Show that the continuity of $\mathbf{f}'$ at the point $\mathbf{a}$ is needed in the inverse function theorem, even in the case $n = 1$: If

$$f(t) = t + 2t^2 \sin\left(\frac{1}{t}\right)$$

for $t \neq 0$, and $f(0) = 0$, then $f'(0) = 1$, f' is bounded in $(-1, 1)$, but f is not one-to-one in any neighborhood of 0.

slu.

Since $\sin(1/t)$ is bounded we have,

$$f(0) = \lim_{t\to 0} t + 2 \cdot t^2 \sin(1/t) = 0$$

By the definition of the derivative,

$$f'(0) = \lim_{t\to 0} \frac{f(0+t) - f(0)}{t} = \lim_{t\to 0} \frac{t + 2t^2 \sin(1/t)}{t} = 1$$

Computing the derivative we get,

$$f'(t) = 1 + 4t \sin(1/t) - 2\cos(1/t)$$

Then for $t \in (-1, 1)$,

$$|f'(t)| = |1 + 4t \sin(1/t) - 2\cos(1/t)| \leq 1 + 4 + 2 = 7$$

So f' is bounded in $(-1, 1)$.

$\forall n \in \mathbb{Z} : n \neq 0$, let $a_n = 2n\pi$ and $b_n = (2n+1)\pi$, then

$$f'(1/a_n) = 1 + 4\frac{1}{2n\pi}\sin(2n\pi) - 2\cos(2n\pi) = 1 - 2 = -1$$
$$f'(1/b_n) = 1 + 4\frac{1}{(2n+1)\pi}\sin((2n+1)\pi) - 2\cos((2n+1)\pi) = 1 + 2 = 3$$

It follows that f has a local maximum on $(1/b_n, 1/a_n)$ and a local minimum on $(1/a_n, 1/b_{n-1})$.

Since every neighborhood of 0 contains infinitely many of such intervals, it follows that f cannot be injective in any neighborhood of 0 $\Diamond$