The last expression is the power series we need. The last step is a switch of summation order, which is justified by the convergence of

$$\left|\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}c_n a^{n-m}(x-a)^m\right| \leqslant \sum_{n=0}^{\infty}|c_n|(|x-a|+|a|)^n$$

and the following claim.

Claim. *Let $\{a_{i,j}\}_{i,j\in\mathbb{N}}$ be a double sequence such that*

$$\sum_{j=1}^{\infty}|a_{i,j}| = b_i < \infty$$

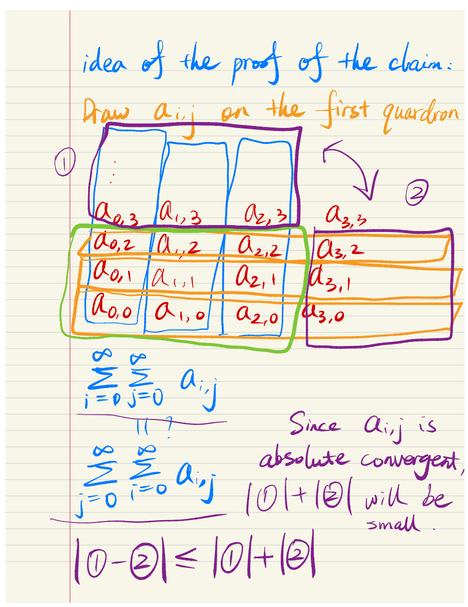
and

$$\sum_{i=1}^{\infty} b_i < \infty.$$

Then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{i,j} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}a_{i,j}.$$

The idea of the proof of the claim is shown by the following picture.



Finally, the last formula in the statement of the theorem then comes from differentiation and then evaluate at $x = a$. $\square$

Theorem (8.5, uniqueness of power series)**.** *Suppose* $\sum_{n=0}^{\infty} a_n x^n$ *and* $\sum_{n=0}^{\infty} b_n x^n$ *converge on* $S = \{x \mid |x| < R\}$ *and*

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n \tag{3}$$

on a set $E \subset S$*. If* E *has a limit point in* S*, then* (3) *holds for all* $x \in S$*.*

Idea of the proof: Suppose E has a limit point in S. First put $c_n = a_n - b_n$ and

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

so that we reduce the proof to showing $f(x) \equiv 0$. Let A be the limit set of E. Using connectivity of S, we reduce the proof to showing that A is open.

Let $x_0 \in A$. So we need to find an interval which contains x_0 and which lies in A. Expand f with respect to $x = x_0$:

$$f(x) = \sum_{n=0}^{\infty} d_n (x - x_0)^n.$$

If we can show that $d_n = 0$ for all n, then we will know that $f(x) = 0$ for all x satisfying $|x - x_0| < R - |x_0|$, which will provide the desired interval.

If some $d_n \neq 0$, then since the function $g(x) = f(x)/(x - x_0)^n$ is continuous and $g(x_0) = d_n \neq 0$, we conclude that $f(x) \neq 0$ for points sufficiently near $x = x_0$, which contradicts the assumption. □

1.2 Exponential functions

If $f(x)$ is defined by a power series

$$f(x) = \sum_{n=0}^{\infty} c_n x^n,$$

then we have seen that

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}.$$

An interesting situation occurs when we have $n c_n = c_{n-1}$, in which case $f'(x) = f(x)$. This suggests that we define

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Remark 1.4. For simplicity, we only discuss real variable in this note and thus use x instead of z. Our textbook contains a more general discussion about $E(z)$.

It turns out that $E(x)$ is another definition of the exponential function e^x, i.e.,

$$E(x) = e^x \tag{4}$$

In order to see this, we first use properties of binomial coefficients to establish (see textbook) $E(x)E(y) = E(x+y)$. Also notice that $E(1) = e$ by definition. Therefore, for every $n \in \mathbb{N}^+$, we have

$$E(n) = E(1 + \cdots + 1) = e^n,$$

which is (4) for positive integer x. For positive rational $p = n/m$, the equality

$$[E(p)]^m = E(mp) = E(n) = e^n$$

then establishes (4) for positive rational x. For negative rationals, one only need to notice that $1 = E(0) = E(-p+p) = E(-p)E(p)$. Finally, for general real numbers, e^x is defined by $e^x = \sup e^p$ where the supremum is taken over all rationals $p < x$. Note that this supremum is in fact a limit process: $e^x = \lim_{p \to x^-} e^p$. Since $E(x)$ is defined by a power series, it is continuous, and therefore we have $E(x) = e^x$. The point of defining e^x by a power series is that many properties of e^x can be easily seen via the series, for instance, continuity. In the sequel, we will use the notation e^x instead of $E(x)$. The following theorem summarizes properties of e^x.

Theorem (8.6)**.** *e^x is continuous, differentiable, and strictly increasing. It grows unboundedly as $x \to \infty$ and satisfies the following equalities*

$$(e^x)' = e^x, \quad e^{x+y} = e^x e^y, \quad \lim_{x \to \infty} x^n e^{-x} = 0.$$

The last equality follows from the summation formula

$$e^x = E(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

and thus e^x grows faster than every power function.

1.3 Logarithmic function

Since e^x is strictly increasing and differentiable on $\mathbb{R}$, it has a differentiable inverse, which is temporarily denoted by $\log x$. The domain of $\log x$ is $E(\mathbb{R}) = (0, \infty)$.

$$\log e^x = x,$$

we see that

$$\log x = \int_1^x \frac{1}{t} dt,$$

which is usually used as an alternative definition of $\log x$ to derived useful properties of $\log x$.

