4. Use Poisson's integral formula for the upper half plane to conclude that

$$\phi(x, y) = e^{-y}\sin x = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\sin t\, dt}{(x - t)^2 + y^2}.$$

5. Show that the function $\phi(x, y)$ given by Poisson's integral formula is harmonic by applying Leibniz's rule, which permits us to write

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} U(t)\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{y}{(x - t)^2 + y^2}\right] dt.$$

6. Let $U(t)$ be a real-valued function that satisfies the conditions for Poisson's integral formula for the upper half plane. If $U(t)$ is an even function, that is, $U(-t) = U(t)$, then show that the harmonic function $\phi(x, y)$ has the property $\phi(-x, y) = \phi(x, y)$.
7. Let $U(t)$ be a real-valued function that satisfies the conditions for Poisson's integral formula for the upper half plane. If $U(t)$ is an odd function, that is, $U(-t) = -U(t)$, then show that the harmonic function $\phi(x, y)$ has the property $\phi(-x, y) = -\phi(x, y)$.
8. Write a report on the Dirichlet problem and include some applications. Resources include bibliographical items 70, 71, 76, 77, 85, 98, 135, and 138.

10.4 Two-Dimensional Mathematical Models

We now turn our attention to problems involving steady state heat flow, electrostatics, and ideal fluid flow that can be solved by conformal mapping techniques. The method uses conformal mapping to transform a region in which the problem is posed to one in which the solution is easy to obtain. Since our solutions will involve only two independent variables, x and y, we first mention a basic assumption needed for the validity of the model.

The physical problems we just mentioned are real-world applications and involve solutions in three-dimensional Cartesian space. Such problems generally would involve the Laplacian in three variables and the divergence and curl of three-dimensional vector functions. Since complex analysis involves only x and y, we consider the special case in which the solution does not vary with the coordinate along the axis perpendicular to the xy plane. For steady state heat flow and electrostatics this assumption will mean that the temperature T, or the potential V, varies only with x and y. For the flow of ideal fluids this means that the fluid motion is the same in any plane that is parallel to the z plane. Curves drawn in the z plane are to be interpreted as cross sections that correspond to infinite cylinders perpendicular to the z plane. Since an infinite cylinder is the limiting case of a "long" physical cylinder, the mathematical model that we present is valid provided that the three-dimensional problem involves a physical cylinder long enough that the effects at the ends can be reasonably neglected.

In Sections 10.1 and 10.2 we learned how to obtain solutions $\phi(x, y)$ for harmonic functions. For applications it is important to consider the family of level curves

(1) $\{\phi(x, y) = K_1 : K_1 \text{ is a real constant}\}$

and the conjugate harmonic function $\psi(x, y)$ and its family of level curves

(2) $\{\psi(x, y) = K_2: K_2 \text{ is a real constant}\}.$

It is convenient to introduce the terminology *complex potential* for the analytic function

(3) $F(z) = \phi(x, y) + i\psi(x, y).$

The following result regarding the orthogonality of the above mentioned families of level curves will be used in developing ideas concerning the physical applications.

Theorem 10.4 (Orthogonal Families of Level Curves) *Let $\phi(x, y)$ be harmonic in a domain D. Let $\psi(x, y)$ be the harmonic conjugate, and let $F(z) = \phi(x, y) + i\psi(x, y)$ be the complex potential. Then the two families of level curves given in (1) and (2), respectively, are orthogonal in the sense that if (a, b) is a point common to the two curves $\phi(x, y) = K_1$ and $\psi(x, y) = K_2$, and if $F'(a + ib) \neq 0$, then these two curves intersect orthogonally.*

Proof Since $\phi(x, y) = K_1$ is an implicit equation of a plane curve, the gradient vector grad ϕ, evaluated at (a, b), is perpendicular to the curve at (a, b). This vector is given by

(4) $\mathbf{N}_1 = \phi_x(a, b) + i\phi_y(a, b).$

Similarly, the vector $\mathbf{N}_2$ defined by

(5) $\mathbf{N}_2 = \psi_x(a, b) + i\psi_y(a, b)$

is orthogonal to the curve $\psi(x, y) = K_2$ at (a, b). Using the Cauchy-Riemann equations, $\phi_x = \psi_y$ and $\phi_y = -\psi_x$, we have

(6) $$\begin{aligned}\mathbf{N}_1 \cdot \mathbf{N}_2 &= \phi_x(a, b)[\psi_x(a, b)] + \phi_y(a, b)[\psi_y(a, b)] \\ &= \phi_x(a, b)[-\phi_y(a, b)] + \phi_y(a, b)[\phi_x(a, b)] = 0.\end{aligned}$$

In addition, since $F'(a + ib) \neq 0$, we have

(7) $\phi_x(a, b) + i\psi_x(a, b) \neq 0.$

The Cauchy-Riemann equations and inequality (7) imply that both $\mathbf{N}_1$ and $\mathbf{N}_2$ are nonzero. Therefore equation (6) implies that $\mathbf{N}_1$ is perpendicular to $\mathbf{N}_2$, and hence the curves are orthogonal.

The complex potential $F(z) = \phi(x, y) + i\psi(x, y)$ has many physical interpretations. Suppose, for example, that we have solved a problem in steady state temperatures; then a similar problem with the same boundary conditions in electrostatics is obtained by interpreting the isothermals as equipotential curves and the heat flow lines as flux lines. This implies that heat flow and electrostatics correspond directly.

