotonically decreases from 2 to 0 on $[-\pi,0]$ and monotonically increases from 0 to 2 on $[0,\pi]$, we have $1-\cos x \ge 1-\cos\delta$ on $[-\pi,-\delta]\cup[\delta,\pi]$. Also, the maximum value of $1-\cos(N+1)x$ is 2, so

$$K_N(x) = \frac{1}{N+1}\cdot\frac{1-\cos(N+1)x}{1-\cos x} \le \frac{1}{N+1}\cdot\frac{2}{1-\cos\delta}$$

for $0 < \delta \le |x| \le \pi$.

By (78) in the text, we have

$$\begin{aligned}
\sigma_N(f;x) &= \frac{1}{N+1}\sum_{n=0}^{N} s_n(f;x) \\
&= \frac{1}{N+1}\sum_{n=0}^{N} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\left(\frac{1}{N+1}\sum_{n=0}^{N} D_N(t)\right)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)\,dt
\end{aligned}$$

Let $\varepsilon > 0$. Since $f(x)$ is uniformly continuous on $[-\pi,\pi]$, there is a $\delta > 0$ such that $\big|f(x-t)-f(t)\big| < \varepsilon/2$ if $|t| \le \delta$. Also, $f(x)$ has a maximum value M on $[-\pi,\pi]$. Using (a), (b), and (c), we have

$$\begin{aligned}
\big|f(x)-\sigma_N(f;x)\big| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\big(f(x)-f(x-t)\big)K_N(t)\,dt\right| \\
&\le \frac{1}{2\pi}\int_{-\pi}^{\pi}\big|f(x)-f(x-t)\big|K_N(t)\,dt \\
&\le \frac{1}{2\pi}\int_{-\pi}^{-\delta} 2M\cdot\frac{1}{N+1}\cdot\frac{2}{1-\cos\delta}\,dx + \frac{1}{2\pi}\int_{-\delta}^{\delta}\frac{\varepsilon}{2}K_N(t)\,dt + \\
&\quad \frac{1}{2\pi}\int_{\delta}^{\pi} 2M\cdot\frac{1}{N+1}\cdot\frac{2}{1-\cos\delta}\,dx \\
&\le \frac{4M}{N+1}\cdot\frac{1}{1-\cos\delta} + \frac{\varepsilon}{2} \\
&\le \varepsilon
\end{aligned}$$

for large enough N. That is, $\sigma_N(f;x) \to f(x)$ uniformly on $[-\pi,\pi]$.

191. Exercise 16: Prove a pointwise version of Fejér's theorem: *If $f \in \mathscr{R}$ and $f(x+)$, $f(x-)$ exist for some x, then*

$$\lim_{N\to\infty}\sigma_N(f;x) = \frac{f(x+)+f(x-)}{2}.$$

Solution: (*analambanomenos*)
Let $\varepsilon > 0$ and let $\delta > 0$ such that $\big|f(x-t)-f(x+)\big| < \varepsilon/2$ for $-\delta < t < 0$ and $\big|f(x-t)-f(x-)\big| < \varepsilon/2$ for $0 < t < \delta$. Since

$$K_N(x) = \frac{1}{N+1}\cdot\frac{1-\cos(N+1)x}{1-\cos x}$$

is an even function, we have from Exercise 15(b) that

$$\frac{1}{2\pi}\int_{-\pi}^{0} K_N(t)\,dt = \frac{1}{2\pi}\int_{0}^{\pi} K_N(t)\,dt = \frac{1}{2}.$$

Hence by the results of Exercise 15,

$$\begin{aligned}
&\left|\sigma_N(f;x) - \frac{f(x+)+f(x-)}{2}\right| \\
&\qquad \le \frac{1}{2\pi}\int_{-\pi}^{0} \bigl|f(x-t)-f(x+)\bigr| K_N(t)\,dt + \frac{1}{2\pi}\int_{0}^{\pi} \bigl|f(x-t)-f(x-)\bigr| K_N(t)\,dt \\
&\qquad \le \frac{1}{2\pi}\frac{1}{N+1}\frac{2}{1-\cos\delta}\left(\int_{-\pi}^{-\delta} \bigl|f(x-t)-f(x+)\bigr|\,dt + \int_{\delta}^{\pi} \bigl|f(x-t)-f(x-)\bigr|\,dt\right) + \\
&\qquad \frac{\varepsilon}{2}\left(\frac{1}{2\pi}\int_{-\delta}^{0} K_N(t)\,dt + \frac{1}{2\pi}\int_{0}^{\delta} K_N(t)\,dt\right)
\end{aligned}$$

The two integrals in the first term are finite, so we can make it as small as we would like as $N \to \infty$, and the second term is less than $\varepsilon/2$. Hence $\sigma_N(f;x) \to \bigl(f(x+)+f(x-)\bigr)/2$ as $N \to \infty$.

192. Exercise 17: Assume f is bounded and monotonic on $[-\pi,\pi)$, with Fourier coefficients c_n as given by (62).

(a) Use Exercise 17 of Chapter 6 to prove that $\{nc_n\}$ is a bounded sequence.

(b) Combine (a) with Exercise 16 and with Exercise 14(e) of Chapter 3, to conclude that

$$\lim_{N\to\infty} s_N(f;x) = \frac{f(x+)+f(x-)}{2}$$

for every x.

(c) Assume only that $f \in \mathscr{R}$ on $[-\pi,\pi]$ and that f is monotonic in some segment $(\alpha,\beta) \subset [-\pi,\pi]$. Prove that the conclusion of (b) holds for every $x \in (\alpha,\beta)$.

Solution: (*analambanomenos*)
(a) Using Exercise 6.17, with $G(x) = (i/n)e^{-inx}$,

$$\begin{aligned}
nc_n &= \frac{n}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \\
&= \frac{i}{2\pi}\bigl(f(\pi)e^{-in\pi} - f(-\pi)e^{in\pi}\bigr) - \frac{i}{2\pi}\int_{-\pi}^{\pi} e^{-inx}\,df \\
|nc_n| &\le \frac{1}{2\pi}\bigl(f(\pi)-f(-\pi)\bigr) + \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\,df \\
&= \frac{1}{\pi}\bigl(f(\pi)-f(-\pi)\bigr)
\end{aligned}$$

(b) Since $|nc_n| \le M = \bigl(f(\pi)-f(-\pi)\bigr)/\pi$, we can apply Exercise 3.14(e) to get

$$\lim_{N\to\infty} s_N(f;x) = \lim_{N\to\infty} \sigma_N(f;x) = \frac{f(x+)+f(x-)}{2}.$$

The last equality is from Exercise 16.

(c) Letting

$$\tilde{f}(x) = \begin{cases} f(\alpha) & -\pi \le x \le \alpha \\ f(x) & \alpha \le x \le \beta \\ f(\beta) & \beta \le x \le \pi \end{cases}$$

then $\tilde{f}$ is monotonically increasing on $[-\pi, \pi]$, so by part (b) we have

$$\lim_{N\to\infty} s_N(\tilde{f}; x) = \frac{\tilde{f}(x+) + \tilde{f}(x-)}{2}$$

for all $-\pi \le x \le \pi$. Hence by the localization theorem, since $f(x) = \tilde{f}(x)$ for $\alpha < x < \beta$, we have

$$\lim_{N\to\infty} s_N(f; x) = \lim_{N\to\infty} s_N(\tilde{f}; x) = \frac{\tilde{f}(x+) + \tilde{f}(x-)}{2} = \frac{f(x+) + f(x-)}{2}$$

for $\alpha < x < \beta$.

193. Exercise 18: Define

$$\begin{aligned} f(x) &= x^3 - \sin^2 x \tan x \\ g(x) &= 2x^2 - \sin^2 x - x \tan x. \end{aligned}$$

Find out, for each of these two functions, whether it is positive or negative for all $x \in (0, \pi/2)$, or whether it changes sign. Prove your answer.

Solution: (*analambanomenos*)
Note that $(d/dx) \tan x = n \tan^{n-1} x + n \tan^{n+1} x$.

$$\begin{aligned} f(x) &= x^3 - (1 - \cos^2 x) \tan x \\ &= x^3 + 2^{-1} \sin 2x - \tan x \\ f'(x) &= 3x^2 + \cos 2x - 1 - \tan^2 x \\ f''(x) &= 6x - 2\sin 2x - 2\tan x - 2\tan^3 x \\ f'''(x) &= 6 - 4\cos 2x - 2 - 8\tan^2 x - 6\tan^4 x \\ f^{(4)}(x) &= 8\sin 2x - 16\tan x - 40\tan^3 x - 24\tan^5 x \\ f^{(5)}(x) &= 16\cos 2x - 16 - 136\tan^2 x - 240\tan^4 x - 120\tan^6 x \\ f^{(6)}(x) &= -32\sin 2x - 272\tan x - 1232\tan^3 x - 1680\tan^5 x - 720\tan^7 x \end{aligned}$$

Note that $f(0) = f'(0) = f''(0) = f'''(0) = f^{(4)}(0) = f^{(5)}(0) = 0$. Since $f^{(6)}(x) < 0$ on $(0, \pi/2)$, $f^{(5)}$ decreases monotonically throughout that interval from $f^{(5)}(0) = 0$ to $f^{(5)}(\pi/2) = -\infty$, hence the same is successively true for $f^{(4)}$, f''', f'', f', and f.

$$\begin{aligned} g(x) &= 2x^2 - \sin^2 x - x\tan x \\ g'(x) &= 4x - \sin 2x - \tan x - x(1 + \tan^2 x) \\ g''(x) &= 4 - 2\cos 2x - 2 - 2\tan^2 x - x(2\tan x + 2\tan^3 x) \\ g'''(x) &= 4\sin 2x - 6\tan x - 6\tan^3 x - x(2 + 8\tan^2 x + 6\tan^3 x) \\ g^{(4)}(x) &= 8\cos 2x - 8 - 32\tan^2 x - 24\tan^4 x - x(16\tan x + 40\tan^3 x + 24\tan^5 x) \\ g^{(5)}(x) &= -16\sin 2x - 80\tan x - 200\tan^3 x - 120\tan^5 x - \\ &\quad x(16 + 136\tan^2 x + 240\tan^3 x + 120\tan^6 x) \end{aligned}$$

Note that $g(0) = g'(0) = g''(0) = g'''(0) = g^{(4)}(0) = 0$. Since $g^{(5)}(x) < 0$ on $(0, \pi/2)$, $g^{(4)}$ decreases monotonically throughout that interval from $g^{(4)}(0) = 0$ to $g^{(4)}(\pi/2) = -\infty$, hence the same is successively true for g''', g'', g', and g.

194. Exercise 19: Suppose f is a continuous function on $\mathbf{R}$, $f(x+2\pi) = f(x)$, and α/π is irrational. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x+n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt$$

for every x.

Solution: (*analambanomenos*)
Following the hint, we first show this for trigonometric polynomials. Let

$$P(x) = \sum_{m=-M}^{M} c_m e^{imx}.$$

Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P(t)\, dt = \sum_{m=-M}^{M} c_m \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{imt}\, dt = c_0$$

and (noting that $e^{im\alpha} \neq 1$ for $m \neq 0$ since α/π is irrational)

$$\begin{aligned}
\frac{1}{N} \sum_{n=1}^{N} P(x+n\alpha) &= \frac{1}{N} \sum_{n=1}^{N} \sum_{m=-M}^{M} c_m e^{im(x+n\alpha)} \\
&= \sum_{m=-M}^{M} c_m e^{im\alpha} \frac{1}{N} \sum_{n=1}^{N} e^{(im\alpha)n} \\
&= c_0 + \sum_{\substack{m=-M \\ m\neq 0}}^{M} c_m e^{im\alpha} \frac{1}{N} \frac{e^{im\alpha} - e^{im\alpha(N+1)}}{1 - e^{im\alpha}} \\
&= c_0 + \sum_{\substack{m=-M \\ m\neq 0}}^{M} c_m e^{im(x+\alpha)} \frac{1}{N} \frac{1 - e^{im\alpha N}}{1 - e^{im\alpha}}.
\end{aligned}$$

Since for $m \neq 0$,

$$\left| \frac{1 - e^{im\alpha N}}{1 - e^{im\alpha}} \right| \leq \frac{2}{|1 - e^{im\alpha}|} < \infty$$

the limit of the sum on the right-hand side above as $N \to \infty$ is 0. Hence for every x,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} P(x+n\alpha) = c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(t)\, dt.$$

For the general case, let $\varepsilon > 0$. By Theorem 8.15 there is a trigonometric polynomial P such that $\bigl|P(x) - f(x)\bigr| < \varepsilon/3$ for all real x. By the result above, there is a positive integer N_0 such that for all $N \geq N_0$

$$\left| \frac{1}{N} \sum_{n=1}^{N} P(x+n\alpha) - \frac{1}{2\pi} \int_{-\pi}^{\pi} P(t)\, dt \right| \leq \frac{\varepsilon}{3}.$$

Hence

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{n=1}^{N} f(x+n\alpha) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt\right| \\
&\qquad \le \frac{1}{N}\sum_{n=1}^{N}\big|f(x+n\alpha)-P(x+n\alpha)\big| + \left|\frac{1}{N}\sum_{n=1}^{N} P(x+n\alpha) - \frac{1}{2\pi}\int_{-\pi}^{\pi} P(t)\,dt\right| + \\
&\qquad\quad \frac{1}{2\pi}\int_{-\pi}^{\pi}\big|P(t)-f(t)\big|\,dt \\
&\qquad \le \frac{1}{N}\left(\frac{N\varepsilon}{3}\right) + \frac{\varepsilon}{3} + \frac{1}{2\pi}\left(\frac{2\pi\varepsilon}{3}\right) \\
&\qquad = \varepsilon
\end{aligned}$$

195. Exercise 20: The following simple computation yields a good approximation to Stirling's formula. For $m = 1, 2, 3, \ldots$, define

$$f(x) = (m+1-x)\log m + (x-m)\log(m+1)$$

if $m \le x \le m+1$, and define

$$g(x) = \frac{x}{m} - 1 + \log m$$

if $m - 1/2 \le x < m + 1/2$. Draw the graphs of f and g. Note that $f(x) \le \log x \le g(x)$ if $x \ge 1$ and that

$$\int_1^n f(x)\,dx = \log(n!) - \frac{\log n}{2} > -\frac{1}{8} + \int_1^n g(x)\,dx.$$

Integrate $\log x$ over $[1, n]$. Conclude that

$$\frac{7}{8} < \log(n!) - \left(n + \frac{1}{2}\right)\log n + n < 1$$

for $n = 2, 3, 4, \ldots$. Thus

$$e^{7/8} < \frac{n!}{(n/e)^n\sqrt{n}} < e.$$

Solution: (*analambanomenos*)
The second derivative of $\log x$ is $-x^{-2}$, which is negative for $x > 0$, so $\log x$ is a "concave" function (i.e., $-\log x$ is convex). The function $f(x)$ is a continuous, piecewise-linear function whose values at the endpoints of the linear segments, $f(m)$ for $m = 1, 2, \ldots$, equal $\log m$, hence $f(x) \le \log x$ by the concavity of $\log x$. Also, $g(x)$ is a continuous, piecewise-linear function which is equal to $\log m$ at $m = 1, 2, \ldots$, and the slope of $g(x)$ is equal to the derivative of $\log x$ at those points. That is, the linear segments of $g(x)$ are tangent to $\log x$ at those points, and so $\log x \le g(x)$, also by the concavity of the $\log x$.

$$\begin{aligned}
\int_1^n f(x)\,dx &= \sum_{m=1}^{n-1}\int_m^{m+1}(m+1-x)\log m + (x-m)\log(m+1)\,dx \\
&= \sum_{m=1}^{n-1}\big(-\tfrac12(m+1-x)^2\log m + \tfrac12(x-m)^2\log(m+1)\big)\big|_{x=m}^{x=m+1} \\
&= \tfrac12\log 1 + \sum_{m=1}^{n-1}(\log m) + \tfrac12\log n \\
&= \log n! - \tfrac12\log n \\
\int_1^n g(x)\,dx &= \int_1^{\frac32}(x-1+\log 2)\,dx + \sum_{m=2}^{n-1}\int_{m-\frac12}^{m+\frac12}\left(\frac{x}{m}-1+\log m\right)dx + \int_{n-\frac12}^{n}\left(\frac{x}{n}-1+\log n\right)dx \\
&= \tfrac12(x-1)^2\big|_{x=1}^{x=\frac32} + \sum_{m=2}^{n-1}\big(\tfrac{1}{2m}x^2 - x + x\log m\big)\big|_{x=m-\frac12}^{x=m+\frac12} + \big(\tfrac{1}{2n}x^2 - x + x\log n\big)\big|_{x=n-\frac12}^{x=n} \\
&= \tfrac18 + \sum_{m=2}^{n-1}\big(1-1+\log m\big) + \big(\tfrac12 - \tfrac{1}{8n} - \tfrac12 + \tfrac12\log n\big) \\
&= \log n! - \tfrac12\log n + \tfrac18\big(1-\tfrac1n\big) \\
&< \int_1^n f(x)\,dx + \tfrac18
\end{aligned}$$

Integrating by parts, we have $\int_1^n \log x\,dx = n\log n - 1\log 1 - \int_1^n 1\,dx = n\log n - n + 1$. Hence

$$-\int_1^n g(x)\,dx < -\int_1^n \log x\,dx < -\int_1^n f(x)\,dx$$
$$-\log n! + \tfrac12\log n - \tfrac18 < -n\log n + n - 1 < -\log n! + \tfrac12\log n$$

Adding $1 + \log n! - \frac12\log n$ to all sides, we get

$$\tfrac78 < \log n! - \big(n+\tfrac12\big)\log n + n < 1.$$

And applying exp to all sides, we get

$$e^{7/8} < \frac{\exp\big(\log n!\big)\exp(n)}{\exp\big(\log(n^{n+1/2})\big)} < e$$
$$e^{7/8} < \frac{n!e^n}{n^n\sqrt{n}} < e$$
$$e^{7/8} < \frac{n!}{(n/e)^n\sqrt{n}} < e$$

196. Exercise 21: Let

$$L_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\big|D_n(t)\big|\,dt \qquad (n = 1, 2, 3, \ldots).$$

Prove that there exists a constant $C > 0$ such that

$$L_n > C\log n \qquad (n = 1, 2, 3, \ldots),$$

or, more precisely, that the sequence

$$\left\{L_n - \frac{4}{\pi^2}\log n\right\}$$

is bounded.

Solution: (*analambanomenos*)
Since D_n is an even function, and making a substitution, we can simplify the integral to

$$\begin{aligned}
L_n(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\big|D_n(x)\big|\,dx \\
&= \frac{1}{\pi}\int_0^{\pi}\frac{\big|\sin\big(n+\frac{1}{2}\big)x\big|}{\sin(x/2)}\,dx \\
&= \frac{2}{\pi}\int_0^{\pi/2}\frac{\big|\sin(2n+1)x\big|}{\sin x}\,dx
\end{aligned}$$

Letting $A = \pi/(2N+1)$, we can break up the interval of integration into the subintervals

$$I_1 = [0, A] \quad I_2 = [A, 2A] \quad \cdots \quad I_n = \big[(n-1)A, nA\big] \quad \big[nA, \pi/2\big]$$

each having length A except the last, which has length $A/2$ and which we can ignore for purposes of the estimate. In the interior of the interval I_m, $\sin(2n+1)x$ is positive for m odd, and negative for m even. Also, $\sin x$, which is positive and monotonically increasing over I_m, has the maximum value $\sin(mA) < mA$. Hence, for m odd, we have

$$\begin{aligned}
\frac{2}{\pi}\int_{(m-1)A}^{mA}\frac{\big|\sin(2n+1)x\big|}{\sin x}\,dx &\ge \frac{2}{\pi}\cdot\frac{1}{mA}\int_{(m-1)A}^{mA}\sin(2n+1)x\,dx \\
&= \frac{2}{\pi}\cdot\frac{2n+1}{m\pi}\cdot\frac{1}{2n+1}\int_{(m-1)\pi}^{m\pi}\sin x\,dx \\
&= \frac{2}{\pi^2}\cdot\frac{1}{m}\Big(\cos\big((m-1)\pi\big) - \cos(m\pi)\Big) \\
&= \frac{4}{\pi^2}\cdot\frac{1}{m}
\end{aligned}$$

and for m even, we have

$$\begin{aligned}
\frac{2}{\pi}\int_{(m-1)A}^{mA}\frac{\big|\sin(2n+1)x\big|}{\sin x}\,dx &\ge \frac{2}{\pi}\cdot\frac{1}{mA}\int_{(m-1)A}^{mA}-\sin(2n+1)x\,dx \\
&= \frac{2}{\pi}\cdot\frac{2n+1}{m\pi}\cdot\frac{1}{2n+1}\int_{(m-1)\pi}^{m\pi}-\sin x\,dx \\
&= \frac{2}{\pi^2}\cdot\frac{1}{m}\Big(\cos(m\pi) - \cos\big((m-1)\pi\big)\Big) \\
&= \frac{4}{\pi^2}\cdot\frac{1}{m}
\end{aligned}$$

Recall that in Exercise 9 it was shown that $1 + (1/2) + \cdots + (1/n) > \log n$. Hence, adding the inequalities above for $m = 1, 2, \ldots, n$, we get

$$L_n \ge \frac{4}{\pi^2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) > \frac{4}{\pi^2}\log n$$

which shows the first part of the Exercise, with $C = 4/\pi^2$. (To solve the second part, you would have to get an upper estimate for L_n).