Theorem. $\log x$ *is a continuous, differentiable, and strictly increasing function defined on* $(0, \infty)$*. It satisfies the following equalities.*

$$\lim_{x\to 0^+} \log x = -\infty, \quad \lim_{x\to\infty} \log x = \infty, \quad \log(xy) = \log x + \log y \textit{ for } x, y > 0.$$

The above equalities follow from the corresponding properties of e^x and the fact that $\log e^x = x$.

Functions $\log x$ and e^x can be used to extend the definition of power functions: For $\alpha \in \mathbb{R}$ and $x > 0$, let

$$x^\alpha = e^{\alpha \log x}.$$

Using chain rule, we obtain the following equalities of differentiation and integration:

$$(x^\alpha)' = \alpha x^{\alpha-1} \text{ for } \alpha \in \mathbb{R};$$

$$\int x^\alpha dx = \frac{1}{\alpha+1} x^{\alpha+1} + C \text{ for } \alpha \neq -1, \quad \int x^{-1} dx = \log x + C.$$

We also have the following equalities

$$\lim_{x\to\infty} \frac{e^x}{x^\alpha} = \infty \text{ and } \lim_{x\to\infty} x^{-\alpha} \log x = 0 \text{ for } \alpha > 0,$$

which follows from the fact that e^x grows faster than x^n for all $n \in \mathbb{N}$.

1.4 Trigonometric functions

Let

$$C(x) = \frac{1}{2}(e^{ix} + e^{-ix}), \quad S(x) = \frac{1}{2i}(e^{ix} - e^{-ix}).$$

In fact, $C(x) = \cos x$ and $S(x) = \sin x$.

From the expansion,

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n,$$

we see that $e^{\bar{z}} = \overline{e^z}$. Therefore, $C(x)$ and $S(x)$ are the real and imaginary parts of e^{ix}, respectively.

It turns out that e^{ix} and thus $C(x)$ and $S(x)$ are periodic functions. Their period is 2π. We first define the number π. We will argue that $C(x)$ has zero on $\mathbb{R}^+$, and the smallest positive zero of $C(x)$ is $\pi/2$.

Note that $C'(x) = -S(x)$ and $S'(x) = C(x)$. Note also that $e^0 = 1$ and thus $C(0) = 1$ and $S(0) = 0$. We claim that there is $x > 0$ such that $C(x) = 0$. Suppose such x does not exist. Then $C(x) > 0$ for all $x > 0$ by continuity. Hence $S'(x) > 0$ and $S(x) > 0$ as $S(0) = 0$. Pick some $c > 0$. Then for any $x > c$, we have $C'(x) = -S(x) < -S(c)$. Hence from $x = c$ and beyond, $C(x)$ is decreasing at a speed at least $S(c)$. Since $S(c) > 0$, we conclude that $C(x) = 0$ for some $x > 0$, a contradiction.

Let x_0 be the first positive number such that $C(x_0) = 0$, which exists due to $C(0) \neq 0$. Denote $4x_0$ by 2π. As $C(\pi/2) = 0$ and $|e^{i\pi/2}| = 1$, we have $S(\pi/2) = \pm 1$. Since $C(x) > 0$ on $(0, \pi/2)$, we have $S(\pi/2) = 1$ and thus $e^{i\pi/2} = i$. Therefore,

$$e^{2\pi i} = (e^{i\pi/2})^4 = 1$$

and

$$e^{i(x+2\pi)} = e^x e^{2\pi i} = e^x$$

which means e^{ix} has period 2π. This establishes parts (a) and (b) of Theorem 8.7.

For part (c) of Theorem 8.7, we first notice that $e^{i\pi/2}$ and $e^{i\pi}$ are not equal to 1. We also notice that on the interval $(0, \pi/2)$, since both $C(x)$ and $S(x)$ are not 0, e^{ix} is neither a real number or a purely imaginary number. The range of e^{ix} on $(\pi/2, \pi), (\pi, 3\pi/2)$, and $(3\pi/2, 2\pi)$ can be obtain from the range of e^{ix} on $(0, \pi/2)$ by multiplying $i, -1$, and $-i$, respectively. Therefore, on each of those intervals, e^{ix} is neither a real number nor a purely imaginary number. In particular, $e^{ix} \neq 1$ on those intervals.

For part (d), part (c) clearly estabilshes uniqueness. For existence, one solves the equation $e^{ix} = z$ by first looking at the real part and then the imaginary part. This finishes the idea of the proof of Theorem 8.17.

1.5 Algebraic completeness of the complex field

Theorem (8.8)**.** *Every single variable polynomial over $\mathbb{C}$ has at least one zero.*

Idea of the proof: Assume that there is a polynomial $P(z)$ which has no zero in $\mathbb{C}$.

Step 1. Pass to a polynomial $Q(z)$ with $\min_{z\in\mathbb{C}} |Q(z)| = Q(1) = 1$.

We notice that there exists $z_0 \in \mathbb{C}$ with $|P(z_0)| = \min_{z\in\mathbb{C}} |P(z)|$ because $\lim_{z\to\infty} |P(z)| = \infty$. And then we put

$$Q(z) = \frac{P(z + z_0)}{P(z_0)}$$

which is well-defined since we assume P has no zero and thus $P(z_0) \neq 0$.

Step 2. Write

$$Q(z) = 1 + b_k z^k + \cdots + b_n z^n, \quad b_k, b_n \neq 0.$$

Consider a k-th root w of $-|b_k|/b_k$ and a small positive number r. We have

$$Q(rw) = 1 - r^k|b_k| + r^{k+1}(b_{k+1}w^{k+1} + \cdots + b_n w^n).$$

And thus

$$|Q(rw)| \leqslant 1 - r^k|b_k| + r^{k+1}(|b_{k+1}w^{k+1}| + \cdots + |b_n w^n|). \tag{5}$$

For r small enough, right-hand side of (5) is less than 1, contradicting properties of Q.