Or suppose we have solved a fluid flow problem; then an analogous problem in heat flow is obtained by interpreting the equipotentials as isothermals and streamlines as heat flow lines. Various interpretations of the families of level curves given in expressions (1) and (2) and correspondences between families are summarized in Table 10.1.

Table 10.1 Interpretations for Level Curves

Physical Phenomenon	$\phi(x, y)$ = constant	$\psi(x, y)$ = constant
Heat flow	Isothermals	Heat flow lines
Electrostatics	Equipotential curves	Flux lines
Fluid flow	Equipotentials	Streamlines
Gravitational field	Gravitational potential	Lines of force
Magnetism	Potential	Lines of force
Diffusion	Concentration	Lines of flow
Elasticity	Strain function	Stress lines
Current flow	Potential	Lines of flow

10.5 Steady State Temperatures

In the theory of heat conduction the assumption is made that heat flows in the direction of decreasing temperature. We also assume that the time rate at which heat flows across a surface area is proportional to the component of the temperature gradient in the direction perpendicular to the surface area. If the temperature $T(x, y)$ does not depend on time, then the heat flow at the point (x, y) is given by the vector

$$(1) \quad \mathbf{V}(x, y) = -K \operatorname{grad} T(x, y) = -K[T_x(x, y) + iT_y(x, y)],$$

where K is the thermal conductivity of the medium and is assumed to be constant. If Δz denotes a straight line segment of length Δs, then the amount of heat flowing across the segment per unit of time is

$$(2) \quad \mathbf{V} \cdot \mathbf{N}\, \Delta s,$$

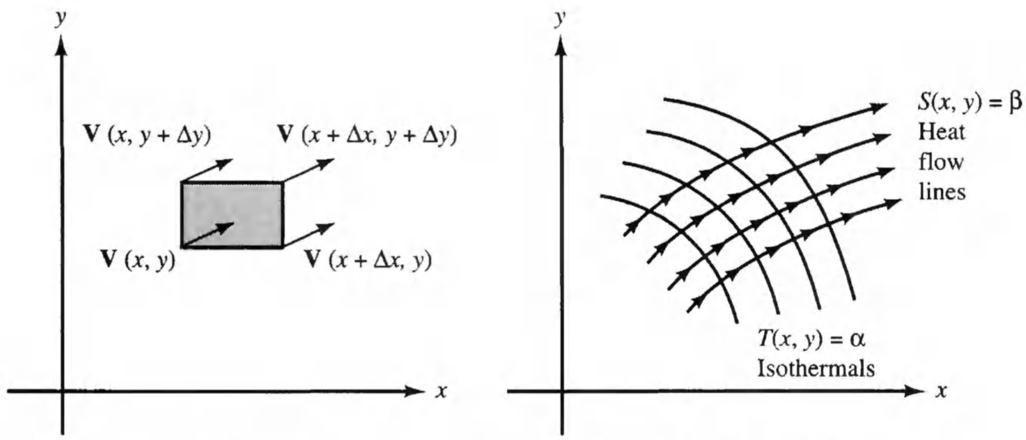
where $\mathbf{N}$ is a unit vector perpendicular to the segment.

If we assume that no thermal energy is created or destroyed within the region, then the net amount of heat flowing through any small rectangle with sides of length Δx and Δy is identically zero (see Figure 10.16(a)). This leads to the conclusion that $T(x, y)$ is a harmonic function. The following heuristic argument is often used to suggest that $T(x, y)$ satisfies Laplace's equation. Using expression (2), we find that the amount of heat flowing out of the right edge of the rectangle in Figure 10.16(a) is approximately

$$\begin{aligned}(3) \quad \mathbf{V} \cdot \mathbf{N}_1\, \Delta s_1 &= -K[T_x(x + \Delta x, y) + iT_y(x + \Delta x, y)] \cdot (1 + 0i)\, \Delta y \\ &= -KT_x(x + \Delta x, y)\, \Delta y,\end{aligned}$$

and the amount of heat flowing out of the left edge is

(4) $\quad \mathbf{V} \cdot \mathbf{N}_2 \, \Delta s_2 = -K[T_x(x, y) + iT_y(x, y)] \cdot (-1 + 0i) \, \Delta y = KT_x(x, y) \, \Delta y.$



(a) The direction of heat flow. (b) Heat flow lines and isothermals.

FIGURE 10.16 Steady state temperatures.

If we add the contributions in equations (3) and (4), the result is

(5) $$-K\left[\frac{T_x(x + \Delta x, y) - T_x(x, y)}{\Delta x}\right] \Delta x \, \Delta y \approx -KT_{xx}(x, y) \, \Delta x \, \Delta y.$$

In a similar fashion it is found that the contribution for the amount of heat flowing out of the top and bottom edges is

(6) $$-K\left[\frac{T_y(x, y + \Delta y) - T_y(x, y)}{\Delta y}\right] \Delta x \, \Delta y \approx -KT_{yy}(x, y) \, \Delta x \, \Delta y.$$

Adding the quantities in equations (5) and (6), we find that the net heat flowing out of the rectangle is approximated by the equation

(7) $\quad -K[T_{xx}(x, y) + T_{yy}(x, y)] \, \Delta x \, \Delta y = 0,$

which implies that $T(x, y)$ satisfies Laplace's equation and is a harmonic function.