197. Exercise 22: If α is real and $-1 < x < 1$, prove Newton's binomial theorem

$$(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n.$$

Show also that

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\,\Gamma(\alpha)} x^n$$

if $-1 < x < 1$ and $\alpha > 0$.

Solution: (*analambanomenos*)
We can show that the series on the right-hand side of the first equation converges by the Ratio Test, Theorem 3.34. Let

$$a_n = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n.$$

Then for $n > \alpha$,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left(\frac{n-\alpha}{n+1} \right) |x| = \lim_{n\to\infty} \left(1 - \frac{\alpha+1}{n+1} \right) |x| = |x| < 1.$$

Hence the right-hand side of the equation defines a function $f(x)$ for $|x| < 1$. Following the hint, by Theorem 8.1 we can differentiate the series term-by-term to get

$$f'(x) = \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} x^{n-1} = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} x^n.$$

Hence

$$\begin{aligned}
(1+x)f'(x) &= \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} x^n + \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} x^{n+1} \\
&= \alpha + \sum_{n=1}^{\infty} \left(\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} \right) x^n \\
&= \alpha + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)(\alpha-n+n)}{n!} x^n \\
&= \alpha + \alpha \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n \\
&= \alpha f(x)
\end{aligned}$$

If we let $g(x) = (1+x)^\alpha$ be the left-hand side of the equation, then $(1+x)g'(x) = (1+x)\alpha(1+x)^{n-1} = \alpha g(x)$, so $g(x)$ also satisfies the differential equation $y' = \alpha y/(1+x)$. Also note that $g(0) = f(0) = 1$. Since

$$\left| \frac{\alpha y_1}{1+x} - \frac{\alpha y_2}{1+x} \right| = \left(\frac{\alpha}{1+x} \right) |y_1 - y_2|$$

we can apply the uniqueness result of Exercise 5.27 to conclude that $g(x) = f(x)$ on any interval $[-1+\epsilon, 1-\epsilon]$, and so on all of $(-1, 1)$.

Note that by Theorem 8.18(a) we have, for $\alpha > 0$,

$$\begin{aligned}
\Gamma(\alpha+n) &= (\alpha+n-1)\Gamma(\alpha+n-1) \\
&= (\alpha+n-1)(\alpha+n-2)\Gamma(\alpha+n-2) \\
&= \cdots = (\alpha+n-1)(\alpha+n-2)\cdots(\alpha+1)\alpha\Gamma(\alpha).
\end{aligned}$$

Hence, for $-1 < x < 1$ and $\alpha > 0$,

$$\begin{aligned}(1-x)^{-\alpha} &= 1 + \sum_{n=1}^{\infty} \frac{-\alpha(-\alpha-1)\cdots(-\alpha-n+1)}{n!}(-1)^n x^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}x^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{\Gamma(n+\alpha)}{n!\,\Gamma(\alpha)}x^n \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\,\Gamma(\alpha)}x^n\end{aligned}$$

198. Exercise 23: Let γ be a continuously differentiable *closed* curve in the complex plane, with parameter interval $[a,b]$, and assume that $\gamma(t) \neq 0$ for every $t \in [a,b]$. Define the *index* of γ to be

$$\operatorname{Ind}(\gamma) = \frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)}{\gamma(t)}\,dt.$$

Prove that $\operatorname{Ind}(\gamma)$ is always an integer.

Compute $\operatorname{Ind}(\gamma)$ when $\gamma(t) = e^{int}$, $a = 0$, $b = 2\pi$.

Explain why $\operatorname{Ind}(\gamma)$ is often called the *winding number* of γ around 0.

Solution: (*analambanomenos*)
Following the hint, for $x \in [a,b]$ define

$$\varphi(x) = \int_a^x \frac{\gamma'(t)}{\gamma(t)}\,dt.$$

Then $\varphi(a) = 0$, and by Theorem 6.20, since γ'/γ is continuous on $[a,b]$, we have $\varphi' = \gamma'/\gamma$ on $[a,b]$. Let $f(x) = \gamma(x)e^{\varphi(x)}$, $x \in [a,b]$. Then

$$f'(x) = \big(\gamma'(x) - \varphi'(x)\gamma(x)\big)e^{\varphi(x)} = 0,$$

so f is constant on $[a,b]$, equal to $f(a) = ae^{\varphi(a)} = a$. Hence $f(b) = be^{\varphi(b)} = a = b$, or

$$e^{\varphi(b)} = e^{2\pi i\operatorname{Ind}(\gamma)} = 1$$

so that $2\pi i\operatorname{Ind}(\gamma) = 2\pi i n$ for some integer n. Hence $\operatorname{Ind}(\gamma) = n$ is an integer.

For $\gamma(t) = e^{int}$, $a = 0$, $b = 2\pi$, we have

$$\operatorname{Ind}(\gamma) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{ine^{int}}{e^{int}}\,dt = \frac{2\pi i n}{2\pi i} = n.$$

For these γ, $\operatorname{Ind}(\gamma)$ measures the number of times γ winds counter-clockwise around 0, with a negative value indicating that γ winds around 0 clockwise $-\operatorname{Ind}(\gamma)$ times. This is true in a general sense for arbitrary γ, which is shown in most introductory Complex Analysis courses. Also, it can be shown that two such curves have the same index if and only if one can be "continuously deformed" to the other through an intermediate set of such curves γ_t.

199. Exercise 24: Let γ be as in Exercise 23, and assume in addition that the range of γ does not intersect the negative real axis. Prove that $\mathrm{Ind}(\gamma) = 0$.

Solution: (*analambanomenos*)
Following the hint, for $0 \le c < \infty$ let $\gamma_c(t) = \gamma(t) + c$. Then γ_c is also a continuous differentiable closed curve in $\mathbf{C}$ such that $\gamma_c(t) \ne 0$ for all $t \in [a, b]$, so we can define $\mathrm{Ind}(\gamma_c)$. Let $\{c_n\}$ be a sequence of nonnegative real numbers converging to c. Then

$$\frac{\gamma_{c_n}'(t)}{\gamma_{c_n}} = \frac{\gamma'(t)}{\gamma(t) + c_n} \to \frac{\gamma'(t)}{\gamma(t)} \text{ as } n \to \infty,$$

and the convergence is uniform for $t \in [a, b]$. Hence by Theorem 7.16

$$\mathrm{Ind}(\gamma_{c_n}) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t) + c_n}\, dt \to \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)}\, dt = \mathrm{Ind}(\gamma),$$

so that $\mathrm{Ind}(\gamma_c)$ is a continuous function of c. Since $\mathrm{Ind}(\gamma_c)$ has only integer values, this forces $\mathrm{Ind}(\gamma_c) = \mathrm{Ind}(\gamma)$ for all $0 \le c < \infty$.

Since $[a, b]$ is compact, $\min \gamma(t) > -\infty$ for $t \in [a, b]$. Hence, for large enough c we have $\min \bigl|\gamma(t) + c\bigr|$ as large as we like. In that case, we have

$$\bigl|\,\mathrm{Ind}(\gamma_c)\bigr| \le \frac{1}{2\pi} \int_a^b \left|\frac{\gamma'(t)}{\gamma(t) + c}\right| dt \le \frac{1}{2\pi} \left(\frac{\max \bigl|\gamma'(t)\bigr|}{\min \bigl|\gamma(t) + c\bigr|} \right) (b - a) \to 0$$

as $c \to \infty$. Hence $\mathrm{Ind}(\gamma) = \mathrm{Ind}(\gamma_c) = 0$ for large enough c.

200. Exercise 25: Suppose γ_1 and γ_2 are curves as in Exercise 23, and

$$\bigl|\gamma_1(t) - \gamma_2(t)\bigr| < \bigl|\gamma_1(t)\bigr| \qquad (a \le t \le b).$$

Prove that $\mathrm{Ind}(\gamma_1) = \mathrm{Ind}(\gamma_2)$.

Solution: (*analambanomenos*)
Following the hint, let $\gamma = \gamma_2/\gamma_1$. This is well-defined since γ_1 is nowhere 0, and so γ is also a continuous differentiable closed curve defined on $[a, b]$ which is nowhere 0, so that we can define $\mathrm{Ind}(\gamma)$. By the condition on γ_1 and γ_2 we have for all $t \in [a, b]$

$$\bigl|1 - \gamma(t)\bigr| = \frac{\bigl|\gamma_1(t) - \gamma_2(t)\bigr|}{\bigl|\gamma_1(t)\bigr|} < \frac{\bigl|\gamma_1(t)\bigr|}{\bigl|\gamma_1(t)\bigr|} = 1.$$

Hence the range of γ does not intersect the negative real axis, so by Exercise 24 we have

$$\begin{aligned}
0 &= \mathrm{Ind}(\gamma) \\
&= \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)}\, dt \\
&= \frac{1}{2\pi i} \int_a^b \frac{\gamma_1(t)\gamma_2'(t) - \gamma_2(t)\gamma_1'(t)}{\gamma_1^2(t)} \cdot \frac{\gamma_1(t)}{\gamma_2(t)}\, dt \\
&= \frac{1}{2\pi i} \int_a^b \left(\frac{\gamma_2'(t)}{\gamma_2(t)} - \frac{\gamma_1'(t)}{\gamma_1)} \right) dt \\
&= \mathrm{Ind}(\gamma_2) - \mathrm{Ind}(\gamma_1)
\end{aligned}$$

so that $\mathrm{Ind}(\gamma_1) = \mathrm{Ind}(\gamma_2)$.

201. Exercise 26: Let γ be a *closed* curve in the complex plane (not necessarily differentiable) with parameter $[0, 2\pi]$, such that $\gamma(t) \neq 0$ for every $t \in [0, 2\pi]$.

Choose $\delta > 0$ so that $\left|\gamma(t)\right| > \delta$ for all $t \in [0, 2\pi]$. If P_1 and P_2 are trigonometric polynomials such that $\left|P_j(t) - \gamma(t)\right| < \delta/4$ for all $t \in [0, 2\pi]$ (their existence is assured by Theorem 8.15), prove that

$$\operatorname{Ind}(P_1) = \operatorname{Ind}(P_2)$$

by applying Exercise 25.

Define this common value to be $\operatorname{Ind}(\gamma)$. Prove that the statements of Exercises 24 and 25 hold without any differentiability assumption.

Solution: (*analambanomenos*)
Since

$$\Big|\left|P_j(t)\right| - \left|\gamma(t)\right|\Big| \leq \left|P_j(t) - \gamma(t)\right| < \frac{\delta}{4}$$

we have

$$\left|P_j(t)\right| \geq \left|\gamma(t)\right| - \frac{\delta}{4} > \frac{3\delta}{4}.$$

Hence $P_1(t)$ and $P_2(t)$, $0 \leq g \leq 2\pi$, define continuous closed differentiable curves in the complex plane which are nowhere 0, so we can define $\operatorname{Ind}(P_j)$. We have

$$\left|P_1(t) - P_2(t)\right| \leq \left|P_1(t) - \gamma(t)\right| + \left|P_2(t) - \gamma(t)\right| < \frac{\delta}{2} < \frac{3\delta}{4} < \left|P_1(t)\right|$$

so by Exercise 25 we have $\operatorname{Ind}(P_1) = \operatorname{Ind}(P_2)$.

Before showing Exercise 24 for the continuous case, I want to show a general theorem which is pretty standard and may have been done in a previous exercise, or even the text, although I couldn't find it. Let X be a metric space with disjoint subsets F and K such that F is closed and K is compact. Then there is a positive minimum distance between them, that is, there is a $\delta > 0$ such that $d(x, y) > \delta$ for all $x \in F$ and $y \in K$. If not, then there are subsequences $\{x_n\} \subset F$ and $\{y_n\} \subset K$ such that $\lim d(x_n, y_n) = 0$. Since K is compact, $\{y_n\}$ has a subsequence $\{y_{n_m}\}$ converging to $y \in K$. Then

$$d(x_{n_m}, y) \leq d(x_{n_m}, y_{n_m}) + d(y_{n_m}, y) \to 0 \text{ as } m \to \infty$$

which, since F is closed, implies that $y \in F$, contradicting the fact that F and K are disjoint. (If you prefer a direct proof, for each $y \in K$ let N_y be an open neighborhood of y disjoint from F. Then $\{N_y\}$ is a cover of K, so you can pass to a finite subcover whose sets have a positive minimum radius and go from there.)

Suppose γ is a closed, continuous curve in the complex plane with domain $[0, 2\pi]$, whose range does not intersect the negative real axis. Since the negative real axis is a closed set and the range of γ is compact, by the preceding paragraph there is a $\delta > 0$ such that if P is a trigonometric polynomial satisfying $\left|P(t) - \gamma(t)\right| < \delta$ for all $t \in [0, 2\pi]$, so that the range of P also does not intersect the negative real axis. Since P is differentiable, $\operatorname{Ind}(P) = 0$ by Exercise 25, hence $\operatorname{Ind}(\gamma) = 0$.

Now suppose γ_1 and γ_2 are closed, continuous curves in the complex plane with domain $[0, 2\pi]$ such that $\left|\gamma_1(t) - \gamma_2(t)\right| < \left|\gamma_1(t)\right|$ for all $t \in [0, 2\pi]$. Then by Theorem 4.16 there is a $\delta > 0$ such that

$$\left|\gamma_1(t)\right| - \left|\gamma_1(t) - \gamma_2(t)\right| > \delta$$

on the compact set $[0, 2\pi]$. Let P_1, P_2 be trigonometric polynomials such that $\left|P_j(t) - \gamma_j(t)\right| < \delta/4$ for all $t \in [0, 2\pi]$, $j = 1, 2$. Then, for all $t \in [0, 2\pi]$,

$$\begin{aligned}
\Big|\left|P_1(t) - P_2(t)\right| - \left|\gamma_1(t) - \gamma_2(t)\right|\Big| &\le \Big|\big(P_1(t) - P_2(t)\big) - \big(\gamma_1(t) - \gamma_2(t)\big)\Big| \\
&= \Big|\big(P_1(t) - \gamma_1(t)\big) - \big(P_2(t) - \gamma_2(t)\big)\Big| \\
&\le \left|P_1(t) - \gamma_1(t)\right| + \left|P_2(t) - \gamma_2(t)\right| \\
&< \frac{\delta}{2} \\
\Big|\left|P_1(t)\right| - \left|\gamma_1(t)\right|\Big| &\le \left|P_1(t) - \gamma_1(t)\right| \\
&< \frac{\delta}{4}
\end{aligned}$$

Hence $\left|P_1(t)\right| - \left|P_1(t) - P_2(t)\right| > \delta/4$ for all $t \in [0, 2\pi]$. Hence, by Exercise 25,

$$\operatorname{Ind}(\gamma_1) = \operatorname{Ind}(P_1) = \operatorname{Ind}(P_2) = \operatorname{Ind}(\gamma_2).$$

202. Exercise 27: Let f be a continuous complex function defined in the complex plane. Suppose there is a positive integer n and a complex number $c \neq 0$ such that

$$\lim_{|z| \to \infty} z^{-n} f(z) = c.$$

Prove that $f(z) = 0$ for at least one complex number z.

Solution: (*analambanomenos*)
Following the hint, suppose that $f(z) \neq 0$ for all z. Then the continuous closed curves $\gamma_r(t) = f(re^{it})$, $r \in [0, \infty)$, do not intersect the origin, so we can define the function $\operatorname{Ind}(\gamma_r)$ for $r \in [0, \infty)$. We now show that this function would have the following three properties, which leads to a contradiction.

(a) $\operatorname{Ind}(\gamma_0) = 0$.

Since $\gamma_0(t) = f(0)$ is a differentiable function, we have $\operatorname{Ind}(\gamma_0) = (2\pi)^{-1} \int \gamma_0'(t)/\gamma_0(t)\, dt = 0$.

(b) $\operatorname{Ind}(\gamma_r) = n$ for all sufficiently large r.

By the assumption, we have

$$\lim_{r \to \infty} \frac{f(re^{it})}{r^n e^{int}} = c$$

That is, for sufficiently large r there is a $\delta > 0$ such that

$$\left|\frac{\gamma_r(t)}{r^n e^{int}} - c\right| < \delta$$
$$\left|\gamma_r(t) - cr^n e^{int}\right| < r^n \delta < r^n |c| = |cr^n e^{int}|.$$

Let $\gamma(t) = cr^n e^{int}$, a differentiable closed curve which is nowhere 0. By the extension of Exercise 25 shown in Exercise 26, we have

$$\operatorname{Ind}(\gamma_r) = \operatorname{Ind}(\gamma) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t)}\, dt = \frac{1}{2\pi i} \int_0^{2\pi i} \frac{in\gamma(t)}{\gamma(t)}\, dt = n.$$

(c) $\operatorname{Ind}(\gamma_r)$ is a continuous function of r on $[0, \infty)$.

Fix $r_0 \in [0, \infty)$. Then the positive continuous function $\left|\gamma_{r_0}(t)\right|$ has a minimum value $\epsilon > 0$ on the compact set $[0, 2\pi]$ by Theorem 4.16. Since f is continuous, by Theorem 4.19 it is uniformly continuous a compact neighborhood of the circle $\{r_0 e^{it}\}$, $0 \le t \le 2\pi$. Hence there is a $\delta > 0$ such that if $|r - r_0| < \delta$ (limited to small positive values of r in case $r_0 = 0$), we have

$$\left|\gamma_r(t) - \gamma_{r_0}(t)\right| < \epsilon < \left|\gamma_{r_0}(t)\right|.$$

Hence, by the extension of Exercise 25 given in Exercise 26, we have

$$\operatorname{Ind}(\gamma_r) = \operatorname{Ind}(\gamma_{r_0}), \text{ for } |r - r_0| < \delta.$$

Note that a continuous, real-valued function on a connected set X with only integer values must be constant. For then the inverse images of the open sets $(z - 0.5, z + 0.5)$ for the integers z define a set of disjoint open sets which cover X, so by the connectedness of X they must all be empty except one. Hence, since $[0, \infty)$ is a connected set, the properties (a), (b), and (c) of the function $\operatorname{Ind}(\gamma_r)$ on $[0, \infty)$ lead to a contradiction.

203. Exercise 28: Let $\bar{D}$ be the closed unit disc in the complex plane. (Thus $z \in \bar{D}$ if and only if $|z| \le 1$.) Let g be a continuous mapping of $\bar{D}$ into the unit circle T. (Thus $\left|g(z)\right| = 1$ for every $z \in \bar{D}$.)

Prove that $g(z) = -z$ for at least one $z \in T$.

Solution: (*analambanomenos*)
Following the hint, for $0 \le r \le 1$, $0 \le t \le 2\pi$, put $\gamma_r(t) = g(re^{it})$, a closed, continuous curve with values on T so that we can define $\operatorname{Ind}(\gamma_r)$. Note that since $|z| > 1/2$ for $z \in T$, if a trigonometric polynomial P satisfies $|P - \gamma| < 1/8$ for any curve with values on T, then $\operatorname{Ind}(\gamma) = \operatorname{Ind}(P)$. Since g is uniformly continuous on the compact set $\bar{D}$, this implies that for every $r \in [0, 1]$ there is a $\delta > 0$ such that if $|r_0 - r| < \delta$, then $\operatorname{Ind}(\gamma_{r_0}) = \operatorname{Ind}(\gamma_r)$, that is, $\operatorname{Ind}(\gamma_r)$ is a continuous function on the connected set $[0, 1]$. As shown at the end of the solution to Exercise 27, this indicates that $\operatorname{Ind}(\gamma_r)$ is constant on $[0, 1]$. Since γ_0 is the curve with the constant value $f(0)$, this constant must be $\operatorname{Ind}(\gamma_0) = 0$.

