Solutions to some homework exercises from
Brown and Churchill 8th edition

Math 165a Undergraduate Complex Analysis

Winter 2020

Professor Kelliher

1. Homework 1

General comments: I graded the three problems whose solutions follow. Each problem was graded on a 3-point scale, according to the following rubric:

- **0**: Missing solution or little to no sense can be made of it.
- **1**: A significant step in solving the problem.
- **2**: Significant progress in solving the problem.
- **3**: An essentially correct answer with maybe some minor blemishes.

I am looking for complete sentences in those exercises asking for a proof. An exercise might be worded "show that" instead of "prove that," but they both mean "prove that." The "show" is used to indicate that it is not as involved a demonstration. It is best to learn this now, before taking the midterm.

Exercise 8-6 page 22

Show that if $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$ then

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2),$$

where principal arguments are used.

Solution: Since Arg is a specific choice of arg, we know that $\operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$ *is* an argument for $z_1 z_2$. Since $\operatorname{Re} z_1, \operatorname{Re} z_2 > 0$, $\operatorname{Arg}(z_1), \operatorname{Arg}(z_2) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Hence, $\operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) \in (-\pi, \pi)$ and so, in fact, it is the principal argument for $z_1 z_2$.

Alternate Solution (slightly wordier): Let $z_j = r_j e^{i\theta_j}$, $j = 1, 2$. Since $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, we know that $r_1, r_2 > 0$ and that we can choose $\theta_1, \theta_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$ with $\theta_1 + \theta_2 \in (-\pi, \pi)$. We see then, by the ranges of θ_1, θ_2, and $\theta_1 + \theta_2$, that $\theta_1 = \operatorname{Arg} z_1$, $\theta_2 = \operatorname{Arg} z_2$, and $\theta_1 + \theta_2 = \operatorname{Arg}(z_1 z_2)$, which gives our result.

Comment: The essential point was to bring in the ranges of the arguments, one way or the other. You didn't need to reprove that $\arg(z_1 z_2) = \arg z_1 + \arg z_2 + 2n\pi$ for some n by going to sines and cosines or even to (directly) use that fact (though it is implicit in both solutions).

Exercise 10-7 page 29

Show that if c is any n^{th} root of unity other than unity itself, then

$$1 + c + c^2 + \cdots + c^{n-1} = 0.$$

Suggestion: Use the first identity in Exercise 9, Sec. 8.

Solution:Applying Exercise 9, Sec. 8 with $z = c \neq 1$ and $n-1$ in place of n, we have

$$1 + c + c^2 + \cdots + c^{n-1} = \frac{1-c^n}{1-c} = 0.$$

Comment: This was a gift. Most of you took it!

Exercise 11-10 page 33

Prove that a finite set of points $z_1, z_2, \ldots, z_n$ cannot have any accumulation points.

Solution: Let z be an arbitrary point in $\mathbb{C}$. We will show that z cannot be an accumulation point by applying the negation of the definition of an accumulation point (see page 32 of the text); that is, we will show there is a *deleted* neighborhood of z that contains no point of $S := \{z_1, z_2, \ldots, z_n\}$.

First suppose that $z \in S$. Thus, $z = z_k$ for some k. Letting

$$\delta = \delta_k = \min_{\{j : z_j \neq z_k\}} |z_k - z_j|,$$

which we note is finite and positive, we see that $\dot{B}_\delta(z)$ contains no point of S. Hence, z is *not* an accumulation point.

So suppose that $z \in \mathbb{C} \setminus S$, the only other possibility. Now let

$$\delta = \delta(z) = \min_{j=1,\ldots,n} |z - z_j|,$$

which we note is finite and positive. Then $\dot{B}_\delta(z)$ (in fact, $B_\delta(z)$) contains no point of S. We see that again, z is *not* an accumulation point. We conclude that there are no accumulation points of S.

One point of clarification could be added. Why are the $\delta's$ finite? Because both are the minimum of a finite number of positive numbers and so are finite and positive.

Comments: Ingredients of this argument appeared in many solutions, but the pieces were usually not organized enough for me to understand the logic behind them. In particular, the two cases were not properly identified, and the need to use an excluded neighborhood seldom mentioned. One can make a unified definition of the δ's in the two cases, but it is more confusing than clarifying to do that. Also, some used as the definition of an accumulation point that any neighborhood of the point must contain an infinite number of points in S. That is fine, except it is a consequence of our definition, not the definition itself; though I mentioned it was true in lecture, I never proved it, so a proof would be needed. Finally, a very picky point, but the exercise does not say that the z_j's are distinct, which is why I wrote the first expression for the first δ the way I did, rather than minimizing over all $j \neq k$.

