2. The following "simple" computation yields a good approximation to Stirling's formula. For each $m \in \mathbb{N}$, define

$$f(x) = (m + 1 - x)\log(m) + (x - m)\log(m + 1)$$

if $m \leq x \leq m + 1$, and define

$$g(x) = \frac{x}{m} - 1 + \log(m)$$

if $m - \dfrac{1}{2} \leq x < m + \dfrac{1}{2}$. Draw the graphs of f and g. Note that $f(x) \leq \log(x) \leq g(x)$ if $x \geq 1$ and that

$$\int_1^n f(x)\ \ dx = \log(n!) - \frac{1}{2}\log(n) > -\frac{1}{8} + \int_1^n g(x)\ \ dx.$$

Integrate $\log(x)$ over $[1, n]$. conclude that

$$\frac{7}{8} < \log(n!) - \left(n + \frac{1}{2}\right)\log(n) + n < 1$$

for $n \in \mathbb{N} - \{1\}$. (*Note:* $\log(\sqrt{2\pi}) \approx 0.918$.) Thus,

$$e^{\frac{7}{8}} < \frac{n!}{(n/e)^n\sqrt{n}} < e.$$

Solution for 2

Proof. Let $y, z \in (0, \infty)$. By the *AM-GM Inequality*, $\sqrt{yz} \leq \frac{y+z}{2}$. Since $\log(u)$ is an increasing function, then $\log(\sqrt{yz}) \leq \log\left(\frac{y+z}{2}\right)$. It follows that

$$-\log\left(\frac{y+z}{2}\right) \leq \frac{-\log(y) - \log(z)}{2}.$$

By *exercise 4.24*, $-\log(u)$ is a convex function. Fix $m \in \mathbb{N}$. If we consider $F\big(\lambda a + (1-\lambda)b\big) \leq \lambda F(a) + (1-\lambda)F(b)$ from the definition of convexity given in *exercise 4.23* and we let $F = -\log$, $a = m + 1$, $b = m$, and $\lambda = x - m$, it follows that

$$-\log(x) \leq (x - m)(-\log(m + 1)) + (m + 1 - x)(-\log(m))$$

and therefore, $f(x) \leq \log(x)$. Next, consider the function $h(u) = u - 1 - \log(u)$ for $u \in (0, \infty)$. Then $h'(u) = 1 - \frac{1}{u}$. Since $h'(u) < 0$ on $(0, 1)$ and $h'(u) > 0$ on $(1, \infty)$, then by *Theorem 5.11*, h strictly decreases on $(0, 1)$ and strictly increases on $(1, \infty)$ meaning that $h(u) \geq h(1) = 0$. Therefore, $\log(u) \leq u - 1$ for all $u \in (0, \infty)$. If we choose $u = \frac{x}{m}$, it follows that $\log(x) \leq g(x)$.

Note that for $x \in [m, m+1]$, $f(x) = \big(\log(m+1) - \log(m)\big)(x - m) + \log(m)$ (the secant line that passes through the points $(m, \log(m))$ and $(m + 1, \log(m + 1))$) and for $x \in \left[m - \frac{1}{2}, m + \frac{1}{2}\right]$, $g(x) = \dfrac{1}{m}(x - m) + \log(m)$ (the line tangent to $\log(x)$ at $x = m$).

For $n \in \mathbb{N}$,

$$\begin{aligned}
\int_1^n f(x)\ dx &= \sum_{k=1}^{n-1} \int_k^{k+1} f(x)\ dx \\
&= \sum_{k=1}^{n-1} \int_k^{k+1} \big(\log(k+1) - \log(k)\big)(x-k) + \log(k)\ dx \\
&= \sum_{k=1}^{n-1} \int_0^1 \big(\log(k+1) - \log(k)\big)u + \log(k)\ du \\
&= \sum_{k=1}^{n-1} \left[\frac{1}{2}\big(\log(k+1) - \log(k)\big)u^2 + \log(k)u\right]_0^1 \\
&= \sum_{k=1}^{n-1} \left(\frac{1}{2}\big(\log(k+1) - \log(k)\big) + \log(k)\right) \\
&= \frac{1}{2}\sum_{k=1}^{n-1} (\log(k+1) + \log(k)) \\
&= \frac{1}{2}\sum_{k=1}^{n-1} \log(k+1) + \frac{1}{2}\sum_{k=1}^{n-1} \log(k) \\
&= \left(\frac{1}{2}\sum_{k=1}^{n-1} \log(k+1)\right) - \frac{1}{2}\log(n) + \frac{1}{2}\log(1) + \frac{1}{2}\sum_{k=2}^{n} \log(k) \\
&= \left(\sum_{k=2}^{n} \log(k)\right) - \frac{1}{2}\log(n) \\
&= \log(n!) - \frac{1}{2}\log(n).
\end{aligned}$$