Put $\psi(t) = e^{it}\gamma_1(t)$, which is also a closed continuous curve with values on T. If $\psi(t) = -1$ for some t, then $g(e^{it}) = \gamma_1(t) = -e^{it}$. So if we assume that $g(z) \ne -z$ for all $z \in T$, then $\psi(z) \ne -1$, which is the intersection of T with the negative real axis. Hence by the extension of Exercise 24 shown in Exercise 26, we have $\operatorname{Ind}(\psi) = 0$. Let P_1 be a trigonometric polynomial such that $|P_1 - \psi| < 1/8$ on T so that $\operatorname{Ind}(P_1) = \operatorname{Ind}(\psi) = 0$. If $P_2(t) = e^{it}P_1(t)$, then

$$\left|P_2(t) - \gamma_1(t)\right| = \left|e^{it}P_1(t) - e^{it}\psi(t)\right| = \left|P_1(t) - \psi(t)\right| < \frac{1}{8}$$

so that

$$\begin{aligned}
\operatorname{Ind}(\gamma_1) &= \operatorname{Ind}(P_2) \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{P_2'(t)}{P_2(t)}\,dt \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{ie^{it}P_1(t) + e^{it}P_1'(t)}{e^{it}P_1(t)}\,dt \\
&= \frac{1}{2\pi i}\int_0^{2\pi} i\,dt + \frac{1}{2\pi i}\int_0^{2\pi} \frac{P_1'(t)}{P_1(t)}\,dt \\
&= 1 + \operatorname{Ind}(P_1) \\
&= 1
\end{aligned}$$

which contradicts $\operatorname{Ind}(\gamma_1) = 0$ shown in the first paragraph.

204. Exercise 29: Prove that every continuous mapping f of $\bar{D}$ into $\bar{D}$ has a fixed point in $\bar{D}$.

Solution: (*analambanomenos*)
Following the hint, assume $f(z) \neq z$ for all $z \in \bar{D}$. For $z \in \bar{D}$ let $g(z) \in T$ be the point which lies on the ray starting at $f(z)$ (but not including $f(z)$) and passes through z. Note that if $z \in T$, then $g(z) = z$. The points on that ray are the points

$$f(z) + r\big(z - f(z)\big), \quad r \in (0, \infty),$$

so the r defining $g(z)$ satisfies

$$\begin{aligned}
1 &= \big|f(z) + r\big(z - f(z)\big)\big|^2 \\
1 &= \Big(f(z) + r\big(z - f(z)\big)\Big)\Big(\overline{f(z)} + r\big(\overline{z} - \overline{f(z)}\big)\Big) \\
0 &= |z - f(z)|^2 r^2 + 2\Re\big(f(z)\overline{z} - |f(z)|^2\big)r + \big(|f(z)|^2 - 1\big)
\end{aligned}$$

This is a quadratic equation in r with coefficients which are continuous functions of z and $f(z)$. From the geometry of the ray, for all $z \in \bar{D}$ there will always be only one positive solution $r(z)$, given by the familar quadratic formula, and so $r(z)$ and $g(z)$ are continuous functions of z. Then $g(z)$, a continuous function which maps $\bar{D}$ into T, satisfies the conditions of Exercise 28, so there must be a $z \in T$ such that $g(z) = -z$, contradicting $g(z) = z$ for all $z \in T$.

205. Exercise 30: Use Stirling's formula to prove that

$$\lim_{x\to\infty} \frac{\Gamma(x+c)}{x^c\Gamma(x)} = 1$$

for every real constant c.

Solution: (*analambanomenos*)
From Stirling's formula we get the limits

$$\lim_{x\to\infty} \frac{x^c\Gamma(x)}{x^c\big(\frac{x-1}{e}\big)^{x-1}\sqrt{2\pi(x-1)}} = 1 \qquad \lim_{x\to\infty} \frac{\Gamma(x+c)}{\big(\frac{x+c-1}{e}\big)^{x+c-1}\sqrt{2\pi(x+c-1)}} = 1.$$

Hence

$$\begin{aligned}
1 &= \lim_{x\to\infty} \frac{\Gamma(x+c)}{x^c\Gamma(x)} \cdot \frac{x^c\left(\frac{x-1}{e}\right)^{x-1}\sqrt{2\pi(x-1)}}{\left(\frac{x+c-1}{e}\right)^{x+c-1}\sqrt{2\pi(x+c-1)}} \\
&= \lim_{x\to\infty} \frac{\Gamma(x+c)}{x^c\Gamma(x)} \cdot \frac{x^ce^c(x-1)^{x-1}}{(x+c-1)^{x+c-1}}\sqrt{1-\frac{c}{x+c-1}} \\
&= \lim_{x\to\infty} \frac{\Gamma(x+c)}{x^c\Gamma(x)} \cdot \frac{x^ce^c(x-1)^{x-1}}{(x+c-1)^{x+c-1}}.
\end{aligned}$$

So to show that the limit of the first factor is 1, it suffices to show that the limit of the second factor is also 1.

By substituting y for $x-1$, we have

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^ce^c(x-1)^{x-1}}{(x+c-1)^{x+c-1}} &= \lim_{y\to\infty} \frac{(y+1)^ce^cy^y}{(y+c)^{y+c}} \\
&= \lim_{y\to\infty} \left(1+\frac{1-c}{y+c}\right)^c \cdot \frac{e^cy^y}{(y+c)^y} \\
&= \lim_{y\to\infty} \frac{e^c}{(1+c/y)^y} \\
&= 1
\end{aligned}$$

The last equality comes from Exercise 4(d).

206. Exercise 31: In the proof of Theorem 7.26 it was shown that

$$\int_{-1}^{1}(1-x^2)^n\,dx \geq \frac{4}{3\sqrt{n}}$$

for $n = 1, 2, 3, \ldots$. Use Theorem 8.20 and Exercise 30 to show the more precise result

$$\lim_{n\to\infty}\sqrt{n}\int_{-1}^{1}(1-x^2)^n\,dx = \sqrt{\pi}.$$

Solution: (*analambanomenos*)

$$\begin{aligned}
\lim_{n\to\infty}\sqrt{n}\int_{-1}^{1}(1-x^2)^n\,dx &= \lim_{n\to\infty}2\sqrt{n}\int_{0}^{1}(1-x^2)^n\,dx && \text{by symmetry} \\
&= \lim_{n\to\infty}\sqrt{n}\int_{0}^{1}t^{-1/2}(1-t)^n\,dt && \text{substituting } t \text{ for } x^2 \\
&= \lim_{n\to\infty}\frac{\sqrt{n}\,\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)} && \text{Theorem 8.20} \\
&= \Gamma\left(\tfrac{1}{2}\right)\lim_{n\to\infty}\frac{n^{3/2}\Gamma(n)}{\Gamma\left(n+\frac{3}{2}\right)} && \text{Theorem 8.18(a)} \\
&= \Gamma\left(\tfrac{1}{2}\right) && \text{Exercise 30} \\
&= \sqrt{\pi} && \text{8.21 (99)}
\end{aligned}$$

9 Functions of Several Variables

207. Exercise 1: If S is a nonempty subset of a vector space X, prove that the span of S is a vector space.

Solution: (*analambanomenos*)
Let $\mathbf{x}, \mathbf{y}$ be in the span of S and let c be a scalar; we need to show that $\mathbf{x}+\mathbf{y}$ and $c\mathbf{x}$ are in the span of S. We have

$$\begin{aligned}\mathbf{x} &= c_1\mathbf{x}_1 + \cdots + c_m\mathbf{x}_m \\ \mathbf{y} &= d_1\mathbf{y}_1 + \cdots + d_n\mathbf{y}_n\end{aligned}$$

for some $\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{y}_1, \ldots, \mathbf{y}_n$ in S and some scalars $c_1, \ldots, c_m, d_1, \ldots, d_n$. Hence

$$\begin{aligned}\mathbf{x}+\mathbf{y} &= c_1\mathbf{x}_1 + \cdots + c_m\mathbf{x}_m + d_1\mathbf{y}_1 + \cdots + d_n\mathbf{y}_n \\ c\mathbf{x} &= cc_1\mathbf{x}_1 + \cdots + cc_m\mathbf{x}_m\end{aligned}$$

shows that $\mathbf{x}+\mathbf{y}$ and $c\mathbf{x}$ are in the span of S.

208. Exercise 2: Prove that BA is linear if A and B are linear transformations. Prove also that A^{-1} is linear and invertible.

Solution: (*analambanomenos*)
Let $A \in L(X,Y)$ and $B \in L(Y,Z)$. Let $\mathbf{x}, \mathbf{y} \in X$ and let c be a scalar. Then

$$\begin{aligned}(BA)(\mathbf{x}+\mathbf{y}) &= B\big(A(\mathbf{x}+\mathbf{y})\big) \\ &= B\big(A(\mathbf{x}) + A(\mathbf{y})\big) \\ &= B\big(A(\mathbf{x})\big) + B\big(A(\mathbf{y})\big) \\ &= (BA)(\mathbf{x}) + (BA)(\mathbf{y}) \\ (BA)(c\mathbf{x}) &= B\big(A(c\mathbf{x})\big) \\ &= B\big(cA(\mathbf{x})\big) \\ &= cB\big(A(\mathbf{x})\big) \\ &= c(BA)(\mathbf{x})\end{aligned}$$

which shows that BA is linear.

Let $A \in L(X)$ be invertible. Let $\mathbf{x}, \mathbf{y} \in X$ and let c be a scalar. Then there are $\mathbf{x}', \mathbf{y}' \in X$ such that

$$A(\mathbf{x}') = \mathbf{x}, \quad A(\mathbf{y}') = \mathbf{y}, \quad A^{-1}(\mathbf{x}) = \mathbf{x}', \quad A^{-1}(\mathbf{y}) = \mathbf{y}'$$

Then

$$\begin{aligned}A^{-1}(\mathbf{x}+\mathbf{y}) &= A^{-1}\big(A(\mathbf{x}') + A(\mathbf{y}')\big) \\ &= A^{-1}\big(A(\mathbf{x}'+\mathbf{y}')\big) \\ &= \mathbf{x}'+\mathbf{y}' \\ &= A^{-1}(\mathbf{x}) + A^{-1}(\mathbf{y}) \\ A^{-1}(c\mathbf{x}) &= A^{-1}\big(cA(\mathbf{x}')\big) \\ &= A^{-1}\big(A(c\mathbf{x}')\big) \\ &= c\mathbf{x}' \\ &= cA^{-1}(\mathbf{x})\end{aligned}$$

which shows that A^{-1} is linear. Since A^{-1} is 1-1 and maps X onto itself, it is also invertible.

209. Exercise 3: Assume $A \in L(X,Y)$ and $A\mathbf{x} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$. Prove that A is then 1-1.

Solution: (*analambanomenos*)
Suppose $A(\mathbf{x}_1) = A(\mathbf{x}_2)$. Then

$$A(\mathbf{x}_1 - \mathbf{x}_2) = A(\mathbf{x}_1) - A(\mathbf{x}_2) = \mathbf{0} \text{ implies that } \mathbf{x}_1 - \mathbf{x}_2 = \mathbf{0}$$

so that $\mathbf{x}_1 = \mathbf{x}_2$. Hence A is 1-1.

210. Exercise 4: Prove that null spaces and ranges of linear transformations are vector spaces.

Solution: (*analambanomenos*)
Let $\mathbf{x}, \mathbf{y} \in \mathscr{N}(A)$ and let c be a scalar. Then

$$\begin{aligned} A(\mathbf{x}+\mathbf{y}) &= A(\mathbf{x}) + A(\mathbf{y}) = \mathbf{0}+\mathbf{0} = \mathbf{0} \text{ so that } \mathbf{x}+\mathbf{y} \in \mathscr{N}(A) \\ A(c\mathbf{x}) &= cA(\mathbf{x}) = c\mathbf{0} = \mathbf{0} \text{ so that } c\mathbf{x} \in \mathscr{N}(A) \end{aligned}$$

which shows that $\mathscr{N}(A)$ is a vector space.

Let $\mathbf{x}, \mathbf{y} \in \mathscr{R}(A)$, so that $\mathbf{x} = A(\mathbf{x}')$ and $\mathbf{y} = A(\mathbf{y}')$, and let c be a scalar. Then

$$\begin{aligned} \mathbf{x}+\mathbf{y} &= A(\mathbf{x}') + A(\mathbf{y}') = A(\mathbf{x}'+\mathbf{y}') \text{ so that } \mathbf{x}+\mathbf{y} \in \mathscr{R}(A) \\ c\mathbf{x} &= cA(\mathbf{x}') = A(c\mathbf{x}') \text{ so that } c\mathbf{x} \in \mathscr{R}(A) \end{aligned}$$

which shows that $\mathscr{R}(A)$ is a vector space.

211. Exercise 5: Prove that to every $A \in L(\mathbf{R}^n, \mathbf{R}^1)$ corresponds a unique $\mathbf{y} \in \mathbf{R}^n$ such that $A\mathbf{x} = \mathbf{x}\cdot\mathbf{y}$. Prove also that $\|A\| = |\mathbf{y}|$.

Solution: (*analambanomenos*)
I'm going to show this by induction on n. For the case $n = 1$, let $A\mathbf{e}_1 = k$, and let $\mathbf{y} = k\mathbf{e}_1$. If $\mathbf{x} \in \mathbf{R}^1$, then $\mathbf{x} = x\mathbf{e}_1$ for some scalar x, so $A\mathbf{x} = xA\mathbf{e}_1 = ck = \mathbf{x}\cdot\mathbf{y}$.

Suppose the assertion is true for the case n and let $A \in L(\mathbf{R}^{n+1}, \mathbf{R}^1)$. Restricting A to the subspace $\mathbf{R}^n$ (spanned by $\mathbf{e}_1, \ldots, \mathbf{e}_n$) yields an element of $L(\mathbf{R}^n, \mathbf{R}^1)$, so by the induction assumption there is a $\mathbf{y}' \in \mathbf{R}^n$ such that $A\mathbf{x}' = \mathbf{x}'\cdot\mathbf{y}'$ for all $\mathbf{x}' \in \mathbf{R}^n$. If $A\mathbf{e}_{n_1} = k$, let $\mathbf{y} = \mathbf{y}' + k\mathbf{e}_{n+1}$. If $\mathbf{x} \in \mathbf{R}^{n+1}$, then $\mathbf{x} = \mathbf{x}' + x\mathbf{e}_{n+1}$ for some $\mathbf{x}' \in \mathbf{R}^n$ and some scalar x. Then, since the scalar product of $\mathbf{e}_{n+1}$ with every element of $\mathbf{R}^n$ is 0, we have

$$\begin{aligned} A\mathbf{x} &= A\mathbf{x}' + xA\mathbf{e}_{n+1} \\ &= \mathbf{x}'\cdot\mathbf{y}' + xk \\ &= (\mathbf{x}' + x\mathbf{e}_{n+1})\cdot\mathbf{y}' + (\mathbf{x}' + x\mathbf{e}_{n+1})\cdot(k\mathbf{e}_{n+1}) \\ &= \mathbf{x}\cdot\mathbf{y} \end{aligned}$$

If $|\mathbf{x}| \le 1$, then $|A\mathbf{x}| = |\mathbf{x}\cdot\mathbf{y}| \le |\mathbf{x}|\,|\mathbf{y}| \le |\mathbf{y}|$, so that $\|A\| \le |\mathbf{y}|$. Let $\tilde{\mathbf{y}} = \mathbf{y}/|\mathbf{y}|$. Then $|\tilde{\mathbf{y}}| = 1$ and

$$|A\tilde{\mathbf{y}}| = \frac{|\mathbf{y}\cdot\mathbf{y}|}{|\mathbf{y}|} = \frac{|\mathbf{y}|^2}{|\mathbf{y}|} = |\mathbf{y}|.$$

Hence $\|A\| = |\mathbf{y}|$.

212. Exercise 6: If $f(0,0) = 0$ and

$$f(x,y) = \frac{xy}{x^2+y^2} \qquad \text{if } (x,y) \neq (0,0),$$

prove that $(D_1 f)(x,y)$ and $(D_2 f)(x,y)$ exist at every point of $\mathbf{R}^2$, although f is not continuous at $(0,0)$.

Solution: (*analambanomenos*)
The usual rules of differentiaion show that the partial derivatives of $\mathbf{R}^2$ at $(x,y) \neq (0,0)$ are

$$D_1 f(x,y) = \frac{y(y^2-x^2)}{(x^2+y^2)^2} \qquad D_2 f(x,y) = \frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

And since f is equal to 0 everywhere along the x and y axes, they also exist and are equal to 0 at $(0,0)$.

For $x \neq 0$, $f(x,x) = x^2/(x^2+x^2) = 1/2$. Since $f(0,0) = 0$, f is not continuous along the line $y = x$ at $(0,0)$.

213. Exercise 7: Suppose that f is a real-valued function defined in an open set $E \subset \mathbf{R}^n$, and that the partial derivatives $D_1 f, \ldots, D_n f$ are bounded in E. Prove that f is continuous in E.

Solution: (*analambanomenos*)
Suppose that $|D_j f| < M$ in E, for $j = 1, \ldots, n$. Following the hint to mimic the proof of Theorem 8.21, fix $\mathbf{x} \in E$ and let $\varepsilon > 0$. Since E is open, there is an open ball $S \subset E$, with center at $\mathbf{x}$ and radius $r < (Mn)^{-1}$. Suppose $\mathbf{h} = \sum h_j \mathbf{e}_j$, $|\mathbf{h}| < r$, put $\mathbf{v}_0 = \mathbf{0}$, and $\mathbf{v}_k = h_1\mathbf{e}_1 + \cdots + h_k\mathbf{e}_k$, for $1 \leq k \leq n$. Then

$$(*) \qquad f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = \sum_{j=1}^{n} \big(f(\mathbf{x}+\mathbf{v}_j) - f(\mathbf{x}+\mathbf{v}_{j-1})\big).$$

Since $|\mathbf{v}_k| < r$ for $1 \leq k \leq n$ and since S is convex, the segments with endpoints $\mathbf{x}+\mathbf{v}_{j-1}$ and $\mathbf{x}+\mathbf{v}_j$ lie in S. Since $\mathbf{v}_j = \mathbf{v}_{j-1} + h_j\mathbf{e}_j$, the mean value theorem, Theorem 5.10, shows that the jth summand in $(*)$ is equal to

$$h_j(D_j f)(\mathbf{x}+\mathbf{v}_{j-1}+\theta_j h_j \mathbf{e}_j)$$

for some $\theta_j \in (0,1)$. By $(*)$, it follows that

$$\begin{aligned}
\big|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})\big| &\leq \sum_{j=1}^{n} \big|f(\mathbf{x}+\mathbf{v}_j) - f(\mathbf{x}+\mathbf{v}_{j-1})\big| \\
&= \sum_{j=1}^{n} \big|h_j(D_j f)(\mathbf{x}+\mathbf{v}_{j-1}+\theta_j h_j\mathbf{e}_j)\big| \\
&< \sum_{j=1}^{n} |h_j| M < (Mn)r < \varepsilon
\end{aligned}$$

which shows that f is continuous at $\mathbf{x}$. Since $\mathbf{x} \in E$ was arbitrary, we have f continuous on E.

214. Exercise 8: Suppose that f is a differentiable real function in an open set $E \subset \mathbf{R}^n$, and that f has a local maximum at a point $\mathbf{x} \in E$. Prove that $f'(\mathbf{x}) = \mathbf{0}$.

Solution: (*analambanomenos*)
You can use Theorem 9.17 to express f' as a sum of the partial derivatives and easily reduce the problem to the the single-variable case, Theorem 5.8. However, I thought I'd use the new definition of derivative (commonly called a Fréchet derivative, by the way) instead.

