Define $f(0, 0) = 0$, and put

$$f(x, y) = x^2 + y^2 - 2x^2y - \frac{4x^6y^2}{(x^4 + y^2)^2}$$

if $(x, y) \neq (0, 0)$.

(*a*) Prove, for all $(x, y) \in R^2$, that

$$4x^4y^2 \leq (x^4 + y^2)^2.$$

Conclude that f is continuous.

(*b*) For $0 \leq \theta \leq 2\pi$, $-\infty < t < \infty$, define

$$g_\theta(t) = f(t \cos \theta, t \sin \theta).$$

Show that $g_\theta(0) = 0$, $g_\theta'(0) = 0$, $g_\theta''(0) = 2$. Each g_θ has therefore a strict local minimum at $t = 0$.

In other words, the restriction of f to each line through $(0, 0)$ has a strict local minimum at $(0, 0)$.

(*c*) Show that $(0, 0)$ is nevertheless not a local minimum for f, since $f(x, x^2) = -x^4$.

pf.

(a) By the AM-GM inequality,

$$x^4 > 0, y^2 > 0 \implies \sqrt{x^4y^2} \leq \frac{x^4 + y^2}{2} \implies 4x^4y^2 \leq (x^4 + y^2)^2.$$

f is differentiable, for $(x, y) \neq (0, 0)$, so it's enough to show f is continuous at $(0, 0)$

$$|f(x, y)| = |x^2 + y^2 - 2x^2y - \frac{4x^6y^2}{(x^4 + y^2)^2}| < |x^2 + y^2| + 2x^2|y| + x^2\left|\frac{4x^4y^2}{(x^4 + y^2)^2}\right|$$

From the first inequality we have,

$$4x^4y^2 \leq (x^4 + y^2)^2 \implies 0 \geq \frac{4x^4y^2}{(x^4 + y^2)^2} \leq 1 \implies 0 \geq x^2\frac{4x^4y^2}{(x^4 + y^2)^2} \leq x^2$$

So as $x \to 0$, $\frac{4x^6y^2}{(x^4+y^2)^2} \to 0$. So as $(x, y) \to 0$, $f(x, y) \to 0$. Therefore, f is continuous.

(b) Plugging in,

$$g_\theta(t) = t^2 - 2t^3 \cos^2(\theta) \sin(\theta) - \frac{4t^4 \cos^6(\theta) \sin^2(\theta)}{(t^2 \cos^4(\theta) + \sin^2(\theta))^2}$$

If $\theta = 0$, $g_0(t) = t^2 \implies g_0(0) = 0, g_0'(0) = 0$, and $g_0''(0) = 2$. We need to consider it separately because if $\theta = 0 = t$, then $(t^2 \cos^4(\theta) + \sin^2(\theta))^2 = 0$.

Now, for $\theta \neq 0$, plugging in 0,

$$g_\theta(0) = 0$$

Differentiating with respect to t,

$$g_\theta'(t) = 2t - 6t^2 \cos^2(\theta) \sin(\theta) - \frac{16t^3 \cos^6(\theta) \sin^2(\theta)}{(t^2 \cos^4(\theta) + \sin^2(\theta))^2}$$

Plugging in 0,

$$g_\theta'(0) = 0$$

Differentiating with respect to t,

$$g_\theta''(t) = 2 - 12t\cos^2(\theta)\sin(\theta) - \frac{48t^2\cos^6(\theta)\sin^2(\theta)}{(t^2\cos^4(\theta) + \sin^2(\theta))^2}$$

Plugging in 0,

$$g_\theta''(0) = 2$$

So each g_θ has a strict local minimum at $t = 0$.

So the restriction of f to each line through $(0,0)$, has a strict local minimum at $(0,0)$.

(c) Computation shows,

$$f(x, x^2) = x^2 + x^4 - 2x^4 - x^2 = -x^4$$

Let $\varepsilon > 0$. Since,

$$f(0,0) = 0 > -\varepsilon^4 = f(\varepsilon, \varepsilon^2)$$

$(0,0)$ can't be a local minimum of f ■