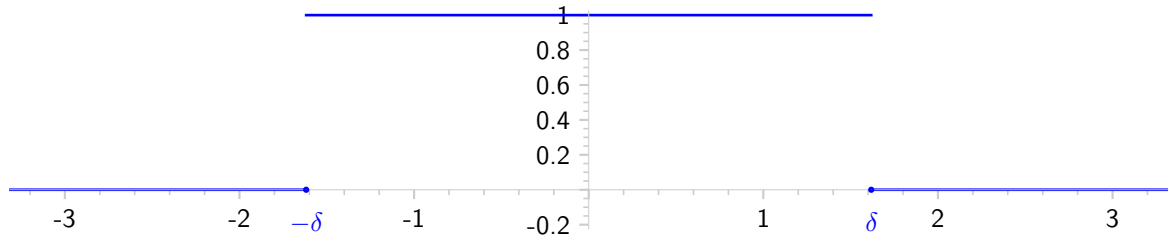


The following is the subset of the plot of  $f$  in  $[-\pi, \pi]$ .



(a)

Following formula (62)

$$c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx$$

Since  $f$  is 1 for  $|x| < \delta$  and 0 for  $\delta < |x| \leq \pi$ , we can see that,

$$\begin{aligned} m \neq 0 \implies c_m &= \frac{1}{2\pi} \int_{-\delta}^{\delta} e^{-imx} dx \\ &= \frac{1}{2\pi} - \frac{e^{-imx}}{im} \Big|_{-\delta}^{\delta} \\ &= \frac{1}{2\pi} - \left( \frac{e^{-im\delta}}{im} - \frac{e^{im\delta}}{im} \right) \\ &= \frac{1}{2\pi} \left( \frac{e^{im\delta} - e^{-im\delta}}{im} \right) \\ &= \frac{1}{m\pi} \left( \frac{e^{im\delta} - e^{-im\delta}}{2i} \right) \\ &= \frac{1}{m\pi} \sin(m\delta) \\ m = 0 \implies c_0 &= \frac{1}{2\pi} \int_{-\delta}^{\delta} e^0 dx \\ &= \frac{\delta - (-\delta)}{2\pi} = \frac{\delta}{\pi} \end{aligned}$$

$$c_{-m} = \frac{\sin(-m\delta)}{-m\pi} = \frac{-\sin(m\delta)}{-m\pi} = \frac{\sin(m\delta)}{m\pi} = \overline{c_m}$$

So,  $f$  is real by the remark in page 186. Notice,  $\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \implies 2\cos(ix) = e^{inx} + e^{-inx}$

Then we can see that if we change  $m$  to  $n$ ,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + \sum_{n=1}^{\infty} c_{-n} e^{-inx} = c_0 + \sum_{n=1}^{\infty} c_n (e^{inx} + e^{-inx}) = c_0 + \sum_{n=1}^{\infty} c_n 2\cos(nx)$$

So  $b_n = 0$  for  $n \in \mathbb{N}$  and  $a_n = 2c_n$ , for  $(n = 0, 1, 2, \dots)$ .<sup>1</sup> That is  $a_0 = \frac{2\delta}{\pi}$ , and  $a_n = \frac{2\sin(n\delta)}{n\pi}$  for  $n \in \mathbb{N}$ . Finally

$$f(x) = \frac{\delta}{\pi} + \sum_{n=1}^{\infty} \frac{2\sin(n\delta)}{n\pi} \cos(nx)$$

<sup>1</sup>This is why Boyce and DiPrima wrote  $a_0/2$  in their definition of Fourier series.

(b)

$$\forall t \in [-\delta, \delta] \quad |f(0+t) - f(0)| = |1-1| = 0 \leq |t|$$

Thus by 8.14  $\lim_{n \rightarrow \infty} S_n(f; 0) = f(0)$ , which means

$$\begin{aligned} \frac{\delta}{\pi} + \sum_{n=0}^{\infty} \frac{2 \sin(n\delta)}{n\pi} \cos(0) = f(0) &\implies \frac{\delta}{\pi} + \sum_{n=0}^{\infty} \frac{2 \sin(n\delta)}{n\pi} = 1 \\ &\implies \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\sin(n\delta)}{n} = 1 - \frac{\delta}{\pi} \\ &\implies \sum_{n=0}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi}{2} - \frac{\delta}{2} \\ &\implies \sum_{n=0}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2} \end{aligned}$$

(c) Formula (84) gives us what we need,

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx &= \sum_{n=-\infty}^{\infty} |c_n|^2 \\ \\ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx &= \frac{1}{2\pi} \int_{-\delta}^{\delta} |1|^2 dx \\ &= \frac{\delta - (-\delta)}{2\pi} \\ &= \frac{\delta}{\pi} \\ &= \frac{\delta^2}{\pi^2} + 2 \sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2 \pi^2} \\ &= c_0^2 + 2 \sum_{n=1}^{\infty} |c_n|^2 \\ &= c_0^2 + \sum_{n=1}^{\infty} |c_n|^2 + \sum_{n=1}^{\infty} |c_n|^2 \\ &= c_0^2 + \sum_{n=1}^{\infty} |c_{-n}|^2 + \sum_{n=1}^{\infty} |c_n|^2 \\ &= \sum_{n=-\infty}^{\infty} |c_n|^2 \\ \\ \frac{\delta}{\pi} &= \frac{\delta^2}{\pi^2} + 2 \sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2 \pi^2} \implies \delta\pi = \delta^2 + 2 \sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2} \\ &\implies \pi = \delta + 2 \sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2 \delta} \\ &\implies \sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2 \delta} = \frac{\pi - \delta}{2} \end{aligned}$$