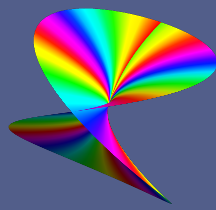
UC RIVERSIDE
Department of Mathematics

MATH 165B - Introduction to Complex Variables

Worksheet 2



Topics: Residues and Poles

Readings: Sec 72, 73, 74 from Brown & Churchill

72. The Three Types of Isolated Singular Points

- Motivating examples: We will look at three examples that illustrate three types of isolated singular point. The point $z_0 = 0$ is an isolated singular point in the three examples and a way of classifying is evident when we look at their Laurent series.

  - $$\begin{aligned}\frac{\sin z}{z} &= \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \cdots\right)\\ &= \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots + (-1)^n \frac{z^{2n}}{(2n+1)!} + \cdots\right)\end{aligned}$$

    No negative terms with powers of z left!

    $z_0 = 0$ is a **removable singularity**

  - $$\begin{aligned}\frac{\sin z}{z^5} &= \frac{1}{z^5}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \cdots\right)\\ \frac{\sin z}{z^5} &= \left(\frac{1}{z^4} - \frac{1}{3!}\frac{1}{z^2} + \frac{1}{5!} - \frac{z^2}{7!} + \cdots + (-1)^n \frac{z^{2n-4}}{(2n+1)!} + \cdots\right)\end{aligned}$$

    A finite number of terms with negative powers of z in the Laurent expansion,

    $z_0 = 0$ is a **pole of order** 4

  - $$\sin\frac{1}{z} = \left(\frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \frac{1}{7!}\frac{1}{z^7} + \cdots + (-1)^n \frac{1}{(2n+1)!}\frac{1}{z^{2n+1}} + \cdots\right)$$

    An infinite number of terms with negative powers of z in the Laurent expansion,

    $z_0 = 0$ is an **essential singularity**

- Reading

  - Read carefully all the examples in the book in these section
  - Look at the visualization of poles in `https://mathworld.wolfram.com/Pole.html`
  - Use Mathematica to reproduce some of these plots. Try other functions with isolated singularities. (*If you already have Mathematica.)*

- (P) Examples of isolated singularities: Give examples of functions $f_1, f_2$and f_3 with the following characteristics

  1. f_1 has a removable singularity at $z_0 = 0$
  2. f_2 has a pole of order 2 at $z_0 = i\pi$
  3. f_3 has a removable singularity at $z_0 = 1$

73. Residues at Poles

- Relevance of the main theorem in this section:

  This theorem makes it easier to compute residues for poles. Residues can be computed without finding the Laurent series first. Read the following theorem from this point of view, noticing that the residue formula does not involve the Lawrence series expansion:

  Theorem. An isolated singular point zo of a function f is a pole of order m if and only if $f(z)$ can be written in the form

  (1) $f(z) = \frac{\phi(z)}{(z-z_0)^m}$ where $\phi(z)$ is analytic and nonzero at zo. Moreover,

  (2) $\mathrm{Res}_{z=z_0} f(z) = \phi(z_0)$ if $m = 1$ and

  (3) $\mathrm{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$ if $m \geq 2$

- (P) Particular cases of the theorem: Rewrite the theorem for functions f_1 and f_2 such that

  - $m = 1$, $z_0 = 0$ and $\phi(z) = \mathrm{e}^z$

  - $m = 3$, $z_0 = 0$ and $\phi(z) = \mathrm{e}^z$

74. Examples of How to Calculate a Residue at a Pole

- Reading: Read all the examples in this section.
- (P) Generating Examples: Notice that the family of functions $f_{z_0}(z) = \frac{1}{z-z_0}$ has a pole of order $m = 1$ for any z_0. Write family of functions with the following characteristics:
  - h_{z_0} has an essential singularity at z_0
  - g_{z_0} has a pole of order 5 at z_0
  - f_m has a pole of order $m \in \mathbb{N}$ at $z_0 = 1$

Homework Problems for Sections 72, 73 and 74

- Page 243: # (a) and (e), #3
- Page 248: #1 (a), (b) and (c), # 2 (b), #3, #6 (b).
- Star Problems: Page 243, #4, and Page 239, #7

The Star Problems are intended for students who are interested in challenging problems, they can substitute regular problems in the assignment.