By Exercise 5, there is a $\mathbf{y} \in \mathbf{R}^n$ such that $f'(\mathbf{x}) = \mathbf{y} \cdot \mathbf{x}$. Let $\mathbf{h} = h\mathbf{y}/|\mathbf{y}|$, and take the limit in the definition of derivative as h approaches 0 through positive numbers. Then we get

$$\begin{aligned}
0 &= \lim_{\mathbf{h}\to\mathbf{0}} \frac{\bigl|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - f'(\mathbf{x})\mathbf{h}\bigr|}{|\mathbf{h}|} \\
&= \lim_{h\to 0+} \frac{\bigl|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - \mathbf{y}\cdot(h\mathbf{y})/|\mathbf{y}|\bigr|}{h|\mathbf{y}|/|\mathbf{y}|} \\
&= \lim_{h\to 0+} \frac{\bigl|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - h|\mathbf{y}|\bigr|}{h} && (\text{since } \mathbf{y}\cdot\mathbf{y} = |\mathbf{y}|^2) \\
&= \lim_{h\to 0+} \frac{f(\mathbf{x}) - f(\mathbf{x}+\mathbf{h}) + h|\mathbf{y}|}{h} && (\text{since } f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) \text{ and } -h|\mathbf{y}| \text{ are } \le 0) \\
&= |\mathbf{y}| + \lim_{h\to 0+} \frac{f(\mathbf{x}) - f(\mathbf{x}+\mathbf{h})}{h}
\end{aligned}$$

Since the term in the limit is non-negative for small h, this forces $|\mathbf{y}| \le 0$, or $\mathbf{y} = \mathbf{0}$. Hence $f'(\mathbf{x}) = 0$.

215. Exercise 9: If $\mathbf{f}$ is a differentiable mapping of a *connected* open set $E \subset \mathbf{R}^n$ into $\mathbf{R}^m$, and if $\mathbf{f}'(\mathbf{x}) = \mathbf{0}$ for every $\mathbf{x} \in E$, prove that $\mathbf{f}$ is constant in E.

Solution: (*analambanomenos*)
Fix $\mathbf{x} \in E$. Since E is open, there is a open ball $S \subset E$ containing $\mathbf{x}$. By the Corollary to Theorem 9.19, $\mathbf{f}$ is constant on S. Hence the set E' of all points $\mathbf{z} \in E$ such that $\mathbf{f}(\mathbf{z}) = \mathbf{f}(\mathbf{x})$ is an open subset of E. Similarly, the set $E - E'$ is also open in E (being the union of open sets on which $\mathbf{f}$ has a constant value not equal to $\mathbf{f}(\mathbf{x})$). By Exercise 2.19(b), E' and $E - E'$ are two separated sets whose union is E. Since E is connected, we must have $E' = E$, so $\mathbf{f}$ is constant on E.

216. Exercise 10: If f is a real function defined in a convex open set $E \subset \mathbf{R}^n$, such that $(D_1 f)(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, prove that $f(\mathbf{x})$ depends only on $x_2, \ldots, x_n$.

Show that the convexity of E can be replaced by a weaker condition, but that some condition is required. For example, if $n = 2$ and E is shaped like a horseshoe, the statement may be false.

Solution: (*analambanomenos*)
Let E satisfy the weaker condition: if $(x, x_2, \ldots, x_n) \in E$ and $(y, x_2, \ldots, x_n) \in E$, then for all $x < z < y$ we have $(z, x_2, \ldots, x_n) \in E$. For such points, define $f_{x_2,\ldots,x_n}(z) = f(z, x_2, \ldots, x_n)$, then $f'_{x_2,\ldots,x_n}(z) = (D_1 f)(z, x_2, \ldots, x_n) = 0$, so by the Mean Value Theorem $f_{x_2,\ldots,x_n}$ must be constant on $[x, y]$. Hence $f(z, x_2, \ldots, x_n)$ must equal this constant value for all z such that $(z, x_2, \ldots, x_n) \in E$, so that $f(\mathbf{x})$ depends only on $x_2, \ldots, x_n$.

To see a counterexample, let $E \subset \mathbf{R}^2$ be the square (x, y), $-1 < x < 1$, $-1 < y < 1$ less the positive

y-axis, $(0,y)$, $0<y<1$. Define

$$f(x,y)=\begin{cases}0 & -1<x<1,\ -1<y<0\\ -y & -1<x<0,\ 0\le y<1\\ y & 0<x<1,\ 0\le y<1\end{cases}$$

Then f is continuous, $(D_1 f)(x,y)=0$ for all $(x,y)\in E$, but the value of f does not depend only on y.

217. Exercise 11: If f and g are differentiable real functions in $\mathbf{R}^n$, prove that

$$\nabla(fg)=f\,\nabla g+g\,\nabla f$$

and that $\nabla(1/f)=-f^{-2}\,\nabla f$ wherever $f\neq 0$.

Solution: (*analambanomenos*)

$$\begin{aligned}\nabla(fg)&=\sum_{i=1}^{n}\big(D_i(fg)\big)\mathbf{e}_i\\ &=\sum_{i=1}^{n}\big(f(D_i g)+g(D_i f)\big)\mathbf{e}_i\\ &=f\sum_{i=1}^{n}(D_i g)\mathbf{e}_i+g\sum_{i=1}^{n}(D_i f)\mathbf{e}_i\\ &=f\,\nabla g+g\,\nabla f\end{aligned}$$

$$\begin{aligned}0&=\nabla(1)\\ &=\nabla(f\cdot f^{-1})\\ &=f\,\nabla(f^{-1})+f^{-1}\,\nabla f\\ f\,\nabla(f^{-1})&=-f^{-1}\,\nabla f\\ \nabla(f^{-1})&=-f^{-2}\,\nabla f\end{aligned}$$

218. Exercise 12: Fix two real numbers a and b, $0<a<b$. Define a mapping $\mathbf{f}=(f_1,f_2,f_3)$ of $\mathbf{R}^2$ into $\mathbf{R}^3$ by

$$\begin{aligned}f_1(s,t)&=(b+a\cos s)\cos t\\ f_2(s,t)&=(b+a\cos s)\sin t\\ f_3(s,t)&=a\sin s\end{aligned}$$

Describe the range K of $\mathbf{f}$. (It is a certain compact subset of $\mathbf{R}^3$.)

(a) Show that there are exactly 4 points $\mathbf{p}\in K$ such that

$$(\nabla f_1)\big(\mathbf{f}^{-1}(\mathbf{p})\big)=\mathbf{0}.$$

Find these points.

(b) Determine the set of all $\mathbf{q}\in K$ such that

$$(\nabla f_3)\big(\mathbf{f}^{-1}(\mathbf{q})\big)=\mathbf{0}.$$

(c) Show that one of the points $\mathbf{p}$ found in part (a) corresponds to a local maximum of f_1, one corresponds to a local minimum, and that the other two are neither (they are so-called "saddle points"). Which of the points $\mathbf{q}$ found in part (b) correspond to maxima or minima?

(d) Let λ be an irrational real number, and define $\mathbf{g}(t) = \mathbf{f}(t, \lambda t)$. Prove that $\mathbf{g}$ is a 1-1 mapping of $\mathbf{R}^1$ onto a dense subset of K. Prove that

$$|\mathbf{g}'(t)|^2 = a^2 + \lambda^2(b + a\cos t)^2.$$

Solution: (*analambanomenos*)
The range of $\mathbf{f}$ is a torus with inner radius $b-a$ and outer radius $b+a$, centered at the origin, whose plane is perpendicular to the z-axis. To see this, the central circle of such a torus is the set of points $(b\cos t, b\sin t, 0)$. Then the circle of radius a which intersects the half-plane that goes through the z-axis and the point $(b\cos t, b\sin t, 0)$ is the circle centered at that point and perpendicular to the x-y-plane, with radius vector $a(\cos t, \sin t, 0)$, which is described by

$$(b\cos t, b\sin t, 0) + a(\cos t, \sin t, 0)\cos s + a(0,0,1)\sin s = \big(f_1(s,t), f_2(s,t), f_3(s,t)\big).$$

Note that $\mathbf{f}$ is a one-to-one mapping of the square S given by (x, y), $0 \le x < 2\pi$, $0 \le y < 2\pi$ onto the torus.

(a) Taking the partial derivatives of f_1 with respect to s and t to get ∇f_1, we want to solve

$$(\nabla f_1)(s,t) = \big(-a\sin s\cos t, -(b + a\cos s)\sin t\big) = (0,0).$$

Restricing ourselves to the square S, the first component equals 0 when either $s = 0$, $s = \pi$, $t = \pi/2$, or $t = 3\pi/2$. Since $b + a\cos s > 0$ for all s, the second component equals 0 only when $t = 0$ or $t = \pi$. Hence ∇f_1 equals $\mathbf{0}$ in S only at the four points

$$(0,0), \quad (0,\pi), \quad (\pi,0), \quad (\pi,\pi),$$

which map onto the four points

$$(b+a,0,0), \quad (-b-a,0,0), \quad (b-a,0,0), \quad (a-b,0,0),$$

respectively.

(b) Taking the partial derivatives of f_3 with respect to s and t to get ∇f_3, we want to solve

$$(\nabla f_3)(s,t) = (a\cos s, 0) = (0,0).$$

which occurs in S at the points (s,t), where $s = \pi/2$ or $s = 3\pi/2$, and $0 \le t < 2\pi$. The points $(\pi/2, t)$, $0 \le t < 2\pi$, are mapped by $\mathbf{f}$ to the circle $(b\cos t, b\sin t, a)$, and the points $(3\pi/2, t)$, $0 \le t < 2\pi$, are mapped to the circle $(b\cos t, b\sin t, -a)$.

(c) At $(b+a,0,0)$, corresponding to $(s,t) = (0,0)$, where $\cos s = \cos t = 1$, f_1 attains its maximum value, $b+a$, and at $(-b-a,0,0)$, corresponding to $(s,t) = (0,\pi)$, where $\cos s = 1$ and $\cos t = -1$, f_1 attains its minimum value, $-b-a$. At $(b-a,0,0)$, corresponding to $(s,t) = (\pi,0)$, where $\cos s = -1$ and $\cos t = 1$, f_1 increases if you fix $t = 0$ and vary s, and decreases if you fix $s = \pi$ and vary t. Similarly, at $(a-b,0,0$, corresponding to $(s,t) = (0,\pi)$, where $\cos s = 1$ and $\cos t = -1$, f_1 decreases if you fix $t = \pi$ and vary s, and increases if you fix $s = \pi$ and vary t.

The points $(b\cos t, b\sin t, a)$, $0 \le t < 2\pi$, correspond to the points $(s,t) = (\pi/2, t)$ where f_3 attains its maximum value of a, and $(b\cos t, b\sin t, -a)$, $0 \le t < 2\pi$, correspond to the points $(s,t) = (3\pi/2, t)$ where f_3 attains its minimum value of $-a$.

(d) To show that the "irrational winding of the torus" (it even has its own Wikipedia page) is dense, I first show a general proposition about continuous functions. Suppose F is a continuous mapping from X to Y (both metric, or even just topological spaces). Then if D is a dense subset of X, then $F(D)$ is a dense subset of $F(Y)$. For if not, then there would be a nonempty open set G in $F(Y)$ disjoint from $F(D)$. Then $F^{-1}(D)$ would be an open subset of X disjoint from D, contradicting the density of D.

Solution: (9.12(d), continued) Hence, to show that the image of $\mathbf{g}$ is dense in the torus K, it suffices to show that its preimage under the mapping $\mathbf{f}$ is dense in $\mathbf{R}^2$. Call this set K', it is the set of points of the form

$$(s + m2\pi, \lambda s + n2\pi), \qquad s \text{ real, } m \text{ and } n \text{ integers.}$$

First I want to show that the intersection of K' with the s-axis is dense on the axis. If $(s, 0)$ is a point in this set, then from the above, we that λs is a multiple of 2π, so $s = (n\lambda^{-1} + m)2\pi$ for some integers m and n. To see that this set is dense, I first show another general proposition.

If α is an irrational number, then the set of points D in $[0, 1]$ which are equivalent to integral multiples of α modulo 1 are dense in that interval. That is, let n be an integer, then $n\alpha$ is in the interval $(m, m + 1)$ for some integer m, so that $x_n = n\alpha - m$ is between 0 and 1. These points are all distinct, since if $x_{n_1} = n_1\alpha - m_1 = x_{n_2} = n_2\alpha - m_2$, for some integers $n_1 \neq n_2$, then $\alpha = (m_1 - m_2)/(n_2 - n_1)$ would be rational. Hence D is an infinite set.

Another simple fact is that if $x_{n_1} < x_{n_2}$ are two points in D then $0 < x_{n_2} - x_{n_1} < 1$, so that $x_{n_2} - x_{n_1} = (n_2 - n_1)\alpha - (m_2 - m_1)$ is also in D. Also, if $0 < jx_n < 1$ for some positive integer j, then $jx_n = jn\alpha - jm$ is also in D.

Let $\varepsilon > 0$ and let k be an integer large enough so that $k^{-1} < \varepsilon$. Divide $[0, 1]$ into k intervals

$$I_j = \left[jk^{-1}, (j + 1)k^{-1}\right] \qquad j = 0, \ldots, (k - 1)$$

(This isn't strictly a partition, but I am ignoring the points on the boundaries, which are rational numbers which cannot be in D.) Since D is infinite, one of these intervals contains at least two points in D, say $x_{n_1} < x_{n_2}$. Then $x_{n_2} - x_{n_1}$ is a point $x_n \in D$ such that $0 < x_n < k^{-1}$. Hence each of the intervals I_j, $j = 0, \ldots, (k - 1)$, contains a point jx_n of D. If $x \in [0, 1]$, then x is in one of the I_j, so there is a point $x_n \in D$ such that $|x - x_n| < k^{-1} < \varepsilon$. Hence D is dense in $[0, 1]$.

Going back to the intersection of K' with the s-axis, it is not hard to see that the intersection of K' with the interval $[0, 2\pi]$ is the image of the set D above, with $\alpha = \lambda^{-1}$, under the mapping $s \mapsto 2\pi s$, hence K' is dense in the interval $[0, 2\pi]$ in the s-axis. Since the other points of K' on the s-axis are translates of this dense set by multiples of 2π, we see that K' is dense in the entire axis.

For other lines in $\mathbf{R}^2$ parallel to the s-axis, $(s, t) \in K'$ if $(s - \lambda^{-1}t, 0) \in K'$, that is, the intersection of K' with this line is the image of the dense interesection of K' with the s-axis under the translation $(s, 0) \mapsto (s + \lambda^{-1}t, t)$. Hence K' is also dense in this line, and so is dense in all of $\mathbf{R}^2$.

If $\mathbf{g}(t_1) = \mathbf{g}(t_2)$, then $\mathbf{f}(t_1, \lambda t_1) = \mathbf{f}(t_2, \lambda t_2)$ which would only happen if $(t_1 - t_2) = n2\pi$ and $\lambda(t_1 - t_2) = m2\pi$ for some integers m and n. But that would imply $\lambda = m/n$ which contradicts the irrationality of λ.

By Theorem 9.17 we have

$$\begin{aligned}\mathbf{f}'(s,t)\mathbf{e}_1 &= \big((D_1f_1)(s,t),(D_1f_2)(s,t),(D_1f_3)(s,t)\big)\\ &= \big(-a\sin s\cos t, -a\sin s\sin t, a\cos s\big)\\ \mathbf{f}'(s,t)\mathbf{e}_2 &= \big((D_2f_1)(s,t),(D_2f_2)(s,t),(D_2f_3)(s,t)\big)\\ &= \big(-(b+a\cos s)\sin t,(b+a\cos s)\cos t,0\big).\end{aligned}$$

If we let $\boldsymbol{\gamma}(t)=(t,\lambda t)$, so that $\boldsymbol{\gamma}'(t)=\mathbf{e}_1+\lambda\mathbf{e}_2$, then we have by the chain rule

$$\begin{aligned}\mathbf{g}'(t) &= \mathbf{f}'\big(\boldsymbol{\gamma}(t)\big)\boldsymbol{\gamma}'(t)\\ &= \mathbf{f}'(t,\lambda t)(\mathbf{e}_1+\lambda\mathbf{e}_2)\\ &= \big(-a\sin t\cos\lambda-\lambda(b+a\cos t)\sin\lambda, -a\sin t\sin\lambda t+\lambda(b+a\cos t)\cos\lambda t, a\cos t\big)\\ \big|\mathbf{g}'(t)\big|^2 &= a^2\sin^2 t\cos^2\lambda t+2\lambda a(b+a\cos t)\sin t\cos\lambda t\sin\lambda t+\lambda^2(b+a\cos t)^2\sin^2\lambda t+\\ &\quad a^2\sin^2 t\sin^2\lambda t-2\lambda a(b+a\cos t)\sin t\sin\lambda t\cos\lambda t+\lambda^2(b+a\cos t)^2\cos^2\lambda t+\\ &\quad a^2\cos^2 t\\ &= a^2+\lambda^2(b+a\cos t)^2.\end{aligned}$$

219. Exercise 13: Suppose that $\mathbf{f}$ is a differentiable mapping of $\mathbf{R}^1$ into $\mathbf{R}^3$ such $\big|\mathbf{f}(t)\big|=1$ for every t. Prove that $\mathbf{f}'(t)\cdot\mathbf{f}(t)=0$. Interpret this result geometrically.

Solution: (*analambanomenos*)
Since $\big|\mathbf{f}(t)|^2=1$ is constant, we have

$$0=\frac{d}{dt}\big|\mathbf{f}(t)\big|^2=\frac{d}{dt}\sum_{i=1}^{3}f_i^2(t)=2\sum_{i=1}^{3}f_i'(t)f_i(t)=2\mathbf{f}'(t)\cdot\mathbf{f}(t).$$

So $\mathbf{f}'(t)$ is perpendicular to $\mathbf{f}(t)$. I suppose this hasn't been shown, but if $\mathbf{a}\cdot\mathbf{b}=0$, then $|\mathbf{a}+\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2$ and you can apply the law of cosines to the triangle formed by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a}+\mathbf{b}$ to conclude that $\mathbf{a}$ and $\mathbf{b}$ form a right angle. Also, the book hasn't mentioned tangent vectors, but this says that tangent vectors of curves on a sphere are perpendicular to the radial vectors, or that the tangent plane at the point on the sphere is perpendicular to the radius.

220. Exercise 14: Define $f(0,0)=0$ and

$$f(x,y)=\frac{x^3}{x^2+y^2}\qquad\text{if }(x,y)\neq(0,0).$$

(a) Prove that D_1f and D_2f are bounded functions in $\mathbf{R}^2$. (Hence f is continuous.)

(b) Let $\mathbf{u}$ be any unit vector in $\mathbf{R}^2$. Show that the directional derivative $D_{\mathbf{u}}f(0,0)$ exists, and that its absolute value is at most 1.

(c) Let γ be a differentiable mapping of $\mathbf{R}^1$ into $\mathbf{R}^2$ (in other words, γ is a differentiable curve in $\mathbf{R}^2$), with $\gamma(0)=(0,0)$ and $\big|\gamma'(0)\big|>0$. Put $g(t)=f\big(\gamma(t)\big)$ and prove that g is differentiable for every $t\in\mathbf{R}^1$. If $\gamma\in\mathscr{C}'$, prove that $g\in\mathscr{C}'$.

(d) In spite of this, prove that f is not differentiable at $(0,0)$.

Solution: (*analambanomenos*)
This function is easier to visualize if we convert to polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, with $r \geq 0$ and $0 \leq \theta < 2\pi$. Then we have $f(r,\theta) = r\cos^3\theta$, so that f is linear in r along rays from the origin, and since $\cos^3(\theta+\pi) = -\cos^3\theta$, f is a linear function along every line through the origin.

(a) Keeping in mind that along the x axis $f(x,0) = x$, and along the y axis $f(0,y) = 0$, we have

$$D_1 f(x,y) = \begin{cases} \dfrac{x^2(x^2+3y^2)}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0) \end{cases}$$
$$D_2 f(x,y) = \begin{cases} \dfrac{2x^3 y}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Converting to polar coordinates and simplifying we have, for $r > 0$ and away from the origin,

$$\begin{aligned} D_1 f(r,\theta) &= \cos^2\theta(\cos^2\theta + 3\sin^2\theta) \\ D_2 f(r,\theta) &= -2\cos^3\theta\sin\theta. \end{aligned}$$

These do not depend on r and are continuous functions of θ, which attain their maximum and minimum values on the compact set $[0, 2\theta]$. Excluding the origin, $D_1 f$ and $D_2 f$ take these maximum and minimum values along certain rays emanating from the origin. Hence they are bounded on $\mathbf{R}^2$.

