165B — Review

RJ Acuña

(862079740)

p.71 (#10) (a) Recall (Sec. 5) that if $z = x + iy$, then

$$x = \frac{z+\bar{z}}{2} \text{ and } y = \frac{z-\bar{z}}{2i} .$$

By *formally* applying the chain rule in calculus to a function $\mathrm{F}(x, y)$ of two real variables, derive the expression

$$\frac{\partial \mathrm{F}}{\partial \bar{z}} = \frac{\partial \mathrm{F}}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial \mathrm{F}}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial \mathrm{F}}{\partial x} + i\frac{\partial \mathrm{F}}{\partial y}\right) .$$

(b) Define the operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

suggested by part (a), to show that if the first-order partial derivatives of the real and imaginary components of a function $f(z) = u(x, y) + iv(x, y)$ satisfy the Cauchy–Riemann equations, then

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[(u_x - v_y) + i(v_x + u_y)\right] = 0$$

Thus derive the *complex form $\partial f/\partial \bar{z} = 0$ of the Cauchy-Riemann equations.*

pf. of (a)

Notice,

$$x : \mathbb{C}^2 \to \mathbb{R}; (z, \bar{z}) \mapsto \frac{z+\bar{z}}{2} \text{ and } y : \mathbb{C}^2 \to \mathbb{R}; (z, \bar{z}) \mapsto \frac{z-\bar{z}}{2i}$$

Then $\mathrm{F} : \mathbb{R}^2 \to \mathbb{R}; (x, y) \mapsto \mathrm{F}(x, y) = \mathrm{F}(x(z, \bar{z}), y(z, \bar{z}))$, so by chain rule we have,

$$\frac{\partial \mathrm{F}}{\partial \bar{z}} = \frac{\partial \mathrm{F}}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial \mathrm{F}}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

Now,

$$\frac{\partial x}{\partial \bar{z}} = \frac{1}{2} \text{ and } \frac{\partial y}{\partial \bar{z}} = -\frac{1}{2i}$$

Also,

$$1 = -i^2 \implies -\frac{1}{i} = -(-i) = i \implies \frac{\partial y}{\partial \bar{z}} = \frac{1}{2}i$$

Plugging in,

$$\frac{\partial \mathrm{F}}{\partial \bar{z}} = \frac{\partial \mathrm{F}}{\partial x}\frac{1}{2} + \frac{\partial \mathrm{F}}{\partial y}\frac{1}{2}i = \frac{1}{2}\left(\frac{\partial \mathrm{F}}{\partial x} + i\frac{\partial \mathrm{F}}{\partial y}\right) \blacksquare$$

pf. of (b)

Let $f = u + iv$. f satisfies the Cauchy-Riemann equations, so

$$u_x = v_y \text{ and } u_y = -v_x \implies u_x - v_y = 0 \text{ and } u_y + v_x = 0 \quad (I)$$

$$\begin{aligned}
\frac{\partial}{\partial \bar{z}}f &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)u + \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)iv \\
&= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) + i\frac{1}{2}\left(\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y}\right) = \frac{1}{2}\left(u_x + iu_y + i(v_x + iv_y)\right) \\
&= \frac{1}{2}\left(u_x + iu_y + iv_x - v_y)\right) = \frac{1}{2}\left(u_x - v_y + i(u_y + v_x)\right) \\
&= \frac{1}{2}\left(0 + i(0)\right) = 0 \quad \text{by } (I) \quad \blacksquare
\end{aligned}$$

p.77 (#7) Let a function f be analytic everywhere in a domain D. Prove that if $f(z)$ is real-valued for all z in D, then $f(z)$ must be constant throughout D.

pf. Let D be a domain,

$$\begin{aligned}
f \text{ is analytic in } D &\implies \forall z \in D, \quad u_x = v_y \text{ and } u_y = -v_x \\
\forall z \in D, f(z) \in \mathbb{R} &\implies Im(f) = v(x,y) = 0 \\
&\implies v_x = v_y = 0 \\
&\implies u_x = u_y = 0 \\
f'(z) = u_x + iu_y &\implies \forall z \in D, \quad f'(z) = 0
\end{aligned}$$