and

$$\begin{aligned}
-\frac{1}{8} + \int_1^n g(x)\ dx &= -\frac{1}{8} + \int_1^{\frac{3}{2}} g(x)\ dx + \sum_{j=2}^{n-1} \int_{j-\frac{1}{2}}^{j+\frac{1}{2}} g(x)\ dx + \int_{n-\frac{1}{2}}^{n} g(x)\ dx \\
&= -\frac{1}{8} + \int_1^{\frac{3}{2}} x - 1\ dx + \sum_{j=2}^{n-1} \int_{j-\frac{1}{2}}^{j+\frac{1}{2}} \frac{1}{j}(x-j) + \log(j)\ dx + \int_{n-\frac{1}{2}}^{n} \frac{1}{n}(x-n) + \log(n)\ dx \\
&= -\frac{1}{8} + \left[\frac{1}{2}x^2 - x\right]_1^{\frac{3}{2}} + \sum_{j=2}^{n-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{j}u + \log(j)\ du + \int_{-\frac{1}{2}}^{0} \frac{1}{n}v + \log(n)\ dv \\
&= -\frac{1}{8} + \left[\frac{1}{2}x^2 - x\right]_1^{\frac{3}{2}} + \sum_{j=2}^{n-1} \left[\frac{1}{2j}u^2 + \log(j)u\right]_{-\frac{1}{2}}^{\frac{1}{2}} + \left[\frac{1}{2n}v^2 + \log(n)v\right]_{-\frac{1}{2}}^{0} \\
&= -\frac{1}{8} + \left[\frac{1}{2}\left(\frac{9}{4} - 1\right) - \frac{1}{2}\right] + \sum_{j=2}^{n-1} \left[\frac{1}{2j}\left(\frac{1}{4} - \frac{1}{4}\right) + \log(j)\left(\frac{1}{2} + \frac{1}{2}\right)\right] + \left[-\frac{1}{8n} + \frac{1}{2}\log(n)\right] \\
&= \left(\sum_{j=2}^{n-1} \log(j)\right) - \frac{1}{8n} + \frac{1}{2}\log(n) \\
&= \left(\sum_{j=2}^{n} \log(j)\right) - \frac{1}{2}\log(n) - \frac{1}{8n} \\
&= \log(n!) - \frac{1}{2}\log(n) - \frac{1}{8n} \\
&< \int_1^n f(x)\ dx.
\end{aligned}$$

Next, note that $f(x) \geq \log(m) \geq 0$. For $x \geq 1$ and $n \in \mathbb{N} - \{1\}$,

$$0 \leq f(x) \leq \log(x) \leq g(x) \Longrightarrow \int_1^n f(x)\ dx \leq \int_1^n \log(x)\ dx \leq \int_1^n g(x)\ dx < \frac{1}{8} + \int_1^n f(x)\ dx.$$

Therefore,

$$\begin{aligned} 0 &\leq \int_1^n \log(x)\ dx - \int_1^n f(x)\ dx < \frac{1}{8} \\ \Longrightarrow 0 &\leq 1 + n\log(n) - n - \Big(\log(n!) - \frac{1}{2}\log(n)\Big) < \frac{1}{8} \\ \Longrightarrow -1 &\leq n\log(n) - n - \Big(\log(n!) - \frac{1}{2}\log(n)\Big) < -\frac{7}{8} \\ \Longrightarrow e^{\frac{7}{8}} &< \frac{n!}{(n/e)^n\sqrt{n}} \leq e. \end{aligned}$$

$\square$