**Your name, LARGE and CLEAR**: ________________________

**Math 165a (Complex Analysis) Midterm Winter 2020**
**WITH SOLUTIONS**

**Using the paper provided**

- **Clearly number** each problem.
- Do **NOT** use **RED INK** (or other red marker).

(1) (a) State, clearly and completely, what it means for the derivative $f'(z)$ of a function $f(z)$ to exist at $z_0$.
**Solution**: $\lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists and is finite.

(b) Let $f = u + iv$. State the Cauchy-Riemann equations in terms of $u$ and $v$.
**Solution**: $\partial_x u = \partial_y v$, $\partial_y u = -\partial_x v$.

(c) Suppose that $f'(z_0)$ exists. What can you say about the Cauchy-Riemann equations?
**Solution**: They hold at $z_0$.

(d) Suppose that $f(z)$ is defined in a neighborhood of $z_0$ and that the Cauchy-Riemann equations are satisfied at $z_0$. What other condition or conditions are needed to insure that $f'(z_0)$ exists?
**Solution**: $\partial_x u, \partial_y v, \partial_y u, \partial_x v$ are continuous at $z_0$.

**Comment**: One could elaborate more on the answers to (c) and (d), but what I wrote in the solutions is all I expected.

(2) Show that if $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$ then

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2),$$

where principal arguments are used.

**Solution**: Since Arg is a specific choice of arg, we know that $\operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$ *is* an argument for $z_1 z_2$. Since $\operatorname{Re} z_1, \operatorname{Re} z_2 > 0$, $\operatorname{Arg}(z_1), \operatorname{Arg}(z_2) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Hence, $\operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) \in (-\pi, \pi)$ and so, in fact, it is the principal argument for $z_1 z_2$.

**Alternate Solution** (slightly wordier): Let $z_j = r_j e^{i\theta_j}$, $j = 1, 2$. Since $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, we know that $r_1, r_2 > 0$ and that we can choose $\theta_1, \theta_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$ with

$\theta_1 + \theta_2 \in (-\pi, \pi)$. We see then, by the ranges of $\theta_1$, $\theta_2$, and $\theta_1 + \theta_2$, that $\theta_1 = \operatorname{Arg} z_1$, $\theta_2 = \operatorname{Arg} z_2$, and $\theta_1 + \theta_2 = \operatorname{Arg}(z_1 z_2)$, which gives our result.

**Comment**: The essential point was to bring in the ranges of the arguments, one way or the other. You didn't need to reprove that $\arg(z_1 z_2) = \arg z_1 + \arg z_2 + 2n\pi$ for some $n$ by going to sines and cosines or even to (directly) use that fact (though it is implicit in both solutions).

(3) Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist, where $g'(z_0) \neq 0$. Use Definition (1), Sec. 19 of the derivative to show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

**Solution**: We have,

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{\dfrac{f(z) - f(z_0)}{z - z_0}}{\dfrac{g(z) - g(z_0)}{z - z_0}} = \frac{\displaystyle\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}}{\displaystyle\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0}} = \frac{f'(z_0)}{g'(z_0)}.$$

In the first equality we used that $f(z_0) = g(z_0) = 0$ so the fractions inside the two limits are identical. In the second equality we used that the limit of a ratio is the ratio of the limit, as long as limits in numerator and denominator exist and the denominator's limit is nonzero. In the final equality, we used the definition of the derivative.

(4) Show that $f(z) = |z|^2$ has a derivative only at the origin.

**Warning**: If you use the Cauchy-Riemann equations for this problem, be careful about applying them at the origin! You might want to use the definition of the derivative instead.

**Solution**: Let $z, h \in \mathbb{C}$ be arbitrary except that $h \neq 0$. Then

$$\begin{aligned}
\frac{f(z+h) - f(z)}{h} &= \frac{|z+h|^2 - |z|^2}{h} = \frac{(z+h)(\overline{z} + \overline{h}) - |z|^2}{h} \\
&= \frac{|z|^2 + z\overline{h} + h\overline{z} + h\overline{h} - |z|^2}{h} \\
&= \frac{z\overline{h} + h\overline{z} + h\overline{h}}{h} = z\frac{\overline{h}}{h} + \overline{z} + \overline{h}.
\end{aligned}$$

So far, we have taken no limits, but we know that $f'(z)$ exists if and only if the limit as $h \to 0$ in $\mathbb{C}$ of the above difference quotient exists. Now, when $z = 0$, we have

$$\frac{f(z+h) - f(z)}{h} = \left[ z\frac{\overline{h}}{h} + \overline{z} + \overline{h} \right]_{z=0} = \overline{h},$$

and the limit exists and equals zero. (I was not expecting a proof of this fact, though I did not mind it.) Hence, $f$ **is** differentiable at the origin.

Now suppose that $z \neq 0$. Then since $\lim_{h\to 0}(\overline{z} + \overline{h}) = \overline{z}$, the limit of the difference quotient exists if and only if

$$\lim_{h\to 0} \frac{\overline{h}}{h}$$

exists. But this limit does not exist, as can be shown, for instance, by taking the limit along the real axis and showing it is not the same as the limit along the imaginary axis, or by using polar form, as some of you did. Let me explain the latter approach.

Since in this limit, $h$ never equals zero, we can write $h = re^{i\theta}$ for some $r > 0$. If we approach the origin along a ray for a fixed value of $\theta$, we obtain

$$\lim_{r\to 0^+} \frac{\overline{re^{i\theta}}}{re^{i\theta}} = \lim_{r\to 0^+} \frac{e^{-i\theta}}{e^{i\theta}} = \lim_{r\to 0^+} e^{-2i\theta}.$$

But this limit depends upon $\theta$; in particular, the limit equals 1 when $\theta = 0$ and the limit equals $-1$ when $\theta = \pi/2$. Hence, $\lim_{h\to 0} \frac{\overline{h}}{h}$ does not exist.

**Alternate Solution** (Using Cauchy-Riemann equations): First observe that since $|z|^2 = x^2 + y^2$, we have $f(z) = u(x,y) + iv(x,y)$, where

$$u(x,y) = x^2 + y^2, \quad v(x,y) = 0.$$

We see, then, that

$$\partial_x u = 2x, \quad \partial_y u = 2y, \quad \partial_x v = \partial_y v = 0,$$

which are continuous for all $x$ and $y$. But the Cauchy-Riemann equations hold only for $x = y = 0$.

Now, the fact that the Cauchy-Riemann equations do not hold for $z = x+iy \neq 0$ immediately tells us that $f'(z)$ does not exist for $z \neq 0$ (see Problem 1($c$)). The fact that $f'(0)$ does exist follows not just from the Cauchy-Riemann equations holding, but from the partial derivatives being continuous at the origin (see Problem 1($d$)).

**Comment**: The warning about using the Cauchy-Riemann approach is the need to quite continuity at the origin. I do not believe that any of you who used this approach mentioned the need for continuity.