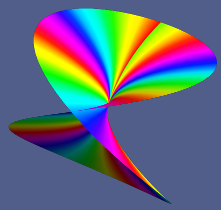
UC RIVERSIDE
Department of Mathematics

MATH 165B - Introduction to Complex Variables

Midterm Exam



Prob. #	Points	Score
1	25 points	
2	25 points	
3	25 points	
4	25 points	
Extra Credit	20 points	
Total	100 points	

Show your work

Problem 1: In each case, write the principal part of the function at its isolated singular point and determine whether that point is a pole, a removable singular point, or an essential singular point:

(a) $\exp\left(\frac{1}{z^2}\right)$

$$\exp\left(\frac{1}{z^2}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z^2}\right)^n = 1 + \sum_{n=1}^{\infty} \frac{1}{n! z^{2n}} \quad \leftarrow \text{ Principal part}$$

$z_0 = 0$ is an essential singularity since the principal part has infinitely many terms.

(b) $\frac{z^3}{1-z}$ Notice that $z_1 = 1$ is a singular point, moreover,

$$z^3 = z^3 - 1 + 1 = (z-1)(1+z+z^2) + 1$$

$$\frac{z^3}{1-z} = \frac{(z-1)(1+z+z^2)+1}{1-z}$$

$$\frac{z^3}{1-z} = -1 - z - z^2 + \frac{1}{1-z} \quad \leftarrow \text{ Principal part}$$

$z_1 = 1$ is a simple pole.

(c) $\frac{\sin 2z}{z}$ Notice that $z_2 = 0$ is a singular point, moreover,

$$\frac{\sin 2z}{z} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n \frac{(2z)^{2n+1}}{(2n+1)!} = \frac{1}{z}\left(2z - \frac{2^3}{3!}z^3 + \frac{2^5}{5!}z^5 + \cdots\right)$$

$$= 2 - \frac{2^3}{3!}z^2 + \frac{2^5}{5!}z^4 + \cdots$$

Therefore, the principal part is zero and $z_2 = 0$ is a removable singularity.

(d) $\dfrac{\cos z - 1}{z^2}$ Notice that $z_3 = 0$ is a singular point, moreover,

$$\frac{\cos z - 1}{z^2} = \frac{1}{z^2}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots - 1\right)$$
$$\frac{\cos z - 1}{z^2} = -1 + \frac{z^2}{4!} - \frac{z^4}{6!} + \cdots$$

Therefore, the principal part is zero and $z_3 = 0$ is a removable singularity.

(e) $\dfrac{1}{(1-z)^3}$. Notice that $z_4 = 1$ is a pole of order 3 and the function is its principal part.

Problem 2: Find

(a) The residue of $f_1(z) = \frac{\pi}{z-z^2}$ at $z = 0$

$z = 0$ is a simple pole since $f_1(z) = \frac{\pi}{z(1-z)} = \frac{1}{z}\frac{\pi}{1-z}$, then by the Theorem in Sec 73,

$$Res_{z=0} f_1(z) = \frac{\pi}{1-0} = \pi$$

(b) The residue of $f_2(z) = z\cos\left(\frac{1}{z}\right)$ at $z = 0$

$f_2(z) = z\cos\left(\frac{1}{z}\right) = z\left(1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} + \cdots\right) = z - \frac{1}{2!z} + \frac{1}{4!z^3} + \cdots$, therefore,

$$Res_{z=0} f_2(z) = -\frac{1}{2!} = -\frac{1}{2}$$

(c) The residue of $f_3(z) = \frac{z-\sin z}{2z}$ at $z = 0$

$f_3(z) = \frac{z-\sin z}{2z} = \frac{1}{2} - \frac{1}{2z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) = \frac{1}{2} - \frac{1}{2} + \frac{z^2}{2\times 3!} - \frac{z^4}{2\times 5!} + \frac{z^6}{2\times 7!} - \cdots = \frac{z^2}{2\times 3!} - \frac{z^4}{2\times 5!} + \frac{z^6}{2\times 7!} - \cdots$, thus,

