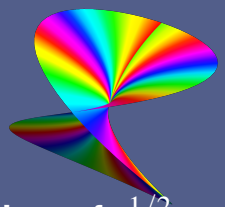
UC RIVERSIDE
UNIVERSITY OF CALIFORNIA
Department of Mathematics

MATH 165B - Introduction to Complex Variables

Worksheet 8



Topics: The Transformation $w = \sin z$ and Mappings by z^2 and Branches of $z^{1/2}$

Readings from Brown & Churchill: Sections 96 and 97

96.The Transformation $w = \sin z$

A very useful identity when we study the transformation $w = \sin z$ is

$$\sin(z) = \sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y) \quad (1)$$

where

$$\cosh(y) = \frac{e^y + e^{-y}}{2}$$
$$\sinh(y) = \frac{e^y - e^{-y}}{2}$$

If you do not recall hyperbolic functions here is a good open source review (at the end of the chapter)

https://openstax.org/books/calculus-volume-1/pages/1-5-exponential-and-logarithmic-function

Then, the transformation $w = \sin z$ is easily described in rectangular coordinates as

$$z = x+iy \quad \rightarrow \quad w = \sin z = u+iv \quad \text{where} \quad \begin{cases} u &= \sin(x)\cosh(y) \\ v &= \cos(x)\sinh(y) \end{cases}$$

- If you do not remember the equations for ellipses and hyperbolas, here are simple reviews
  https://openstax.org/books/precalculus/pages/10-1-the-ellipse#49153 and
  https://openstax.org/books/precalculus/pages/10-2-the-hyperbola
- Find the image of three vertical lines of your choice ($z = x_i + it, i = 1,2,3$ and $t \in \mathbb{R}$) under $w = \sin z$ using (1)
- Find the image of three horizontal lines of your choice ($z = t + y_i, i = 1,2,3$ and $t \in \mathbb{R}$) under $w = \sin z$ using (1)
- (P) What properties do you observe when you map vertical and horizontal lines under $w = \sin z$?
- Read Examples 1 & 2

96. Mappings by z^2 and Branches of $z^{1/2}$

The mapping $w = z^2$ can be view in rectangular coordinates as

$$z = x+iy \quad \rightarrow \quad w = (x+iy)^2 = x^2 - y^2 + i2xy = u+iv \quad \text{where} \quad \begin{cases} u &= x^2 - y^2 \\ v &= 2xy \end{cases}$$

- Find the image of three vertical lines of your choice ($z = x_i + it, i = 1,2,3$ and $t \in \mathbb{R}$) under $w = z^2$
- Find the image of three horizontal lines of your choice ($z = t + y_i, i = 1,2,3$ and $t \in \mathbb{R}$) under $w = z^2$
- (P) What properties do you observe when you map vertical and horizontal lines under $w = z^2$?
- Read Example 1

Recall that it is natural to use polar coordinates to express the **principal branch** of the square root, :

$$z = r\exp(i\theta) \quad \rightarrow \quad w = z^{1/2} = \sqrt{r}\exp\left(i\frac{\theta}{2}\right) \quad \text{where } r > 0 \text{ and } -\pi < \theta < \pi$$

1. Find the image of three rays of your choice of the form $z = r\exp(i\theta_i), i = 1,2,3$ and $r > 0$ under $w = \sqrt{r}\exp\left(i\frac{\theta}{2}\right)$

2. Find the image of three curves of your choice $z = r_i\exp(i\theta), i = 1,2,3$ and $-\pi < \theta < \pi$ under $w = \sqrt{r}\exp\left(i\frac{\theta}{2}\right)$

3. (P) What properties do you observe when you map your curves in (1.) and (2.) under $w = \sqrt{r}\exp\left(i\frac{\theta}{2}\right)$?

4. Read Example 2

HOMEWORK PROBLEMS FOR SECTION 95 and 96

1. Page 334: #1, #2, #4
2. Page 340: #1, #3
3. Star Problems: Page 334: #7, #8, Page 340: #7, #8

The Star Problems are intended for students who are interested in challenging problems, they can substitute regular problems in the assignment.