165B — Midterm Exam

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Problem 1

In each case, write the principal part of the function at its isolated singular point and determine whether that point is a pole, a removable singular point, or an essential singular point:

(a) $\exp\left(\frac{1}{z^2}\right)$

slu.

$\exp\left(\frac{1}{z^2}\right)$ has an isolated singular point at 0, we will see why from it's series expansion about 0.

We know that, $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Thus,

$$\exp\left(\frac{1}{z^2}\right) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z^2}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!z^{2n}} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!z^{2n}}$$

The principal part is then,

$$\sum_{n=1}^{\infty} \frac{1}{n!z^{2n}}$$

Since the principal part has infinitely many non-zero coefficients. It follows that, 0 is an essential singularity of $\exp\left(\frac{1}{z^2}\right)$ ♦

(b) $\frac{z^3}{1-z}$

We can see that $\frac{z^3}{1-z}$ is not analytic at $z = 1$. Then we perform polynomial long division,

$$\begin{array}{r l}
 & -z^2 - z - 1 \\ \cline{2-2}
-z+1) & z^3 \\
 & -z^3 + z^2 \\ \cline{2-2}
 & z^2 \\
 & -z^2 + z \\ \cline{2-2}
 & z \\
 & -z + 1 \\ \cline{2-2}
 & 1
\end{array}$$

And we can conclude that,

$$\frac{z^3}{1-z} = -z^2 - z - 1 + \frac{1}{1-z}$$

So the principal part is,

$$\frac{1}{1-z}$$

So 1 is a simple pole ♦

(c) $\frac{\sin 2z}{z}$

slu.

$$\frac{\sin 2z}{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^n (2z)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} z^{2n}}{(2n+1)!}$$

Since $\sin 2z$ is analytic, but $\frac{1}{z}$ is not analytic at 0. 0 is the isolated singular point of $\frac{\sin 2z}{z}$, then we compute the following expansion.

$$\frac{\sin 2z}{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^n (2z)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} z^{2n}}{(2n+1)!}$$

Thus, the principal part is 0, therefore 0 is a removable singularity of $\frac{\sin 2z}{z}$ ◊

(d) $\frac{\cos z-1}{z^2}$

slu.

$\cos z-1$ is analytic, but but $\frac{1}{z^2}$ is not analytic at 0. 0 is the isolated singular point of $\frac{\cos z-1}{z^2}$, then we compute the following expansion.

$$\frac{\cos z-1}{z} = \frac{1}{z^2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}-1\right) = \frac{1}{z^2}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}-1\right) = \frac{1}{z^2}\left(\sum_{n=1}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}\right) = \sum_{n=1}^{\infty}\frac{(-1)^n z^{2n-2}}{(2n)!}$$

Since there are no negative powers, the principal part is 0, therefore 0 is a removable singularity of $\frac{\cos z-1}{z^2}$ $\quad\Diamond$

(e) $\frac{1}{(1-z)^3}$

slu. Since it's already in it's Laurent series representation we can see that 1 is a pole of order 3 of $\frac{1}{(1-z)^3}$ $\quad\Diamond$

Problem 2

Find

(a) The residue of $f_1(z)=\frac{\pi}{z-z^2}$ at $z=0$

slu. $f_1(z)=\frac{\pi}{z(1-z)}=\frac{\frac{\pi}{1-z}}{z} \implies \operatorname*{Res}_{z=0} f_1(z)=\frac{pi}{1-0}=\pi \quad \Diamond$

(b) The residue of $f_2(z)=z\cos\left(\frac{1}{z}\right)$ at $z=0$

slu.

$$f_2(z)=z\cos\left(\frac{1}{z}\right)=z\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{z}\right)^{2n}}{(2n)!}=z\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!z^{2n}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!z^{2n-1}}$$

$$\implies f_2(z)=\frac{(-1)^0}{0!z^{-1}}+\frac{(-1)^1}{2!z^{2-1}}+\sum_{n=}^{\infty}\frac{(-1)^n}{(2n)!z^{2n-1}}=z-\frac{1}{2z}+\sum_{n=}^{\infty}\frac{(-1)^n}{(2n)!z^{2n-1}}$$

$$\implies \operatorname*{Res}_{z=0} f_2(z)=-\frac{1}{2} \quad \blacklozenge$$

(c) The residue of $f_3(z)=\frac{z-\sin z}{2z}$ at $z=0$

slu. $f_3(z)=\frac{z-\sin z}{2z}=\frac{\frac{z-\sin z}{2}}{z}$, since $\frac{z-\sin z}{2}$ is analytic at $0 \implies \operatorname*{Res}_{z=0} f_3(z)=\frac{0-\sin 0}{2}=0 \quad \Diamond$