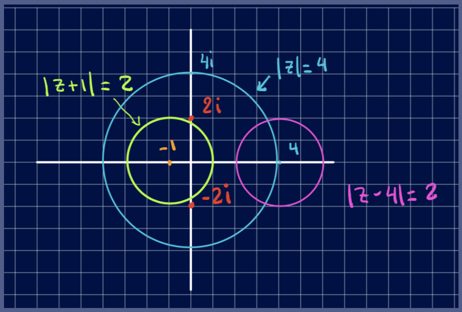
$$Res_{z=0} f_3(z) = 0$$

(d) A function f_4 with a simple pole at $z = 0$ such that the residue of f_4 at $z = 0$ is π.

$$f_4(z) = \frac{\pi}{z}$$

(e) A function f_5 with a pole of order 3 at $z = 0$ such that the residue of f_5 at $z = 0$ is 17.

$$f_5(z) = \frac{1}{z^3} + \frac{17}{z}$$



Problem 3: Consider the integral

$$\int_C \frac{2z^3+3}{(z+1)\,(z^2+4)}dz$$

taken counterclockwise around the curve C.

(a) Find the value of the integral when the curve C is the circle $\cancel{|z-1|=2}$ $|z+1|=2.$

$$\int_C \frac{2z^3+3}{(z+1)\,(z^2+4)}dz = 2\pi i Res_{z=-1}\frac{2z^3+3}{(z+1)\,(z^2+4)} = 2\pi i\frac{2(-1)^3+3}{((-1)^2+4)} = 2\pi i\frac{1}{5}$$

(b) Find the value of the integral when the curve C is the circle $|z|=4$

If we denote $f(z)=\frac{2z^3+3}{(z+1)(z^2+4)}$, then f can be decompose (check) as

$$f(z)=\frac{-\frac{11}{10}-\frac{29}{20}i}{z-(-2i)}+\frac{\frac{1}{5}}{z-(-1)}+\frac{-\frac{11}{10}+\frac{29}{20}i}{z-2i}+2$$

This partial fraction decomposition (I use the command **Apart** in Mathematica) give us the residues. Observe that one can also use the theorem in Sec 73 to compute them. Then

$$\begin{aligned}
Res_{z=-1}f(z) &= \frac{1}{5}\\
Res_{z=-2i}f(z) &= -\frac{11}{10}-\frac{29}{20}i\\
Res_{z=2i}f(z) &= -\frac{11}{10}+\frac{29}{20}i
\end{aligned}$$

From the picture above, we see that these poles are inside the curve $|z|=4$, then by the Residue Theorem,

$$\begin{aligned}
\int_C \frac{2z^3+3}{(z+1)\,(z^2+4)}dz &= 2\pi i\left(Res_{z=-1}f(z)+Res_{z=-2i}f(z)+Res_{z=2i}f(z)\right)\\
&= 2\pi i\left(\frac{1}{5}-\frac{11}{10}-\frac{29}{20}i-\frac{11}{10}+\frac{29}{20}i\right) = -4\pi i
\end{aligned}$$

(c) Give a curve C such that the value of the integral is 0.

Consider: $\mathbf{C}=\{z:|z-4|=2\}$. By Cauchy's Theorem, the integral is zero.