If the domain in which $T(x, y)$ is defined is simply connected, then a conjugate harmonic function $S(x, y)$ exists, and

(8) $\quad F(z) = T(x, y) + iS(x, y)$

is an analytic function. The curves $T(x, y) = K_1$ are called *isothermals* and are lines connecting points of the same temperature. The curves $S(x, y) = K_2$ are called the *heat flow lines*, and one can visualize the heat flowing along these curves from points of higher temperature to points of lower temperature. The situation is illustrated in Figure 10.16(b).

Boundary value problems for steady state temperatures are realizations of the Dirichlet problem where the value of the harmonic function $T(x, y)$ is interpreted as the temperature at the point (x, y).

EXAMPLE 10.14 Suppose that two parallel planes are perpendicular to the z plane and pass through the horizontal lines $y = a$ and $y = b$ and that the temperature is held constant at the values $T(x, a) = T_1$ and $T(x, b) = T_2$, respectively, on these planes. Then T is given by

$$(9) \quad T(x, y) = T_1 + \frac{T_2 - T_1}{b - a}(y - a).$$

Solution It is reasonable to make the assumption that the temperature at all points on the plane passing through the line $y = y_0$ is constant. Hence $T(x, y) = t(y)$, where $t(y)$ is a function of y alone. Laplace's equation implies that $t''(y) = 0$, and an argument similar to that in Example 10.1 will show that the solution $T(x, y)$ has the form given in equation (9).

The isothermals $T(x, y) = \alpha$ are easily seen to be horizontal lines. The conjugate harmonic function is

$$S(x, y) = \frac{T_1 - T_2}{b - a} x,$$

and the heat flow lines $S(x, y) = \beta$ are vertical segments between the horizontal lines. If $T_1 > T_2$, then the heat flows along these segments from the plane through $y = a$ to the plane through $y = b$ as illustrated in Figure 10.17.

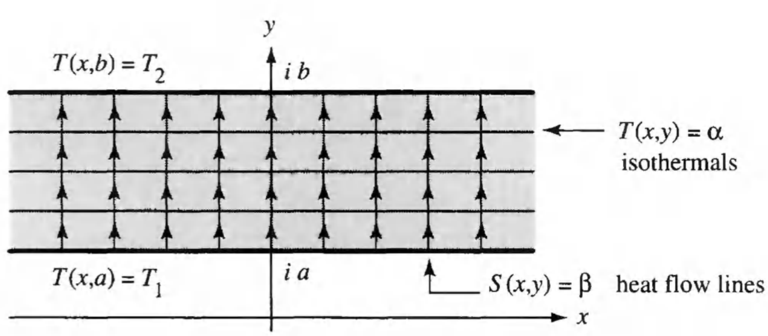


FIGURE 10.17 The temperature between parallel planes where $T_1 > T_2$.

EXAMPLE 10.15 Find the temperature $T(x, y)$ at each point in the upper half plane $\mathrm{Im}(z) > 0$ if the temperature along the x axis satisfies

$$(10) \quad T(x, 0) = T_1 \quad \text{for } x > 0 \quad \text{and} \quad T(x, 0) = T_2 \quad \text{for } x < 0.$$

Solution Since $T(x, y)$ is a harmonic function, this is an example of a Dirichlet problem. From Example 10.2 it follows that the solution is

$$(11) \quad T(x, y) = T_1 + \frac{T_2 - T_1}{\pi} \operatorname{Arg} z.$$

The isothermals $T(x, y) = \alpha$ are rays emanating from the origin. The conjugate harmonic function is $S(x, y) = (1/\pi)(T_1 - T_2) \ln|z|$, and the heat flow lines $S(x, y) = \beta$ are semicircles centered at the origin. If $T_1 > T_2$, then the heat flows counterclockwise along the semicircles as shown in Figure 10.18.

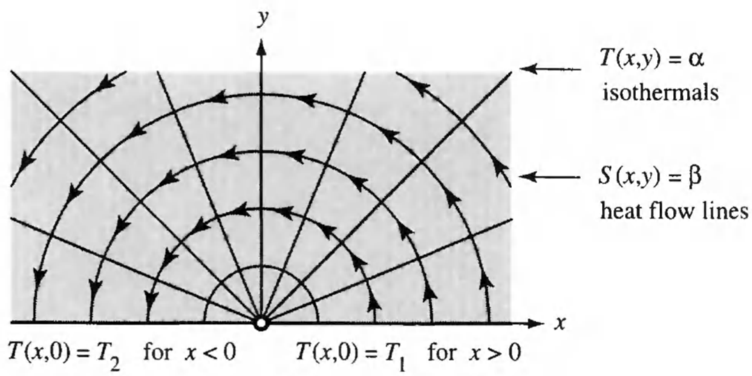


FIGURE 10.18 The temperature $T(x, y)$ in the upper half plane where $T_1 > T_2$.