(d) A function f_4 with a simple pole at $z=0$ such that the residue of f_4 at $z=0$ is π.

slu. $f_4(z)=\frac{\pi}{z} \quad \Diamond$

(d) A function f_5 with a pole of order 3 at $z=0$ such that the residue of f_5 at $z=0$ is 17.

slu. $f_5(z)=\frac{17}{z}+\frac{27891}{z^3} \quad \Diamond$

Problem 3

Consider the integral
$$\int_C \frac{2z^3+3}{(z+1)(z^2+4)}dz$$
taken counterclockwise around the curve C

(a) Find the value of the integral when the curve C is the circle $|z+1|=2$

slu.

$$(z+1)(z^4+4)=0 \implies z+1=0 \text{ or } z^2+4=0$$
$$z+1=0 \implies z_0=-1$$
$$z^4+4=0 \implies z^2=-4=-1\cdot 4=4e^{i(\pi+2k\pi)},\ k\in\mathbb{Z} \implies z=\sqrt{4}e^{i(\frac{\pi}{2}+k\pi)}=\pm 2i$$
$$z^2+4=0 \implies z_1=2i, z_2=-2i$$
$$z_0=-1 \implies |-1+1|=0<2 \implies z_0\in C$$
$$z_1=2i \implies |2i+1|=|1+2i|=\sqrt{5}$$
$$4<5 \implies \sqrt{4}=2<\sqrt{5} \implies z_1\notin C$$
$$z_2=-2i \implies |-2i+1|=|1-2i|=\sqrt{5}>2 \implies z_2\notin C$$

Note, that finding all the zeros of z^2+4 gives us the following factorization,
$$z^4+4=(z-2i)(z+2i)$$
$$\text{Then } f(z)=\frac{2z^3+3}{(z+1)(z^2+4)}=\frac{2z^3+3}{(z+1)(z-2i)(z+2i)}$$
$$\implies \operatorname*{Res}_{z=-1} f(z)=\frac{2(-1)^3+3}{(-1-2i)(-1+2i)}=\frac{1}{5}$$

So Cauchy's residue theorem gives us that,
$$\int_C \frac{2z^3+3}{(z+1)(z^2+4)}dz=2\pi i\left(\frac{1}{5}\right)=\frac{2\pi}{5}i$$

(b) find the value of the integral when the curve C is the circle $|z|=4$

Since, $|\pm 2i|=2<4$, and $|-1|=1<4$, we need to compute a few more residues.

$$\begin{aligned}\implies \operatorname*{Res}_{z=2i} f(z)&=\frac{2(2i)^3+3}{(2i+1)(2i+2i)}=\frac{2(2i)^3+3}{(2i+1)2(2i)}=\frac{(2i)^2}{1+2i}-\frac{3i}{4(1+2i)}\\&=\frac{-4(1-2i)}{5}-\frac{3i(1-2i)}{20}=\frac{-16+32i-3i-6}{20}\\&=\frac{-22+29i}{20}=\frac{29}{20}i-\frac{11}{10}\end{aligned}$$

$$\begin{aligned}\implies \operatorname*{Res}_{z=-2i} f(z)&=\frac{2(-2i)^3+3}{(-2i+1)(-2i-2i)}=\frac{2(-2i)^3+3}{2(1-2i)(-2i)}=\frac{(-2i)^2}{1-2i}+\frac{3}{2(1-2i)(-2i)}\\&=\frac{(-2i)^2(1+2i)}{5}+\frac{3(1+2i)}{10(-2i)}=\frac{-4(1+2i)}{5}-\frac{3i(1+2i)}{10(-2)}=\frac{-16(1+2i)}{20}+\frac{3i(1+2i)}{20}\\&=\frac{-16(1+2i)+3i(1+2i)}{20}=\frac{(-16+3i)(1+2i)}{20}=\frac{(-16-6)+(3-32)i}{20}=\frac{-22-29i}{20}\\&=-\frac{29}{20}i-\frac{11}{10}\end{aligned}$$

So Cauchy's residue theorem gives us that,

$$\int_C \frac{2z^3+3}{(z+1)(z^4+4)}dz = 2\pi i\left(\frac{1}{5}-\frac{11}{10}+\frac{29}{20}i-\frac{11}{10}-\frac{29}{20}i\right)$$
$$= 2\pi i\left(\frac{2-22}{10}\right) = -4\pi i \quad \blacksquare$$

(c) Give a curve C such that the value of the integral is 0

slu.