2. Homework 2

Exercise 18-1 page 55

Use Definition (2) Section 15 of a limit to prove that

(a) $\lim_{z \to z_0} \operatorname{Re} z = \operatorname{Re} z_0$;

(b) $\lim_{z \to z_0} \overline{z} = \overline{z_0}$;

(c) $\lim_{z \to 0} \dfrac{\overline{z}^2}{z} = 0$.

Solution: **(a)** Let $\epsilon > 0$ and set $\delta = \epsilon$. Suppose that $0 < |z - z_0| < \delta$. Then

$$|\operatorname{Re} z - \operatorname{Re} z_0| \leq \sqrt{|\operatorname{Re} z - \operatorname{Re} z_0|^2 + |\operatorname{Im} z - \operatorname{Im} z_0|^2} = |z - z_0| < \delta = \epsilon.$$

(b) Let $\epsilon > 0$ and set $\delta = \epsilon$. Suppose that $0 < |z - z_0| < \delta$. Then

$$|\overline{z} - \overline{z_0}| = |\overline{z - z_0}| = |z - z_0| < \delta = \epsilon.$$

(c) Letting $f(z) = \overline{z}^2/z$, first observe that

$$|f(z) - f(0)| = |f(z)| = |z|\,.$$

Hence, for any $\epsilon > 0$ we can let $\delta = \epsilon$, as we did in (a) and (b). Then supposing that $0 < |z - 0| < \delta$, we see that

$$|f(z) - f(0)| = |z| = |z - 0| < \delta = \epsilon.$$

Exercise 18-5 page 55

Show that the limit of the function

$$f(z) = \left(\frac{z}{\bar{z}}\right)^2$$

as z tends to 0 does not exist. Do this by letting nonzero points $z = (x, 0)$ and $z = (x, x)$ approach the origin. [Note that it is not sufficient to simply consider points $z = (x, 0)$ and $z = (0, y)$, as it was in Example 2, Sec. 15.]

Solution: First, for any $z \neq 0$, write $z = re^{i\theta}$. Then

$$f(z) = \left(\frac{re^{i\theta}}{re^{-i\theta}}\right)^2 = \left(e^{2i\theta}\right)^2 = e^{4i\theta}.$$

Letting nonzero points $z = (x, 0)$ and $z = (x, x)$ approach the origin is equivalent to choosing $\theta = 0$ and $\theta = \pi/4$. For $\theta = 0$, $f(z)$ always equals 1 and for $\theta = \pi/4$, $f(z)$ always equals $e^{4i\pi/4} = -1$. Hence, the limit cannot exist.

Finally note that choosing $z = (0, y)$ corresponds to $\theta = \pi/2$, in which case $f(z)$ always equals $e^{4i\pi/2} = 1$, so that would be insufficient to show that the limit does not exist.

Exercise 18-7 page 55

Use Definition (2) Section 15 of a limit to prove that

$$\text{if } \lim_{z \to z_0} f(z) = w_0 \text{ then } \lim_{z \to z_0} |f(z)| = |w_0|.$$

Suggestion: Observe how the first of inequalities (9) Section 4 enables one to write

$$||f(z)| - |w_0|| \leq |f(z) - w_0|.$$

Solution: Let $\epsilon > 0$. Since $\lim_{z \to z_0} f(z) = w_0$ we know that there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon.$$

So assume that $0 < |z - z_0| < \delta$. Then by the suggestion (which is the reverse triangle inequality),

$$||f(z)| - |w_0|| \leq |f(z) - w_0| = \epsilon,$$

which shows that $\lim\limits_{z \to z_0} |f(z)| = |w_0|$.

Exercise 18-11 page 55

With the aid of the theorem in Section 17, show that when

$$T(z) = \frac{az+b}{cz+d} \quad (ad - bc \neq 0),$$

(a) $\lim_{z\to\infty} T(z) = \infty$ if $c = 0$;

(b) $\lim_{z\to\infty} T(z) = \dfrac{a}{c}$ and $\lim_{z\to -d/c} T(z) = \infty$ if $c \neq 0$.