Therefore, by the theorem in Section 24, f is constant throughout D ■

p.81 (#7) Let the function $f(z) = u(x,y) + iv(x,y)$ be analytic in a domain D, and consider the families of level curves $u(x,y) = c_1$ and $v(x,y) = c_2$, where c_1 and c_2 are arbitrary real constants. Prove that these families are orthogonal. More precisely, show that if $z_0 = (x_0, y_0)$ is a point in D which is common to two particular curves $u(x,y) = c_1$ and $v(x,y) = c_2$ and if $f'(z_0) \neq 0$, then the lines tangent to those curves at (x_0, y_0) are perpendicular.

pf.

Let D be a domain. Fix, $z_0 = (x_0, y_0) \in \mathbb{C}$, and $c_1, c_2 \in \mathbb{R}$.

$$\begin{aligned}
f \text{ is analytic in} D &\implies \forall z \in D, \exists f'(z) = u_x + iv_x \\
f \text{ is analytic in} D &\implies u_x = v_y \text{ and } u_y = -v_x \\
f'(z_0) \neq 0 &\implies u_x(x_0, y_0) \neq 0 \text{ and } v_x(x_0, y_0) \neq 0 \\
&\implies v_y(x_0, y_0) \neq 0 \text{ and } -u_y(x_0, y_0) \neq 0 \\
&\implies u_y(x_0, y_0) \neq 0
\end{aligned}$$

Since, u and v are the components of f, they're continuously differentiable maps $\mathbb{R}^2 \to \mathbb{R}$. The equations $u(x,y) = c_1$ and $v(x,y) = c_2$, define curves. Since, $u_y(x_0, y_0) \neq 0$ and $u_y(x_0, y_0) \neq 0$. Then, the level curves $u(x,y) - c_1 = 0$ and $v(x,y) - c_2 = 0$, both satisfy the implicit function theorem. Therefore, we can express $y = f(x)$, where $f : \mathbb{R} \to \mathbb{R}$, for each curve. Such that, $u(x_0, f_1(x_0)) = c_1$, and $v(x_0, f_2(x_0)) = c_2$, for $(x_0, y_0 = f_1(x_0) = f_2(x_0))$ common to both curves. Therefore,

$$\frac{\partial}{\partial x}u = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{df_1}{dx} = 0 = \frac{\partial}{\partial x}c_1 \text{ , and similarly } \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{df_2}{dx} = 0 \quad (1) \text{ .}$$

Two curves are orthogonal if the product of their slopes is -1. Thus,

$$\frac{df_1}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \text{ and } \frac{df_2}{dx} = -\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}$$

Since f is analytic in D containing z_0,

$$u_x = v_y \text{ and } u_y = -v_x$$

So,

$$\frac{df_1}{dx}\frac{df_2}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \cdot -\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = \frac{u_x v_x}{u_y v_y} = \frac{v_y v_x}{-v_x v_y} = -1$$

Therefore, the families of functions defined by $u(x,y) = c_1$ and $v(x,y) = c_2$ are orthogonal ■

p.92 (#9) Show $\overline{\exp(iz)} = \exp(i\bar{z})$ if and only if $z = n\pi$ $(n = 0, \pm 1, \pm 2, \dots)$.

pf.

($\implies$) Let $x, y \in \mathbb{R} : z = x + iy$. Suppose $\overline{\exp(iz)} = \exp(i\bar{z})$,

