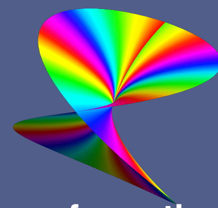
Department of Mathematics

MATH 165B - Introduction to Complex Variables

Worksheet 6



Topics: Linear Fractional Transformations (also known as Möbius Transformations or Bilinear Trasformations)

Readings from Brown & Churchill:

- Section 93: The Möbius transformation is defined. The proof that this transformation is the composition of a linear and the reciprocal transformations is given.
- Section 94: This sections introduces a useful property of the ”Möbius transformation: the preservation of ”cross-ratios”. This property is a very convenient tool in applications where we need to find a transformation with particular characteristics

93. Linear Fractional (Möbius) Transformations

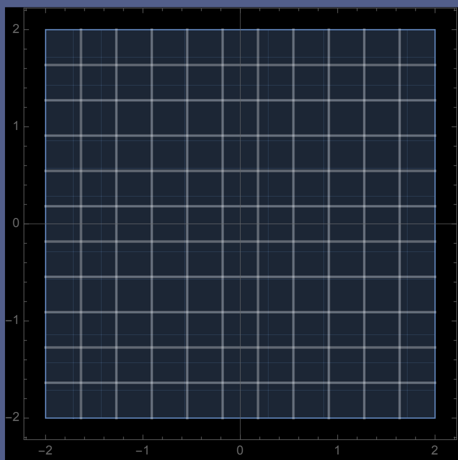
- Following the procedure at the beginning of the section, compute of the inverse of the transformation with coefficients $a=-1$, $b=i$, $c=1$ and $d=i$. That is, write down the Möbius transformation $w=T(z)$ with these coefficients and its corresponding inverse $z=T^{-1}(w)$.
- (P) Find 4 examples of Möbius Transformations T_1, T_2, T_3 and T_4 such that
  - $T_1(0)=0$
  - $T_2(0)=\infty$
  - $T_3(\infty)=0$
  - $T_4(\infty)=\infty$
- Show that the transformation T of the first item can be expressed also as:

$$T(z)=-1+2i\frac{1}{i+z}$$

- The following plots below correspond to the image of the region $[-2,2]\times[-2,2]$ under the transformations

$$\begin{aligned}T_1(z)&=z\\ T_2(z)&=\frac{1}{i+z}\\ T_3(z)&=-1+2iz\\ T(z)&=-1+2i\frac{1}{i+z}\end{aligned}$$

Label the plots and order them according to the composition of transformations that will result in the transformation $T(z)$.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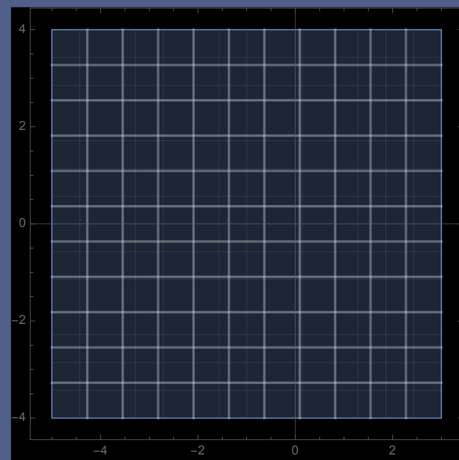
−1
−2
−2
−1
0
1
2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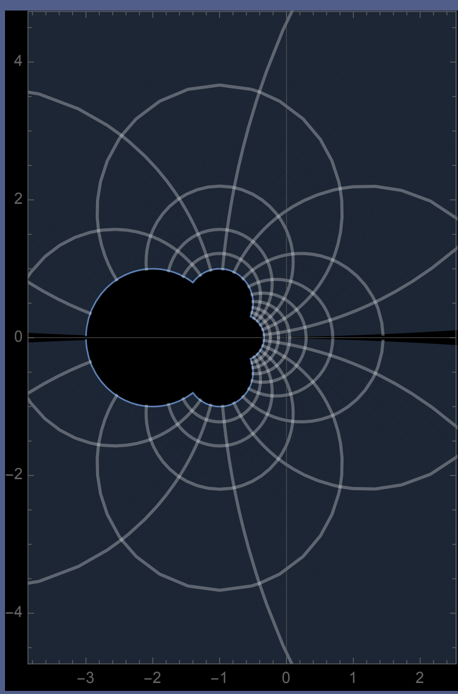
0
−2
−4
−4
−2
0
2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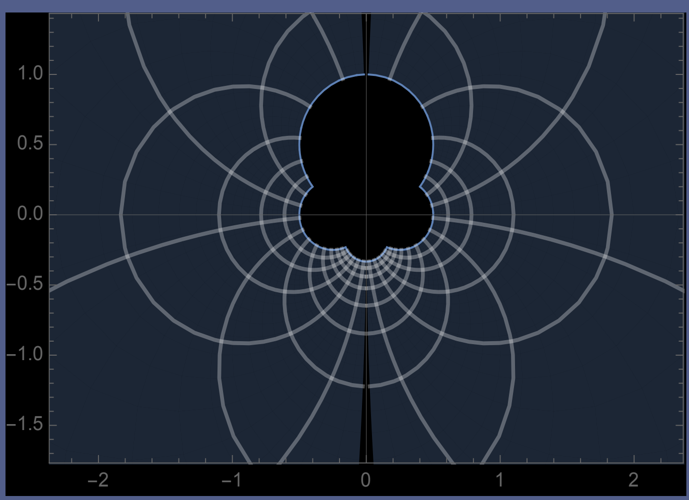
0
−2
−4
−3
−2
−1
0
1
2


1.0
0.5
0.0
−0.5
−1.0
−1.5
−2
−1
0
1
2

94. An Implicit Form

- For the transformation $T(z) = \dfrac{i-z}{i+z}$, consider
  $z_1 = -1, z_2 = 0, z_3 = 1$ and find the values of $w_1 = T(z_1), \quad w_2 = T(z_2), \quad w_3 = T(z_3)$
  Verify that $w = T(z) = \dfrac{i-z}{i+z}$ satisfies the identity:
  $$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$
- Read Examples 1 and 2 from the textbook
- (P) Describe a general method to find a Möbius Transformations that satisfies $w_i = T(z_i),\ i = 1, 2, 3$ for given points z_i and w_i using the cross-ratio identity.

HOMEWORK PROBLEMS FOR SECTION 93 and 94

1. Page 324: #2, #4, #5 #6, #7
2. Star Problems: Page 325: #10, #11, #12

The Star Problems are intended for students who are interested in challenging problems, they can substitute regular problems in the assignment.