Let C be the circle $|z-20|=1$

Since, $|1-20| = 19 > 1$, and $|\pm 2i - 20| = \sqrt{404} > \sqrt{400} = 20 > 1$. So the interior of C, not C contain any of the singular points of the integrand, thus the integrand is analytic in and on the circle of radius 1 centered at 20. Therefore,

$$\int_C \frac{2z^3+3}{(z+1)(z^4+4)}dz = 0 \quad \diamondsuit$$

Problem 4

Problem 4: Show that the image of the right half plane Re$(z) > \frac{1}{2}$, under the mapping $w = \frac{1}{z}$, is the disk $|w-1| < 1$.

slu.

Let z in the right half plane Re$(z) > \frac{1}{2}$. Then $\exists x, y \in \mathbb{R} : z = x + iy : x > \frac{1}{2}$.

If $w = u + iv$, then

$$w = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} \implies u = \frac{x}{x^2+y^2} \text{ and } v = \frac{-y}{x^2+y^2}$$

$$w = \frac{1}{z} \implies z = \frac{1}{w} \implies x = \frac{u}{u^2+v^2} \text{ and } y = \frac{-v}{u^2+v^2}$$

The general equation for circles an lines in the plane is as follows,

$$A(x^2+y^2) + Bx + Cy + D = 0$$

And it gets mapped by w to the u, v plane into the following equation,

$$D(u^2+v^2) + Bu - Cv + A = 0$$

First consider the boundary of the right half plane Re$(z) > \frac{1}{2}$. Which is the line $x = \frac{1}{2}$

That is, $A = 0, B = 1, C = 0, D = -\frac{1}{2}$ so the image is,

$$-\frac{1}{2}(u^2+v^2) + u - 0v + 0 = 0 \implies u = \frac{1}{2}(u^2+v^2) \implies 2u = u^2+v^2$$
$$\implies 0 = u^2 - 2u + v^2 \implies 1 = u^2 - 2u + 1 + v^2 \implies 1 = (u-1)^2 + v^2$$

Which is the circle of radius 1 centered at 1.

Now, if z is in the right half plane Re$(z) > \frac{1}{2}$. $\forall z :$ Re$(z) > \frac{1}{2}$, $\exists \varepsilon > 0 : z$ lies in the line $x = \frac{1}{2} + \varepsilon$. So the image under w lies in the circle,

$$u = \left(\frac{1}{2}+\varepsilon\right)(u^2+v^2) = \frac{1+\varepsilon}{2}(u^2+v^2) \implies u^2+v^2 = \frac{2}{1+\varepsilon}u \implies u^2 - \frac{2}{1+\varepsilon}u + v^2 = 0$$
$$\implies u^2 - \frac{2}{1+\varepsilon}u + \frac{1}{(1+\varepsilon)^2} + v^2 = \frac{1}{(1+\varepsilon)^2} \implies \left(u - \frac{1}{1+\varepsilon}\right)^2 + v^2 = \frac{1}{(1+\varepsilon)^2}$$

Which is the circle of radius $\frac{1}{1+\varepsilon}$ centered at $\frac{1}{1+\varepsilon}$. That is the circle $|w - \frac{1}{1+\varepsilon}| = \frac{1}{1+\varepsilon}$. The circle $|w - \frac{1}{1+\varepsilon}| = \frac{1}{1+\varepsilon}$ has the following parametric representation,

$$w = \frac{1}{1+\varepsilon} + \frac{1}{1+\varepsilon}\exp i\theta \quad (0 \le \theta < 2\pi)$$

We want to show that the distance from $\frac{1}{1+\varepsilon} + \frac{1}{1+\varepsilon}\exp(i\theta)$ to 1 is less than or equal to 1.