EXAMPLE 10.16 Find the temperature $T(x, y)$ at each point in the upper half-disk H: $\text{Im}(z) > 0$, $|z| < 1$ if the temperature at points on the boundary satisfies

$$(12) \quad \begin{aligned} T(x, y) &= 100 \quad \text{for } z = e^{i\theta}, 0 < \theta < \pi, \\ T(x, 0) &= 50 \quad \text{for } -1 < x < 1. \end{aligned}$$

Solution As discussed in Example 10.9, the function

$$(13) \quad u + iv = \frac{i(1 - z)}{1 + z} = \frac{2y}{(x + 1)^2 + y^2} + i\frac{1 - x^2 - y^2}{(x + 1)^2 + y^2}$$

is a one-to-one conformal mapping of the half-disk H onto the first quadrant Q: $u > 0$, $v > 0$. The conformal map (13) gives rise to a new problem of finding the temperature $T^*(u, v)$ that satisfies the boundary conditions

$$(14) \quad T^*(u, 0) = 100 \quad \text{for } u > 0 \quad \text{and} \quad T^*(0, v) = 50 \quad \text{for } v > 0.$$

If we use Example 10.2, the harmonic function $T^*(u, v)$ is given by

$$(15) \quad T^*(u, v) = 100 + \frac{50 - 100}{\pi/2} \text{Arg } w = 100 - \frac{100}{\pi} \text{Arctan } \frac{v}{u}.$$

Substituting the expressions for u and v in mapping (13) into equation (15) yields the desired solution

$$T(x, y) = 100 - \frac{100}{\pi} \operatorname{Arctan} \frac{1 - x^2 - y^2}{2y}.$$

The isothermals $T(x, y) =$ constant are circles that pass through the points ± 1 as shown in Figure 10.19.

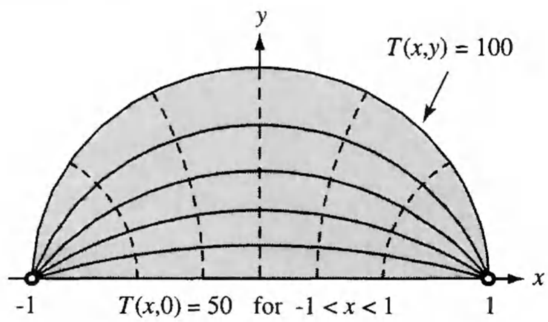


FIGURE 10.19 The temperature $T(x, y)$ in a half-disk.

We now turn our attention to the problem of finding the steady state temperature function $T(x, y)$ inside the simply connected domain D whose boundary consists of three adjacent curves C_1, C_2, and C_3, where $T(x, y) = T_1$ along C_1, $T(x, y) = T_2$ along C_2, and the region is insulated along C_3. Zero heat flowing across C_3 implies that

(16) $\quad \mathbf{V}(x, y) \cdot \mathbf{N}(x, y) = -K\mathbf{N}(x, y) \cdot \operatorname{grad} T(x, y) = 0,$

where $\mathbf{N}(x, y)$ is perpendicular to C_3. This means that the direction of heat flow must be parallel to this portion of the boundary. In other words, C_3 must be part of a heat flow line $S(x, y) =$ constant and the isothermals $T(x, y) =$ constant intersect C_3 orthogonally.

This problem can be solved if we can find a conformal mapping

(17) $\quad w = f(z) = u(x, y) + iv(x, y)$

from D onto the semi-infinite strip G: $0 < u < 1$, $v > 0$ so that the image of the curve C_1 is the ray $u = 0$, $v > 0$; the image of the curve C_2 is the ray given by $u = 1$, $v > 0$; and the thermally insulated curve C_3 is mapped onto the thermally insulated segment $0 < u < 1$ of the u axis, as shown in Figure 10.20.

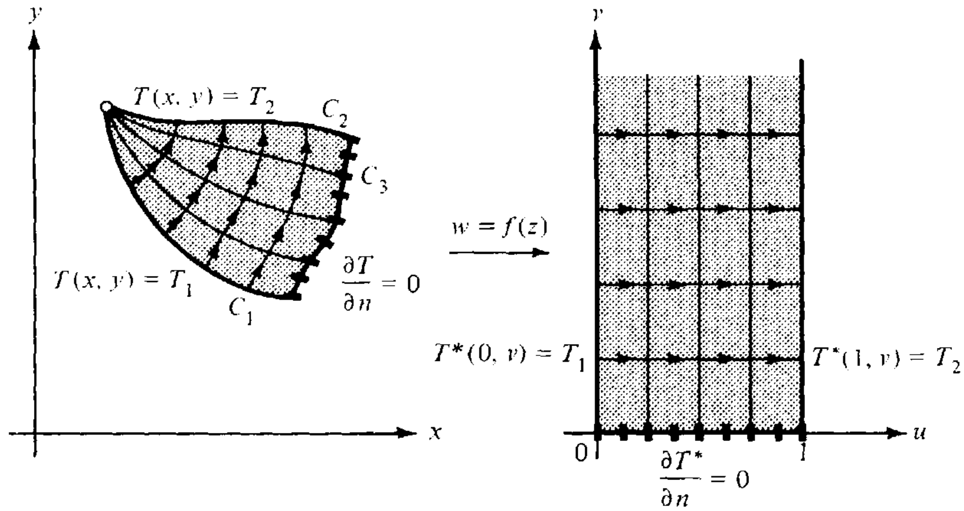


FIGURE 10.20 Steady state temperatures with one boundary portion insulated.

The new problem in G is to find the steady state temperature function $T^*(u, v)$ so that along the rays we have the boundary values

(18) $T^*(0, v) = T_1$ for $v > 0$ and $T^*(1, v) = T_2$ for $v > 0$.

