165A — Homework 2

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1 For each of the functions below, describe the domain of definition that is understood:

(a) $f(z) = \frac{1}{1-z^2}$;

Rational functions are defined whenever the denominator is not zero.

$1 - z^2 = 0 \implies z = \pm i$, so dom$(f) = \mathbb{C}\backslash\{\pm i\}$.

(b) $f(z) = \text{Arg}(\frac{1}{z})$;

For the same reasons, dom$(f) = \mathbb{C}\backslash\{0\}$.

(c)$f(z) = \frac{z}{z-\bar{z}}$;

$z - \bar{z} = 0 \implies \frac{z-\bar{z}}{2} = \Re(z) = 0.$

However, $f(0) = 0$ and $\Re(0) = 0$, so dom$(f) = \{z \in \mathbb{C} | z \neq 0 \implies \Re(z) \neq 0\}$.

(d) $f(z) = \frac{1}{1-|z|^2}$;

$1 - |z|^2 = 0 \implies |z| = 1$,so dom$(f) = \{z \in \mathbb{C}| \quad |z| \neq 1\}$.

3 Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$. Use the expressions (see Sec. 5)

$$x = \frac{z + \bar{z}}{2}, \text{ and } y = \frac{z - \bar{z}}{2i}$$

to write $f(z)$ in terms of z, and simplify the result.

slu.

$f(z) = x^2 - y^2 - 2y + i(2x - 2xy) = x^2 - y^2 - 2y + 2ix - 2ixy,$

$x^2 = \frac{z+\bar{z}}{2}^2 = \frac{z^2+2z\bar{z}+\bar{z}^2}{4}$ and $y^2 = \frac{z-\bar{z}}{2i}^2 = \frac{z^2-2z\bar{z}+\bar{z}^2}{-4} = \frac{-z^2+2z\bar{z}-\bar{z}^2}{4} \implies x^2 - y^2 = \frac{z^2+2z\bar{z}+\bar{z}^2-(-z^2+2z\bar{z}-\bar{z}^2)}{4}$

$\implies x^2 - y^2 = \frac{z^2+\bar{z}^2}{2},$

$-2y = -2\frac{z-\bar{z}}{2i} = \frac{-z+\bar{z}}{i}$ and $(\frac{i}{i} = 1 = i(-i) \implies \frac{1}{i} = -i)$

$\implies -2y = -i(-z + \bar{z}) = iz - i\bar{z},$

$2ix = 2i\frac{z+\bar{z}}{2} = iz + i\bar{z} \implies 2ixy = (iz + i\bar{z})(\frac{z-\bar{z}}{2i}) = \frac{z^2-z\bar{z}+z\bar{z}-\bar{z}^2}{2} = \frac{z^2-\bar{z}^2}{2}$

$\implies 2ix - 2ixy = iz + i\bar{z} - \frac{z^2-\bar{z}^2}{2}.$

$\implies f(z) = \frac{z^2+\bar{z}^2}{2} + iz - i\bar{z} + iz + i\bar{z} - \frac{z^2-\bar{z}^2}{2}.$

$\implies f(z) = \bar{z}^2 + 2iz \quad \Diamond$

4 Write the function

$$f(z) = z + \frac{1}{z} \quad (z \neq 0)$$

in the form $f(z) = u(r,\theta) + iv(r,\theta)$.

slu.

Let $z = r(\cos\theta + i\sin\theta)$,

$\implies \frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)}\frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta} = \frac{\cos\theta - i\sin\theta}{r(\cos^2\theta + \sin^2\theta)} = \frac{\cos\theta - i\sin\theta}{r}$

$\implies z + \frac{1}{z} = r(\cos\theta + i\sin\theta) + \frac{\cos\theta - i\sin\theta}{r} = (r + \frac{1}{r})\cos\theta + i(r - \frac{1}{r})\sin\theta\Diamond$