$$\left|\frac{1}{1+\varepsilon} + \frac{1}{1+\varepsilon}\exp(i\theta) - 1\right| = \left|\frac{1}{1+\varepsilon} - 1 + \frac{1}{1+\varepsilon}(\cos\theta + i\sin\theta)\right| = \left|\frac{1}{1+\varepsilon} - 1 + \frac{1}{1+\varepsilon}\cos\theta + i\frac{1}{1+\varepsilon}\sin\theta\right|$$

$$= \left|\frac{1}{1+\varepsilon}(1+\cos\theta) - 1 + i\frac{1}{1+\varepsilon}\sin\theta\right| = \sqrt{\left(\frac{1}{1+\varepsilon}(1+\cos\theta) - 1\right)^2 + \frac{1}{(1+\varepsilon)^2}\sin^2\theta}$$

$$= \sqrt{\frac{1}{(1+\varepsilon)^2}(1+\cos\theta)^2 - 2\frac{1}{1+\varepsilon}(1+\cos\theta) + 1 + \frac{1}{(1+\varepsilon)^2}\sin^2\theta}$$

$$= \sqrt{\frac{1}{(1+\varepsilon)^2}(1+2\cos\theta+\cos^2\theta) - 2\frac{1}{1+\varepsilon}(1+\cos\theta) + 1 + \frac{1}{(1+\varepsilon)^2}\sin^2\theta}$$

$$= \sqrt{\frac{1}{(1+\varepsilon)^2} + \frac{1}{(1+\varepsilon)^2}(1+2\cos\theta) - 2\frac{1}{1+\varepsilon}(1+\cos\theta) + 1}$$

$$= \sqrt{\frac{1}{(1+\varepsilon)^2} + \frac{1}{(1+\varepsilon)^2} + \frac{2}{(1+\varepsilon)^2}\cos\theta - \frac{2}{1+\varepsilon} - \frac{2}{1+\varepsilon}\cos\theta + 1}$$

$$= \sqrt{\frac{2}{(1+\varepsilon)^2} + \left(\frac{2}{(1+\varepsilon)^2} - \frac{2}{1+\varepsilon}\right)\cos\theta + 1 - \frac{2}{1+\varepsilon}}$$

$$= \frac{\sqrt{2 + (2 - 2(1+\varepsilon))\cos\theta + (1+\varepsilon)^2 - 2(1+\varepsilon)}}{1+\varepsilon} = \frac{\sqrt{2 + (2-2-2\varepsilon)\cos\theta + 1 + 2\varepsilon + \varepsilon^2 - 2 - 2\varepsilon}}{1+\varepsilon}$$

$$= \frac{\sqrt{(-2\varepsilon)\cos\theta + 1 + \varepsilon^2}}{1+\varepsilon} = \frac{\sqrt{1 - 2\varepsilon\cos\theta + \varepsilon^2}}{1+\varepsilon}$$

Since $-1 < \cos\theta < 1$ when $0 \le \theta < 2\pi$, it follows that,

$$0 < \frac{\sqrt{1 - 2\varepsilon\cos\theta + \varepsilon^2}}{1+\varepsilon} \le \frac{\sqrt{\varepsilon^2 + 2\varepsilon + 1}}{1+\varepsilon} = \frac{\sqrt{(1+\varepsilon)^2}}{1+\varepsilon} = \frac{1+\varepsilon}{1+\varepsilon} = 1$$

The distance from the centers is 1 only when $\theta = \pi$. So, 0 is a common point of the circles. Otherwise, the distance is less than 1.

So the circle $|w - \frac{1}{1+\varepsilon}| = \frac{1}{1+\varepsilon}$ lies inside the circle $|w - 1| = 1$. Since every $w(z)$: Re z, lies in the image of one of such lines. Then the right half plane Re$(z) > \frac{1}{2}$ lies inside the disk $|w-1| < 1$ ■

Extra Credit Problem

Show that all four zeros of the polynomial $g(z) = z^4 - 7z - 1$ lie in the disk $|z| < 2$

slu. Let C be the circle $|z| = 2$. Let $p(z) = z^4$, $q(z) = -7z - 1$. Notice, $g = p + q$ then if $z \in C$,

$$|p(z)| < |z|^4 = 2^4 = 16 \text{ and } |q(z)| < 7|z| + 1 = 2 \cdot 7 + 1 = 15$$

$$\implies \forall z \in \mathbb{C}, |q(z)| < |p(z)|$$

Both $p(z)$, and $q(z)$ are polynomials, so they're entire. So they're analytic in the closed disk $|z|leq2$. Therefore by Rouché's theorem, $p(z)$ and $g(z)$ have the same number of zeros counting multiplicities inside C. So, all four zeros of g lie in the disk $|z| < 2 \quad \Diamond$

Extra Credit STAR PROBLEM

Show that the parabola $2x = 1 - y^2$ is mapped onto the cardioid $\rho = 1 + \cos\phi$ by the reciprocal transformation $w = \frac{1}{z}$.

slu.