The condition that a segment of the boundary is insulated can be expressed mathematically by saying that the normal derivative of $T^*(u, v)$ is zero. That is,

(19) $$\frac{\partial T^*}{\partial n} = T_v^*(u, 0) = 0$$

where n is a coordinate measured perpendicular to the segment.

It is easy to verify that the function

(20) $T^*(u, v) = T_1 + (T_2 - T_1)u$

satisfies the conditions (19) and (20) for the region G. Therefore using (17), we find that the solution in D is

(21) $T(x, y) = T_1 + (T_2 - T_1)u(x, y)$.

The isothermals $T(x, y) =$ constant, and their images under $w = f(z)$ are illustrated in Figure 10.20.

EXAMPLE 10.17 Find the steady state temperature $T(x, y)$ for the domain D consisting of the upper half plane $\text{Im}(z) > 0$ where $T(x, y)$ has the boundary conditions

(22) $T(x, 0) = 1$ for $x > 1$ and $T(x, 0) = -1$ for $x < -1$ and

$$\frac{\partial T}{\partial n} = T_y(x, 0) = 0 \quad \text{for } -1 < x < 1.$$

Solution The mapping $w = \text{Arcsin } z$ conformally maps D onto the semi-infinite strip $v > 0$, $-\pi/2 < u < \pi/2$, where the new problem is to find the steady state temperature $T^*(u, v)$ that has the boundary conditions

$$(23)\quad T^*\left(\frac{\pi}{2}, v\right) = 1 \quad \text{for } v > 0 \quad \text{and} \quad T^*\left(\frac{-\pi}{2}, v\right) = -1 \quad \text{for } v > 0$$
$$\text{and} \quad \frac{\partial T^*}{\partial n} = T_v^*(u, 0) = 0 \quad \text{for } \frac{-\pi}{2} < u < \frac{\pi}{2}.$$

By using the result of Example 10.1 it is easy to obtain the solution

$$(24)\quad T^*(u, v) = \frac{2}{\pi} u.$$

Therefore the solution in D is

$$(25)\quad T(x, y) = \frac{2}{\pi} \text{Re(Arcsin } z).$$

If an explicit solution is required, then we can use formula (14) in Section 9.4 to obtain

$$(26)\quad T(x, y) = \frac{2}{\pi} \arcsin\left[\frac{\sqrt{(x+1)^2 + y^2} - \sqrt{(x-1)^2 + y^2}}{2}\right],$$

where the real function $\arcsin t$ has range values satisfying $-\pi/2 < \arcsin t < \pi/2$, see Figure 10.21.

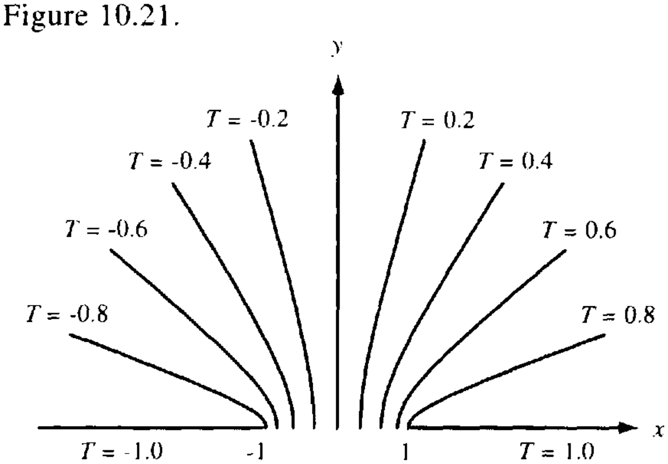


FIGURE 10.21 The temperature $T(x, y)$ with $T_y(x, 0) = 0$ for $-1 < x < 1$, and boundary values $T(x, 0) = -1$ for $x < -1$ and $T(x, 0) = 1$ for $x > 1$.

EXERCISES FOR SECTION 10.5

1. Show that $H(x, y, z) = 1/\sqrt{x^2 + y^2 + z^2}$ satisfies Laplace's equation $H_{xx} + H_{yy} + H_{zz} = 0$ in three-dimensional Cartesian space but that $h(x, y) = 1/\sqrt{x^2 + y^2}$ *does not* satisfy equation $h_{xx} + h_{yy} = 0$ in two-dimensional Cartesian space.
2. Find the temperature function $T(x, y)$ in the infinite strip bounded by the lines $y = -x$ and $y = 1 - x$ that satisfies the boundary values in Figure 10.22.

$$T(x, -x) = 25 \quad \text{for all } x,$$
$$T(x, 1 - x) = 75 \quad \text{for all } x.$$

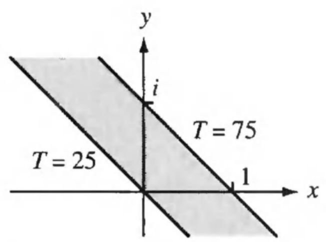


FIGURE 10.22 Accompanies Exercise 2.