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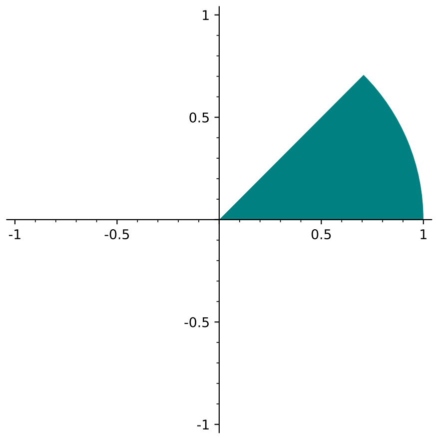
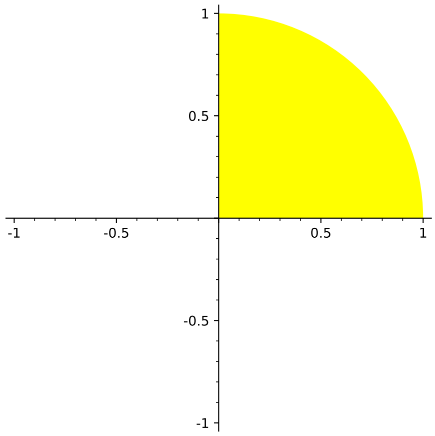
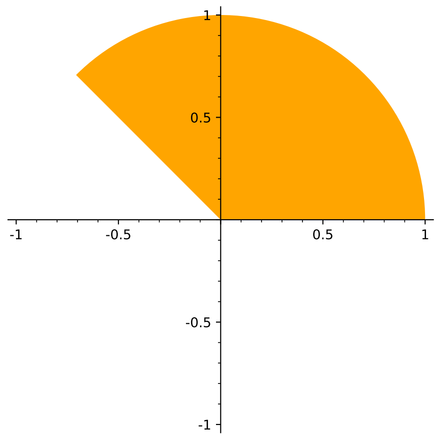
3 Sketch the region onto which the sector $r \leq 1, 0 \leq \theta \leq \pi/4$ is mapped by the transformation (a) $w = z^2$;(b) $w = z^3$; (c) $w = z^4$.

slu.

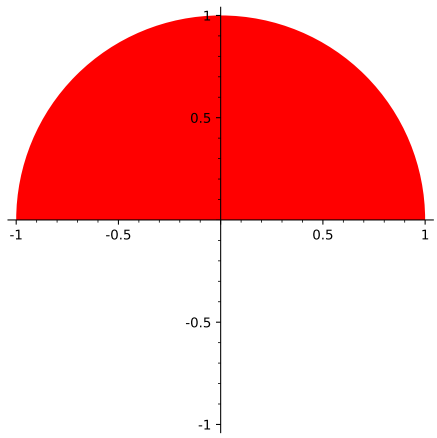




under z^2 the region maps to



under z^3 the region maps to



under z^4 the region maps to

4 Show that the lines $ay = x(a \neq 0)$ are mapped onto the spirals $\rho = \exp(a\varphi)$ under the transformation $w = \exp z$, where $w = \rho \exp(i\varphi)$.

slu.

Let $z = x + iy$.

$ay = x \implies z = ay + iy$

$\implies |z| = \sqrt{a^2y^2 + y^2} = |y|\sqrt{a^2+1}$.

$\implies \gamma := \sqrt{a^2+1}$ is fixed.

and

$x = ay$

$\implies$

$\phi :=$ atan$(y/ay) =$ atan$(1/a)$.

$\implies \phi$ is fixed.

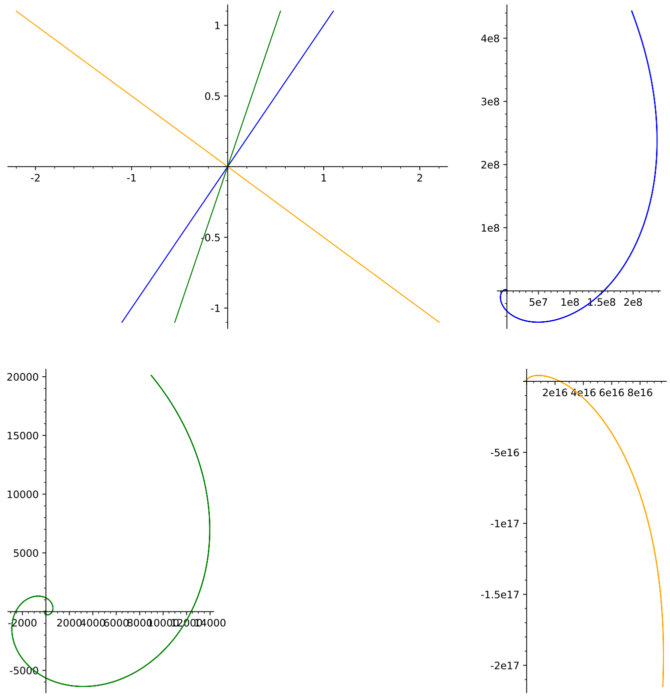
$\implies z = |y|\gamma\cos(\phi) + i|y|\gamma\sin(\phi)$

