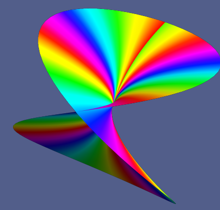
UCRIVERSIDE
Department of Mathematics

MATH 165B - Introduction to Complex Variables
Worksheet 3



Topic: Zeros of an Analytic Function
Readings: Section 75 from Brown & Churchill

75. Zeros of Analytic Functions

- Definition: Recall that an analytic function at a point z_0 has a zero of order m, m a positive integer iff

$$\begin{aligned} f(z_0) &= 0 \\ f'(z_0) &= f^{(2)}(z_0) = \cdots = f^{(m-1)}(z_0) = 0 \\ f^{(m)}(z_0) &\neq 0 \end{aligned}$$

- Motivating example:

  Let $f(z) = (z-1)^3(z-i)^2z^4$. Notice that $f(1) = 0$ and $z_0 = 1$ is a zero of order $m = 3$ since the Taylor expansion of f centered at $z_0 = 1$ is

$$\begin{aligned} f(z) = &-2i(z-1)^3 + (2-10i)(z-1)^4 + (9-20i)(z-1)^5 + (16-20i)(z-1)^6 + \\ &(14-10i)(z-1)^7 + (6-2i)(z-1)^8 + (z-1)^9 \end{aligned}$$

  The expansion was computed using the Mathematica command:

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Series[(z - 1)∧ 3*(z - I)∧2 * z∧4, {z, 1, 10}]
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  Notice that $f(1) = f'(1) = f''(1) = 0$ but $\frac{f^{(3)}(1)}{3!} = -2i \neq 0$. Thus $f^{(3)}(1) = 3!(-2i) = -12i \neq 0$ and 1 is a zero of order $m = 3$.

- Reading
  - Read carefully Theorem 1 with the example above in mind, what is the function $g(z)$ in the motivating example.
  - Use the ComplexPlot3D Mathematica command to plot the function f of the motivating example and of all the examples of Theorem 1 in the textbook. Try other functions with zeros and to help you generate the examples below. (*If you already have Mathematica 12.)*

- (P) Examples of zeros: Give examples of functions $f_1, f_2$and f_3 with the following characteristics

  1. f_1 has a zero of order 4 at $z_1 = 0$

  2. f_2 has a zero of order 2 at $z_2 = i\pi$ and a zero of order 3 at $z_4 = -2$

3. f_3 has a zero of order 3 $z_3 = 1$ and a pole of order 2 at $z_5 = 0$

- (P) Particular cases of Theorem 2: Read carefully Theorem 2 and find epsilons for z_1, z_2 and z_3 for each of the functions f_1, f_2 and f_3 given by the theorem. That is, an ε that determines a neighborhood where the only zero is z_1, z_2 or z_3.

HOMEWORK PROBLEMS FOR SECTION 75

1. Locate the zeros of the following functions and determine their order
   (a) $(1+z^2)^4$
   (b) $z^3 \sin z$
   (c) $1+\mathrm{Exp} z$
   (d) z^6+1
2. Let f be analytic with a zero of order k at z_0. Show that f' has a zero of order $k-1$ at z_0.

Star Problem: Show that if $f(z)$ and $g(z)$ are analytic at $z=\alpha$ and have zeros of order m and n, respectively, at $z=\alpha$, then their product $h(z) = f(z)g(z)$has a zero of order $m+n$ at $z=\alpha$.

The Star Problems are intended for students who are interested in challenging problems, they can substitute regular problems in the assignment.