$$\begin{aligned}
iz = -y + ix &\implies \exp(iz) = \exp(-y + ix) = \exp(-y)\exp(ix) \\
&\implies \overline{\exp(iz)} = \overline{\exp(-y)\exp(ix)} \\
&\implies \overline{\exp(iz)} = \exp(-y)\overline{\exp(ix)} \\
&\implies \overline{\exp(iz)} = \exp(-y)(\overline{\cos(x) + i\sin(x)}) \\
&\implies \overline{\exp(iz)} = \exp(-y)(\cos(x) - i\sin(x)) \\
i\bar{z} = i(x - iy) = y + ix &\implies \exp(i\bar{z}) = \exp(y + ix) = \exp(y)\exp(ix) \\
&\implies \exp(i\bar{z}) = \exp(y)(\cos(x) + i\sin(x)) \\
\overline{\exp(iz)} = \exp(i\bar{z}) &\implies \exp(-y)(\cos(x) - i\sin(x)) = \exp(y)(\cos(x) + i\sin(x)) \\
&\implies \exp(-y)\cos(x) - i\exp(-y)\sin(x) = \exp(y)\cos(x) + i\exp(y)\sin(x) \\
&\implies \exp(-y)\cos(x) = \exp(y)\cos(x) \quad \text{and} \quad -\exp(-y)\sin(x) = \exp(y)\sin(x) \\
&\implies \cos(x) = \exp(2y)\cos(x) \quad \text{and} \quad -\sin(x) = \exp(2y)\sin(x) \\
&\implies (1 - \exp(2y))\cos(x) = 0 \quad \text{and} \quad (-1 - \exp(2y))\sin(x) = 0 \\
-1 - \exp(2y) = 0 &\implies -1 = \exp(2y) \\
2y \in \mathbb{R} &\implies \exp(2y) > 0 \\
&\implies -1 - \exp(2y) \neq 0 \\
&\implies \sin(x) = 0 \implies x = n\pi, \quad n \in \mathbb{Z} \\
x = n\pi, n \in \mathbb{Z} &\implies \cos(x) = \pm 1 \\
&\implies (1 - \exp(2y)) \pm 1 = 0 \\
&\implies 1 - exp(2y) = 0 \\
&\implies \exp(2y) = 1 \\
&\implies 2y = 0 \\
2 \neq 0 &\implies y = 0 \\
&\implies z = n\pi, (n = 0, \pm 1, \pm 2, \dots)
\end{aligned}$$

($\impliedby$) Suppose, $z = n\pi, (n = 0, \pm 1, \pm 2, \dots)$,

$$\overline{\exp(iz)} = \overline{\exp(n\pi i)} = \overline{\cos(n\pi) + i\sin(n\pi)} = \cos(n\pi) - isin(n\pi) = \cos(n\pi) - 0i = \cos(n\pi)$$

$$z = n\pi \implies \bar{z} = n\pi \implies i\bar{z} = n\pi \implies \exp(i\bar{z}) = \cos(n\pi) + i\sin(n\pi) = \cos(n\pi) + 0i = \cos(n\pi)$$

So,

$$\overline{\exp(iz)} = \exp(i\bar{z})$$

The conclusion follows $\blacksquare$

p.97 (# 9) Show that

(a) the function $f(z) = \text{Log}(z-i)$ is analytic everywhere except on the portion $x \leq 0$ of the line $y = 1$;

(b) the function

$$f(z) = \frac{\text{Log}(z+4)}{z^2+i}$$

is analytic everywhere except at the points $\pm(1-i)/\sqrt{2}$ and on the portion $x \leq -4$ of the real axis.

slu. of (a)

Consider the function $t : \mathbb{C} \to \mathbb{C}; z \mapsto z - i$, at glance we can see that $f = \text{Log} \circ t$. We know that Log is analytic on the portion $x \leq 0$ of the x−axis. Therefore, since $t^{-1}(\{0\}) = 0 + 1i$, f must be analytic on the portion $x \leq 0$ of the line $y = 1$ ♦

slu. of (b)

Consider the function $t : \mathbb{C} \to \mathbb{C}; z \mapsto z + 4$, let $g = \text{Log} \circ t$. We know that Log is analytic on the portion $x \leq 0$ of the x−axis. Therefore, since $t^{-1}(\{0\}) = -4 + 0i$, g must be analytic on the portion $x \leq -4$ of the x−axis.