(b) We can express $\mathbf{u}$ as $\cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$ for some $0 \leq \theta < 2\pi$. Then, from the above expression of f in polar coordinates, we have $f(t\mathbf{u}) = t\cos^3\theta$ for all $t \in \mathbf{R}^1$ so that $D_{\mathbf{u}} f(0,0) = \cos^3\theta$, whose absolute value is less than 1.

(c) Since f is differentiable away from the origin, we only need to show that $g'(0)$ exists. Since $\gamma(t)$ is differentiable at $t = 0$, there are functions $\delta_1(t)$ and $\delta_2(t)$ such that, for $j = 1, 2$, $\gamma_j(t) = t\gamma_j'(0) + t\delta_j(t)$ and $\lim \delta_j(t) = 0$ as $t \to 0$. Hence

$$\begin{aligned} g'(0) &= \lim_{t\to 0} \frac{g(t)}{t} \\ &= \lim_{t\to 0} \frac{f\big(\gamma_1(t), \gamma_2(t)\big)}{t} \\ &= \lim_{t\to 0} \frac{f\big(t\gamma_1'(0) + t\delta_1(t), t\gamma_2'(0) + t\delta_2(t)\big)}{t} \\ &= \lim_{t\to 0} \frac{1}{t} \cdot \frac{t^3\big(\gamma_1'(0) + \delta_1(t)\big)^3}{t^2\Big(\big(\gamma_1'(0) + \delta_1(t)\big)^2 + \big(\gamma_2'(0) + \delta_2(t)\big)^2\Big)} \\ &= \frac{\gamma_1'(0)^3}{\big|\gamma'(0)\big|^2} \end{aligned}$$

Also, if $\gamma_r(t)$ and $\gamma_\theta(t)$ are the polar coordinates of $\gamma(t)$ for $t \neq 0$, then we have

$$\begin{aligned}
\lim_{t\to 0} \cos\gamma_\theta(t) &= \lim_{t\to 0} \frac{\gamma_1(t)}{\left|\gamma(t)\right|} \\
&= \lim_{t\to 0} \frac{t\bigl(\gamma'(0)+\delta_1(t)\bigr)}{t\sqrt{\bigl(\gamma_1'(0)+\delta_1(0)\bigr)^2+\bigl(\gamma_2'(0)+\delta_2(0)\bigr)^2}} \\
&= \frac{\gamma_1'(0)}{\left|\gamma'(0)\right|} \\
\lim_{t\to 0} \sin\gamma_\theta(t) &= \lim_{t\to 0} \frac{\gamma_2(t)}{\left|\gamma(t)\right|} \\
&= \lim_{t\to 0} \frac{t\bigl(\gamma'(0)+\delta_2(t)\bigr)}{t\sqrt{\bigl(\gamma_1'(0)+\delta_1(0)\bigr)^2+\bigl(\gamma_2'(0)+\delta_2(0)\bigr)^2}} \\
&= \frac{\gamma_2'(0)}{\left|\gamma'(0)\right|}.
\end{aligned}$$

For t where $\gamma(t) \neq \mathbf{0}$, we can apply the special case of the chain rule, Theorem 9.15, worked out in Example 9.18 to get

$$\begin{aligned}
g'(t) &= D_1 f\bigl(\gamma_1(t),\gamma_2(t)\bigr)\gamma_1'(t) + D_2 f\bigl(\gamma_1(t),\gamma_2(t)\bigr)\gamma_2'(t) \\
&= \cos^2\gamma_\theta(t)\bigl(\cos^2\gamma_\theta(t)+3\sin^2\gamma_\theta(t)\bigr)\gamma_1'(t) - 2\bigl(\cos^3\gamma_\theta(t)\sin\gamma_\theta(t)\bigr)\gamma_2'(t)
\end{aligned}$$

Now assume that $\gamma \in \mathscr{C}'$. Applying the above limits we get

$$\begin{aligned}
\lim_{t\to 0} g'(t) &= \frac{\gamma_1'(0)^2}{\left|\gamma'(0)\right|^2}\left(\frac{\gamma_1'(0)^2}{\left|\gamma'(0)\right|^2} + 3\frac{\gamma_2'(0)^2}{\left|\gamma'(0)\right|^2}\right)\gamma_1'(0) - 2\left(\frac{\gamma_1'(0)^3}{|\gamma'(0)|^3}\frac{\gamma_2'(0)}{\left|\gamma'(0)\right|}\right)\gamma_2'(0) \\
&= \frac{\gamma_1'(0)^3\bigl(\gamma_1'(0)^2+\gamma_2'(0)^2\bigr)}{\left|\gamma'(0)\right|^4} \\
&= \frac{\gamma_1'(0)^3}{\left|\gamma'(0)\right|^2} \\
&= g'(0).
\end{aligned}$$

Hence g is continuous at 0, and so $g \in \mathscr{C}'$.

(d) Let $\mathbf{u} = \bigl(\cos(\pi/4), \sin(\pi/4)\bigr)$, a unit vector, and suppose f were differentiable at $(0,0)$. Then by formula (40) in Example 9.18 in the text, the directional derivative would be

$$D_{\mathbf{u}} f(0,0) = D_1(0,0)\cos(\pi/4) + D_2(0,0)\sin(\pi/4) = \cos(\pi/4) = \sqrt{2}/2.$$

However, by part (b), $D_{\mathbf{u}} f(0,0) = \cos^3(\pi/4) = \sqrt{2}/4$. Hence f cannot be differentiable at $(0,0)$.

221. Exercise 15: Define $f(0,0) = 0$, and put

$$f(x,y) = x^2 + y^2 - 2x^2 y - \frac{4x^6y^2}{(x^4+y^2)^2} \qquad \text{if } (x,y) \neq (0,0).$$

(a) Prove, for all $(x,y) \in \mathbf{R}^2$, that

$$4x^4y^2 \leq (x^4+y^2)^2.$$

Conclude that f is continuous.

(b) For $0 \le \theta \le 2\pi$, $-\infty < t < \infty$, define

$$g_\theta(t) = f(t\cos\theta, t\sin\theta).$$

Show that $g_\theta(0) = 0$, $g'_\theta(0) = 0$, $g''_\theta(0) = 2$. Each g_θ has therefore a strict local minimum at $t = 0$. In other words, the restriction of f to each line through $(0,0)$ has a strict local minimum at $(0,0)$.

(c) Show that $(0,0)$ is nevertheless not a local minimum for f, since $f(x, x^2) = -x^4$.

Solution: (*analambanomenos*)
(a) We have

$$(x^4+y^2)^2 - 4x^4y^2 = x^8 - 2x^4y^2 + y^4 - 4x^4y^2 = x^8 - 2x^4y^2 + y^4 = (x^4-y^2)^2 \ge 0$$

so that $4x^4y^2 \le (x^4+y^2)^2$. To show that f is continuous, it suffices to show this at $(0,0)$, since f is differentiable otherwise. We have

$$\big|f(x,y)\big| \le x^2 + y^2 + 2x^2|y| + x^2\frac{4x^4y^2}{(x^4+y^2)^2} \le 2x^2 + y^2 + 2x^2|y|$$

which goes to $f(x,y) = 0$ as $(x,y) \to 0$.

(b) We have $g_\theta(0) = f(0,0) = 0$, and for $t \ne 0$,

$$\begin{aligned} g_\theta(t) &= t^2\cos^2\theta + t^2\sin^2\theta - 2t^3\cos^2\theta\sin\theta - \frac{4t^8\cos^6\theta\sin^2\theta}{(t^4\cos^4\theta + t^2\sin^2\theta)^2} \\ &= t^2 - 2t^3\cos^2\theta\sin\theta - \frac{4t^4\cos^6\theta\sin^2\theta}{(t^2\cos^4\theta + \sin^2\theta)^2} \end{aligned}$$

The denominator in the last term is nonzero unless $\theta = 0$ and $t = 0$. Excluding the case $\theta = 0$ this expression for $g_\theta(t)$ is true for all values of t including 0, so we can differentiate it using the usual rules of calculus to get

$$\begin{aligned} g'_\theta(t) &= 2t - 6t^2\cos^2\theta\sin\theta - \frac{16t^3\cos^6\theta\sin^4\theta}{(t^2\cos^2\theta + \sin^2\theta)^3} \\ g''_\theta(t) &= 2 - 12t\cos^2\theta\sin\theta + \frac{48t^2\cos^6\theta\sin^4\theta(t^2\cos^2\theta - \sin^2\theta)}{(t^2\cos^2\theta + \sin^2\theta)^4} \end{aligned}$$

Hence, for $\theta \ne 0$ we have $g'_\theta(0) = 0$ and $g''_\theta(0) = 2$. For the special case $\theta = 0$, the formula for $g_\theta(t)$ simplifies to $g_0(t) = t^2$, so that we also have $g'_0(0) = 0$ and $g''_0(0) = 2$.

(c) We have

$$f(x,x^2) = x^2 + x^4 - 2x^2x^2 - \frac{4x^6x^4}{(x^4+x^4)^2} = x^2 + x^4 - 2x^4 - x^2 = -x^4.$$

222. Exercise 16: Show that the continuity of $\mathbf{f}'$ at the point $\mathbf{a}$ is needed in the inverse function theorem, even in the case $n = 1$: If

$$f(t) = t + 2t^2\sin\left(\frac{1}{t}\right)$$

for $t \ne 0$, and $f(0) = 0$, then $f'(0) = 1$, f' is bounded in $(-1,1)$, but f is not one-to-one in any neighborhood of 0.

Solution: (*analambanomenos*)
The derivative of f at 0 is

$$f'(0) = \lim_{t\to 0} \frac{f(t)}{t} = \lim_{t\to 0} \big(1 + 2t\sin(t^{-1})\big) = 1$$

since $\big|2t\sin(t^{-1})\big| \le 2t \to 0$ as $t \to 0$.

For $t \in (-1, 1)$, $t \ne 0$, we have

$$\begin{aligned} f'(t) &= 1 + 4t\sin(t^{-1}) - 2\cos(t^{-1}) \\ \big|f'(t)\big| &\le 1 + 4\big|t\sin(t^{-1})\big| + 2\big|\cos(t^{-1})\big| \\ &\le 3 + 4|t| \\ &\le 7 \end{aligned}$$

If n is a positive integer, let $a_n = 2n\pi$, $b_n = (2n+1)\pi$. Then, since $\sin a_n = \sin b_n = 0$, $\cos a_n = 1$, and $\cos b_n = -1$, we have $f'(a_n^{-1}) = -1$ and $f'(b_n^{-1}) = 3$. Hence f has a local maximum in (b_n^{-1}, a_n^{-1}) and a local minimum in (a_n^{-1}, b_{n+1}^{-1}) and so cannot be one-to-one in either interval. Since a neighborhood of 0 contains an infinite number of such intervals, f cannot be one-to-one in any neighborhood of 0.

223. Exercise 17: Let $\mathbf{f} = (f_1, f_2)$ be the mapping of $\mathbf{R}^2$ into $\mathbf{R}^2$ given by

$$f_1(x, y) = e^x \cos y, \qquad f_2(x, y) = e^x \sin y.$$

(a) What is the range of $\mathbf{f}$?

(b) Show that the Jacobian of $\mathbf{f}$ is not zero at any point of $\mathbf{R}^2$. Thus every point of $\mathbf{R}^2$ has a neighborhood in which $\mathbf{f}$ is one-to-one. Nevertheless, $\mathbf{f}$ is not one-to-one on $\mathbf{R}^2$.

(c) Put $\mathbf{a} = (0, \pi/3)$, $\mathbf{b} = \mathbf{f}(\mathbf{a})$, let $\mathbf{g}$ be the continuous inverse of $\mathbf{f}$, defined in a neighborhood of $\mathbf{b}$, such that $\mathbf{g}(\mathbf{b}) = \mathbf{a}$. Find an explicit formula for $\mathbf{g}$, compute $\mathbf{f}'(\mathbf{a})$ and $\mathbf{g}'(\mathbf{b})$, and verity the formula (52).

(d) What are the images under $\mathbf{f}$ of lines parallel to the coordinate axes?

Solution: (*analambanomenos*)
(d) Fix $y = y_0$. Then $\mathbf{f}$ maps (x, y_0), $x \in \mathbf{R}$, to the ray from (but not including) the origin through the point $(\cos y_0, \sin y_0)$ on the unit circle. The negative values of x map to points on the ray inside the circle, and the positive values of x map to points outside the unit circle.

Fix $x = x_0$. Then $\mathbf{f}$ maps (x_0, y), $v \in \mathbf{R}$, to the circle of radius e^x. Note that $\mathbf{f}(x_0, y) = \mathbf{f}(x_0, y + 2n\pi)$ for all integers n.

(a) The results of (d) show that the range of $\mathbf{f}$ is $\mathbf{R}^2$ minus the origin.

(b) Since

$$\begin{aligned} D_1 f_1(x, y) &= e^x \cos y & D_1 f_2(x, y) &= e^x \sin y \\ D_2 f_1(x, y) &= -e^x \sin y & D_1 f_2(x, y) &= e^x \cos y \end{aligned}$$

the Jacobian of $\mathbf{f}$ at (x, y) is $e^{2x}(\cos^2 y + \sin^2 y) = e^{2x} \ne 0$ for all x. From (d) we have $\mathbf{f}(x, y) = \mathbf{f}(x, y + 2n\pi)$ for all integers n, so $\mathbf{f}$ is not one-to-one on $\mathbf{R}^2$.

(c) Let $w = f_1(x, y) = e^x \cos y$, $z = f_2(x, y) = e^x \sin y$. Then, in a neighborhood of $\mathbf{b} = (1/2, \sqrt{3}/2)$ where $w \neq 0$, we have $w^2 + z^2 = e^{2x}$, $z/w = \tan y$, so

$$\mathbf{g}(w, z) = \left(\log \sqrt{w^2 + z^2}, \arctan(z/w)\right)$$

$$\mathbf{f}'(x, y) = \begin{pmatrix} e^x \cos y & e^x \sin y \\ -e^x \sin y & e^x \cos y \end{pmatrix}$$

$$\mathbf{g}'(w, z) = \begin{pmatrix} w/(w^2 + z^2) & -z/(w^2 + z^2) \\ z/(w^2 + z^2) & w/(w^2 + z^2) \end{pmatrix}$$

$$\mathbf{f}'(\mathbf{a}) \circ \mathbf{g}'(\mathbf{b}) = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

224. Exercise 18: Answer analogous questions for the mapping defined by

$$u = x^2 - y^2, \qquad v = 2xy.$$

Solution: (*analambanomenos*)
Let the mapping be denoted by $\mathbf{F} = (f_1, f_2)$. Then $\mathbf{F}$ maps the points $(x, 0)$ and $(-x, 0)$ on the x-axis to $(x^2, 0)$ on the positive u-axis, and for a fixed $y_0 \neq 0$, $\mathbf{F}$ maps the lines (x, y_0) and $(x, -y_0)$ to the parabola $u = v^2/(4y_0)^2 - y_0^2$, which is symmetric with respect to the u-axis and passes through and otherwise lies to the left of the point $(-y_0^2, 0)$. If $y_0 > 0$, then the points (x, y_0), $x > 0$, are mapped to the upper branch of the parabola, and the points (x, y_0), $x < 0$, are mapped to the lower branch, and the opposite is true if $y_0 < 0$.

Similarly, $\mathbf{F}$ maps the point $(0, y)$ and $(0, -y)$ on the y-axis to $(-y^2, 0)$ on the u-axis, and for a fixed $x_0 \neq 0$, $\mathbf{F}$ maps the lines (x_0, y) and $(-x_0, y)$ to the parabola $u = -v^2/(4x_0)^2 + x_0^2$, which is symmetric with respect to the u-axis and passes through and otherwise lies to the right of the point $(x_0^2, 0)$. If $x_0 > 0$, then the points (x_0, y), $y > 0$, are mapped to the upper branch of the parabola, and the points (x_0, y), $y < 0$, are mapped to the lower branch, and the opposite is true if $x_0 < 0$.

Other than mapping the origin in the x-y plane to the origin in the u-v plane, $\mathbf{F}$ maps two distinct points in the x-y plane to each point in the u-v plane outside the origin. If we let $w = \sqrt{u^2 + v^2}$ be the distance from the origin to the point (u, v), then

$$\left(\sqrt{\frac{w+u}{2}}, \sqrt{\frac{w-u}{2}}\right) \quad \text{and} \quad \left(-\sqrt{\frac{w+u}{2}}, -\sqrt{\frac{w-u}{2}}\right)$$

are mapped by $\mathbf{F}$ to the point (u, v) if v is positive, and

$$\left(\sqrt{\frac{w+u}{2}}, -\sqrt{\frac{w-u}{2}}\right) \quad \text{and} \quad \left(-\sqrt{\frac{w+u}{2}}, \sqrt{\frac{w-u}{2}}\right)$$

are mapped by $\mathbf{F}$ to the point (u, v) if v is negative.

Since

$$\begin{aligned} D_1 f_1(x, y) &= 2x & D_2 f_1(x, y) &= -2y \\ D_1 f_2(x, y) &= 2y & D_2 f_2(x, y) &= 2x \end{aligned}$$

the Jacobian of $\mathbf{f}$ at (x,y) is $4(x^2+y^2)$ which is nonzero except at the origin. If we exclude the origin in both planes, then $\mathbf{F}$ is locally one-to-one, but globally two-to-one.

Letting $\mathbf{b}=(3,4)$, then $\mathbf{F}$ maps the point $\mathbf{a}=(2,1)$ to $\mathbf{b}$. Again letting $w=\sqrt{u^2+v^2}$ be the distance from the origin to the point (u,v), locally $\mathbf{F}$ has the inverse function

$$\begin{aligned}
\mathbf{G}(u,v) &= \left(\sqrt{\frac{w+u}{2}}, \sqrt{\frac{w-u}{2}}\right) \\
\mathbf{F}'(x,y) &= \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix} \\
\mathbf{G}'(u,v) &= \begin{pmatrix} \frac{u+w}{4w}\sqrt{\frac{2}{w+u}} & \frac{v}{4w}\sqrt{\frac{2}{w+u}} \\ \frac{u-w}{4w}\sqrt{\frac{2}{w-u}} & \frac{v}{4w}\sqrt{\frac{2}{w-u}} \end{pmatrix} \\
\mathbf{F}'(\mathbf{a})\circ\mathbf{G}'(\mathbf{b}) &= \begin{pmatrix} 4 & -2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1/5 & 1/10 \\ -1/10 & 1/5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

225. Exercise 19: Show that the system of equations

$$\begin{aligned}
3x+y-z+u^2 &= 0 \\
x-y+2z+u &= 0 \\
2x+2y-3z+2u &= 0
\end{aligned}$$

can be solved for x,y,u in terms of z; for x,z,u in terms of y; for y,z,u in terms of x; but not for x,y,z in terms of u.

Solution: (*analambanomenos*)
Let

$$\begin{aligned}
\mathbf{f}(x,y,z,u) &= \big(f_1(x,y,z,u), f_2(x,y,z,u), f_3(x,y,z,u)\big) \\
&= (3x+y-z+u^2, x-y+2z+u, 2x+2y-3z+2u.
\end{aligned}$$

Then the matrix of $\mathbf{f}'(x,y,z,u)$ is

$$\big(D_j f_i(x,y,z,u)\big) = \begin{pmatrix} 3 & 1 & -1 & 2u \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{pmatrix}.$$

Note that $\mathbf{f}(\mathbf{0})=\mathbf{0}$. The x,y,u part of $\mathbf{f}'$ has determinant $8u-12\neq 0$ near $\mathbf{0}$, so by the implicit function theorem, there is a solution of $\mathbf{f}\big(x(z),y(z),z,u(z)\big)=\mathbf{0}$ near $\mathbf{0}$.

Similarly, the determinant of the x,z,u part of $\mathbf{f}'$ is equal to $21-14u\neq 0$ near $\mathbf{0}$, so there is a solution of $\mathbf{f}\big(x(y),y,z(y),u(y)\big)=\mathbf{0}$ near $\mathbf{0}$. And the determinant of the y,z,u part of $\mathbf{f}'$ is equal to $3-2u\neq 0$ near $\mathbf{0}$, so there is a solution of $\mathbf{f}\big(x,y(x),z(x),u(x)\big)=\mathbf{0}$ near $\mathbf{0}$.