3. Find the temperature function $T(x, y)$ in the first quadrant $x > 0$, $y > 0$ that satisfies the boundary values in Figure 10.23. *Hint*: Use $w = z^2$.

$$T(x, 0) = 10 \quad \text{for } x > 1,$$
$$T(x, 0) = 20 \quad \text{for } 0 < x < 1,$$
$$T(0, y) = 20 \quad \text{for } 0 \le y < 1,$$
$$T(0, y) = 10 \quad \text{for } y > 1.$$

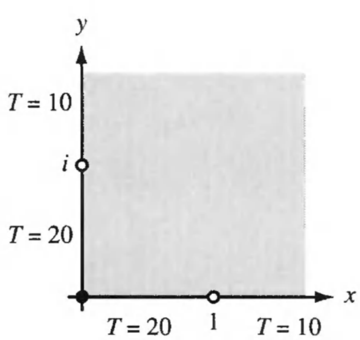


FIGURE 10.23 Accompanies Exercise 3.

4. Find the temperature function $T(x, y)$ inside the unit disk $|z| < 1$ that satisfies the boundary values in Figure 10.24. *Hint*: Use $w = i(1 - z)/(1 + z)$.

$$T(x, y) = 20 \quad \text{for } z = e^{i\theta}, 0 < \theta < \frac{\pi}{2},$$

$$T(x, y) = 60 \quad \text{for } z = e^{i\theta}, \frac{\pi}{2} < \theta < 2\pi.$$

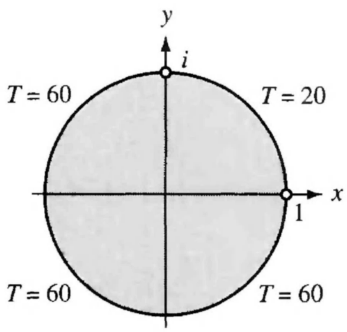


FIGURE 10.24 Accompanies Exercise 4.

5. Find the temperature function $T(x, y)$ in the semi-infinite strip $-\pi/2 < x < \pi/2$, $y > 0$ that satisfies the boundary values in Figure 10.25. *Hint*: Use $w = \sin z$.

$$T\left(\frac{\pi}{2}, y\right) = 100 \quad \text{for } y > 0,$$
$$T(x, y) = 0 \quad \text{for } \frac{-\pi}{2} < x < \frac{\pi}{2},$$
$$T\left(\frac{-\pi}{2}, y\right) = 100 \quad \text{for } y > 0.$$

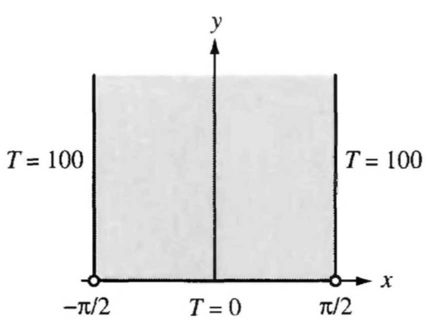


FIGURE 10.25 Accompanies Exercise 5.

6. Find the temperature function $T(x, y)$ in the domain $r > 1$, $0 < \theta < \pi$ that satisfies the boundary values in Figure 10.26. *Hint*: $w = i(1 - z)/(1 + z)$.

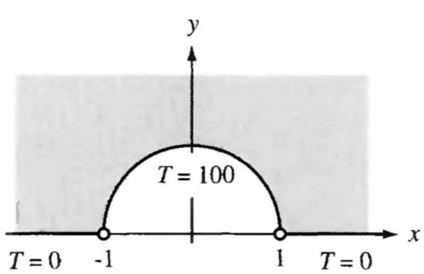


FIGURE 10.26 Accompanies Exercise 6.

$$T(x, 0) = 0 \quad \text{for } x > 1,$$
$$T(x, 0) = 0 \quad \text{for } x < -1,$$
$$T(x, y) = 100 \quad \text{if } z = e^{i\theta},\ 0 < \theta < \pi.$$

7. Find the temperature function $T(x, y)$ in the domain $1 < r < 2$, $0 < \theta < \pi/2$ that satisfies the boundary conditions in Figure 10.27. *Hint*: Use $w = \text{Log } z$.

$$T(x, y) = 0 \quad \text{for } r = e^{i\theta},\ 0 < \theta < \frac{\pi}{2},$$

$$T(x, y) = 50 \quad \text{for } r = 2e^{i\theta},\ 0 < \theta < \frac{\pi}{2},$$

$$\frac{\partial T}{\partial n} = T_y(x, 0) = 0 \quad \text{for } 1 < x < 2,$$

$$\frac{\partial T}{\partial n} = T_x(0, y) = 0 \quad \text{for } 1 < y < 2.$$

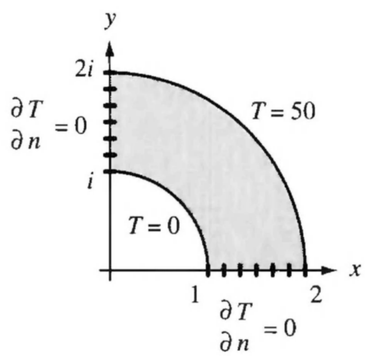


FIGURE 10.27 Accompanies Exercise 7.

8. Find the temperature function $T(x, y)$ in the domain $0 < r < 1$, $0 < \text{Arg } z < \alpha$ that satisfies the boundary conditions in Figure 10.28. *Hint*: Use $w = \text{Log } z$.