Since the function $h(z) = \frac{1}{z^2+i}$ is a rational function, it is analytic whenever the denominator is not zero. Furthermore, it fails to be analytic precisely where the denominator is zero.

$$\begin{aligned}
z^2 + i = 0 &\implies z = \pm\sqrt{-i} \\
&\implies z = \pm\sqrt{\exp\left(\frac{3\pi i}{2}\right)} \\
&\implies z = \pm\exp\left(\frac{3\pi i}{4}\right) \\
&\implies z = \pm\left[\cos\left(\frac{3\pi i}{4}\right) + i\sin\left(\frac{3\pi i}{4}\right)\right] \\
&\implies z = \pm\left[-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right] \\
&\implies z = \pm(1-i)/\sqrt{2}
\end{aligned}$$

Since, $f = gh$ it is not analytic where g or h are not analytic the conclusion follows ♦

p.108 (#16) With the aid of expression (14), Sec. 34, show that the roots of the equation $\cos z = 2$ are

$$z = 2n\pi + i\cosh^{-1}(2) \quad (n = 0, \pm1, \pm2, \dots)$$

Then express them in the form

$$z = 2n\pi \pm i\ln(2+\sqrt{3}) \quad (n = 0, \pm1, \pm2, \dots)$$

slu.

Let $x, y \in \mathbb{R} : z = x + iy$.

By expression (14) in Section 34,

$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y) = 2$$

Now, if $\Re(z) \neq 2n\pi$, then $\sin(x) \neq 0 \implies \Im(\cos(z)) \neq 0$ since $\forall y \in \mathbb{R}, \sinh(y) \neq 0$.

So, $\Im(2) = 0 \implies \Re(z) = 2n\pi$, which forces $\cos(\Re(z)) = 1$. Which is required because

$$2 \geq 1 \text{ and } \forall x \in \mathbb{R}, -1 \leq \cos(x) \leq 1.$$

Thus $\Im(z)$ must be such that, $\cosh(\Im(z)) = 2$, i.e: $\Im(z) = \cosh^{-1}(2)$.

So, $z = 2n\pi + i\cosh^{-1}(2) \quad (n = 0, \pm1, \pm2, \dots)$.

Remains to show that,

$$\cosh^{-1}(2) = \pm\ln(2+\sqrt{3})$$

$\cosh(\pm\ln(2+\sqrt{3})) = \frac{e^{\pm\ln(2+\sqrt{3})} + e^{-\pm\ln(2+\sqrt{3})}}{2} = \frac{(2+\sqrt{3})^{\pm1} + (2+\sqrt{3})^{\mp1}}{2}$

So,

$$\cosh(\ln(2+\sqrt{3})) = \frac{(2+\sqrt{3}) + (2+\sqrt{3})^{-1}}{2} = \frac{(2+\sqrt{3})^{-1} + (2+\sqrt{3})}{2} = \cosh(-\ln(2+\sqrt{3}))$$

Finally we need,

$$\frac{(2+\sqrt{3}) + (2+\sqrt{3})^{-1}}{2} = 2$$

$\frac{(2+\sqrt{3})+(2+\sqrt{3})^{-1}}{2} = 1 + \frac{\sqrt{3}}{2} + \frac{1}{(2+\sqrt{3})2} = 1 + \frac{(2+\sqrt{3})\sqrt{3}+1}{(2+\sqrt{3})2} = 1 + \frac{2\sqrt{3}+3+1}{(2+\sqrt{3})2} = 1 + \frac{4+2\sqrt{3}}{4+2\sqrt{3}} = 1 + 1 = 2$

So, $z = 2n\pi \pm i\ln(2+\sqrt{3}) \quad (n = 0, \pm1, \pm2, \dots)$ ♦

p.126 (#5) Suppose that a function $f(z)$ is analytic at a point $z_0 = z(t_0)$ lying on a smooth arc $z = z(t)(a \le t \le b)$. Show that if $w(t) = f(z(t))$, then

$$w'(t) = f'[z(t)]z'(t)$$

when $t = t_0$.

pf.

$$\begin{aligned} w'(t) &= \lim_{t\to t_0} \frac{w(t) - w(t_0)}{t - t0} \\ &= \lim_{t\to t_0} \frac{f(z(t)) - f(z(t_0))}{t - t_0} \\ z \text{ is a smooth arc } &\implies \forall t \in (a,b), \exists z'(t) \\ &\implies 1 = \frac{z'(t)}{z'(t)} \\ \text{limit laws } &\implies 1 = \frac{\lim_{t\to t_0} \frac{z(t)-z(t_0)}{t-t_0}}{\lim_{t\to t_0} \frac{z(t)-z(t_0)}{t-t_0}} = \lim_{t\to t_0} \frac{z(t) - z(t_0)}{z(t) - z(t_0)} \end{aligned}$$