However, the determinant of the x,y,z part of $\mathbf{f}'$ is equal to 0, so the implicit function theorem cannot be applied. If you try to solve for x,y,z in terms of u, you just get an equation in u which has no solution near $\mathbf{0}$.

226. Exercise 20: Take $n=m=1$ in the implicit function theorem, and interpret the theorem (as well as

its proof) graphically.

Solution: (*analambanomenos*)
If the real-valued function $f(x,y)$ is smooth and nonconstant in a region of $\mathbf{R}^2$, then the solution of $f(x,y)=0$ is locally a smooth curve. If $D_1f(x_0,y_0)\neq 0$ at a point of the curve, then the curve doesn't have a vertical tangent at (x_0,y_0), and so it will be the graph of a function $y=g(x)$ near $x=x_0$, so that $f\big(x,g(x)\big)=0$. Similarly, if $D_2f(x_0,y_0)\neq 0$, then the curve doesn't have a horizontal tangent at (x_0,y_0), and so it will be the graph of a function $x=h(y)$ near $y=y_0$, so that $f\big(h(y),y\big)=0$.

227. Exercise 21: Define f in $\mathbf{R}^2$ by

$$f(x,y)=2x^3-3x^2+2y^3+3y^2.$$

(a) Find the four points of $\mathbf{R}^2$ at which the gradient of f is zero. Show that f as exactly one local maximum and one local minimum in $\mathbf{R}^2$.

(b) Let S be the set of all $(x,y)\in\mathbf{R}^2$ at which $f(x,y)=0$. Find those points of S that have no neighborhoods in which the equation $f(x,y)=0$ can be solved for y in terms of x (or for x in terms of y). Describe S as precisely as you can.

Solution: (*analambanomenos*)
(a) We have $D_1f(x,y)=6x(x-1)$ so $D_1f(x,y)$ if $x=0$ or $x=1$. Also, $D_2f(x,y)=6y(6+1)$ so $D_2f(x,y)=0$ if $y=0$ or $y=-1$. Hence the gradient of f equals 0 at the four points $(0,0)$, $(1,0)$, $(0,-1)$, and $(1,-1)$. To tell whether these are local maxima or minima, or saddle points, the easiest way is to apply the second derivative test for multivariable functions, which involves finding the eigenvalues of the matrix of second derivatives (the "Hessian") at these points, but that wasn't demonstrated in the text, so we need to show this more directly.

At $(1,0)$, for small values d_1 and d_2, we have

$$f(1+d_1,d_2)-f(1,0)=d_1^2(2d_1+3)+d_2^2(2d_2+3)$$

which is positive for small values of d_1 and d_2. Hence f has a local minimum at $(1,0)$.

Similarly, at $(0,-1)$ we have

$$f(d_1,-1+d_2)-f(0,-1)=d_1^2(2d_1-3)+d_2^2(2d_2-3)$$

which is negative for small values of d_1 and d_2. Hence f has a local maximum at $(1,0)$.

At $(0,0)$ $f(x,0)=2x^3-3x^2$ can be shown to have a local maximum at $x=0$, using the usual calculus techniques. However, $f(0,y)=2y^3+3y^2$ has a local minimum at $y=0$. Hence $(0,0)$ is a saddle point for f.

Similarly, at $(1,-1)$ $f(x,-1)=2x^3-3x^2+1$ has a local minimum at $x=1$, but $f(1,y)=2y^3+3y^2-1$ has a local maximum at $y=-1$. Hence $(1,-1)$ is also a saddle point for f.

(b) Note that

$$f(x,y)=(x+y)\big(2(x^2-xy+y^2)-3(x-y)\big).$$

The first factor shows that f is equal to zero along the diagonal $y=-x$. The zero set of the second factor of degree 2 must be some sort of conic section. Converting to polar coordinates we get

$$r(\theta)=\frac{3}{2}\left(\frac{\cos\theta-\sin\theta}{1-\cos\theta\sin\theta}\right)$$

which can be seen to describe a ellipse symmetric with the diagonal line $y = -x$ and intersecting it at the points $(0,0)$ and $(1,-1)$.

The points of the zero set along the diagonal satisfy the relation $y = -x$ except at the two intersection points, where we cannot describe x or y as single-valued functions of each other.

Let $g(x,y) = 2(x^2 - xy + y^2) - 3(x - y)$. Along the zero set of $g(x,y)$, x cannot be expressed as a function of y where $D_1 g(x,y) = 4x - 2x - 3 = 0$. (I am using the elliptical zero set as described above; the general case is more complicated.) That is, we are looking for the intersection of the zero set of g with the line $y = 2x - (3/2)$, which occurs at the points $(0,-3/2)$ and $(1,1/2)$.

Similarly, y cannot be expressed as a function of x where $D_2 g(x,y) = -2x + 4y + 3 = 0$, so we are looking for the intersection of the zero set of g with the line $x = 2y + (3/2)$, which occurs at the points $(3/2,0)$ and $(-1/2,-1)$.

228. Exercise 22: Give a similar discussion for

$$f(x,y) = 2x^3 + 6xy^2 - 3x^2 + 3y^2.$$

Solution: (*analambanomenos*)
We have

$$\begin{aligned} D_1 f(x,y) &= 6(x^2 + y^2 - x) \\ D_2 f(x,y) &= (12x + 6)y \end{aligned}$$

so $D_2 f(x,y) = 0$ if $y = 0$ or $x = -1/2$. If $y = 0$, then $D_1 f(x,y) = 6x(x-1) = 0$ if $x = 0$ or $x = 1$. However, if $x = -1/2$, then $D_1 f(x,y) = 6(3/4 + y^2) > 0$ for all y. Hence the gradient of f equals $\mathbf{0}$ only at $(0,0)$ and $(1,0)$.

At $(1,0)$, let d_1, d_2 be near 0. Then

$$\begin{aligned} f(1+d_1, d_2) - f(1,0) &= 2(1+d_1)^3 + 6(1+d_1)d_2^2 - 3(1+d_1)^2 + 3d_2^2 + 1 \\ &= (d_1^2 + 3d_2^2)(2d_1 + 3) \end{aligned}$$

which is positive for small values of d_1 and d_2, hence f has a local minimum at $(1,0)$.

At $(0,0)$, the values of f along the x-axis, $f(x,0) = 2x^3 - 3x^2$, have a local maximum at $x = 0$, while the values of f along y-axis, $f(0,y) = 3y^2$ have a local minimum at $y = 0$. Hence f has a saddle point at $(0,0)$.

Solving $f(x,y) = 0$ for y, we get

$$y = \pm x\sqrt{\frac{3-2x}{3(2x+1)}}$$

The graph of this looks like the folium of Descartes, only with a vertical asymptote of $x = -1/2$ and symmetrical with the x-axis, with a double point at the origin and intersecting the x-axis at 0 and $3/2$.

At the origin, a double-point, we cannot solve for x in terms of y, or vice versa. Otherwise we can solve for x in terms of y except where $D_2 f(x,y) = 6y(2x+1) = 0$, which on the zero set only occurs at the x-axis intercepts of the origin and $(3/2,0)$. We can solve for y in terms of x away from the origin except where $D_1 f(x) = 6(x^2 + y^2 - x) = 0$. Inserting the expression for y above in this equation and solving for x, we see that this happens only at origin and the points

$$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\sqrt{2\sqrt{3}-3}\right) \quad \text{and} \quad \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\sqrt{2\sqrt{3}-3}\right)$$

229. Exercise 23: Define f in $\mathbf{R}^3$ by

$$f(x, y_1, y_2) = x^2 y_1 + e^x + y_2.$$

Show that $f(0,1,-1) = 0$, $D_1 f(0,1,-1) \neq 0$, and that there exists therefore a differentiable function g in some neighborhood of $(1,-1)$ in $\mathbf{R}^2$, such that $g(1,-1) = 0$ and

$$f\big(g(y_1, y_2), y_1, y_2\big) = 0.$$

Find $D_1 g(1,-1)$, and $D_2 g(1,-1)$.

Solution: (*analambanomenos*)
Note that

$$\begin{aligned} D_1 f(x, y_1, y_2) &= 2xy_1 + e^x \\ D_2 f(x, y_1, y_2) &= x^2 \\ D_3 f(x, y_1, y_2) &= 1. \end{aligned}$$

Hence

$$\begin{aligned} f(0,1,-1) &= 0 + 1 - 1 = 0 \\ D_1 f(0,1,-1) &= 0 + 1 = 1 \\ D_2 f(0,1,-1) &= 0 \\ D_3 f(0,1,-1) &= 1. \end{aligned}$$

We can apply the implicit function theorem with $m = 1$, $n = 2$, where

$$\begin{aligned} A_x &= \begin{pmatrix} 1 \end{pmatrix} \\ A_y &= \begin{pmatrix} 0 & 1 \end{pmatrix} \end{aligned}$$

to conclude that there is a function g in some neighborhood of $(1,-1)$ such that

$$\begin{aligned} f\big(g(y_1, y_2), y_1, y_2\big) &= 0 \\ g'(1,-1) &= -A_x^{-1} A_y \\ &= -\begin{pmatrix} 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \end{pmatrix} \end{aligned}$$

so that $D_1 g(1,-1) = 0$ and $D_2 g(1,-1) = -1$.

230. Exercise 24: For $(x, y) \neq (0,0)$, define $\mathbf{f} = (f_1, f_2)$ by

$$f_1(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \qquad f_2(x, y) = \frac{xy}{x^2 + y^2}.$$

Compute the rank of $\mathbf{f}'(x, y)$, and find the range of $\mathbf{f}$.

Solution: (*analambanomenos*)
The Jacobian of $\mathbf{f}$ is

$$\begin{aligned}\begin{vmatrix} D_1 f_1(x,y) & D_2 f_1(x,y) \\ D_1 f_2(x,y) & D_2 f_2(x,y) \end{vmatrix} &= \frac{1}{(x^2+y^2)^4} \begin{vmatrix} 4xy^2 & -4x^2y \\ y(y^2-x^2) & x(x^2-y^2) \end{vmatrix} \\ &= \frac{4x^4y^2 - 4x^2y^4 + 4x^2y^4 - 4x^4y^2}{(x^2+y^2)^4} \\ &= 0 \end{aligned}$$

so the rank of $\mathbf{f}'$ is less than 2. Since $\mathbf{f}$ is nonconstant, the rank of $\mathbf{f}'$ must be more than 0, hence the rank of $\mathbf{f}'$ is 1.

Converting to polar coordinates, we get

$$\mathbf{f}(r\cos\theta, r\sin\theta) = \left(\cos(2\theta), \frac{\sin(2\theta)}{2}\right)$$

so we see that the range of $\mathbf{f}$ is an ellipse centered at the origin and intersecting the coordinate axes at $(\pm 1, 0)$ and $(0, \pm 1/2)$.

231. Exercise 25: Suppose $A \in L(\mathbf{R}^n, \mathbf{R}^m)$, let r be the rank of A.

(a) Define S as in the proof of Theorem 9.32. Show that SA is a projection in $\mathbf{R}^n$ whose null space is $\mathscr{N}(A)$ and whose range is $\mathscr{R}(S)$.

(b) Use (a) to show that

$$\dim \mathscr{N}(A) + \dim \mathscr{R}(A) = n.$$

Solution: (*analambanomenos*)
(a) If r equals the rank of A, and $\mathscr{R}(A)$ is spanned by the independent set $\mathbf{y}_1, \dots, \mathbf{y}_r$, then $\mathbf{y}_i = A\mathbf{z}_i$, $i = 1, \dots, r$ for some independent set $\mathbf{z}_1, \dots, \mathbf{z}_r$ in $\mathbf{R}^n$, and S is a map of $\mathscr{R}(A)$ into $\mathbf{R}^n$ defined as

$$S(c_1\mathbf{y}_1 + \cdots + c_r\mathbf{y}_r) = c_1\mathbf{z}_1 + \cdots + c_r\mathbf{z}_r.$$

Note that S is a one-to-one map of $\mathscr{R}(A)$ into $\mathbf{R}^n$. Following the hint, note that

$$AS(c_1\mathbf{y}_1 + \cdots + c_r\mathbf{y}_r) = A(c_1\mathbf{z}_1 + \cdots + c_r\mathbf{z}_r) = c_1\mathbf{y}_1 + c_r\mathbf{y}_r,$$

that is, $AS(\mathbf{y}) = \mathbf{y}$ for $\mathbf{y} \in \mathscr{R}(A)$. Hence, for $\mathbf{x} \in \mathbf{R}^n$,

$$SASA(\mathbf{x}) = S\Big(AS\big(A(\mathbf{x})\big)\Big) = S\big(A(\mathbf{x})\big) = SA(\mathbf{x})$$

so $SASA$ is a projection on $\mathbf{R}^n$.

If $SASA(\mathbf{x}) = SA(\mathbf{x}) = \mathbf{0}$, then $A(\mathbf{x}) = \mathbf{0}$ since S is one-to-one, so the null space of $SASA$ is $\mathscr{N}(A)$. The range of $SASA$ is clearly a subset of $\mathscr{R}(S)$, and if $\mathbf{z} = c_1\mathbf{z}_1 + \cdots + c_r\mathbf{z}_r \in \mathscr{R}(S)$, then

$$SASA(\mathbf{z}) = SA(\mathbf{z}) = S(c_1\mathbf{y}_1 + \cdots + c_r\mathbf{y}_r) = \mathbf{z}$$

so that the range of SASA is $\mathscr{R}(S)$.

(b) From the discussion in 9.31, you can conclude that if P is a projection in $\mathbf{R}^n$, then $n = \dim \mathscr{N}(P) + \dim \mathscr{R}(P)$. Hence from part (a), we have

$$\begin{aligned} n &= \dim \mathscr{N}(SASA) + \dim \mathscr{R}(SASA) \\ &= \dim \mathscr{N}(A) + \dim \mathscr{R}(S) \\ &= \dim \mathscr{N}(A) + \dim \mathscr{R}(A) \end{aligned}$$

where the last equality follows from the fact that S is a one-to-one map on $\mathscr{R}(A)$.

232. Exercise 26: Show that the existence (and even the continuity) of $D_{12}f$ does not imply the existence of $D_1 f$. For example, let $f(x,y) = g(x)$, where is nowhere differentiable.

Solution: (*analambanomenos*)
Letting $f(x,y) = g(x)$ be the function given in the example, then $D_2 f(x,y) = 0$, so $D_{12}f(x,y) = 0$, for all (x,y). However, $D_1 f(x,y) = g'(x)$ does not exist. For g you can use the function defined in Theorem 7.18.

233. Exercise 27: Put $f(0,0) = 0$, and

$$f(x,y) = \frac{xy(x^2 - y^2)}{x^2 + y^2} \qquad \text{if } (x,y) \neq (0,0).$$

Prove that

(a) f, $D_1 f$, $D_2 f$ are continuous in $\mathbf{R}^2$;

(b) $D_{12}f$ and $D_{21}f$ exist at every point of $\mathbf{R}^2$, and are continuous except at $(0,0)$;

(c) $D_{12}f(0,0) = 1$, and $D_{21}f(0,0) = -1$.

Solution: (*analambanomenos*)
(a) Converting to polar coordinates for $(x,y) \neq (0,0)$, we have

$$\begin{aligned} f(r\cos\theta, r\sin\theta) &= \frac{r^4 \cos\theta \sin\theta(\cos^2\theta - \sin^2\theta)}{r^2} \\ &= \frac{r^2 \sin 2\theta \cos 2\theta}{2} \\ &= \frac{r^2 \sin 4\theta}{4} \\ \big|f(x,y)\big| &\leq \frac{r^2}{4} = \frac{x^2 + y^2}{4} \end{aligned}$$

which converges to $0 = f(0,0)$ as $(x,y) \to (0,0)$.

We have

$$D_1 f(0,0) = \lim_{h \to 0} \frac{f(h,0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0.$$

For $(x, y) \neq (0, 0)$, we have

$$\begin{aligned} D_1 f(x,y) &= \frac{(x^2+y^2)(3x^2y-y^3)-2x(x^3y-xy^3)}{(x^2+y^2)^2} \\ &= \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2} \\ D_1 f(r\cos\theta, r\sin\theta) &= \frac{r^5(\cos^4\theta\sin\theta+4\cos^2\theta\sin^3\theta-\sin^5\theta)}{r^4} \\ \left|D_1 f(x,y)\right| &\leq 6r = 6\sqrt{x^2+y^2} \end{aligned}$$

which converges to $0 = D_1 f(0, 0)$ as $(x, y) \to (0, 0)$.

We have

$$D_2 f(0,0) = \lim_{h\to 0}\frac{f(0,h)}{h} = \lim_{h\to 0}\frac{0}{h} = 0.$$

For $(x, y) \neq (0, 0)$, we have

$$\begin{aligned} D_2 f(x,y) &= \frac{(x^2+y^2)(x^3-3xy^2)-2y(x^3y-xy^3)}{(x^2+y^2)^2} \\ &= \frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2} \\ D_2 f(r\cos\theta, r\sin\theta) &= \frac{r^5(\cos^5 - 4\cos^3\theta\sin^2\theta-\cos\theta\sin^4\theta)}{r^4} \\ \left|D_1 f(x,y)\right| &\leq 6r = 6\sqrt{x^2+y^2} \end{aligned}$$

which converges to $0 = D_2 f(0, 0)$ as $(x, y) \to (0, 0)$.

(b) For $(x, y) \neq (0, 0)$, we have

$$\begin{aligned} D_{12} f(x,y) &= \frac{(x^2+y^2)(5x^4-12x^2y^2-y^4)-4x(x^5-4x^3y^2-xy^4)}{(x^2+y^2)^3} \\ &= \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3} \\ D_{12} f(r\cos\theta, r\sin\theta) &= \frac{r^6(\cos^6\theta+9\cos^4\theta\sin^2\theta-\sin^6\theta)}{r^6} \\ &= \cos^6\theta+9\cos^4\theta\sin^2\theta-\sin^6\theta \end{aligned}$$

So $D_{12} f$ has a constant value along the rays emanating from the origin. Since this value is not a constant function of θ, we see that $D_{12} f(x, y)$ does not converge to a limit as $(x, y) \to (0, 0)$, and so $D_{12} f(x, y)$ cannot be continuous at the origin. Also, for $(x, y) \neq (0, 0)$ we can apply Theorem 9.41 and conclude that $D_{21} f(x, y) = D_{12} f(x, y)$ is also not continuous at the origin.

(c) We have

$$\begin{aligned} D_{12} f(0,0) &= \lim_{h\to 0}\frac{D_2 f(h,0)}{h} = \lim_{h\to 0}\frac{h^5/h^4}{h} = \lim_{h\to 0} 1 = 1 \\ D_{21} f(0,0) &= \lim_{h\to 0}\frac{D_1 f(0,h)}{h} = \lim_{h\to 0}\frac{-h^5/h^4}{h} = \lim_{h\to 0} -1 = -1 \end{aligned}$$

234. Exercise 28: For $t \geq 0$, put

$$\varphi(x,t) = \begin{cases} x & 0 \leq x \leq \sqrt{t} \\ -x + 2\sqrt{t} & \sqrt{t} \leq x \leq 2\sqrt{t} \\ 0 & \text{otherwise,} \end{cases}$$

and put $\varphi(x,t) = \varphi\big(x, |t|\big)$ if $t < 0$. Show that φ is continuous on $\mathbf{R}^2$, and $D_2\varphi(x,0) = 0$ for all x. Define

$$f(t) = \int_{-1}^{1} \varphi(x,t)\,dx.$$

Show that $f(t) = t$ if $|t| < \frac{1}{4}$. Hence

$$f'(0) \neq \int_{-1}^{1} D_2\varphi(x,0)\,dx.$$

Solution: (*analambanomenos*)
Away from the origin, φ is continuous since the definitions agree at the points $(x,0)$, $(x,\sqrt{t})$ and $(x,2\sqrt{t})$. Since $\big|\varphi(x,t)\big| \leq \sqrt{|t|}$, we have for $\varepsilon > 0$, $-\varepsilon < x < \varepsilon$, and $-\varepsilon < t < \varepsilon$, $\big|\varphi(x,t)\big| < \sqrt{\varepsilon}$, so φ is also continuous at the origin.

For $x \leq 0$ we have $(x,t) = 0$ for all t, and for $x > 0$ we have $(x,t) = 0$ for $-\frac{1}{2}x^2 \leq t \leq \frac{1}{2}x^2$, so $D_2(x,0) = 0$ for all x.