Let $z \in \mathbb{C} : \exists x, y \in \mathbb{R} : z = x + iy$, then

$$x = \frac{z + \bar{z}}{2} \text{ and } y = \frac{z - \bar{z}}{2i}$$

$$\implies y^2 = \left(\frac{z - \bar{z}}{2i}\right)^2 = \frac{z^2 - 2z\bar{z} + \bar{z}^2}{-4}$$

$$\implies -y^2 = -\left(\frac{z - \bar{z}}{2i}\right)^2$$

$$2x = 1 - y^2 \iff 2\frac{z + \bar{z}}{2} = 1 - \left(\frac{z - \bar{z}}{2i}\right)^2$$

Let $z = \rho e^{i\phi}$, $(0 \leq \phi < 2\pi) \implies \bar{z} = \rho e^{-i\phi}$ and plug in,

$$\implies 2\frac{\rho e^{i\phi} + \rho e^{-i\phi}}{2} = 1 - \left(\frac{\rho e^{i\phi} - \rho e^{-i\phi}}{2i}\right)^2 \implies 2\rho\frac{e^{i\phi} + e^{-i\phi}}{2} = 1 - \rho^2\left(\frac{e^{i\phi} - e^{-i\phi}}{2i}\right)^2$$

$$\implies 2\rho\cos\phi = 1 - \rho^2\sin^2\phi \implies 2\rho\cos\phi = 1 - \rho^2(1 - \cos^2\phi)$$

$$\implies \rho^2(1 - \cos^2\phi) + 2\rho\cos\phi = 1 \implies \rho^2 + 2\rho\cos\phi - \rho^2\cos^2\phi = 1$$

$$\implies (\rho - \cos\phi)^2 = 1 \implies \rho - \cos\phi = 1$$

$$\implies \rho = 1 + \cos\phi \quad \blacksquare$$

Appendix

I misread the exam and solved the following problem, I'll leave it here for your amusement.

Consider the integral

$$\int_C \frac{2z^3+3}{(z+1)(z^4+4)}dz$$

taken counterclockwise around the curve C

(a) Find the value of the integral when the curve C is the circle $|z+1|=2$

slu.

$$(z+1)(z^4+4)=0 \implies z+1=0 \text{ or } z^4+4=0$$
$$z+1=0 \implies z_0=-1$$
$$z^4+4=0 \implies z^4=-4=-1\cdot 4=4e^{i(\pi+2k\pi)},\ k\in\mathbb{Z} \implies z=\sqrt{2}e^{i(\frac{\pi}{4}+\frac{k\pi}{2})}=\pm(1\pm i)$$
$$z^4+4=0 \implies z_1=1+i, z_2=-1+i, z_3=-1-i, z_4=1-i$$
$$z_0=-1 \implies |-1+1|=0<2 \implies z_0\in C$$
$$z_1=1+i \implies |1+i+1|=|2+i|=\sqrt{5}$$
$$4<5 \implies \sqrt{4}=2<\sqrt{5} \implies z_1\notin C$$
$$z_2=-1+i \implies |-1+i+1|=|i|=1<2 \implies z_2\in C$$
$$z_3=-1-i \implies |-1-i+1|=|-i|=1<2 \implies z_3\in C$$
$$z_4=1-i \implies |1-i+1|=|2-i|=\sqrt{5}>2 \implies z_4\notin C$$

Note, that finding all the zeros of z^4+4 gives us the following factorization,

$$z^4+4=(z-(1+i))(z-(-1+i))(z-(-1-i))(z-(1-i))$$

Then $f(z)=\dfrac{2z^3+3}{(z+1)(z^4+4)}=\dfrac{2z^3+3}{(z+1)(z-(1+i))(z-(-1+i))(z-(-1-i))(z-(1-i))}$

$$\begin{aligned}\implies \operatorname*{Res}_{z=-1} f(z) &= \frac{2(-1)^3+3}{(-1-(1+i))(-1-(-1+i))(-1-(-1-i))(-1-(1-i))}\\ &= \frac{1}{(-2-i)(-i)(i)(-2+i)} = \frac{1}{((-2)^2-i^2)}\\ &= \frac{1}{5}\end{aligned}$$

