151C lecture note

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This is the lecture note for the course MATH 151 Advanced Calculus III. Rather than a self-contained reference of the course materials, this lecture note is simply a complementary resources to our textbook. We will omit most (if not all) proofs and only focus on the ideas. The readers are referred to the textbook for detailed proofs.

1 Special functions (Chapter 8)

1.1 Power series

The notion of a power series is an attempt to generalize polynomials in order to express many functions in terms of "generalized polynomials". The following is a motivating example.

Example 1.1.

$$f(x) = \frac{1}{1-x}.$$

For every $x < 1$, we notice the formula

$$1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1-x}$$

Letting $n \to \infty$, the left-hand side becomes an infinite sum $1 + x + x^2 + \cdots$, while the right-hand side becomes $1/(1-x)$ as $x^n \to 0$. We have therefore expressed $f(x)$ as a "generalized polynomials":

$$f(x) = 1 + x + x^2 + \cdots$$

This idea of expressing functions as an infinite sum of monomials can be applied to analyze a vast class of functions. In this subsection, we will be interested in functions defined by a power sequences.

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \tag{1}$$

Here are some natural questions.

Question 1.2.

1. Is f continuous? Is f differentiable?

2. Given a function f, how can we find c_n and thus express f as a power series?

3. If f and g are expressed as power series, how to express $f+g$ and fg as power series?

4. One can rewrite the equality (1) as

$$f(x) = \sum_{n=0}^{\infty} c_n (x-0)^n$$

and interpret it as expending f as a power series with respect to $x = 0$. What if we want to expend f with respect to another point, say $x = a$?

5. Is the expansion (1) unique? Is it possible to have

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

but for some n, $a_n \neq b_n$?

Theorem (8.1)**.** *If the power series*

$$\sum_{n=0}^{\infty} c_n x^n \tag{2}$$

converges for $|x| < R$, then it converges uniformly on $|x| \leqslant R - \epsilon$ for any small $\epsilon > 0$. Therefore, if we define

$$f(x) = \sum_{n=0}^{\infty} c_n x^n,$$

then $f(x)$ is continuous on $|x| < R$.

Idea of the proof: First, Given any small ϵ, the assumption implies $|c_n|(R - \epsilon/2)^n \to 0$ as $n \to 0$. Therefore,

$$|c_n x^n| \leqslant |c_n|(R-\epsilon)^n = |c_n|(R - \frac{\epsilon}{2})^n \left(\frac{R-\epsilon}{R-\epsilon/2}\right)^n \leqslant C\left(\frac{R-\epsilon}{R-\epsilon/2}\right)^n$$

for some $C > 0$.

Therefore, the error between the partial sum of the series and its limit f is bounded by a geometric series, which implies uniform convergence. It follows from uniform convergence that f is continuous. ☐

Remark 1.3. If functions f and g are expressed as power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad g(x) = \sum_{n=0}^{\infty} b_n x^n$$

which converge on $|x| < R$, then $f + g$ can be expanded as a power series:

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n.$$

Moreover, we see from the proof above that for every fixed x with $|x| < R$, the power series of f and g are absolutely convergent. Therefore,

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n x^n,$$

where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$. Here we use Theorem 3.50 which says that if we have two absolutely convergent serires $\sum_{n=0}^{\infty} e_n$ and $\sum_{n=0}^{\infty} f_n$, then the series

$$\sum_{n=0}^{\infty} g_n$$

converges to the product of $\sum_{n=0}^{\infty} e_n$ and $\sum_{n=0}^{\infty} f_n$, where $g_n = \sum_{i=0}^{n} e_i f_{n-i}$ (so in our application, $e_n = a_n x^n, f_n = b_n x^n$). We emphasize that the conclusion of Theorem 3.50 no longer holds if absolute convergence was dropped. See the example right before Theorem 3.50.

Theorem 8.1, applied to term-wise derivative of the series, yields

Corollary. *If f is as above, then $f^{(k)}$ can be computed by term-wise differentiation.*

Idea of the proof: The corollary is proved using a mathematical induction, which first proves the statement for $k = 1$ and then the induction step goes from $f^{(K)}$ to $f^{(K+1)}$. The induction hypothesis asserts that $f^{(K)}$ is a power series with the desired radius of convergence, and $f^{(K+1)}$ results from $f^{(K)}$ by taking derivative. So the whole induction process can be reduced to the following statement:

If $f(x)$ is defined by the power series $\sum_{n=0}^{\infty} c_n x^n$, which has convergence radius R, then $f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$.