So,

$$\begin{aligned} w'(t) &= \lim_{t\to t_0} \frac{f(z(t)) - f(z(t_0))}{t - t_0} \cdot \lim_{t\to t_0} \frac{z(t) - z(t_0)}{z(t) - z(t_0)} \\ &= \lim_{t\to t_0} \frac{f(z(t)) - f(z(t_0))}{z(t) - z(t_0)} \cdot \lim_{t\to t_0} \frac{z(t) - z(t_0)}{t - t_0} \\ &= \lim_{t\to t_0} \frac{f(z(t)) - f(z(t_0))}{z(t) - z(t_0)} \cdot z'(t_0) \end{aligned}$$

$\forall t \in (a,b), \exists z'(t) \implies z$ is continuous on $[a,b] \implies z(t) \to z(t_0)$ as $t \to t_0$.

Since $z = z(t)$ and $z_0 = z(t_0)$,

$$\lim_{t\to t_0} \frac{f(z(t)) - f(z(t_0))}{z(t) - z(t_0)} = \lim_{z(t)\to z(t_0)} \frac{f(z(t)) - f(z(t_0))}{z(t) - z(t_0)} = f'(z(t_0)) = f'(z_0)$$

Finally,

$$w'(t) = f'[z(t)]z'(t)$$

when $t = t_0$ ■

p.135 (#11) (a) Suppose that a function $f(z)$ is continuous on a smooth arc C, which has a parametric representation $z = z(t)(a \le t \le b)$; that is, $f[z(t)]$ is continuous on the interval $a \le t \le b$. Show that if $\phi(\tau)(\alpha \le \tau \le \beta)$ is the function described in Sec. 39, then

$$\int_a^b f[z(t)]z'(t)dt = \int_\alpha^\beta f[Z(\tau)]Z'(\tau)d\tau$$

where $Z(\tau) = z[\phi(\tau)]$.

pf. of (a)

$\alpha = \phi(a)$, and $\beta = \phi(b)$

$$\int_a^b f[z(t)]z'(t)dt = \int_\alpha^\beta f[z(\phi(\tau))]z'(\phi(\tau))d\phi(\tau) = \int_\alpha^\beta f[Z(\tau)]Z'(\tau)d\tau \quad ■$$

(b) Point out how it follows that the identity obtained in part (a) remains valid when C is any contour, not necessarily a smooth one, and $f(z)$ is piecewise continuous on C. Thus show that the value of the integral of $f(z)$ along C is the same when the representation $z = Z(\tau)(\alpha \le \tau \le \beta)$ is used, instead of the original one.

pf. of (b) Notice, that the integral of a contour is the sum of the integrals along the smooth arcs that make up the contour, thus the result follows. ■

p.140 (#6) Let C_ρ denote a circle $|z| = \rho(0 < \rho < 1)$, oriented in the counterclockwise direction, and suppose that $f(z)$ is analytic in the disk $|z| \le 1$. Show that if $z^{-1/2}$ represents any particular branch of that power of z, then there is a nonnegative constant M, *independent* of ρ, such that

$$\left| \int_{C_\rho} z^{-1/2} f(z) dz \right| \le 2\pi M \sqrt{\rho}$$

Thus show that the value of the integral here approaches 0 as ρ tends to 0.

pf.

f is analytic in the disk $|z| \le 1$, then $|f|$ is bounded by a positive number M independent of ρ. So,

$$|z^{-1/2} f(z)| = |z|^{-1/2} |f(z)| \le \frac{M}{\sqrt{\rho}}$$

$$\implies \left| \int_{C_\rho} z^{-1/2} f(z) \right| \le \frac{M}{\sqrt{\rho}} 2\pi\rho = 2\pi M \sqrt{\rho}$$

Thus, the integral goes to zero as ρ goes to zero ■

p.160 (#6) Let C denote the positively oriented boundary of the half disk$0 \le r \le 1, 0 \le \theta \le \pi$, and let $f(z)$ be a continuous function defined on that half disk by writing $f(0) = 0$ and using the branch

$$f(z) = \sqrt{r} e^{i\theta/2} \quad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2} \right)$$