$w = \exp(z) = \exp(|y|\gamma\cos(\phi) + i|y|\gamma\sin(\phi)) =$

$\exp(|y|\gamma\cos(\phi))\exp(i|y|\gamma\sin(\phi)) =$

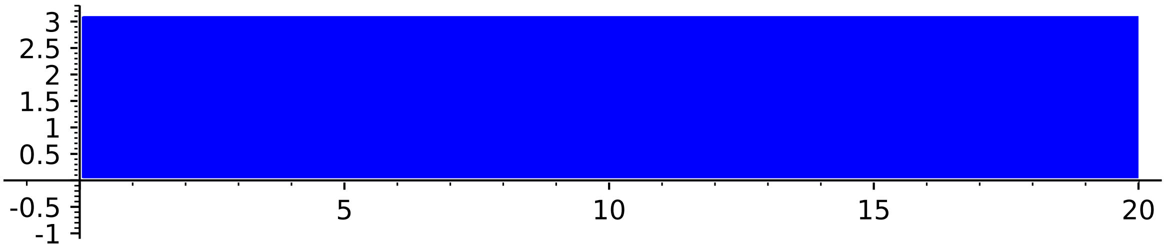
$\exp(|y|\gamma\cos(\phi))(\cos(|y|\gamma\sin(\phi)) + i\sin(|y|\gamma\sin(\phi))) =$

$\exp(|y|\sqrt{a^2+1}\cos(\text{atan}(1/a)))(\cos(|y|\sqrt{a^2+1}\sin(\text{atan}(1/a))) + i\sin(|y|\sqrt{a^2+1}\sin(\text{atan}(1/a))))$



They are spirals because as $|y| \to 0$, $\exp(A|y|) \to 1, \forall A$ constants. But, it does go through all of the angles, because cos and sin are cyclic over the positive reals.

7 Find the image of the semi-infinite strip $x \geq 0$, $0 \leq y < \pi$, under the trasformation $w = \exp z$.

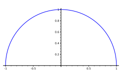
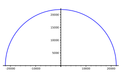
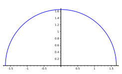
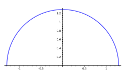


Let $z = x + iy \implies w = \exp(z) = \exp(x + iy) = \exp(x)\exp(iy)$

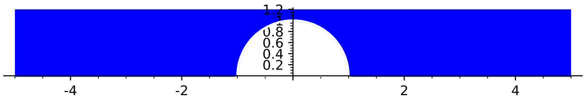
$x \geq 0 \implies$ the radius $\rho = \exp(x)$ of w is increasing, and $\rho \geq 1$.

$0 \leq y < \pi \implies$ the angle of w runs from 0 to π, not including π.

We can compute the following to see the pattern.

: $\rho = 1$, : $\rho = \exp(10)$, : $\rho = \exp(1/2)$, : $\rho = \exp(1/4)$.

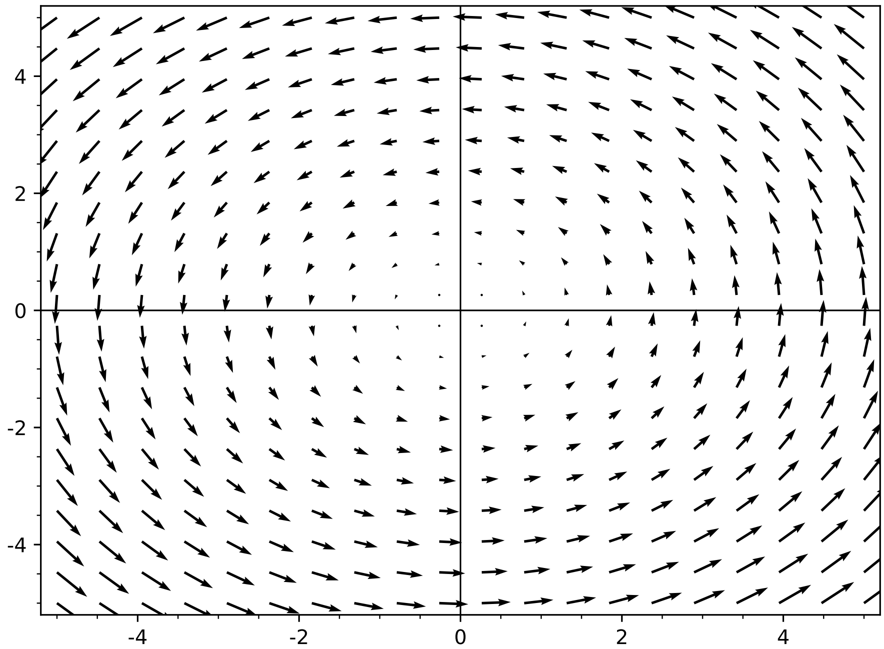
So, the map wraps the semi-infinite strip into the upper half-plane, minus the open unit circle. And not including the ray that starts from $-1 + 0i$ along the x—axis to $-\infty + 0i$.



