Decide whether the following is true or false and prove your conclusion.

Statement: Let $f : \mathbb{R}^m \to \mathbb{R}$ be a continuous function such that for every $x \in \mathbb{R}^m$ and every unit vector $e \in \mathbb{R}^m$, the directional derivative of f at x in the direction of e exists. Then f is differentiable.

slu.

It is false!

Consider function,

$$f(x,y) = \begin{cases} 0 & \text{, if } (x,y) = (0,0) \\ \frac{x^3}{x^2+y^2}, & \text{else} \end{cases}$$

Since the limit,

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0$$

f is continuous, as $(0,0)$ is the only point where it could fail to be continuous.

Let $\mathrm{u} = (u_1, u_2) \in \mathbb{R}^2$ be a unit vector. If $\mathrm{x} = (x,y) \in \mathbb{R}^2 : \mathrm{x} \neq (0,0)$, then formula (40), gives us,

$$\begin{aligned}(D_u f)(x,y) &= (D_1 f)(x,y)u_1 + (D_2 f)(x,y)u_2 \\ &= \frac{x^4 + 3x^2y^2}{(x^2+y^2)^2}u_1 - \frac{2x^3y}{(x^2+y^2)^2}u_2\end{aligned}$$

Since $(x,y) \neq (0,0) \implies x^2 + y^2 \neq 0$, it follows that for all $(x,y) \neq (0,0)$ the directional derivative exists in the direction of every unit vector u.

At $(0,0)$, equation (39) gives us,

$$\begin{aligned}(D_u f)(0,0) &= \lim_{t\to 0} \frac{f(tu_1, tu_2) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{\frac{(tu_1)^3}{(tu_1)^2+(tu_2)^2} - 0}{t} \\ &= \lim_{t\to 0} \frac{u_1^3}{u_1^2+u_2^2} \\ &= u_1^3\end{aligned}$$

So, the directional derivatives of f along every unit vector $\mathrm{u} \in \mathbb{R}^2$ exist for all points $\mathrm{x} \in \mathbb{R}^2$.

Let $\{\mathrm{e}_1, \mathrm{e}_2\}$ be the standard basis of $\mathbb{R}^2$.

If, the derivative exists at $(0,0)$, then

$$f'(0,0) = [(D_{\mathrm{e}_1} f)(0,0), (D_{\mathrm{e}_2} f)(0,0)] = [1,0]$$

Then, $[1,0]$, must satisfy equation (14) at $(0,0)$, but if $\mathrm{h} = (h_1, h_2)$

$$\begin{aligned}\lim_{\mathrm{h}\to 0} \frac{|f(\mathrm{h}) - f(0) - [1,0]\mathrm{h}|}{|\mathrm{h}|} &= \lim_{\mathrm{h}\to 0} \frac{|h_1^3 - h_1^3 - h_1h_2^2|}{(h_1^2+h_2^2)^{3/2}} \\ &= \lim_{\mathrm{h}\to 0} \frac{|h_1|h_2^2}{(h_1^2+h_2^2)^{3/2}} \\ &= \lim_{h_1=h_2=h\to 0} \frac{h^3}{(2h^2)^{3/2}} \\ &= \frac{1}{2^{3/2}} \neq 0\end{aligned}$$

So, f is not differentiable at $(0,0)$. So, f contradicts the proposition