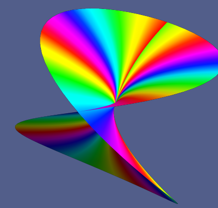
Department of Mathematics

# MATH 165B - Introduction to Complex Variables

Worksheet 10

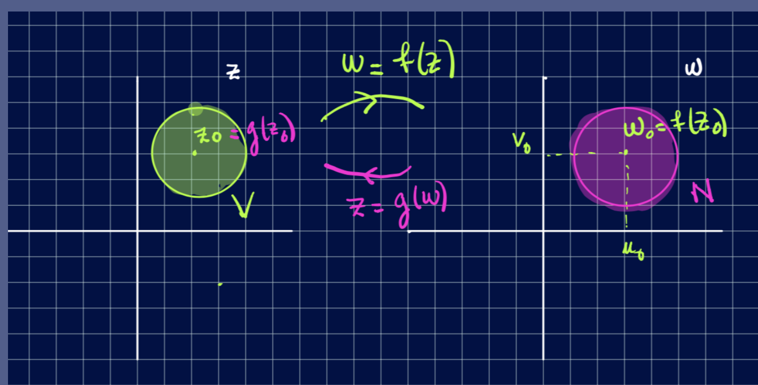


## Topics: Conformal Mappings: Local Inverses and Harmonic Conjugates

**Readings from Brown & Churchill:** Sections 101 and 102

### 103. Local Inverse

- In this section we will see that a conformal transformation $w = f(z)$ at a point $z_0$ has a "local inverse" . That is, if $w_0 = f(z_0)$, then there exists a unique transformation $z = g(w)$ which is defined and analytic in a neighborhood $N$ of $w_0$, such that $g(w_0) = z_0$ and $f[g(w)] = w$ for all points $w$ in $N$. See picture below:




- Read the example in page 361
- (P) Find an example in one real variable (e.g $f(x) = x^2$ ) of a function without a "global" inverse but with a "local" inverse when $f'(x) \neq 0$ and with a corresponding conformal mapping (e.g. $f(z) = z^2$ in this case) with similar behavior.

### 102. Harmonic Conjugates

In this section, we will see a method for giving a harmonic function $u(x,y)$, find another harmonic function $v(x,y)$ such that

$$f(x+iy) = u(x,y) + iv(x,y)$$

- Recall that if

$$f(x+iy) = u(x,y) + iv(x,y)$$

is analytic in a domain $D$, then the real-valued functions $u$ and $v$ are harmonic in that domain. That is, they have continuous partial derivatives of the first and second order in $D$ and satisfy Laplace's equation there:

$$u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0$$

- Recall also that $u$ and $v$ satisfy the Cauchy-Riemann equations

$$u_x = v_y, \tag{1}$$
$$u_y = -v_x \tag{2}$$

- Assume that the domain is simply connected and since the function $u$ is harmonic, it has continuous first derivatives, then we can use the methods of vector calculus to find a potential from equations (1) and (2) to find the function $v$. Compare the method in the textbook with the method for finding a potential for a conservative vector field in

  `https://openstax.org/books/calculus-volume-3/pages/6-3-conservative-vector-fields`

- (P) Express your preference between the two methods observed in the previous item. Why?

**HOMEWORK PROBLEMS FOR SECTION 103 and 104**

1. Page 362: #6 a) and c)
2. Find the analytic function $f(z) = u(x,y) + iv(x,v)$ given the following.
   (a) $u(x,y) = y^2 - 3x^2y$
   (b) $v(x,y) = e^{+}\sin x$
   (c) $v(x,y) = \sin x \cosh y$
3. Let $v$ he the harmonic conjugate of $u$. Show that $-u$ is is the harmonic conjugate of $v$.
4. Let $v$ he the harmonic conjugate of $u$. Show that $h = w^2 - y^2$ is a harmonic function.

Star Problems:
Let $f(x) = f(re^{m}) = u(r,\theta) + iv(r,\theta)$ be analytic in a domain $P$ that does not contain the origin. Use the polar form of the Cauchy-Riemann equations $u_\theta = -rv_r$, and $v_\theta = ru_r$, and differentiate them with respect to $\theta$ and then with respect to $r$. Use the results to establish the polar form of Laplace's equation:

$$r^2u_{rr}(r,\theta) + ru_r(r,\theta) + u_{\theta\theta}(r,\theta) = 0$$

Moreover,

(a) Use the previous polar form of Laplace's equation to show that $u(r,\theta) = r^n \cos n\theta$ and $v(r,\theta) = r^n \sin n\theta$ are harmonic functions.

(b) Use the polar form of Laplace's equation given above to show that

$$u(r,\theta) = \left(r + \frac{1}{r}\right)\cos\theta \quad \text{and} \quad v(r,\theta) = \left(r - \frac{1}{r}\right)\sin\theta$$

are harmonic functions

The Star Problems are intended for students who are interested in challenging problems, they can substitute regular problems in the assignment.