8 One interpretation of a function $w = f(z) = u(x, y) + iv(x, y)$ is that of a vector field in the domain of definition of f. The function assigns a vector w, with components $u(x, y)$ and $v(x, y)$, to each point z at which it is defined. Indicate graphically the vector fields represented by (a) $w = iz$; (b) $w = z/|z|$.

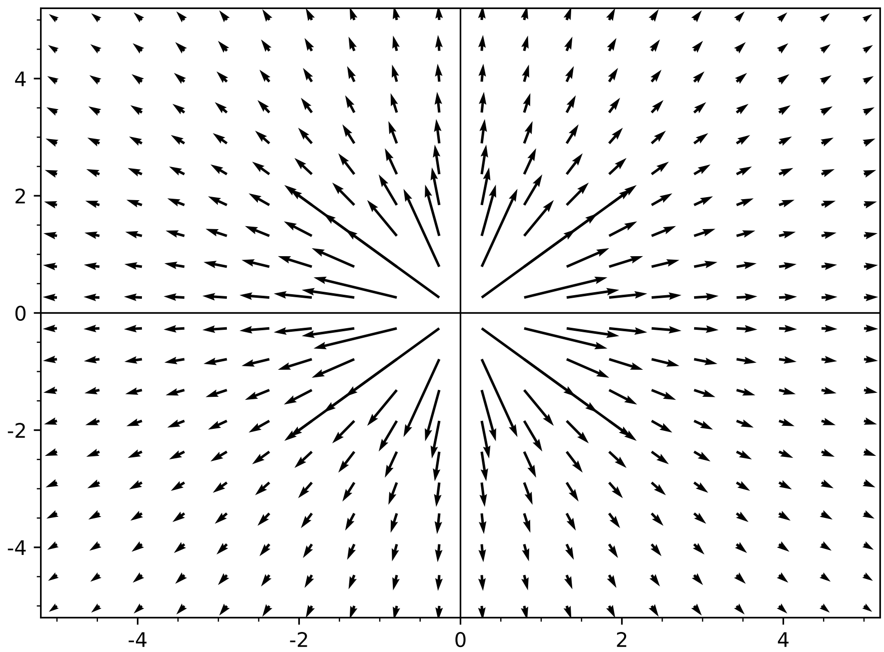
(a) First we want to express $w = u + iv$.

$z = x + iy$ and $w = iz \implies w = ix + i^2 y = -y + ix \implies u = -y$ and $v = x$.



(b) First we want to express $w = u + iv$.

$z = x + iy$ and $w = z/|z| \implies w = (x + iy)/(x^2 + y^2) = \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2} \implies u = \frac{x}{x^2+y^2}$ and $v = \frac{y}{x^2+y^2}$.



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1 Use definition (2), Sec. 15, of limit to prove that

(a) $\lim_{z\to z_0} \Re(z) = \Re(z_0)$;

slu.

WTS $\forall\epsilon > 0 : \exists\delta > 0 : |z - z_0| < \delta \implies |\Re(z) - \Re(z_0)| < \epsilon$

$|\Re(z) - \Re(z_0)| < \epsilon \iff |\frac{z+\bar{z}}{2} - \frac{z_0\bar{z_0}}{2}| < \epsilon \iff |z + \bar{z} - z_0 - \bar{z_0}|/2 < \epsilon \iff |z - z_0 + \bar{z} - \bar{z_0}| < 2\epsilon \iff |z - z_0 + \bar{z} - \bar{z_0}| < |z - z_0| + |\bar{z} - \bar{z_0}| = \epsilon$.

