If α is real and $-1 < x < 1$, prove Newton's binomial theorem,

$$(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n$$

pf.

Let $f(x) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n$, by the quotient test we have,

$$\begin{aligned}
R &= \lim_{n\to\infty} \frac{\left|\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}\right|}{\left|\frac{\alpha(\alpha-1)\cdots(\alpha-(n+1)+1)}{(n+1)!}\right|} \\
&= \lim_{n\to\infty} \left| \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1)+1)(\alpha-n+1)(n+1)n!}{\alpha(\alpha-1)\cdots(\alpha-(n-1))(\alpha-n)n!} \right| \\
&= \lim_{n\to\infty} \left| \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)(n+1)}{\alpha(\alpha-1)\cdots(\alpha-n+1)(\alpha-n)} \right| \\
&= \lim_{n\to\infty} \frac{n+1}{|\alpha-n|} = 1
\end{aligned}$$

So f converges for $|x| < 1$

$$\begin{aligned}
f'(x) &= \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} x^{n-1} \\
&\implies f'(x) + xf'(x) = \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} x^{n-1} + x\sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} x^{n-1} \\
&\implies (1+x)f'(x) = \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} x^{n-1} + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} x^{n} \\
k = n-1 \quad &\implies (1+x)f'(x) = \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k)}{k!} x^{k} + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} x^{n} \\
&\implies (1+x)f'(x) = \alpha + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k)}{k!} x^{k} + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} x^{n} \\
&\implies (1+x)f'(x) = \alpha + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} x^{n} \\
&\implies (1+x)f'(x) = \alpha + \alpha\sum_{n=1}^{\infty} \frac{(\alpha-1)\cdots(\alpha-n)}{n!} + \frac{(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!} x^{n} \\
&\implies (1+x)f'(x) = \alpha + \alpha\sum_{n=1}^{\infty} \frac{(\alpha-1)\cdots(\alpha-n) + n(\alpha-1)\cdots(\alpha-n+1)}{n!} x^{n} \\
&\implies (1+x)f'(x) = \alpha + \alpha\sum_{n=1}^{\infty} \frac{(\alpha-1)\cdots(\alpha-n+1)(\alpha-n) + n(\alpha-1)\cdots(\alpha-n+1)}{n!} x^{n} \\
&\implies (1+x)f'(x) = \alpha + \alpha\sum_{n=1}^{\infty} \frac{(\alpha-1)\cdots(\alpha-n+1)[(\alpha-n)+n]}{n!} x^{n} \\
&\implies (1+x)f'(x) = \alpha + \alpha\sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^{n} \\
&\implies (1+x)f'(x) = \alpha\left(1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^{n}\right) \\
&\implies (1+x)f'(x) = \alpha f(x)
\end{aligned}$$

Write $f'(x) = \frac{dy}{dx}$, and $f(x) = y$, now we have a separable first order ordinary differential equation,

$$(1+x)\frac{dy}{dx} = \alpha y \implies \int \frac{dy}{y} = \int \frac{\alpha}{1+x} dx \implies \log y = \alpha \log(1+x) + K \implies y = e^K e^{\alpha \log(1+x)} = C(1+x)^\alpha$$

If $K = 0 \implies C = 1$, thus for $-1 < x < 1$,

$$(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n$$

Now, for the other equality note by 8.18 (a) $0 < \alpha < \infty$ implies,

$$\begin{aligned}
\Gamma(n+\alpha) &= \Gamma(1+n-1+\alpha) \\
&= (n-1+\alpha)\Gamma(n-1+\alpha) \\
\Gamma(n-1+\alpha) &= \Gamma(1+n-2+\alpha) \\
&= (n-2+\alpha)\Gamma(n-2+\alpha) \\
\Gamma(n-2+\alpha) &= \Gamma(1+n-3+\alpha) \\
&= (n-3+\alpha)\Gamma(n-3+\alpha) \\
&\vdots \\
\Gamma(n-(n-2)+\alpha) &= \Gamma(1+1+\alpha) \\
&= (1+\alpha)\Gamma(1+\alpha) \\
\Gamma(n-(n-1)+\alpha) &= \Gamma(1+\alpha) \\
&= \alpha\Gamma(\alpha)
\end{aligned}$$

Therefore $\Gamma(n+\alpha) = (n-1+\alpha)(n-2+\alpha)(n-3+\alpha)\cdots(2+\alpha)(1+\alpha)\alpha\Gamma(\alpha)$ is a reasonable formula, which we want to prove by induction.

The base case is just 8.18 (a), now assume the formula above holds for all $k \leq n$.

$$\Gamma(n+1+\alpha) = \Gamma(1+n+\alpha) = (n+\alpha)\Gamma(n+\alpha) = (n+\alpha)(n-1+\alpha)(n-2+\alpha)(n-3+\alpha)\cdots(2+\alpha)(1+\alpha)\alpha\Gamma(\alpha)$$

The third equality holds by 8.18 (a) and the fourth by plugging in our assumption for n. Thus the induction is complete and the formula works for all natural numbers and $0 < \alpha < \infty$. Then,

$$\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)} = (n-1+\alpha)(n-2+\alpha)(n-3+\alpha)\cdots(2+\alpha)(1+\alpha)\alpha$$

Note,

$$-\alpha(-\alpha-1)\cdots(-\alpha-n+2)(-\alpha-n+1) = -(n-1+\alpha)\cdot-(n-2+\alpha)\cdots-(1+\alpha)-\alpha$$

There are n terms in the product since you start at $(n-1+\alpha)$ and you end at $(n-n+\alpha) = \alpha$, so,

$$-\alpha(-\alpha-1)\cdots(-\alpha-n+2)(-\alpha-n+1) = (-1)^n(n-1+\alpha)(n-2+\alpha)(n-3+\alpha)\cdots(2+\alpha)(1+\alpha)\alpha = (-1)^n\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}$$

Since $(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n$, plugging in $-\alpha$ and $-x$ for $|-x| = |x| < 1$, we get,

$$(1+(-x))^{-\alpha} = 1 + \sum_{n=1}^{\infty} \frac{-\alpha(-\alpha-1)\cdots(-\alpha-n+1)}{n!}(-x)^n = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n\Gamma(n+\alpha)}{n!\Gamma(\alpha)}(-1)^n x^n$$

So, the $(-1)^n$ cancel each other and we have for $-1 < x < 1$ and $\alpha > 0$,

$$(1-x)^{-\alpha} = 1 + \sum_{n=1}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} x^n$$