To prove this statement, we notice that the series $\sum_{n=1}^{\infty} n c_n x^{n-1}$ is uniformly convergent on $(-R+\epsilon, R-\epsilon)$ for every small ϵ. Fix one such ϵ. Then the partial sums

$$\sum_{n=0}^{N} n c_n x^{n-1}$$

form a sequence which satisfies the hypothesis of Theorem 7.17, which in turn asserts that

$$f'(x) = (\lim_{N\to\infty} \sum_{n=0}^{N} c_n x^n)' = \lim_{N\to\infty} (\sum_{n=0}^{N} c_n x^n)' = \lim_{N\to\infty} \sum_{n=0}^{N} nc_n x^{n-1}.$$

Therefore the desired formula holds for $x \in (-R+\epsilon, R-\epsilon)$. Letting $\epsilon \to 0$ gives the result.

The above assertion can alternatively be proved using integral. Since $\sum_{n=1}^{\infty} nc_n x^{n-1}$ converges, we may denote its limit by $g(x)$. For every $x \in (-R, R)$, pick $\epsilon > 0$ small enough so that $|x| < R-\epsilon$. Then since $\sum_{n=1}^{\infty} nc_n x^{n-1}$ converges uniformly on $(-R+\epsilon, R-\epsilon)$, we have

$$\int_{-R+\epsilon}^{x} \sum_{n=1}^{\infty} \int_{-R+\epsilon}^{x} nc_n x^{n-1} = \sum_{n=1}^{\infty} nc_n x^{n-1} = f(x) - f(-R+\epsilon).$$

The fundamental theorem of calculus then gives $f'(x) = \sum_{n=1}^{\infty} \int_{-R+\epsilon}^{x} nc_n x^{n-1}$, as desired. □

We analyze the endpoint behavior of power series. If (2) converges at an endpoint, say $x = R$, then $f(x)$ is also continuous at $x = R$. For simplicity, we take $R = 1$ in the following theorem.

Theorem (8.2)**.** *Suppose*

$$\sum_{n=0}^{\infty} c_n$$

converges. Define

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \text{ for } |x| < 1.$$

Then

$$\lim_{x\to 1} f(x) = \sum_{n=0}^{\infty} c_n.$$

Idea of the proof: Put $s_n = c_0 + \cdots + c_n$, $s_{-1} = 0$ and thus $s_n - s_{n-1} = c_n$. Put $s = \lim_{n\to\infty} s_n = c_0 + c_1 + c_2 + \cdots$. The natural idea is to compute the difference $f(x) - s$ directly:

$$f(x) - s = \sum_{n=0}^{\infty} c_n(x^n - 1).$$

The problem with this approach is that it is unclear whether $\sum_{n=0}^{\infty} c_n(x^n - 1)$ converges because we do not assume that $\sum_{n=0}^{\infty} c_n$ converges absolutely. In general, if one multiplies a non-absolutely convergent series with a bounded sequence term-by-term, then the result can be not convergent at all. This is because a non-absolutely convergent series relies heavily on the cancellation of positive and negative terms to provide a convergence, which can be ruined by multiplying with something else.

The way around this is to first sum the series $\sum_{n=0}^{\infty} c_n$. Rewrite the function $f(x)$:

$$f(x) = (1-x)\sum_{n=0}^{\infty} s_n x^n.$$

Use $s = s(1-x)\sum_{n=0}^{\infty} x^n$ to estimate $|f(x) - s|$:

$$|f(x) - s| \leqslant (1-x)\sum_{n=0}^{N} |s_n - s| x^n + (1-x)\sum_{n=N+1}^{\infty} |s_n - s| x^n.$$

Pick N large enough so that $|s_n - s|$ is very small for all $n > N$. Then if x is very close to 1, the first term will also be small.

This methods works because in the formula $f(x) - s = (1-x)\sum_{n=0}^{\infty}(s_n - s)x^n$, it is the absolutely convergent series $(1-x)\sum_{n=0}^{\infty} x^n$ that is multiplies by bounded terms $s_n - s$. □

Theorem ((8.4, Taylor's theorem))**.** *If*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

converges for $|x| < R$ and a is a number with $|a| < R$. Then f can be expended as the following power series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

with convergence at least for $|x - a| < R - |a|$.

Idea of the proof: First, we do not worry about the coefficients $f^{(n)}(a)/n!$ and just try to obtain a power series with respect to $x = a$ using the binomial theorem:

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{n=0}^{\infty} c_n (x - a + a)^n \\
&= \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} \binom{n}{m} c_n a^{n-m}\right] (x-a)^m \\
&= \sum_{m=0}^{\infty} \left[\sum_{n=m}^{\infty} \binom{n}{m} c_n a^{n-m}\right] (x-a)^m
\end{aligned}$$