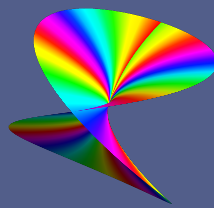
UC RIVERSIDE
Department of Mathematics

MATH 165B - Introduction to Complex Variables

Worksheet 1



Topics: Residues and Poles

Readings: Sec 68, 69, 70 from Brown & Churchill

68. Isolated Singular Points

- Concept

  A point z_0 is called a singular point of a function f if

  - f is not analytic at z_0 and
  - f is analytic at some point in every neighborhood of z_0.

  A singular point z_0 is isolated if,

  - z_0 is a singular point
  - there is a deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0 where f is analytic.

- (P) Examples of Isolated Singular Points: Give three examples of isolated singular points. Check that the conditions of the definition is satisfied.

- (P) Example of a Singular Point that is NOT Isolated: Give an example of a singular point that is not isolated. Justify.

69. Residues

- Motivation for the definition of Residue and the Residue Theorem

  Let $z_0 = i$ be an isolated singular point of a function f, such that $f(z)$ has the following Laurent series representation on $0 < |z-i| < 2$:

$$f(z) = \cdots \frac{b_n}{(z-i)^n} + \cdots + \frac{b_2}{(z-i)^2} + \frac{b_1}{z-i} + a_0 + a_1(z-i) + a_2(z-i)^2 + \cdots + a_n(z-i)^n + \cdots \quad (1)$$

  If we integrate f along the curve $|z-i| = 1$ positively oriented:

$$\int_C f(z)dz = \int_C \left(\cdots \frac{b_n}{(z-i)^n} + \cdots + \frac{b_2}{(z-i)^2} + \frac{b_1}{z-i} + a_0 + a_1(z-i) + a_2(z-i)^2 + \cdots + a_n(z-i)^n + \cdots \right) dz$$

  Integrating "term by term":

$$\begin{aligned} \int_C f(z)dz &= \cdots \int_C \frac{b_n}{(z-i)^n}dz + \cdots + \int_C \frac{b_2}{(z-i)^2}dz + \int_C \frac{b_1}{z-i}dz + \\ &= + \int_C a_0\, dz + \int_C a_1(z-i)dz + \int_C a_2(z-i)^2 dz + \cdots + \int_C a_n(z-i)^n dz + \cdots \end{aligned}$$

  Verify that

$$\begin{aligned} \int_C \frac{b_n}{(z-i)^n}dz &= 0 \ \text{if} \ n \neq 1 \\ \int_C \frac{b_1}{z-i}dz &= 2\pi i\, b_1 \\ \int_C a_n(z-i)^n dz &= 0 \ \text{if} \ n \geq 0 \end{aligned}$$

  Then

$$\boxed{\int_C f(z)dz = 2\pi i\, b_1} \qquad (2)$$

  The complex number b_1, the coefficient of $1/(z-i)$ in expansion (1), is called the residue of f at the isolated singular point $z_0 = i$, and we will use the notationn

$$b_1 = \operatorname{Res}_{z=i} f(z)$$

  and the equation (2) can be expressed as:

$$\boxed{\int_C f(z)dz = 2\pi i \operatorname{Res}_{z=i} f(z)} \qquad (3)$$

  Equation (3) is known as the "Residue Theorem" that we will study in the Section 70.

- (P) Examples Write 3 different Laurent series expansion of functions f at isolated singular points at z_0 of your choice and identify the corresponding residues $\mathrm{Res}_{z=z_0} f(z)$.

70. Cauchy's Residue Theorem

- Application of the "Residue Theorem"

  Let

  $$f(z) = \frac{2}{(z-1)^4} + \frac{57}{z-1}$$

  and C be the circle $|z| = 2$ positively oriented. Then, we could use (3) to compute the integral,

  $$\int_C f(z)dz = \int_C \left(\frac{2}{(z-1)^4} + \frac{57}{z-1} \right) dz = 2\pi i \operatorname{Res}_{z=1} f(z) = 2\pi i\, 57$$

- Simplest version of the "Residue Theorem" Write the statement of the theorem for only ONE singular point. Draw the corresponding picture.

- (P) Examples Give examples of three functions (f_1, f_2 and f_3) and how the residue theorem can be used to compute the integral of each of these functions on the positively oriented curve C defined $|z| = 10$. In addition, the functions must to satisfy the following conditions:

  - f_1 has two singular points inside the curve C
  - f_2 has one singular point inside the curve and one point outside the curve C
  - f_3 has two singular points inside and two outside the curve C

  Draw the curve and the location of the points for each example.

Homework Problems for Sections 68, 69 and 70

- Page 239: #1 (a), (b) and (c), # 2 (a), (b) and (c).
- Star Problem: Page 239, #4

The Star Problems are intended for students who are interested in challenging problems, they could substitute regular problems in the assignment.