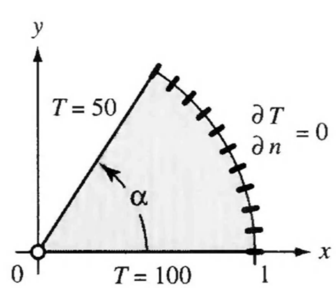


FIGURE 10.28 Accompanies Exercise 8.

$$T(x, 0) = 100 \quad \text{for } 0 < x < 1,$$
$$T(x, y) = 50 \quad \text{for } z = re^{i\alpha},\ 0 < r < 1,$$
$$\frac{\partial T}{\partial n} = 0 \quad \text{for } z = e^{i\theta},\ 0 < \theta < \alpha.$$

9. Find the temperature function $T(x, y)$ in the first quadrant $x > 0$, $y > 0$ that satisfies the boundary conditions in Figure 10.29. *Hint*: Use $w = \text{Arcsin } z^2$.

$$\begin{aligned} T(x, 0) &= 100 && \text{for } x > 1, \\ T(0, y) &= -50 && \text{for } y > 1, \\ \frac{\partial T}{\partial n} = T_y(x, 0) &= 0 && \text{for } 0 < x < 1, \\ \frac{\partial T}{\partial n} = T_x(0, y) &= 0 && \text{for } 0 < y < 1. \end{aligned}$$

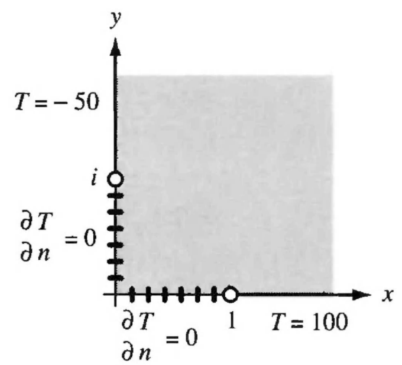


FIGURE 10.29 Accompanies Exercise 9.

10. Find the temperature function $T(x, y)$ in the infinite strip $0 < y < \pi$ that satisfies the boundary conditions in Figure 10.30. *Hint*: Use $w = e^z$.

$$\begin{aligned} T(x, 0) &= 50 && \text{for } x > 0, \\ T(x, \pi) &= -50 && \text{for } x > 0, \\ \frac{\partial T}{\partial n} = T_y(x, 0) &= 0 && \text{for } x < 0, \\ \frac{\partial T}{\partial n} = T_y(x, \pi) &= 0 && \text{for } x < 0. \end{aligned}$$

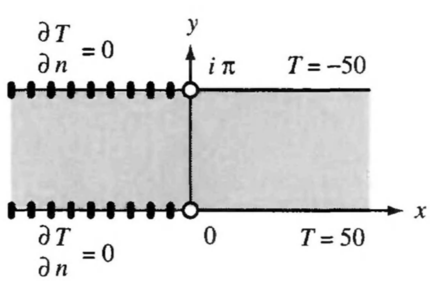


FIGURE 10.30 Accompanies Exercise 10.

11. Find the temperature function $T(x, y)$ in the upper half plane $\text{Im}(z) > 0$ that satisfies the boundary conditions in Figure 10.31. *Hint*: Use $w = 1/z$.

$$\begin{aligned} T(x, 0) &= 100 && \text{for } 0 < x < 1, \\ T(x, 0) &= -100 && \text{for } -1 < x < 0, \\ \frac{\partial T}{\partial n} = T_y(x, 0) &= 0 && \text{for } x > 1, \\ \frac{\partial T}{\partial n} = T_y(x, 0) &= 0 && \text{for } x < -1. \end{aligned}$$

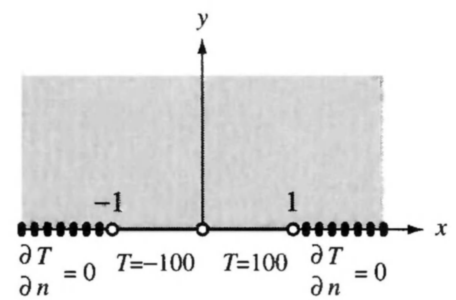


FIGURE 10.31 Accompanies Exercise 11.

12. Find the temperature function $T(x, y)$ in the first quadrant $x > 0$, $y > 0$ that satisfies the boundary conditions in Figure 10.32.

$$\begin{aligned} T(x, 0) &= 50 && \text{for } x > 0, \\ T(0, y) &= -50 && \text{for } y > 1, \\ \frac{\partial T}{\partial n} = T_x(0, y) &= 0 && \text{for } 0 < y < 1. \end{aligned}$$

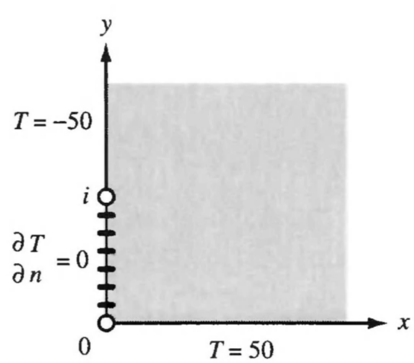


FIGURE 10.32 Accompanies Exercise 12.

13. For the temperature function

$$T(x, y) = 100 - \frac{100}{\pi} \arctan \frac{1 - x^2 - y^2}{2y}$$

in the upper half-disk $|z| < 1$, $\mathrm{Im}(z) > 0$, show that the isothermals $T(x, y) = \alpha$ are portions of circles that pass through the points $+1$ and -1 as illustrated in Figure 10.33.