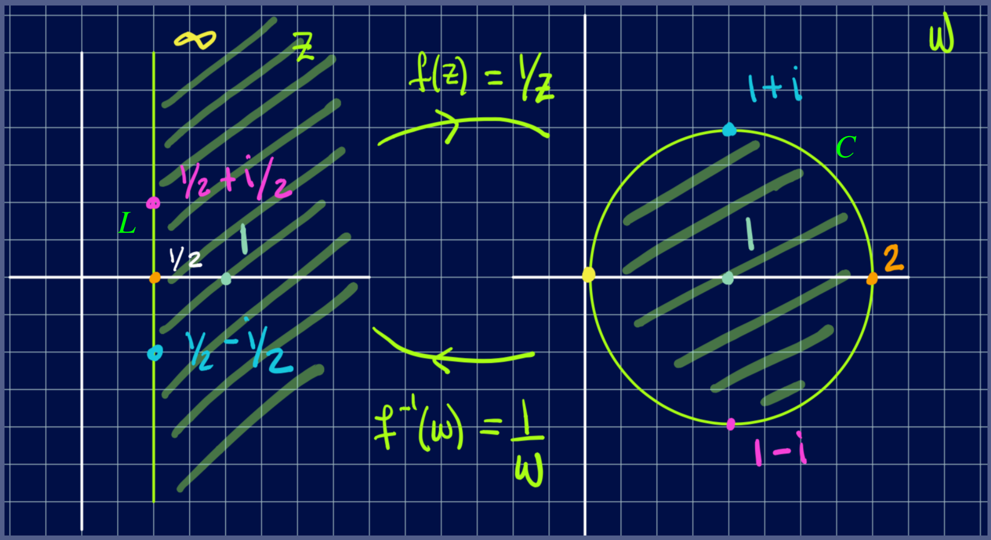
Problem 4: Show that the image of the right half plane $\mathrm{Re}(z) > \frac{1}{2}$, under the mapping $w = \dfrac{1}{z}$, is the disk $|w-1| < 1$.

Recall that the transformation $w = \dfrac{1}{z}$ maps "circles" to "circles". Then since the line $L = \{z : \mathrm{Re}(z) = \frac{1}{2}\}$ is also a "circle", its image is a circle. To determine the image L under $w = \dfrac{1}{z}$, notice that the points

$$\begin{aligned} z_1 &= \tfrac{1}{2} & & & w_1 &= 2 \\ z_2 &= \tfrac{1}{2} + \tfrac{i}{2} & &\longrightarrow & w_2 &= 1 - i \\ z_3 &= \tfrac{1}{2} - \tfrac{i}{2} & & & w_3 &= 1 + i \\ & \infty & & & & 0 \end{aligned}$$

Then, by basic geometry, it is clear that the image of L is the circle $C = \{w : |w-1| = 1\}$. See figure below. *(Note: this was not my first approach to solve this problem.)*

Moreover the image of 1 is also 1, then the right half plane $\mathrm{Re}(z) > \frac{1}{2}$, under the mapping $w = \dfrac{1}{z}$, is the disk $|w-1| < 1$.



Extra Credit Problem
Show that all four zeros of the polynomial $g(z) = z^4 - 7z - 1$ lie in the disk $|z| < 2$

Let $f(z) = z^4$ and $h(z) = -7z - 1$, $g(z) = f(z) + h(z)$. Now on the circle $|z| = 2$, we notice that

$$|f(z)| = |z^4| = |z^4| = 2^4 = 16 \quad \text{and} \quad |h(z)| = |-7z - 1| < |-7z| + 1 = 7|z| + 1 = 15$$

Therefore on the circle $|z| = 2$, $|f(z)| < |h(z)|$ and we can apply Rouché's Theorem. Then, since $f(z)$ has four zeros counting multiplicity on $|z| < 2$, so thus $g(z)$.

Extra Credit STAR PROBLEM
Show that the parabola $2x = 1 - y^2$ is mapped onto the cardioid $\rho = 1 + \cos\phi$ by the reciprocal transformation $w = \dfrac{1}{z}$.

There are several ways of solving this problem. One way is to consider the equivalent problem that $z = \dfrac{1}{w}$ maps the cardioid $\rho = 1 + \cos\phi$ onto the parabola $2x = 1 - y^2$. We will present an outline of this approach. On the cardioid:

$$w = \rho e^{i\phi} = (1 + \cos\phi)e^{i\phi} \quad \to z = \frac{1}{w} = \frac{1}{(1 + \cos\phi)e^{i\phi}} = \frac{1}{1 + \cos\phi}e^{-i\phi} = x + iy$$

Then

$$x = \frac{1}{1 + \cos\phi}\cos\phi$$
$$y = -\frac{1}{1 + \cos\phi}\sin\phi$$

It is easy to check now that $2x = 1 - y^2$.