Suppose that $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuously differentiable. Prove that there exists some non-negative integer r and a non-empty open set $E \subset \mathbb{R}^m$ such that

$$\text{rant } f'(\mathrm{x}) = r \text{ for all } \mathrm{x} \in E.$$

pf.

Proposition 2 of (Lewis, 2009), establishes that the rank of a linear map is lower semicontinuous.

Proposition 2 means in particular that set $\{A \in L(\mathbb{R}^m, \mathbb{R}^n) |\, \text{rank } (A) > r\}$ is open for all $r \geq 0$.

Let $r \geq 0$, then

$\forall \mathrm{x} \in \mathbb{R}^m,\, 0 \leq \text{ rank } f'(\mathrm{x}) \leq n \implies \exists \mathrm{x}_0 \in \mathbb{R}^m : \forall \mathrm{x} \in \mathbb{R}^m : \text{ rank } (f'(\mathrm{x})) \leq \text{ rank } (f'(\mathrm{x}_0)) = r$

So, since f is continuously differentiable, we have that,

$$\exists \delta > 0 : \forall \varepsilon > 0,\, |x - x_0| < \delta \implies \|f'(\mathrm{x}) - f'(\mathrm{x}_0)\| < \varepsilon$$

Then the lower semicontinuity of rank tells us that,

$$\|f'(\mathrm{x}) - f'(\mathrm{x}_0)\| < \varepsilon \implies \text{rank } (f'(\mathrm{x}_0)) \leq \text{rank } (f'(\mathrm{x}))$$

But since x_0 is a global maximum of f', it follows that,

$$\forall \mathrm{x} \in \mathbb{R}^m : |\mathrm{x} - \mathrm{x}_0| < \delta \implies \text{rank } (f'(\mathrm{x})) = r \quad \blacksquare$$

0.1 Bibliography

Lewis, Andrew. *Semicontinuity of rank and nullity and some consequences*. Online, 2009. URL https://mast.queensu.ca/ andrew/notes/pdf/2009a.pdf.