If $0 \leq t < \frac{1}{4}$, then

$$\begin{aligned} f(t) &= \int_{-1}^{1} \varphi(x,t)\,dx \\ &= \int_0^{\sqrt{t}} x\,dx + \int_{\sqrt{t}}^{2\sqrt{t}} -x + 2\sqrt{t}\,dx \\ &= \frac{(\sqrt{t})^2}{2} - 0 - \frac{(2\sqrt{t})^2}{2} + 2\sqrt{t}\big(2\sqrt{t}\big) + \frac{(\sqrt{t})^2}{2} - 2\sqrt{t}\big(\sqrt{t}\big) \\ &= t \end{aligned}$$

and if $-\frac{1}{4} < t \leq 0$, then

$$\begin{aligned} f(t) &= \int_{-1}^{1} \varphi(x,t)\,dx \\ &= \int_0^{\sqrt{-t}} -x\,dx + \int_{\sqrt{-t}}^{2\sqrt{-t}} x - 2\sqrt{-t}\,dx \\ &= -\frac{(\sqrt{-t})^2}{2} + 0 + \frac{(2\sqrt{-t})^2}{2} - 2\sqrt{-t}\big(2\sqrt{-t}\big) - \frac{(\sqrt{-t})^2}{2} + 2\sqrt{-t}\big(\sqrt{-t}\big) \\ &= t \end{aligned}$$

Hence $f'(0) = 1$, while $\int_{-1}^{1} D_2\varphi(x,0)\,dx = 0$.

235. Exercise 29: Let E be an open set in $\mathbf{R}^n$. The classes $\mathscr{C}'(E)$ and $\mathscr{C}''(E)$ are defined in the text. By induction $\mathscr{C}^{(k)}(E)$ can be defined as follows, for all positive integers k: To say that $f \in \mathscr{C}^{(k)}(E)$ means that the partial derivatives $D_1 f, \ldots, D_n f$ belong to $\mathscr{C}^{(k-1)}(E)$.

Assume $f \in \mathscr{C}^{(k)}(E)$, and show (by repeated application of Theorem 9.41) that the kth-order derivative

$$D_{i_1 i_2 \cdots i_k} f = D_{i_1} D_{i_2} \cdots D_{i_k} f$$

is unchanged if the subscripts $i_1, \ldots, i_k$ are permuted. For instance, if $n \geq 3$, then $D_{1213} f = D_{3112} f$ for every $f \in \mathscr{C}^{(4)}$.

Solution: (*analambanomenos*)
If we let $g = D_{i_{n+1} \cdots i_k} f$, then by Theorem 9.41 we have

$$\begin{aligned} D_{i_1 \cdots i_k} f &= D_{i_1 \cdots i_{n-2}}(D_{i_{n-1} i_n} g) \\ &= D_{i_1 \cdots i_{n-2}}(D_{i_n i_{n-1}} g) \\ &= D_{i_1 \cdots i_{n-2} i_n i_{n-1} \cdots i_k} f, \end{aligned}$$

that is, we can swap any two adjacent indices and the derivative remains unchanged. We can apply this result to show that we can swap any two indices:

$$\begin{aligned} D_{i_1 \cdots i_m \cdots i_n \cdots i_k} f &= D_{i_1 \cdots i_m \cdots i_n i_{n-1} i_{n+1} \cdots i_k} f \\ &= (\text{keep swapping adjacent indices until } i_n \text{ comes before } i_m) \\ &= D_{i_1 \cdots i_n i_m \cdots i_{n-1} i_{n+1} \cdots i_k} f \\ &= D_{i_1 \cdots i_n i_{m+1} i_m \cdots i_{n-1} i_{n+1} \cdots i_k} f \\ &= (\text{keep swapping adjacent indices until } i_m \text{ comes after } i_{n-1}) \\ &= D_{i_1 \cdots i_{m-1} i_n i_{m+1} \cdots i_{n-1} i_m i_{n+1} \cdots i_k} f \end{aligned}$$

Since any permutation is the result of pairwise swaps (this is usually shown in elementary abstract algebra courses when discussing the permutation group), we see that in the case of $f \in \mathscr{C}^{(k)}$, we can permute the order of partial differentiation without changing the derivative.

Solution: (*analambanomenos*)
(a) I am going to show this by induction on k. For the case $k = 1$, we have by Theorem 9.15 and Theorem 9.17

$$h'(t) = f'\big(\mathbf{p}(t)\big)\mathbf{p}'(t) = \sum_{i=1}^{n} D_i f\big(\mathbf{p}(t)\big) x_i$$

which is the assertion in the case $k = 1$. Now assume the assertion is true for the case $k - 1$. Then we have

$$\begin{aligned} h^{(k)}(t) &= \frac{d}{dt} h^{(k-1)}(t) \\ &= \frac{d}{dt} \sum_{i_1, \ldots, i_{k-1}=1}^{n} D_{i_1 \cdots i_{k-1}} f\big(\mathbf{p}(t)\big) x_{i_1} \cdots x_{i_{k-1}} \\ &= \sum_{i_1, \ldots, i_{k-1}=1}^{n} \frac{d}{dt} D_{i_1 \cdots i_{k-1}} f\big(\mathbf{p}(t)\big) x_{i_1} \cdots x_{i_{k-1}} \\ &= \sum_{i_1, \ldots, i_{k-1}=1}^{n} \left(\sum_{i_k=1}^{n} D_{i_k} D_{i_1 \cdots i_{k-1}} f\big(\mathbf{p}(t)\big) x_{i_k} \right) x_{i_1} \cdots x_{i_{k-1}} \\ &= \sum_{i_1, \ldots, i_k=1}^{n} D_{i_1 \cdots i_k} f\big(\mathbf{p}(t)\big) x_{i_1} \cdots x_{i_k} \end{aligned}$$

where the last equality follows from Exercise 29.

236. Exercise 30: Let $f \in \mathscr{C}^{(m)}(E)$, where E is an open subset of $\mathbf{R}^n$. Fix $\mathbf{a} \in E$, and suppose $\mathbf{x} \in \mathbf{R}^n$ is so close to $\mathbf{0}$ that the points $\mathbf{p}(t) = \mathbf{a} + t\mathbf{x}$ lie in E whenever $0 \leq t \leq 1$. Define

$$h(t) = f\big(\mathbf{p}(t)\big)$$

for all $t \in \mathbf{R}^1$ for which $\mathbf{p}(t) \in E$.

(a) For $1 \leq k \leq m$, show (by repeated application of the chain rule) that

$$h^{(k)}(t) = \sum D_{i_1 \cdots i_k} f\big(\mathbf{p}(t)\big) x_{i_1} \cdots x_{i_k}.$$

The sum extends over all ordered k-tuples $(i_1, \ldots, i_k)$ in which each i_j is one of the integers $1, \ldots, n$.

(b) By Taylor's theorem (5.15),

$$h(1) = \sum_{k=0}^{m-1} \frac{h^{(k)}(0)}{k!} + \frac{h^{(m)}(t)}{m!}$$

for some $t \in (0, 1)$. Use this to prove Taylor's theorem in n variables showing that the formula

$$f(\mathbf{a} + \mathbf{x}) = \sum_{k=0}^{m-1} \frac{1}{k!} \sum D_{i_1 \cdots i_k} f(\mathbf{a}) x_{i_1} \cdots x_{i_k} + r(\mathbf{x})$$

represents $f(\mathbf{a} + \mathbf{x})$ as the sum of its so-called "Taylor polynomial of degree $m-1$," plus a remainder that satisfies

$$\lim_{\mathbf{x} \to \mathbf{0}} \frac{r(\mathbf{x})}{|\mathbf{x}|^{m-1}} = 0.$$

Each of the inner sums extends over all ordered k-tuples $(i_1, \ldots, i_k)$, as in part (a); as usual, the zero-order derivative of f is simply$f$, so that the constant term of the Taylor polynomial of f at $\mathbf{a}$ is $f(\mathbf{a})$.

(c) Exercise 29 shows that repetition occurs in the Taylor polynomial as written in part (b). For instance, D_{113} occurs three times, as D_{113}, D_{131}, D_{311}. The sum of the corresponding three terms can be written in the form

$$3D_1^2 D_3 f(\mathbf{a}) x_1^2 x_3.$$

Prove (by calculating how often each derivative occurs) that the Taylor polynomial in (b) can be written in the form

$$\sum \frac{D_1^{s_1} \cdots D_n^{s_n} f(\mathbf{a})}{s_1! \cdots s_n!} x_1^{s_1} \cdots x_n^{s_n}.$$

Here the summation extends over all ordered n-tuples $(s_1, \ldots, s_n)$ such that each s_i is a nonnegative integer, and $s_1 + \cdots + s_n \leq m - 1$.

Solution: (*analambanomenos*)
(a) I am going to show this by induction on k. For the case $k = 1$, we have by Theorem 9.15 and Theorem 9.17

$$h'(t) = f'\big(\mathbf{p}(t)\big)\mathbf{p}'(t) = \sum_{i=1}^{n} D_i f\big(\mathbf{p}(t)\big) x_i$$

which is the assertion in the case $k=1$. Now assume the assertion is true for the case $k-1$. Then we have

$$\begin{aligned}
h^{(k)}(t) &= \frac{d}{dt} h^{(k-1)}(t) \\
&= \frac{d}{dt} \sum_{i_1,\ldots,i_{k-1}=1}^{n} D_{i_1\cdots i_{k-1}} f\big(\mathbf{p}(t)\big) x_{i_1} \cdots x_{i_{k-1}} \\
&= \sum_{i_1,\ldots,i_{k-1}=1}^{n} \frac{d}{dt} D_{i_1\cdots i_{k-1}} f\big(\mathbf{p}(t)\big) x_{i_1} \cdots x_{i_{k-1}} \\
&= \sum_{i_1,\ldots,i_{k-1}=1}^{n} \left(\sum_{i_k=1}^{n} D_{i_k} D_{i_1\cdots i_{k-1}} f\big(\mathbf{p}(t)\big) x_{i_k} \right) x_{i_1} \cdots x_{i_{k-1}} \\
&= \sum_{i_1,\ldots,i_k=1}^{n} D_{i_1\cdots i_k} f\big(\mathbf{p}(t)\big) x_{i_1} \cdots x_{i_k}
\end{aligned}$$

where the last equality follows from Exercise 29.

(b) Plugging in the results of part (a), we get, for some $t \in (0,1)$.

$$\begin{aligned}
f(\mathbf{a}+\mathbf{x}) &= h(1) \\
&= \sum_{k=0}^{m-1} \frac{h^{(k)}(0)}{k!} + \frac{h^{(m)}(t)}{m!} \\
&= \sum_{k=0}^{m-1} \frac{1}{k!} \sum_{i_1,\ldots,i_k=1}^{n} D_{i_1\cdots i_k} f(\mathbf{a}) x_{i_1} \cdots x_{i_k} + \frac{1}{m!} \sum_{i_1,\ldots,i_m=1}^{n} D_{i_1\cdots i_k} f(\mathbf{a}+t\mathbf{x}) x_{i_1} \cdots x_{i_m}
\end{aligned}$$

Since $f \in \mathscr{C}^{(m)}(E)$, there is a bound M such that $\big|D_{i_1\cdots i_m} f(\mathbf{a}+t\mathbf{x})\big| < M$ for all $t \in (0,1)$ and all partial derivatives of f of order m. Hence,

$$\begin{aligned}
\big|r(\mathbf{x})\big| &\le \frac{1}{m!} \sum_{i_1,\ldots,i_m=1}^{n} \big|D_{i_1\cdots i_k} f(\mathbf{a}+t\mathbf{x})\big| \cdot |x_{i_1}| \cdots |x_{i_m}| \\
&\le \frac{1}{m!} m! \big(M|\mathbf{x}|^m\big) \\
\lim_{\mathbf{x}\to\mathbf{0}} \frac{\big|r(\mathbf{x})\big|}{|\mathbf{x}|^{m-1}} &= \lim_{\mathbf{x}\to\mathbf{0}} M|\mathbf{x}| = 0
\end{aligned}$$

(c) By simple combinatorics, the number of ways to arrange k distinct objects in an ordered sequence is $k!$. If s of these objects are identical, this reduces the number of distinct ordered sequences by a factor of $s!$, since there are $s!$ ways of rearranging the identical objects in a given sequence. Hence the number of times a given partial derivative $D_1^{s_1} \cdots D_n^{s_n}$ of order $k = s_1 + \cdots + s_n$ occurs in the Taylor polynomial is $k!/(s_1! \cdots s_n!)$, so we can rewrite the Taylor polynomial as

$$\begin{aligned}
\sum_{k=0}^{m-1} \frac{1}{k!} \sum_{i_1,\ldots,i_k=1}^{n} D_{i_1\cdots i_k} f(\mathbf{a}) x_{i_1} \cdots x_{i_k} &= \sum_{k=0}^{n} \frac{1}{k!} \sum_{s_1+\cdots+s_n=k} \frac{k! D_1^{s_1} \cdots D_n^{s_n} f(\mathbf{a})}{s_1! \cdots s_n!} x_1^{s_1} \cdots x_n^{s_n} \\
&= \sum_{k=0}^{n} \sum_{s_1+\cdots+s_n=k} \frac{D_1^{s_1} \cdots D_n^{s_n} f(\mathbf{a})}{s_1! \cdots s_n!} x_1^{s_1} \cdots x_n^{s_n}
\end{aligned}$$

237. Exercise 31: Suppose $f \in \mathscr{C}^{(3)}$ in some neighborhood of a point $\mathbf{a} \in \mathbf{R}^2$, the gradient of f is $\mathbf{0}$ at $\mathbf{a}$, but not all second-order derivatives of f are 0 at $\mathbf{a}$. Show how one can then determine from the Taylor polynomial of f at $\mathbf{a}$ (of degree 2) whether f has a local maximum, or a local minimum, or neither, at the point $\mathbf{a}$. Extend this to $\mathbf{R}^n$ in place of $\mathbf{R}^2$.

Solution: (*analambanomenos*)
I am going to give the results, but not prove them. You can find a proof in any good advanced calculus text, and it's easy to find online. (For example, see Theorem 16.4 of Loomis and Sternberg's Advanced Calculus, which is legally available online now. Actually, why don't you just read the whole book, it wouldn't be difficult for you at this point, and it would introduce you to Differential Geometry and Mechanics. They actually used to assign it in advanced Freshman Calculus courses long ago, which must have been a good way to generate a lot of pre-meds.)

For a function f satisfying the above conditions, we have from Exercise 30 that f is approximately the following quadratic function in two variables:

$$f(\mathbf{a}) + \frac{D_{11}f(\mathbf{a})}{2}(x_1 - a_1)^2 + D_{12}f(\mathbf{a})(x_1 - a_1)(x_2 - a_1) + \frac{D_{22}f(\mathbf{a})}{2}(x_2 - a_1)^2.$$

Let $D = \big(D_{11}f(\mathbf{a})\big)\big(D_{22}f(\mathbf{a})\big) - \big(D_{12}f(\mathbf{a})\big)^2$. Then the above function will have a local maximum if and only if D is positive and $D_{11}f(\mathbf{a})$ is negative. It will have a local minimum if and only if D is positive and $D_{11}f(\mathbf{a})$ is positive. This will also hold for f near $\mathbf{a}$.

For n variables, we need to consider the eigenvalues of the Hessian matrix $\big(D_{ij}f(\mathbf{a})\big)$. There will be n real eigenvalues since the matrix is symmetric. Then f has a local maximum at $\mathbf{a}$ if and only if the eigenvalues are all negative, and it will have a local minimum at $\mathbf{a}$ if and only if they are all positive.

10 Integration of Differential Forms

238. Exercise 2: For $i = 1, 2, 3, \ldots$, let $\varphi_i \in \mathscr{C}(\mathbf{R}^1)$ have support in $(2^{-i}, 2^{-i+1})$, such that $\int \varphi_i = 1$. Put

$$f(x,y) = \sum_{i=1}^{\infty} \big(\varphi_i(x) - \varphi_{i+1}(x)\big)\varphi_i(y)$$

Then f has compact support in $\mathbf{R}^2$, f is continuous except at $(0,0)$, and

$$\int dy \int f(x,y)\,dx = 0 \qquad \text{but} \qquad \int dx \int f(x,y)\,dy = 1.$$

Observe that f is unbounded in every neighborhood of $(0,0)$.

Solution: (*analambanomenos*)
The $\varphi_i(x)\varphi_i(y)$ has support in the square $2^{-i} < x < 2{-}1+1$, $2^{-i} < y < 2{-}1+1$, and the $\varphi_{i+1}(x)\varphi_i(y)$ term has support in the rectangle $2^{-i-1} < x < 2{-}1$, $2^{-i} < y < 2{-}1+1$, so f has compact support in the square $0 < x < 1$, $0 < y < 1$. Each $(x,y) \neq (0,0)$ has a neighborhood small enough so that at most three of the terms in the sum are nonzero. Since these terms are continuous, f is continous away from the origin.

Let M_i be the maximum value of φ_i, attained at $x_i \in (2^{-i}, 2^{-i+1})$. Since $1 = \int \varphi_i < M_i 2^{-i}$, we have $M_i > 2^i$. Hence $f(x_i, x_i) = M_i^2 > 2^{i+1}$ diverges to ∞ as $i \to \infty$, so f is not continuous at $(0,0)$ and is unbounded in every neighborhood of $(0,0)$.

We have

$$\begin{aligned}
\int dy \int f(x,y)\,dx &= \sum_{i=1}^{\infty} \left(\int \varphi_i(y)\,dy\right)\left(\int \varphi_i(x)\,dx - \int \varphi_{i+1}(x)\,dx\right) \\
&= \sum_{i=1}^{\infty} 1 \cdot 0 = 0 \\
\int dx \int f(x,y)\,dy &= \left(\int \varphi_1(x)\,dx\right)\left(\int \varphi_1(y)\,dy\right) + \\
&\qquad \sum_{i=2}^{\infty} \left(\int \varphi_i(x)\,dx\right)\left(\int \varphi_i(y)\,dy - \int \varphi_{i-1}(y)\,dy\right) \\
&= 1 \cdot 1 + \sum_{i=2}^{\infty} 1 \cdot 0 = 1
\end{aligned}$$

239. Exercise 3: (a) If $\mathbf{F}$ is as in Theorem 10.7, put $\mathbf{A} = \mathbf{F}'(\mathbf{0})$, $\mathbf{F}_1(\mathbf{x}) = \mathbf{A}^{-1}\mathbf{F}(\mathbf{x})$. Then $\mathbf{F}_1'(\mathbf{0}) = I$. Show that

$$\mathbf{F}_1(\mathbf{x}) = \mathbf{G}_n \circ \cdots \circ \mathbf{G}_1(\mathbf{x})$$

in some neighborhood of $\mathbf{0}$, for certain primitive mappings $\mathbf{G}_1, \ldots, \mathbf{G}_n$. This gives another version of Theorem 10.7:

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}'(\mathbf{0})\mathbf{G}_n \circ \cdots \circ \mathbf{G}_1(\mathbf{x}).$$

(b) Prove that the mapping $(x,y) \to (y,x)$ of $\mathbf{R}^2$ onto $\mathbf{R}^2$ is not the composition of any two primitive mappings, in any neighborhood of the origin. This shows that the flips B_i cannot be omitted from the statement of Theorem 10.7.