Show that

$$\int_C f(z) dz = 0$$

by evaluating separately the integrals of $f(z)$ over the semicircle and the two radii which make up C. Why does the Cauchy–Goursat theorem not apply here?

p.170 (#2) Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when (a) $g(z) = \frac{1}{z^2+4}$; (b) $g(z) = \frac{1}{(z^2+4)^2}$.

slu. of (a)

$$z^2 + 4 = 0 \implies z = \pm 2i$$

$$g(z) = \frac{1}{z^2 + 4} \implies g(z) = \frac{1}{(z - 2i)(z + 2i)} = \frac{\frac{1}{z+2i}}{z - 2i}$$

$$|2i - i| = |i| = 1 < 2 \implies 2i \text{ is inside the circle } |z - i| = 2$$

Let C be the circle $|z - i| = 2$,

Let $f(z) = \dfrac{1}{z + 2i} \implies 2\pi i f(2i) = \displaystyle\int_C \frac{f(z)}{z - 2i} dz$ by the Cauchy integral formula.

Then,

$$\int_C g(z)dz = 2\pi i \frac{1}{2i+2i} = \frac{\pi}{2} \quad \Diamond$$

slu. of (b)

With f and C as in slu. of (a). Let $h(z) = f(z)^2$,

$$k(z) = \frac{1}{(z^2+4)^2} = \frac{1}{[(z+2i)(z-2i)]^2} = \frac{1}{(z+2i)^2(z-2i)^2} = \frac{(f(z))^2}{(z-2i)^2} = \frac{h(z)}{(z-2i)^2}$$

So by formula (6) in Section 51 we have,

$$\int_C \frac{1}{(z^2+4)^2}dz = \frac{2\pi i}{1!}h'(2i)$$

$$h'(2i) = 2\frac{1}{z+2i}(-\frac{1}{(z+2i)^2})|_{2i} = -\frac{2}{(4i)^3} = -\frac{2}{4\cdot 16\cdot -i} = \frac{1}{32i}$$

So,

$$\int_C \frac{1}{(z^2+4)^2}dz = \frac{2\pi i}{32i} = \frac{\pi}{16} \quad \Diamond$$

p.170 (#10) Let f be an entire function such that $|f(z)| \leq A|z|$ for all z, where A is a fixed positive number. Show that $f(z) = a_1 z$, where a_1 is a complex constant.

pf.

$$|f^{(n)}(z)| \leq \frac{n!M_R}{R^n} \implies |f^{(2)}(z)| \leq \frac{2M_R}{R^2} \leq \frac{2A(|z|+R)}{R^2} \forall z \in \mathbb{C}$$

Since f is entire, then the radius R where it is analytic is arbitrarily large. So let $R \to \infty$, gives the right hand side of the last inequality is zero.

$$|f^{(2)}(z_0)| \leq 0 \implies f^{(2)}(z) = 0 \quad \forall z \in \mathbb{C} \implies \exists a_1, a_2 \in \mathbb{C} : f(z) = a_1 z + a_2$$

But, $|f| \leq A|z| \quad \forall z \in \mathbb{C} \implies a_2 = 0 \implies f(z) = a_1 z \quad \blacksquare$

p.178 (#9) Let z_0 be a zero of the polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \quad (a_n \neq 0)$$

of degree $n(n \geq 1)$. Show in the following way that

$$P(z) = (z - z_0)Q(z)$$

where $Q(z)$ is a polynomial of degree $n-1$.

(a) Verify that

$$z^k - z_0^k = (z-z_0)(z^{k-1} + z^{k-2}z_0 + \cdots + z z_0^{k-2} + z_0^{k-1}) \quad (k = 2, 3, \cdots)$$

pf. of (a)

The geometric progression has the property,

$$\sum_{i=0}^{k-1} w^i = \frac{1-w^k}{1-w} \implies 1 - w^k = (1-w)\sum_{i=0}^{k-1} w^i$$

$$\text{Let } w = \frac{z_0}{z} \implies 1 - \frac{z_0}{z}^k = (1 - \frac{z_0}{z})\sum_{i=0}^{k-1}\left(\frac{z_0}{z}\right)^i$$

$$\implies z^k - z_0^k = z(1 - \frac{z_0}{z})z^{k-1}\sum_{i=0}^{k-1}\left(\frac{z_0}{z}\right)^i$$

$$\implies z^k - z_0^k = (z - z_0)\sum_{i=0}^{k-1} z^{k-1-i}z_0^i \quad (k = 2, 3, ...) \quad \blacksquare$$