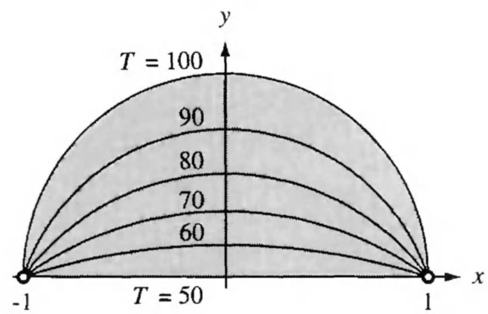


FIGURE 10.33 Accompanies Exercise 13.

14. For the temperature function

$$T(x, y) = \frac{300}{\pi}\,\mathrm{Re}(\mathrm{Arcsin}\,z)$$

in the upper half plane $\mathrm{Im}(z) > 0$, show that the isothermals $T(x, y) = \alpha$ are portions of hyperbolas that have foci at the points ± 1 as illustrated in Figure 10.34.

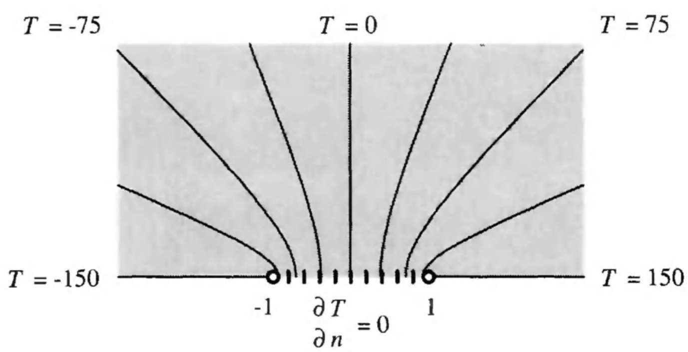


FIGURE 10.34 Accompanies Exercise 14.

15. Find the temperature function in the portion of the upper half plane $\mathrm{Im}(z) > 0$ that lies inside the ellipse

$$\frac{x^2}{\cosh^2 2} + \frac{y^2}{\sinh^2 2} = 1$$

and satisfies the boundary conditions given in Figure 10.35. *Hint*: Use $w = \mathrm{Arcsin}\,z$.

$$\begin{aligned} T(x, y) &= 80 && \text{for } (x, y) \text{ on the ellipse,} \\ T(x, 0) &= 40 && \text{for } -1 < x < 1, \\ \frac{\partial T}{\partial n} = T_y(x, 0) &= 0 && \text{when } 1 < |x| < \cosh 2. \end{aligned}$$

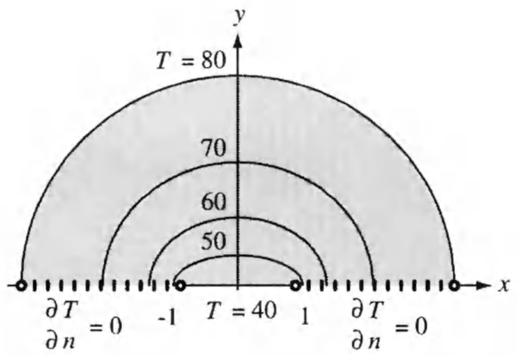


FIGURE 10.35 Accompanies Exercise 15.

10.6 Two-Dimensional Electrostatics

A two-dimensional electrostatic field is produced by a system of charged wires, plates, and cylindrical conductors that are perpendicular to the z plane. The wires, plates, and cylinders are assumed to be so long that the effects at the ends can be neglected as mentioned in Section 10.4. This sets up an electric field $\mathbf{E}(x, y)$ that can be interpreted as the force acting on a unit positive charge placed at the point (x, y). In the study of electrostatics the vector field $\mathbf{E}(x, y)$ is shown to be *conservative* and is derivable from a function $\phi(x, y)$, called the *electrostatic potential*, as expressed by the equation

(1) $\mathbf{E}(x, y) = -\text{grad}\ \phi(x, y) = -\phi_x(x, y) - i\phi_y(x, y)$.

If we make the additional assumption that there are no charges within the domain D, then Gauss' law for electrostatic fields implies that the line integral of the outward normal component of $\mathbf{E}(x, y)$ taken around any small rectangle lying inside D is identically zero. A heuristic argument similar to the one for steady state temperatures with $T(x, y)$ replaced by $\phi(x, y)$ will show that the value of the line integral is

(2) $-[\phi_{xx}(x, y) + \phi_{yy}(x, y)]\ \Delta x\ \Delta y$.

Since the quantity in expression (2) is zero, we conclude that $\phi(x, y)$ is a harmonic function. We let $\psi(x, y)$ denote the harmonic conjugate, and

(3) $F(z) = \phi(x, y) + i\psi(x, y)$

is the complex potential (not to be confused with the electrostatic potential).

The curves $\phi(x, y) = K_1$ are called the *equipotential curves*, and the curves $\psi(x, y) = K_2$ are called the *lines of flux*. If a small test charge is allowed to move under the influence of the field $\mathbf{E}(x, y)$, then it will travel along a line of flux. Boundary value problems for the potential function $\phi(x, y)$ are mathematically the same as those for steady state heat flow, and they are realizations of the Dirichlet problem where the harmonic function is $\phi(x, y)$.