Solution: **(a)** Applying the referenced theorem,

$$\lim_{z\to\infty} T(z) = \infty \iff \lim_{z\to 0} \frac{1}{T(1/z)} = 0.$$

Then the claimed limit does hold, since

$$\lim_{z\to 0} \frac{1}{T(1/z)} = \lim_{z\to 0} \frac{cz^{-1}+d}{az^{-1}+b} = \lim_{z\to 0} \frac{c+dz}{a+bz} = \frac{c}{a} = 0.$$

In the last step we used that $c = 0$, while $a \neq 0$, since otherwise $ad - bc = 0$.

(b) Applying the referenced theorem,

$$\lim_{z\to\infty} T(z) = \frac{a}{c} \iff \lim_{z\to 0} T(z^{-1}) = \frac{a}{c}.$$

Then the claimed limit does hold, since

$$\lim_{z\to 0} T(z^{-1}) = \lim_{z\to 0} \frac{az^{-1}+b}{cz^{-1}+d} = \lim_{z\to 0} \frac{a+bz}{c+dc} = \frac{a}{c},$$

where, of course, we used that $c \neq 0$.

Applying the referenced theorem one final time,

$$\lim_{z\to -d/c} T(z) = \infty \iff \lim_{z\to -d/c} \frac{1}{T(z)} = 0.$$

Then the claimed limit does hold, since

$$\lim_{z\to -d/c} \frac{1}{T(z)} = \lim_{z\to -d/c} \frac{cz+d}{az+b} = \frac{-d+d}{-ad/c+b} = -\frac{0}{ad-bc} = 0.$$

Here we used that $c \neq 0$ and also that $ad - bc \neq 0$ to avoid division by zero.

Exercise 20-4 page 62

Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist, where $g'(z_0) \neq 0$. Use Definition (1), Sec. 19 of the derivative to show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

Solution: We have,

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{\dfrac{f(z) - f(z_0)}{z - z_0}}{\dfrac{g(z) - g(z_0)}{z - z_0}} = \frac{\lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}}{\lim\limits_{z \to z_0} \dfrac{g(z) - g(z_0)}{z - z_0}} = \frac{f'(z_0)}{g'(z_0)}.$$

In the first equality we used that $f(z_0) = g(z_0) = 0$ so the fractions inside the two limits are identical. In the second equality we used that the limit of a ratio is the ratio of the limit, as long as limits in numerator and denominator exist and the denominator's limit is nonzero. In the final equality, we used the definition of the derivative.

Exercise 23-8 page 71

Let a function $f(z) = u + iv$ be differentiable at a nonzero point $z = re^{i\theta}$. Use the expressions for $\partial_x u$ and $\partial_x v$ found in Exercise 7, together with the polar form (6), Sec. 23, of the Cauchy-Riemann equations to rewrite the expression

$$f'(z_0) = \partial_x u + i\partial_x v$$

in Sec. 22 as

$$f'(z_0) = e^{-i\theta}(\partial_r u + i\partial_r v)(r_0, \theta_0).$$

Solution: From Exercise 7,

$$\partial_x f = \partial_r f \cos\theta - \partial_\theta f \frac{\sin\theta}{r}, \quad \partial_y f = \partial_r f \sin\theta + \partial_\theta f \frac{\cos\theta}{r},$$

for any differentiable function f (observing that (2) of Sec. 23, upon which Exercise 7 builds, does not assume anything more about u), while the Cauchy-Riemann equations in polar form are

$$r\partial_r u = \partial_\theta v, \quad \partial_\theta u = -r\partial_r v.$$

Using these, we see that

$$\begin{aligned}
f'(z_0) &= \partial_x u + i\partial_x v = \partial_r u\cos\theta - \partial_\theta u\frac{\sin\theta}{r} + i\left(\partial_r v\cos\theta - \partial_\theta v\frac{\sin\theta}{r}\right)\\
&= \partial_r u\cos\theta + r\partial_r v\frac{\sin\theta}{r} + i\left(\partial_r v\cos\theta - r\partial_r u\frac{\sin\theta}{r}\right)\\
&= \partial_r u(\cos\theta - i\sin\theta) + \partial_r v(\sin\theta + i\cos\theta)\\
&= \partial_r u e^{-i\theta} + i\partial_r v(\cos\theta - i\sin\theta) = e^{-i\theta}(\partial_r u + i\partial_r v),
\end{aligned}$$

where we note all these calculations are evaluated at (r_0, θ_0).