(b) Use the factorization in part (a) to show that

$$P(z) - P(z_0) = (z - z_0)Q(z)$$

where $Q(z)$ is a polynomial of degree $n-1$, and deduce the desired result from this.

pf. of (b)

$$P(z) = \sum_{k=0}^{n} a_n z^n \text{ and } P(z_0) = \sum_{k=0}^{n} a_n z_0^n$$

Therefore,

$$\begin{aligned}
P(z) - P(z_0) &= \sum_{k=0}^{n} a_k(z^k - z_0^k) \\
&= a_0(z^0 - z_0^0) + a_1(z - z_0) + \sum_{k=2}^{n} a_k(z^k - z_0^k) \\
&= a_0(1-1) + a_1(z - z_0) + \sum_{k=2}^{n} a_k(z^k - z_0^k) \\
&= a_1(z - z_0) + \sum_{k=2}^{n} a_k(z - z_0)\sum_{i=0}^{k-1} z^{k-1-i}z_0^i \\
&= (z - z_0)[a_1 + \sum_{k=2}^{n} a_k \sum_{i=0}^{k-1} z^{k-1-i}z_0^i]
\end{aligned}$$

Then, $Q(z) = a_1 + \sum_{k=2}^{n} a_k \sum_{i=0}^{k-1} z^{k-1-i}z_0^i$ the degree of Q is the largest power of z in Q. This happens when $k = n$, and $i = 0$. Therefore, the degree of Q is $n-1$, since $a_n \neq 0$.

Since, z_0 is a zero of P, so $P(z_0) = 0$. The result follows,

$$P(z) = (z - z_0)Q(z)$$

where Q is a polynomial of degree $n-1$ $\blacksquare$

p.188 (# 9) Let a sequence $z_n \quad (n = 1, 2, ...)$ converge to a number z. Show that there exists a positive number M such that the inequality $|z_n| \leq M$ holds for all n. Do this in each of the following ways.

(a) Note that there is a positive integer n_0 such that

$$|z_n| = |z + (z_n - z)| \leq |z| + 1$$

whenever $n > n_0$.

(b)Write $z_n = x_n + iy_n$ and recall from the theory of sequences of real numbers that the convergence of x_n and y_n $(n = 1, 2, \dots)$ implies that $|x_n| \le M_1$ and $|y_n| \le M_2$ $(n = 1, 2, \dots)$ for some positive numbers M_1 and M_2.

pf. by method (a)

$z_n \to z$ as $n \to \infty \implies \forall \varepsilon \ge 0, \exists N \in \mathbb{N} : n > N \implies |z - z_n| < \varepsilon$.

Put $\varepsilon = 1$. Then, by reverse triangle inequality,

$$||z| - |z_n|| < |z - z_n| < 1 \implies -1 < |z| - |z_n| \implies |z_n| < |z| + 1$$

Let $M = \max\{|z_0|, \dots, |z_N|, |z| + 1\}$, since $|z_n| \ge 0$ and $|z| + 1 > 0 \implies M > 0$

So, $|z_n| \le M$ holds for all n ■

pf. by method (b)

Let $z_n = x_n + iy_n$ and $M = M_1 + M_1$, and x_n, y_n convergent real sequences such that $|x_n| \le M_1$ and $|y_n| \le M_2$ for all n.Then,

$|z_n| = |x_n + iy_n| \le |x_n| + |iy_n| = |x_n| + |i||y_n| = |x_n| + |y_n| \le M_1 + M_2 = M$ holds for all n ■

p.195 (#13) Show that when $0 < |z| < 4$

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}$$

slu. Let $z \in \mathbb{C}$:$0 < |z| < 4 \implies z \ne 0 \implies \exists \frac{1}{z} : z\frac{1}{z} = 1$

