Suppose f is a continuous function on R^1,$f(x+2\pi) = f(x)$, and α/π is irrational.

Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x+n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt$$

for every x. *Hint:* do it first for $f(x) = e^{ikx}$.

pf.

f is continuous on $\mathbb{R} \implies f \in \mathscr{R}$, furthermore $f(x+2\pi) = f(x)$ so the period of f is 2π.

Since α/π is irrational $\nexists k \in \mathbb{Z} : n\alpha = k\pi \quad (n = 1, 2, \ldots, N)$. So $f(x+n\alpha) \neq f(x+k\pi)$.

Furthermore $\{f(x+n\alpha)\}_{n=1}^{N} \subset f([-\pi, \pi])$, because $f(-\pi) = f(\pi)$ and f is continuous with period 2π.

Furthermore, $f(x+n\alpha)$ are distinct points in $f([-\pi, \pi])$.

Since, $f \in \mathscr{R}$ it follows that a fine enough partition by sample points is equal to the integral.

Let P be the evenly distributed partition of $[-\pi, \pi]$. Then $\Delta x_n = \frac{\pi-(-\pi)}{N} = \frac{2\pi}{N}$.

$$\int_{-\pi}^{\pi} f(x)dx = \lim_{n\to\infty} \sum_{n=1}^{N} f(x+n\alpha)\frac{2\pi}{N} \implies \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x+n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt \quad \blacksquare$$