$$\begin{aligned}\implies \operatorname*{Res}_{z=-1+i} f(z) &= \frac{2(-1+i)^3+3}{(-1+i+1)(-1+i-(1+i))(-1+i-(-1-i))(-1+i-(1-i))}\\ &= \frac{2(-1+i)^3+3}{i(-2)(2i)(-2+2i)} = \frac{2(-1+i)(-2i)+3}{i(-2)(2i)(-2+2i)} = \frac{2(2i+2)+3}{i(-2)(2i)(-2+2i)}\\ &= \frac{7+4i}{i(-2)(2i)(-2+2i)} = \frac{7+4i}{4(-2+2i)} = \frac{7+4i}{-8+8i} = \frac{(7+4i)(-8-8i)}{128}\\ &= \frac{-56+32+(-32-56)i}{128} = \frac{-24-88i}{128} = -\frac{24}{128}-\frac{88}{128}i\\ &= -\frac{3}{16}-\frac{11}{16}i\end{aligned}$$

$$\implies \operatorname*{Res}_{z=-1-i} f(z) = \frac{2(-1-i)^3+3}{(-1-i+1)(-1-i-(1+i))(-1-i-(-1+i))(-1-i-(1-i))}$$
$$= \frac{2(-1-i)^3+3}{-i(-2-2i)(-2i)(-2)} = \frac{2(-1-i)^3+3}{i(2+2i)(2i)(2)} = \frac{2(-1-i)^3+3}{-4(2+2i)} = \frac{2(-1-i)^3+3}{-8-8i}$$
$$= \frac{2(-1-i)^3+3}{8(-1-1i)} = \frac{(-1-i)^2}{4} + \frac{3}{8(-1-i)} = \frac{2i}{4} + \frac{3(-1+i)}{16} = \frac{i}{2} + \frac{3(-1+i)}{16} = -\frac{3}{16} + \frac{11}{16}i$$

So Cauchy's residue theorem gives us that,

$$\int_C \frac{2z^3+3}{(z+1)(z^4+4)}dz = 2\pi i\left(\frac{1}{5} - \frac{3}{16} - \frac{11}{16}i - \frac{3}{16} + \frac{11}{16}i\right)$$
$$= 2\pi i\left(\frac{1}{5} - \frac{6}{16}\right) = 2\pi i\left(\frac{16-30}{80}\right)$$
$$= \pi i\left(\frac{-14}{40}\right)$$
$$= -\frac{7\pi}{20}i$$

(b) find the value of the integral when the curve C is the circle $|z| = 4$

Since, $|\pm(1\pm i)| = \sqrt{2} < 4$, and $|-1| < 4$, we need to compute a few more residues.

$$\implies \operatorname*{Res}_{z=1+i} f(z) = \frac{2(1+i)^3+3}{(1+i+1)(1+i-(-1+i))(1+i-(-1-i))(1+i-(1-i))}$$
$$= \frac{2(1+i)^3+3}{(2+i)(2)(2+2i)(2i)} = \frac{(1+i)^2}{4i(2+i)} - \frac{3i}{8(2+i)(1+i)} = \frac{2i}{4i(2+i)} - \frac{3i}{8(1+3i)}$$
$$= \frac{2-i}{10} - \frac{3i(1-3i)}{80} = \frac{2-i}{10} - \frac{9+3i}{80} = \frac{16-8i-(9+3i)}{80}$$
$$= \frac{7}{80} - \frac{11}{80}i$$

$$\implies \operatorname*{Res}_{z=1-i} f(z) = \frac{2(1-i)^3+3}{(1-i+1)(1-i-(-1+i))(1-i-(-1-i))(1-i-(1+i))}$$
$$= \frac{2(1-i)^3+3}{(2-i)(2-2i)(2)(-2i)} = \frac{2(1-i)^3+3}{(2-i)(1-i)(-8i)} = \frac{-2i}{(2-i)(-4i)} + \frac{3i}{8(2-i)(1-i)}$$
$$= \frac{1}{2(2-i)} + \frac{3i}{8(1-3i)} = \frac{2+i}{10} + \frac{3i(1+3i)}{80} = \frac{2+i}{10} - \frac{9-3i}{80} = \frac{16+8i-9+3i}{80}$$
$$= \frac{7}{80} + \frac{11}{80}i$$

So Cauchy's residue theorem gives us that,

$$\int_C \frac{2z^3+3}{(z+1)(z^4+4)}dz = 2\pi i\left(\frac{16-30}{80} + \frac{7}{80} - \frac{11}{80}i + \frac{7}{80} + \frac{11}{80}i\right)$$
$$= 2\pi i\left(\frac{16+14-30}{80}\right) = 0 \quad \blacksquare$$

(c) Give a curve C such that the value of the integral is 0

slu. Let C be the circle $|z| = 4$ $\quad\Diamond$