Put $f(0,0)=0$, and

$$f(x,y)=\frac{xy(x^2-y^2)}{x^2+y^2}$$

if $(x,y)\neq(0,0)$. Prove that
(a) f, $D_1 f$, $D_2 f$ are continuous in R^2;
(b) $D_{12}f$ and $D_{21}f$ exist at every point of R^2, and are continuous except at $(0,0)$;
(c) $(D_{12}f)(0,0)=1$, and $(D_{21}f)(0,0)=-1$.

slu.

Let,$r,\theta\in\mathbb{R}:r>0$, and $0<\theta\leq 2\pi$, then for $(x,y)\in\mathbb{R}^2:(x,y)\neq(0,0)$, let

$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases} \implies f(r,\theta)=r^2\cos(\theta)\sin(\theta)(\cos^2(\theta)-\sin^2(\theta))=\frac{r^2}{2}\sin(2\theta)\cos(2\theta)=\frac{r^2\sin(4\theta)}{4}$$

Since,

$$-1\leq\sin(4\theta)\leq 1 \implies |f(x,y)|\leq\frac{r^2}{4}.$$

Since as $(x,y)\to 0$, $r\to 0$, it follows that,

$$\lim_{(x,y)\to(0,0)} f(x,y)=0.$$

So, f is continuous.

Now, the partials at $(0,0)$, are

$$(D_1 f)(0,0)=\lim_{h\to 0}\frac{f(h,0)}{h}=\frac{0}{h}=0=\frac{0}{h}=\lim_{h\to 0}\frac{f(0,h)}{h}=(D_2 f)(0,0)$$

For convenience rewrite f,

$$f(x,y)=\frac{x^3y-xy^3}{x^2+y^2}.$$

For $(x,y)\neq(0,0)$, the partials are,

$$(D_1 f)(x,y)=\frac{(3x^2y-y^3)(x^2+y^2)-(x^3y-xy^3)(2x)}{(x^2+y^2)^2},$$

and

$$(D_2 f)(x,y)=\frac{(x^3-3xy^2)(x^2+y^2)-(x^3y-xy^3)(2y)}{(x^2+y^2)^2}$$

With the change of coordinates used for f,

$$(D_1 f)(x,y)=r[3\cos^2(\theta)\sin^2(\theta)-\sin^3(\theta)-2\cos^4(\theta)\sin(\theta)-2\cos^2(\theta)\sin^3(\theta)]=r\phi_1(\theta),$$

and

$$(D_2 f)(x,y)=r[\cos^3(\theta)-3\cos(\theta)\sin^2(\theta)-2\cos^3(\theta)\sin(\theta)-2\cos(\theta)\sin^4(\theta)]=r\phi_2(\theta).$$

Both ϕ_1, and ϕ_2 are bounded, so in the same way as f it follows that,

$$\text{as } (x,y)\to(0,0), (D_1 f)(x,y)\to 0, \text{ and } (D_2 f)(x,y)\to 0$$

So both partials are continuous. Completing part (a).

Now, for the mixed partials at $(0,0)$,

$$(D_{21}f)(0,0) = \lim_{h\to 0} \frac{(D_1 f)(0,h) - (D_1 f)(0,0)}{h} = \lim_{h\to 0} \frac{-h^5}{h^5} = -1$$

$$(D_{12}f)(0,0) = \lim_{h\to 0} \frac{(D_2 f)(h,0) - (D_2 f)(0,0)}{h} = \lim_{h\to 0} \frac{h^5}{h^5} = 1$$

Which, establishes part (c).

Now, for $(x,y) \neq (0,0)$, for convenience first rewrite,

$$(D_1 f)(x,y) = \frac{x^4 y + 4x^2 y^3 - y^5}{(x^2+y^2)^2},$$

and

$$(D_2 f)(x,y) = \frac{x^5 - 4x^3 y^2 - xy^4}{(x^2+y^2)^2}.$$

So, taking the mixed partials,

$$\begin{aligned}(D_{21}f)(x,y) &= \frac{(x^4 y + 4x^2 y^3 - y^5)\cdot 2(x^2+y^2)(2y) - (x^4 + 12x^2y^2 - 5y^4)(x^2+y^2)^2}{(x^2+y^2)^4} \\ &= \frac{9x^2y^4 - x^6 - 9x^4y^2 + y^6}{(x^2+y^2)^3},\end{aligned}$$

and

$$\begin{aligned}(D_{12}f)(x,y) &= \frac{(x^5 - 4x^3y^2 - xy^4)\cdot 2(x^2+y^2)(2x) - (5x^4 - 12x^2y^2 - y^4)(x^2+y^2)^2}{(x^2+y^2)^4} \\ &= \frac{-9x^4y^2 - x^6 + 9x^2y^4 + y^6}{(x^2+y^2)^3}.\end{aligned}$$

So, $(x,y) \neq (0,0) \implies D_{21}f = D_{12}f$.

Let $\mathrm{h} \in \mathbb{R}^2 : \mathrm{h} = (h_1, h_2)$, so the limit as $\mathrm{h} \to 0$ along the x-axis, is

$$\lim_{h_1\to 0} \frac{-h_1^6}{(h_1^2)^3} = -1,$$

and the limit as $\mathrm{h} \to 0$ along the y-axis, is

$$\lim_{h_2\to 0} \frac{h_2^6}{(h_2^2)^3} = 1.$$

Since D_{21} is a rational function it can only fail to be continuous where the denominator is zero. So, D_{12} is continuous everywhere except at $(0,0)$. Thus, (b) is established completing the problem $\Diamond$