Now, $|\bar{z} - \bar{z_0}| = |z \bar{-} z_0| = |z - z_0|$, so all of the above is if and only if, $|\Re(z) - \Re(z_0)| < 2|z - z_0| < 2\delta = 2\epsilon$.

Put $\delta = \epsilon$. So, that shows it. $\Box$

(b) $\lim_{z\to z_0} \bar{z} = \bar{z_0}$;

slu.

WTS $\forall\epsilon > 0 : \exists\delta > 0 : |z - z_0| < \delta \implies |\bar{z} - \bar{z_0}| < \epsilon$.

Put $z = x + iy$, and $z_0 = x_0 + iy_0$

$|\bar{z} - \bar{z_0}| < \epsilon \iff |x - iy - (x_0 - iy_0)| < \epsilon$

$\iff |x - x_0 - iy + iy_0| < |x - x_0| + |i(-y + y_0)| = |x - x_0| + |i||y_0 - y| = |x - x_0| + |y - y_0| = \epsilon$.

$|z - z_0| = |x + iy - x_0 - iy_0| < |x - x_0| + |i(y - y_0)| = |x - x_0| + |y - y_0| < \delta$

Put $\delta = \epsilon$, and we are done! $\Box$

(c) $\lim_{z\to 0} \frac{\bar{z}^2}{z} = 0$;

slu.

WTS $\forall\epsilon > 0 : \exists\delta > 0 : |z - 0| < \delta \implies |\frac{\bar{z}^2}{z} - 0| < \epsilon$.

$|\frac{\bar{z}^2}{z} - 0| < \epsilon \iff |\frac{\bar{z}^2}{z}| = \frac{|\bar{z}^2|}{|z|} = \frac{|\bar{z^2}|}{|z|} = \frac{|z^2|}{|z|} = \frac{|z|^2}{|z|} = |z| < \epsilon$

So, put $\delta = \epsilon \quad \Box$

3 . Let n be a positive integer and let $P(z)$ and $Q(z)$ be polynomials, where $Q(z_0) \neq 0$. Use Theorem 2 in Sec. 16, as well as limits appearing in that section, to find

(a) $\lim_{z\to z_0 \frac{1}{z^n}} \quad (z_0 \neq 0)$;

slu. $\lim_{z\to z_0} 1 = 1$, and $\lim_{z\to z_0} z^n = z_0^n$.

Given $z_0 \neq 0 \implies z_0^n \neq 0$ for all positive integers.

By theorem 2.(10) $\lim_{z\to z_0} \frac{1}{z^n} = \frac{1}{z_0^n} \quad \Diamond$

(b) $\lim_{z\to i \frac{iz^3-1}{z+i}}$;

slu.

$\lim_{z\to i} i = i$ and $\lim_{z\to i} z^3 = i^3$

By theorem 2.(9) $\lim_{z\to i} iz^3 = i * i^3 = 1$

$\lim_{z\to i} -1 = -1$, so by theorem 2.(8) $\lim_{z\to i} iz^3 - 1 = 1 - 1 = 0$.

$\lim_{z\to i} z = i$, so by theorem 2.(8) $\lim_{z\to i} z + i = i + i = 2i \neq 0$.

So, by theorem 2.(10) $\lim_{z\to i} \frac{iz^3-1}{z+i} = 0 \quad \Diamond$

(c) $\lim_{z\to z_0} \frac{P(z)}{Q(z)} = \frac{P(z_0)}{Q(z_0)}$, by repeated applications of theorem 2, we get limits of polynomials are evaluations, and observing $Q(z_0) \neq 0$ we see the limit exists and is that $\Diamond$

5 Show that the limit of the function

$$f(z) = (\frac{z}{\bar{z}})^2$$

as z tends to 0 does not exist. Do this by letting non-zero points $z = (x, 0)$, and $z = (x, x)$ approach the origin. [Note that it is not sufficient to simply consider points $z = (x, 0)$ and $z = (0, y)$, as it was in Example 2, Sec. 15.]

slu.

Let $z = x + 0i \implies f(z) = (\frac{x}{x})^2 = 1$, so as $x \to 0$, $f(x + 0i) \to 1$.

Let $z = x + ix \implies f(z) = (\frac{x+ix}{x-ix})^2 = (\frac{x(1+i)}{x(1-i)})^2 = (\frac{1+i}{1-i})^2 = (\frac{1+i}{1-i}\frac{1+i}{1+i})^2 = (\frac{(1+i)^2}{1+1})^2 = (\frac{2i}{2})^2 = -1.$, so as $x + ix \to 0$, $f(x + ix) \to -1$.