Exercise 23-10 page 71

(a) Recall (Sec.5) that if $x = x + iy$ then

$$x = \frac{z+\overline{z}}{2} \text{ and } y = \frac{z-\overline{z}}{2i}.$$

By *formally* applying the chain rule in calculus to a function $F(x, y)$ of two real variables, derive the expression

$$\frac{\partial F}{\partial \overline{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right).$$

(b) Define the operator

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

suggested by part (a), to show that if the first-order partial derivatives of the real and imaginary components of a function $f(z) = u(x, y) + iv(x, y)$ satisfy the Cauchy-Riemann equations, then

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left[(\partial_x u - \partial_y v) + i(\partial_x v + \partial_y u)\right] = 0.$$

Thus derive the *complex form* $\partial f/\partial\overline{z} = 0$ *of the Cauchy-Riemann equations.*

Solution: **(a)** We are being asked to apply the chain rule,

$$\frac{\partial F}{\partial \overline{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \overline{z}},$$

to obtain the given expression for $\frac{\partial F}{\partial \overline{z}}$. To do this, we first calculate,

$$\frac{\partial x}{\partial \overline{z}} = \frac{1}{2}, \quad \frac{\partial y}{\partial \overline{z}} = -\frac{1}{2i} = \frac{i}{2}.$$

Note that in calculating these two partial derivatives, we act as though z and $\overline{z}$ are independent variables and use the expressions for x and y in terms of z and $\overline{z}$. This is what the text means by *formally* applying the chain rule. Now we see that

$$\frac{\partial F}{\partial \overline{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{\partial F}{\partial x}\frac{1}{2} + \frac{\partial F}{\partial y}\frac{i}{2} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right).$$

(b) We have,

$$\begin{aligned}\frac{\partial f}{\partial \overline{z}} &= \frac{\partial u}{\partial \overline{z}} + i\frac{\partial v}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y}\right)\\ &= \frac{1}{2}(\partial_x u - \partial_y v) + \frac{i}{2}(\partial_y u + \partial_x v) = 0\end{aligned}$$

by the Cauchy-Riemann equations.

4. Homework 4

Exercise 25-2 page 77

With the aid of the Theorem in Section 21 show that each of these functions is nowhere analytic:

(*a*) $f(z) = xy + iy$; (*b*) $f(z) = 2xy + i(x^2 - y^2)$; (*c*) $f(z) = e^y e^{ix}$.

Solution: I will note up front, that this is a subtle exercise.

The Theorem in Section 21 says, in short, that if a function is differentiable at a point then it must satisfy the Cauchy-Riemann equations at that point (and hence, the partial derivatives must exist else the Cauchy-Riemann equations cannot be satisfied).

Now, in all three cases, writing $f = u + iv$, we see that u and v are infinitely differentiable, so we don't need to worry about that. For ease of reference, let us refer to $\partial_x u = \partial_y v$ as $(*)$ and $\partial_y u = -\partial_x v$ as $(**)$.

Here is the situation case-by-case, where in each case we search for all points $z = x + iy$ for which both $(*)$ and $(**)$ hold:

(a) Here, $u = xy$, $v = y$, so

$$\begin{aligned}\partial_x u = y, \quad \partial_y v = 1,\\ \partial_y u = x, \quad \partial_x v = 0.\end{aligned}$$

Then $(**) \implies x = 0$, $(*) \implies y = 1$. Hence, $f'(z)$ exists only for $z = 0 + 1i = i$.

(b) Here, $u = 2xy$, $v = x^2 - y^2$ so

$$\begin{aligned}\partial_x u = 2y, \quad \partial_y v = -2y,\\ \partial_y u = 2x, \quad \partial_x v = 2x.\end{aligned}$$

Then $(**) \implies x = 0$, $(*) \implies y = 0$. Hence, $f'(z)$ exists only at the origin.

(c) Writing $f(z) = e^y \cos x + ie^y \sin x$, we see that $u = e^y \cos x$, $v = e^y \sin x$. So

$$\begin{aligned} \partial_x u &= -e^y \sin x, & \partial_y v &= e^y \sin x, \\ \partial_y u &= e^y \cos x, & \partial_x v &= e^y \cos x. \end{aligned}$$

Then since e^y never equals zero, $(**) \implies \cos x = 0$, while $(*) \implies \sin x = 0$. Since there are no real values x for which $\cos x$ and $\sin x$ both equal zero, $f'(z)$ exists nowhere.

Now the subtle point: for f to be analytic at a point z we need, by the original definition, the derivative of f to exist in a *neighborhood* of z. In none of the three cases is there such a neighborhood of any point at which the derivative exists (in the last case there are no points at all where the derivative exists to begin with); therefore, none of the three functions are analytic anywhere.

