Suppose that f is a real-valued function defined in an open set $E \subset R^n$, and that the partial derivatives $D_1 f, \ldots, D_n f$ are bounded in E. Prove that f is continuous in E.

Hint: **Proceed as in the proof of Theorem 9.21.**

pf.

Let $\{\mathrm{e}_i\}_{i=1}^n$ is be standard basis of $\mathbb{R}^n$.

Let $f : E \subset \mathbb{R}^n \to \mathbb{R}$, such that

$$\exists M_i \in \mathbb{R} : \forall \mathrm{x} \in E, |(D_i f)(\mathrm{x})| \leq M_i \quad (1 \leq i \leq n).$$

Let $M = \max\{M_i\}_{i=1}^n$, let $\varepsilon > 0$, let $\mathrm{h} = \sum_{i=1}^n h_i \mathrm{e}_i : \left|\sum_{i=1}^n h_i\right| < \frac{\varepsilon}{M}$.

Let $\mathrm{v}_0 = 0$, and $v_k = \sum_{i=1}^k h_i \mathrm{e}_i \quad (1 \leq k \leq n)$, then

$$f(\mathrm{x} + \mathrm{h}) - f(\mathrm{x}) = \sum_{i=1}^n f(\mathrm{x} + \mathrm{v_i}) - f(\mathrm{x} + \mathrm{v_{i-1}}) \tag{1}$$

Since $\mathrm{v_i} = \mathrm{v_{i-1}} + h_i \mathrm{e}_i$, the Mean Value Theorem (5.10) shows that

$$f(\mathrm{x} + \mathrm{v_i}) - f(\mathrm{x} + \mathrm{v_{i-1}}) = h_i (D_i f)(\mathrm{x} + \mathrm{v_{i-1}} + \theta_i h_i \mathrm{e}_i)$$

for some $\theta_i \in (0, 1)$. Then we can write,

$$|f(\mathrm{x} + \mathrm{h}) - f(\mathrm{x})| = |\sum_{i=1}^n h_i (D_i f)(\mathrm{x} + \mathrm{v_{i-1}} + \theta_i h_i \mathrm{e_i})| \leq \sum_{i=1}^n |h_i| M_i < M \left|\sum_{i=1}^n h_i\right| < \varepsilon$$

So f is continuous on E $\blacksquare$