If the limit exists it is unique, therefore the limit doesn't exist □

7 7. Use definition (2), Sec. 15, of limit to prove that if $\lim_{z\to z_0} f(z) = w_0$, then $\lim_{z\to z_0} |f(z)| = |w_0|$

Suggestion: Observe how the first of inequalities (9), Sec. 4, enables one to write $||f(z)|-|w_0|| \leq |f(z)-w_0|$

pf.

$\lim_{z\to z_0} f(z) = w_0$ means that,

$\forall \epsilon > 0 : \exists \delta > 0 : |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$

by the reverse triangle inequality,

$||f(z)| - |w_0|| \leq |f(z) - w_0| < \epsilon$, so we are done, because the same δ works ■

10 Use the theorem in Sec. 17, to show that

(a) $\lim_{z\to\infty} \frac{4z^2}{(1-z)^2} = 4$;

slu.

$f(z) = \frac{4z^2}{(1-z)^2} = \frac{4z^2}{z^2-2z+1}\frac{1/z^2}{1/z^2} = \frac{4}{1-\frac{2}{z}+\frac{1}{z^2}}$

$\implies f(1/z) = \frac{4}{1-\frac{2}{1/z}+\frac{1}{(1/z)^2}} = \frac{4}{1-2z+z^2}$

$\lim_{z\to 0} f(1/z) = 4$, so by the theorem $\lim_{z\to\infty} f(z) = 4$ ◊

(b) $\lim_{z\to 1} \frac{1}{(z-1)^3} = \infty$;

slu.

Consider, $\lim_{z\to 1} \frac{1}{\frac{1}{(z-1)^3}} = (z-1)^3 = 0 \implies \lim_{z\to 1} \frac{1}{(z-1)^3} = \infty$ by the theorem in Sec. 17 ◊

(c) $\lim_{z\to\infty} \frac{z^2+1}{z-1} = \infty$;

slu. $f(z) = \frac{z^2+1}{z-1} \implies f(1/z) = \frac{(1/z)^2+1}{(1/z)-1} \implies 1/f(1/z) = \frac{(1/z)-1}{1/z^2+1} = \frac{\frac{1-z}{z}}{\frac{1+z^2}{z^2}} = \frac{\frac{1-z}{z}}{\frac{1+z^2}{z^2}} = \frac{(1-z)z^2}{(1+z^2)z} = \frac{z-z^2}{1+z^2}$

$\lim_{z\to 0} \frac{z-z^2}{1+z^2} = 0 \implies \lim_{z\to\infty} \frac{z^2+1}{z-1} = \infty$ ◊

11 With the aid of theorem in Sec. 17, show that when

$$T(z) = \frac{az+b}{cz+d} \quad ad - bc \neq 0,$$

(a) $\lim_{z\to\infty} T(z) = \infty$ if $c = 0$;

slu.

$c = 0 \implies T(z) = \frac{az+b}{d} = \frac{a}{d}z + \frac{b}{d}$

$\implies T(1/z) = \frac{a}{dz} + \frac{b}{d} = \frac{a+bz}{dz}$

$\implies 1/T(1/z) = \frac{dz}{a+bz}$

$\lim_{z\to 0} \frac{dz}{a+bz} = 0 \implies \lim_{z\to\infty} T(z) = \infty \quad \lozenge$

(b) $\lim_{z\to\infty} T(z) = \frac{a}{c}$, and $\lim_{z\to -d/c} T(z) = \infty$ if $c \neq 0$;

slu.

$c \neq 0 \implies T(z) = \frac{az+b}{cz+d}$

$\implies T(1/z) = \frac{a(1/z)+b}{c(1/z)+d} = \frac{\frac{a+bz}{z}}{\frac{c+dz}{z}} = \frac{a+bz}{c+dz}$

$\lim_{z\to 0} T(1/z) = \frac{a}{c} \implies \lim_{z\to\infty} T(z) = \frac{a}{c}$.

$\implies 1/T(z) = \frac{cz+d}{az+b}$

$\implies \lim_{z\to -d/c} 1/T(z) = \frac{c(-d/c)+d}{a(-d/c)+b} = 0$, which works because $ad - bc \neq 0 \implies ad \neq bc$.

$\implies \lim_{z\to -d/c} T(z) = \infty \quad \lozenge$