Exercise 25-7 page 77

Let a function f be analytic everywhere in a domain D. Prove that if $f(z)$ is real-valued for all z in D then $f(z)$ must be constant throughout D.

Solution: Because f is real-valued, $\overline{f} \equiv f$, so both f and $\overline{f}$ are real-valued. It follows from Example 3 in Section 25 (which we made a lemma in lecture) that f must be constant throughout D.

Alternate Solution (more direct): We know that $f(z) = u(x, y)$, where u is real-valued. Since f is analytic, we must have $\partial_x u = \partial_y v = 0$ and $\partial_y u = -\partial_x v = 0$. Hence, $f'(z) = \partial_x f \equiv 0$; the result then follows, since we know that a function whose complex derivative vanishes everywhere in a domain is constant in that domain.

We could also finish the proof by using that $\nabla u \equiv 0$ on D, which means that u is constant on D, using a result from real variable theory.

Exercise 31-9 page 97

Show that

(a) the function $f(z) = \operatorname{Log}(z - i)$ is analytic everywhere except on the portion $x \le 0$ of the line $y = 1$;

(b) the function

$$f(z) = \frac{\operatorname{Log}(z + 4)}{z^2 + i}$$

is analytic everywhere except at the points $\pm(1 - i)/\sqrt{2}$ and on the portion $x \le -4$ of the real axis.

Solution: One could try to attack this problem by directly applying the Cauchy-Riemann equations. But the Cauchy-Riemann equations do not understand branch cuts, they are satisfied even right on a branch cut. (Consider $f(z) = \operatorname{Log} z$ and you will discover that only at $z = 0$ are the Cauchy-Riemann equations not satisfied.) So that approach is doomed to failure, unless you treat the domains properly. But if you treat the domains properly, you don't even need the Cauchy-Riemann equations!

Instead, let us view the functions in (a) and (b) as being built up from other, simpler functions.

For any complex-valued function f on $\mathbb{C}$, let us call the *domain of analyticity of* f the set of all points in the domain of f at which f is analytic. Note that the domain of analyticity of f is necessarily a subset of the domain of f. Also, the domain of analyticity of f is open (think about why) but need not be connected, so the term *domain* in this context is a little misleading, but commonly used.

For (a), we can write, $f(z) = (\operatorname{Log} \circ \phi)(z)$, where $\phi = z - i$. Note that the domain of f is all of $\mathbb{C}$, but

$$\text{domain of analyticity of } f = \mathbb{C} \setminus \{z = x + iy \in \mathbb{C} \colon x \leq 0,\, y = 1\}\,.$$

This is because the domain of analyticity of $\phi = \mathbb{C}$, but Log's domain of analyticity excludes the negative real axis—and by the chain rule, f is differentiable at any point z for which ϕ is differentiable and $\operatorname{Log}(\phi(z))$ is differentiable.

(b) is similar, though now we write $f(z) = (\operatorname{Log} \circ \phi)(z)/g(z)$, where $\phi(z) = z + 4$ and $g(z) = z^2 + i$. Now,

$$\text{domain of analyticity of } \operatorname{Log} \circ \phi = \mathbb{C} \setminus \{z = x \in \mathbb{C} \colon x \leq -4\}\,,$$

much as in (a). Then as long as also $z^2 + i \neq 0$, avoiding division by zero, we will be in the domain of analyticity of f. Since the roots of $z^2 + i$ are $\pm(1 - i)/\sqrt{2}$, this gives the stated domain of analyticity.

5. Homework 5

Exercise 39-5 page 125

Suppose that a function $f(z)$ is analytic at a point $z_0 = z(t_0)$ lying on a smooth arc $z = z(t)$ $(a \leq t \leq b)$. Show that if $w(t) = f[z(t)]$ then

$$w'(t) = f'[z(t)]z'(t)$$

when $= t_0$.
Suggestion: Write $f(z) = u(x, y) + iv(x, y)$ and $z(t) = x(t) + iy(t)$, so that

$$w(t) = u[x(t), y(t)] + iv[x(t), y(t)].$$

Then apply the chain rule in calculus for functions of two real variables to write

$$w' = (\partial_x u\, x' + \partial_y u\, y') + i(\partial_x v\, x' + \partial_y v\, y'),$$

and use the Cauchy-Riemann equations.

