165A — Homework 7

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3 Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z} ds \quad (|z| \neq 3),$$

then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z| > 3$?

slu.

$|2| = 2 < 3 \implies 2 \in \text{int}(C) \implies g(2) = 2\pi i(2(2)^2 - 2 - 2) = 2\pi i(8 - 4) = 8\pi i$

$\forall z \in \mathbb{C} : |z| \neq 3 \implies s - z \neq 0 \implies \frac{2s^2-s-2}{s-z}$ is analytic on $\mathbb{C}\backslash\{z \in \mathbb{C} | |z| \neq 3\}$

C is a closed curve and $C \subset \mathbb{C}\backslash\{z \in \mathbb{C}| \quad |z| \neq 3\}$ ♦

10 Let f be an entire function such that $|f(z)| \leq A|z|$ for all z, when A is a fixed positive number. Show that $f(z) = a_1 z$, where a_1 is a complex constant.

slu.

$$|f^{(n)}(z)| \leq \frac{n! M_R}{R^n} \implies |f^{(2)}(z)| \leq \frac{2M_R}{R^2} \leq \frac{2A(|z| + R)}{R^2} \forall z \in \mathbb{C}$$

Since f is entire, then the radius R where it is analytic is arbitrarily large. So let $R \to \infty$, gives the right hand side of the last inequality is zero.

$$|f^{(2)}(z_0)| \leq 0 \implies f^{(2)}(z) = 0 \quad \forall z \in \mathbb{C} \implies \exists a_1, a_2 \in \mathbb{C} : f(z) = a_1 z + a_2$$

But, $|f| \leq A|z| \quad \forall z \in \mathbb{C} \implies a_2 = 0 \implies f(z) = a_1 z$ ♦

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3 Let a function f be continuous on a closed bounded region R, and let it be analytic and not constant throughout the interior of R. Assuming that $f(z) \neq 0$ anywhere in R, prove that $|f(z)|$ has a *minimum value m* in R which occurs on the boundary of R and never in the interior. Do this by applying the corresponding result for maximum values (Sec. 54) to the function $g(z) = 1/f(z)$.

pf.

Note, that $\forall z \in R, f(z) \neq 0 \implies g(z) = 1/f(z)$ is well-defined analytic and non-constant throughout the interior of R. Then the maximum of $|g(z)|$ is always reached and it is located somewhere in ∂R, by the corollary in (Sec. 54).

Let $m = \min |f(z)| \implies \forall z \in R, m \leq |f(z)| \implies \forall z \in R, |g(z)| = \frac{1}{|f(z)|} \leq \frac{1}{m} = \max |g(z)|$ ■