Solution: (*analambanomenos*)
(a) (Much of this solution just repeats the proof of Theorem 10.7.) Assume $1 \le m \le n-1$, and make the following induction hypothesis (which evidently holds for $m = 1$):

V_m is a neighborhood of $\mathbf{0}$, $\mathbf{F}_m \in \mathscr{C}'(V_m)$, $\mathbf{F}_m(\mathbf{0}) = \mathbf{0}$, $\mathbf{F}_m'(\mathbf{0}) = I$, and for $\mathbf{x} \in V_m$,

$$P_{m-1}\mathbf{F}_m(\mathbf{x}) = P_{m-1}\mathbf{x} \tag{$*$}$$

By $(*)$, we have

$$\mathbf{F}_m(\mathbf{x}) = P_{m-1}\mathbf{x} + \sum_{i=m}^{n} \alpha_i(\mathbf{x})\mathbf{e}_i,$$

where $\alpha_m, \ldots, \alpha_n$ are real $\mathscr{C}'$-functions in V_m. Hence, for $j = m, \ldots, n$,

$$\mathbf{e}_j = \mathbf{F}_m'(\mathbf{0})\mathbf{e}_j = \sum_{i=m}^{n} (D_j\alpha_i)(\mathbf{0})\mathbf{e}_i.$$

Since $\mathbf{e}_m, \ldots \mathbf{e}_n$ are independent, we must have

$$(D_m\alpha_m)(\mathbf{0}) = 1 \qquad (D_{m+1}\alpha_m)(\mathbf{0}) = \cdots = (D_n\alpha_m)(\mathbf{0}) = 0. \tag{$**$}$$

Define, for $\mathbf{x} \in V_m$,

$$\mathbf{G}_m(\mathbf{x}) = \mathbf{x} + \big(\alpha_m(\mathbf{x}) - x_m\big)\mathbf{e}_m$$

Then $\mathbf{G}_m \in \mathscr{C}'(V_m)$, $\mathbf{G}_m$ is primitive, and $\mathbf{G}'_m(\mathbf{0}) = I$ by $(**)$. The inverse function theorem shows therefore that there is an open set U_m, with $\mathbf{0} \in U_m \subset V_m$, such that $\mathbf{G}_m$ is a 1-1 mapping of U_m onto a neighborhood V_{m+1} of $\mathbf{0}$, in which $\mathbf{G}_m^{-1}$ is continuously differentiable, and

$$\mathbf{G}_m^{-1\prime}(\mathbf{0}) = \mathbf{G}'_m(\mathbf{0})^{-1} = I.$$

Define $\mathbf{F}_{m+1}(\mathbf{y})$, for $\mathbf{y} \in V_{m+1}$, by

$$\mathbf{F}_{m+1}(\mathbf{y}) = \mathbf{F}_m \circ \mathbf{G}_m^{-1}(\mathbf{y}).$$

Then $\mathbf{F}_{m+1} \in \mathscr{C}'(V_{m+1})$, $\mathbf{F}_{m+1}(\mathbf{0}) = \mathbf{0}$, and $\mathbf{F}'_{m+1}(\mathbf{0}) = I$ by the chain rule. Also, for $\mathbf{x} \in U_m$,

$$\begin{aligned} P_m\mathbf{F}_{m+1}\big(\mathbf{G}_m(\mathbf{x})\big) &= P_m\mathbf{F}_m(\mathbf{x}) \\ &= P_m\big(P_{m-1}\mathbf{x} + \alpha_m(\mathbf{x})\mathbf{e}_m + \cdots\big) \\ &= P_{m-1}\mathbf{x} + \alpha_m(\mathbf{x})\mathbf{e}_m \\ &= P_m\mathbf{G}_m(\mathbf{x}) \end{aligned}$$

so that, for $\mathbf{y} \in V_{m+1}$, $P_m\mathbf{F}_{m+1}(\mathbf{y}) = P_m\mathbf{y}$. Our induction hypothesis holds therefore with $m+1$ in place of m.

Note that, for $\mathbf{y} = \mathbf{G}_m(\mathbf{x})$, we have

$$\mathbf{F}_{m+1}\big(\mathbf{G}_m(\mathbf{x})\big) = \mathbf{F}_m(\mathbf{x}).$$

If we apply this with $m = 1, \ldots, n-1$, we successively obtain

$$\mathbf{F}_1 = \mathbf{F}_2 \circ \mathbf{G}_1 = \mathbf{F}_3 \circ \mathbf{G}_2 \circ \mathbf{G}_1 = \cdots = \mathbf{F}_n \circ \mathbf{G}_{n-1} \circ \cdots \circ \mathbf{G}_1$$

in some neighborhood of $\mathbf{0}$. By $(*)$, $\mathbf{F}_n$ is primitive, so we can let $\mathbf{G}_n = \mathbf{F}_n$.

(b) Let $\mathbf{F}$ be the mapping $(x, y) \to (y, x)$ and suppose $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$ in some neighborhood of the origin, where

$$\mathbf{G}_1(x, y) = \big(f(x, y), y\big) \qquad \mathbf{G}_2(u, v) = \big(u, g(u, v)\big)$$

are primitive mappings. Then we would have

$$\begin{aligned} (y, x) &= \mathbf{G}_2 \circ \mathbf{G}_1(x, y) \\ &= \mathbf{G}_2\big(f(x, y), y\big) \\ &= \big(f(x, y), g\big(f(x, y), y\big)\big) \end{aligned}$$

so that

$$y = f(x, y) \qquad x = g\big(f(x, y), y\big) = g(y, y)$$

which is impossible. Trying $\mathbf{F} = \mathbf{G}_1 \circ \mathbf{G}_2$ leads to a similar contradiction.

240. Exercise 4: For $(x, y) \in \mathbf{R}^2$, define

$$\mathbf{f}(x, y) = (e^x \cos y - 1, e^x \sin y).$$

Prove that $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$, where

$$\begin{aligned} \mathbf{G}_1(x, y) &= (e^x \cos y - 1, y) \\ \mathbf{G}_2(u, v) &= \big(u, (1 + u) \tan v\big) \end{aligned}$$

are primitive in some neighborhood of $(0,0)$. Compute the Jacobians of $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{F}$ at $(0,0)$. Define

$$\mathbf{H}_2(x,y) = (x, e^x \sin y)$$

and find

$$\mathbf{H}_1(u,v) = \big(h(u,v), v\big)$$

so that $\mathbf{F} = \mathbf{H}_1 \circ \mathbf{H}_2$ in some neighborhood of $(0,0)$.

Solution: (*analambanomenos*)
We have

$$\begin{aligned}\mathbf{G}_2 \circ \mathbf{G}_1(x,y) &= \mathbf{G}_2(e^x \cos y - 1, y) \\ &= (e^x \cos y - 1, e^x \cos y \tan y) \\ &= (e^x \cos y - 1, e^x \sin y) \\ &= \mathbf{F}(x,y)\end{aligned}$$

The derivative matrices are

$$\begin{aligned}\mathbf{G}_1'(x,y) &= \begin{pmatrix} e^x \cos y & -e^x \sin y \\ 0 & 1 \end{pmatrix} \\ \mathbf{G}_2'(u,v) &= \begin{pmatrix} 1 & 0 \\ \tan v & (1+u)\cos^{-2} v \end{pmatrix}\end{aligned}$$

so that $\mathbf{G}_1'(0,0) = \mathbf{G}_2'(0,0) = I$, hence $J_{\mathbf{G}_1}(0,0) = J_{\mathbf{G}_2}(0,0) = 1$. By the chain rule and the properties of determinants, we also have $J_{\mathbf{F}}(0,0) = 1$.

Let $h(u,v) = \sqrt{v^2 - e^{2u}} - 1$. Then, for (x,y) near the origin,

$$\begin{aligned}\mathbf{H}_1 \circ \mathbf{H}_2(x,y) &= \mathbf{H}_1(x, e^x \sin y) \\ &= \left(\sqrt{e^{2x}\sin^2 y - e^{2x}} - 1, e^x \sin y\right) \\ &= (e^x \cos y - 1, e^x \sin y) \\ &= \mathbf{F}(x,y)\end{aligned}$$

241. Exercise 5: Formulate and prove an analogue of Theorem 10.8, in which K is a compact subset of an arbitrary metric space.

Solution: (*analambanomenos*)
We want to show: *Suppose K is a compact subset of a metric space X, and $\{V_\alpha\}$ is an open cover of K. Then there exists $\psi_1, \ldots, \psi_s \in \mathscr{C}(X)$ such that*

(a) $0 \le \psi_i \le 1$ *for* $1 \le i \le s$;

(b) *each* ψ_i *has its support in some* V_α, *and*

(c) $\psi_i(x) + \cdots + \psi_s(x) = 1$ *for every* $x \in K$.

Repeating the proof of Theorem 10.8 in the text and following the hint, associate with each $x \in K$ an index $\alpha(x)$ so that $x \in V_{\alpha(x)}$. Then there are open balls $B(x)$ and $W(x)$ centered at x, with

$$\overline{B(x)} \subset W(x) \subset \overline{W(x)} \subset V_{\alpha(x)}.$$

Since K is compact, there are points $x_1, \dots, x_s$ in K such that

$$K \subset B(x_1) \cup \cdots \cup B(x_s).$$

By Exercise 4.22, there are functions $\varphi_1 \dots, \varphi_s \in \mathscr{C}(X)$ such that $\varphi_i(x) = 1$ on $\overline{B(x_i)}$, $\varphi_i(x) = 0$ outside $W(x_i)$, and $0 \le \varphi_i(x) \le 1$ on X, namely,

$$\varphi_i(x) = \frac{\rho_{i1}(x)}{\rho_{i1}(x) + \rho_{i2}(x)}$$

where $\rho_{i1}(x)$ is the distance from x to the complement of $W(x_i)$, a closed set, and $\rho_{i2}(x)$ is the distance from x to $\overline{B(x_i)}$. Letting $\psi_1 = \varphi_1$, and

$$\psi_{i+1} = (1 - \varphi_1) \cdots (1 - \varphi_i)\varphi_{i+1}$$

for $i = 1, \dots, s-1$, the remainder of the proof follows exactly as in the proof of Theorem 10.8.

242. Exercise 6: Strengthen the conclusion of Theorem 10.8 by showing that the functions ψ_i can be made differentiable, and even infinitely differentiable.

Solution: (*analambanomenos*)
Following the hint, recall that Exercise 8.1 defined an infinitely differentiable function on $\mathbf{R}^1$ such that $f(x) = 0$ for $x \le 0$ and $0 \le f(x) < 1$ for all x. Let $a < b$. Then the function $g_{a,b}(x) = f(x-a)f(b-x)$ is also infinitely differentiable, equals 0 for $x \le a$ and $x \ge b$, and $0 \le g_{a,b}(x) < 1$ for all x. Since it has compact support, we can define a function

$$h_{a,b}(x) = \frac{1}{A}\int_x^\infty g_{a,b}(t)\,dt \qquad \text{where} \qquad A = \int_{-\infty}^\infty g_{a,b}(t)\,dt$$

which is infinitely differentiable, equals 1 for $x \le a$, equals 0 for $x \ge b$, and $0 \le h_{a,b}(x) \le 1$ for all x.

Now let $\mathbf{x} \in \mathbf{R}^n$, and let $B(\mathbf{x})$ and $W(\mathbf{x})$ be open balls centered at $\mathbf{x}$ with radii $a < b$, respectively. Define the function $r(\mathbf{y})$ for $\mathbf{y} \in \mathbf{R}^n$ which is the distance between $\mathbf{x}$ and $\mathbf{y}$, that is,

$$r(\mathbf{y}) = \sqrt{\sum (x_i - y_i)^2}$$

which is infinitely differentiable for $\mathbf{y} \ne \mathbf{x}$. Then the function $\varphi = h_{a,b} \circ r$ is infinitely differentiable on $\mathbf{R}^n$, equals 1 on $\overline{B(\mathbf{x})}$, equals 0 on $W(\mathbf{x})$, and $0 \le \varphi(\mathbf{y}) \le 1$ for all $\mathbf{y}$. We can use these functions in the proof of Theorem 10.8 to get infinitely differentiable functions ψ_i.

243. Exercise 7: (a) Show that the simplex Q^k is the smallest convex subset of $\mathbf{R}^k$ that contains $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$.

(b) Show that affine mappings take convex sets to convex sets.

Solution: (*analambanomenos*)
(a) First we need to show that Q^k is convex. Let $\mathbf{x}, \mathbf{y} \in Q^k$, so that the components satisfy

$$x_i \ge 0, \quad y_i \ge 0, \quad \sum x_i \le 1, \quad \sum y_i \le 1.$$

Let $0 \le \lambda \le 1$, and let $\mathbf{z} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$. Then

$$z_i = \lambda x_i + (1-\lambda)y_i \qquad \sum z_i = \lambda \sum x_i + (1-\lambda)\sum y_i$$

so that z_i lies between x_i and y_i, and $\sum z_i$ lies between $\sum x_i$ and $\sum y_i$. Hence $\mathbf{z} \in Q^k$.

Let C be a convex subset of $\mathbf{R}^k$ containing $\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_k$; we need to show that $Q^k \subset C$. We can consider $Q^i \subset Q^j$ for $i < j$ by letting the components with index greater than i be 0. I am going to show that $Q^i \subset C$, $i = 1, \ldots, k$ by induction. Let $\mathbf{x} \in Q^1$. Then $\mathbf{x} = x_1\mathbf{e}_1 + (1-x_1)\mathbf{0}$ for $0 \le x_1 \le 1$, so that $\mathbf{x} \in C$. Now suppose that $Q^{i-1} \subset C$ and let $\mathbf{x} \in Q^i$. Then $x_1 + \ldots + x_i \le 1$ implies

$$\frac{(x_1 + \ldots + x_{i-1})}{1 - x_i} \le 1.$$

so that

$$\mathbf{x}' = (1 - x_i)^{-1}(x_1, \ldots, x_{i-1}, 0, \ldots, 0) \in Q^{i-1} \subset C.$$

Hence $\mathbf{x} = (1 - x_i)\mathbf{x}' + x_i\mathbf{e}_i \in C$ since $0 \le x_i \le 1$, which shows that $Q^i \subset C$.

(b) Let X, Y be vector spaces and let $\mathbf{f} = \mathbf{f}(\mathbf{0}) + A$, for some $A \in L(X, Y)$, be an affine mapping from X to Y. Let C be a convex subset of X, and let $\mathbf{y}_1 = \mathbf{f}(\mathbf{x}_1)$, $\mathbf{y}_2 = \mathbf{f}(\mathbf{x}_2)$ be elements of $\mathbf{f}(C)$ for some $\mathbf{x}_1 \in C$ and $\mathbf{x}_2 \in C$. Then for $0 \le \lambda \le 1$, we have $\lambda\mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2 \in C$, so that

$$\lambda\mathbf{y}_1 + (1 - \lambda)\mathbf{y}_2 = \mathbf{f}(\mathbf{0}) + \lambda A(\mathbf{x}_1) + (1 - \lambda)A(\mathbf{x}_2) = \mathbf{f}(\mathbf{0}) + A\big(\lambda\mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2\big) \in \mathbf{f}(C).$$

Hence $\mathbf{f}(C)$ is convex.

244. Exercise 8: Let H be the parallelogram in $\mathbf{R}^2$ whose vertices are $(1, 1)$, $(3, 2)$, $(4, 5)$, $(2, 4)$. Find the affine map T which sends $(0, 0)$ to $(1, 1)$, $(1, 0)$ to $(3, 2)$, $(0, 1)$ to $(2, 4)$. Show that $J_T = 5$. Use T to convert the integral

$$\alpha = \int_H e^{x-y}\,dx\,dy$$

to an integral over I^2 and thus compute α.

Solution: (*analambanomenos*)
Since $(3, 2) = (1, 1) + (2, 1)$ and $(2, 4) = (1, 1) + (1, 3)$, the linear part of the affine map is $A(u, v) = (2u + v, u + 3v)$, so

$$\begin{aligned}
T(u, v) &= (1, 1) + A(u, v) = (2u + v + 1, u + 3v + 1) \\
J_T &= \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5 \\
\int_H e^{x-y}\,dx\,dy &= \int_0^1 \int_0^1 e^{(2u+v+1)-(u+3v+1)} J_T\,du\,dv \\
&= 5\bigg(\int_0^1 e^u\,du\bigg)\bigg(\int_0^1 e^{-2v}\,dv\bigg) \\
&= \tfrac{5}{2}(e - e^{-1} + e^{-2} - 1)
\end{aligned}$$

11 The Lebesgue Theory

245. Exercise 1: If $f \ge 0$ and $\int_E f\,d\mu = 0$, prove that $f(x) = 0$ almost everywhere on E.

Solution: (*analambanomenos*)
Following the hint, let E_n be the set of all $x \in E$ at which $f(x) > n^{-1}$, $n = 1, 2, \ldots$, and let K_{E_n} be the characteristic function of E_n. Then $0 \le n^{-1}K_{E_n} < f$ on E, and so by Remark 11.23(c)

$$0 = \int_E f\,d\mu > \frac{1}{n}\int_E K_{E_n}\,d\mu = \frac{1}{n}\,\mu(E_n) \ge 0$$

so that $\mu(E_n) = 0$ for all n. If $f(x) > 0$, then $x \in E_n$ for some positive integer n, so $A = \cup E_n$ is the set at which $f(x) > 0$. Since $\mu(A) = \lim \mu(E_n) = 0$ by Theorem 11.3, $f(x) = 0$ almost everywhere on E.

246. Exercise 2: If $\int_A f\,d\mu = 0$ for every measurable subset A of a measurable set E, then $f(x) = 0$ almost everywhere on E.

Solution: (*analambanomenos*)
Let A^+ be the measurable set where $f(x) \ge 0$ on E. Then $\int_{A^+} f\,d\mu = 0$ implies that $f(x) = 0$ almost everywhere on A^+ by Exercise 1. Similarly, if A^- is the measurable set where $f(x) \le 0$ on E, then $\int_{A^-}(-f)\,d\mu = 0$ implies that $f(x) = 0$ almost everywhere on A^-. Since $E = A^+ \cup A^-$, $f(x) = 0$ almost everywhere on E.

247. Exercise 3: If $\{f_n\}$ is a sequence of measurable functions, prove that the set of points x at which $\{f_n(x)\}$ converges is measurable.

Solution: (*analambanomenos*)
If h is a measurable function, then for every real number a

$$h^{-1}(a) = \{x \,|\, h(x) = a\} = \{x \,|\, h(x) \ge a\} \cap \{x \,|\, h(x) \le a\}$$

is a measurable set by Theorem 11.15. Also, if g_1 and g_2 are measurable functions then $h = g_1 - g_2$ is also a measurable function by Theorem 11.18. Hence $h^{-1}(0)$, the set where $g_1(x) = g_2(x)$, is a measurable set.

Letting

$$g_1(x) = \limsup_n f_n(x) \qquad g_2(x) = \liminf_n f_n(x),$$

then these are measurable functions by Theorem 11.17, hence the set where $g_1(x) = g_2(x)$ is measurable. But this is precisely the set of points at which $\{f_n(x)\}$ converges.

248. Exercise 4: If $f \in \mathscr{L}(\mu)$ on E and g is bounded and measurable on E, then $fg \in \mathscr{L}(\mu)$ on E.

Solution: (*analambanomenos*)
By Theorem 11.18 fg is measurable, and if $|g(x)| < C$ for some real number C, then

$$\int_E |fg|\,d(\mu) \le C\int_E |f|\,d(\mu) < \infty$$

so that $fg \in \mathscr{L}(\mu)$.

249. Exercise 5: Put, for $0 \le x \le 1$,

$$g(x) = \begin{cases} 0 & \left(0 \le x \le \frac{1}{2}\right) \\ 1 & \left(\frac{1}{2} < x \le 1\right) \end{cases}$$
$$f_{2k}(x) = g(x)$$
$$f_{2k+1}(x) = g(1-x)$$

Show that for $0 \le x \le 1$

$$\liminf_{n\to\infty} f_n(x) = 0$$

but

$$\int_0^1 f_n(x)\,dx = \frac{1}{2}.$$

Solution: (*analambanomenos*)
For each $0 \le x \le 1$, $\{f_n(x)\}$ is a sequence alternating between 0 and 1, so $\liminf f_n(x) = 0$. Let $A = \left[0, \frac{1}{2}\right]$ and $B = \left(\frac{1}{2}, 1\right]$. Then, for n even $f_n = K_B$, the characteristic function of B, and for n odd $f_n = K_A$. Hence

$$\int_0^1 f_{2k}(x)\,dx = m(B) = \frac{1}{2}$$
$$\int_0^1 f_{2k+1}(x)\,dx = m(A) = \frac{1}{2}$$

This shows that strict inequality is possible in the conclusion of Fatou's Lemma.

250. Exercise 6: Let

$$f_n(x) = \begin{cases} \dfrac{1}{n} & (|x| \le n) \\ 0 & (|x| > n). \end{cases}$$

Then $f_n(x) \to 0$ uniformly on $\mathbf{R}$, but

$$\int_{-\infty}^{\infty} f_n\,dx = 2 \quad n = 1, 2, \ldots.$$

Thus uniform convergence does not imply dominated convergence in the sense of Theorem 11.32. However, on sets of finite measure, uniformly convergent sequences of bounded functions do satisfy Theorem 11.32.