Furthermore, $0 < \left|\frac{z}{4}\right| < 1$, so we can write,

$$\frac{1}{4z - z^2} = \frac{1}{4(z - \frac{z^2}{4})} = \frac{\frac{1}{4}}{z(1 - \frac{z}{4})} = \frac{1}{4z}\frac{1}{1 - \frac{z}{4}} = \frac{1}{4z}\sum_{k=0}^{\infty}\left(\frac{z}{4}\right)^k = \sum_{k=0}^{\infty}\frac{z^{k-1}}{4^{k+1}} = \frac{1}{4z} + \sum_{k=1}^{\infty}\frac{z^{k-1}}{4^{k+1}}$$

Let $n = k - 1 \implies n + 2 = k + 1$ and if $k = 1$, then $n = 0$. Thus,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}} \quad \blacklozenge$$

p.205 (# 8) (a) Let a denote a real number, where $-1 < a < 1$, and derive the Laurent series representation

$$\frac{a}{z - a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n} \quad (|a| < |z| < \infty)$$

pf. of (a) $|a| < |z| < \infty \implies 0 < \left|\frac{a}{z}\right| < 1$

$$\implies \frac{a}{z - a} = \frac{a}{z}\frac{1}{1 - \frac{a}{z}} = \frac{a}{z}\sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^k = \sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^{k+1}$$

Let $n = k + 1$, when $k = 0, n = 1$. Thus,

$$\frac{a}{z - a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n} \quad (|a| < |z| < \infty)$$

(b) After writing $z = e^{i\theta}$ in the equation obtained in part (a), equate real parts and then imaginary parts on each side of the result to derive the summation formulas

$$\sum_{n=1}^{\infty} a^n \cos(n\theta) = \frac{a\cos(\theta) - a^2}{1 - 2a\cos(\theta) + a^2} \text{ and } \sum_{n=1}^{\infty} a^n \sin(n\theta) = \frac{a\sin(\theta)}{1 - 2a\cos(\theta) + a^2}$$

where $-1 < a < 1$.

pf. of (b) Let $z = e^{i\theta}$ then,

$$\begin{aligned}
\frac{a}{e^{i\theta} - a} &= \frac{a}{\cos(\theta) + i\sin(\theta) - a} \\
&= \frac{a}{\cos(\theta) - a + i\sin(\theta)} \\
&= \frac{a}{\cos(\theta) + i\sin(theta) - a} \\
&= \frac{a}{\cos(\theta) - a + i\sin(\theta)} \frac{\cos(\theta) - a - i\sin(theta)}{\cos(\theta) - a - i\sin(\theta)} \\
&= \frac{a\cos(\theta) - a^2 - ia\sin(\theta)}{\cos^2(\theta) - 2a\cos(\theta) + a^2 + \sin^2(\theta)} \\
&= \frac{a\cos(\theta) - a^2 - ia\sin(\theta)}{1 - 2a\cos(\theta) + a^2} \\
&= \frac{a\cos(\theta) - a^2}{1 - 2a\cos(\theta) + a^2} - i\frac{a\sin(\theta)}{1 - 2a\cos(\theta) + a^2} \\
\sum_{n=1}^{\infty} \left(\frac{a}{e^{i\theta}}\right)^n &= \sum_{n=1}^{\infty} \frac{a^n}{e^{n\theta}} \\
&= \sum_{n=1}^{\infty} a^n e^{-n\theta} \\
&= \sum_{n=1}^{\infty} a^n(\cos(n\theta) - i\sin(n\theta)) \\
&= \sum_{n=1}^{\infty} a^n\cos(n\theta) - a^n i\sin(n\theta) \\
&= \sum_{n=1}^{\infty} a^n\cos(n\theta) - i\sum_{n=1}^{\infty} a^n\sin(n\theta)
\end{aligned}$$

By equating the real and imaginary parts the result follows ■

p.219 (# 7) Use the result in Exercise 6 to show that if

$$f(z) = \frac{\text{Log}(z)}{z - 1} \text{ when } (z \neq 1)$$

and $f(1) = 1$, then f is analytic throughout the domain

$$0 < |z| < \infty, -\pi < \text{Arg}(z) < \pi$$