Solution: We know that if $w(t) = x(t) + iy(t)$,

$$w'(t) = x'(t) + iy'(t) = \lim_{h \to 0} \frac{w(t+h) - w(t)}{h}.$$

The first equality is the text's *definition* of $w'(t)$, the second follows immediately from it (and is better as a definition), as we commented on in lecture. Then, as we did in lecture,

$$\begin{aligned} w'(t_0) &= \lim_{h \to 0} \frac{f(z(t_0 + h)) - f(z(t_0))}{h} \\ &= \lim_{h \to 0} \frac{f(z(t_0 + h)) - f(z(t_0))}{z(t_0 + h) - z(t_0)} \lim_{h \to 0} \frac{z(t_0 + h) - z(t_0)}{h} \\ &= f'(z(t_0))z'(t_0). \end{aligned}$$

The second limit held directly by the definition of the derivative. For the first limit, I just indicated the idea in class, but let me say a few more words here.

Because z is continuous at t_0, we can write $z(t_0 + h) - z(t_0) = r(h)$, where $\lim_{h \to 0} r(h) = 0$. Then,

$$\lim_{h \to 0} \frac{f(z(t_0 + h)) - f(z(t_0))}{z(t_0 + h) - z(t_0)} = \lim_{h \to 0} \frac{f(z(t_0) + r(h)) - f(z(t_0))}{r(h)} = f'(z(t_0)).$$

If this last limit is still not entirely convincing, note that because f is analytic at $z_0 = z(t_0)$, we have

$$\lim_{h \to 0} \frac{f(z(t_0) + r(h)) - f(z(t_0))}{r(h)} = \lim_{h \to 0} \frac{f(z_0 + r(h)) - f(z_0)}{r(h)}.$$

In this last form, it is perhaps clearer that we are taking the limit of the difference quotient defining $f'(z_0)$, approaching z_0 along the curve $h \mapsto z_0 + r(h)$. But complex derivatives, when they exist, are independent of the "path of approach," so we conclude that the limit is $f'(z_0) = f'(z(t_0))$.

The text's suggestion is a direct brute force calculation. I will leave you to it.

Exercise 42-8 page 135

With the aid of the result in Exercise 3, Sec. 38, evaluate the integral

$$\int_C z^m \overline{z}^n \, dz,$$

where m and n are integers and C is the unit circle, $|z| = 1$, taken counterclockwise.

Solution: Exercise 3, Sec. 38 tells us that for any integers m, n,

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} \, d\theta = \begin{cases} 0, & m \neq n, \\ 2\pi, & m = n. \end{cases}$$

For the sake of completeness, let us first work this exercise. We have, for $m \neq n$,

$$\begin{aligned} \int_0^{2\pi} e^{im\theta} e^{-in\theta} \, d\theta = \int_0^{2\pi} e^{i(m-n)\theta} \, d\theta &= \frac{1}{i(m-n)} \left[e^{i(m-n)\theta} \right]_0^{2\pi} \\ &= \frac{1}{i(m-n)} [1 - 1] = 0. \end{aligned}$$

Here, we used the observations in lecture (or see 38 then 37) that the fundamental theorem of calculus applies to any integral of this type (that is, an integral of a complex-valued function of a real variable) and that $(d/dt) e^{z_0 t} = z_0 e^{z_0} t)$.

If $m = n$ then the integrand is 1 and we see that the integral gives 2π.

Now let us turn to the exercise at hand. Letting $z(\theta) = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, we have, by the definition of a complex contour integral,

$$\begin{aligned} \int_C z^m \overline{z}^n \, dz = \int_0^{2\pi} e^{im\theta} e^{-in\theta} (e^{i\theta})' \, d\theta &= i \int_0^{2\pi} e^{i(m-n+1)\theta} \, d\theta \\ &= \begin{cases} 0, & m - n + 1 \neq 0, \\ 2\pi i, & m - n + 1 = 0. \end{cases} \end{aligned}$$

Here, we applied Exercise 3, Sec. 38 with $m - n + 1$ in place of m and 0 in place of n. We also used that if $z = e^{i\theta}$ then

$$\overline{z} = \cos\theta - i \sin\theta = e^{-i\theta},$$

so that $\overline{z}^n = e^{-in\theta}$.

Notice that when $n = 0$, the result reduces to

$$\int_C z^m \, dz = \begin{cases} 0, & m \neq -1, \\ 2\pi i, & m = -1, \end{cases}$$

something with which we are by now very familiar.