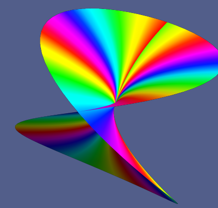
UC RIVERSIDE
Department of Mathematics

MATH 165B - Introduction to Complex Variables

Worksheet 5



Topics: Argument Principle and Rouché's Theorem

Readings from Brown & Churchill:

- Section 86: Theorem and Example
- Section 87: Theorem and Examples 1 & 2

86. Argument Principle

In a similar way that we could ”think” of analytic functions as generalizations of polynomials, we will define the concept of a meromorphic function as a generalization of rational functions.

- Definition of Meromorphic Functions

  A function f is said to be meromorphic in a domain D if it is analytic throughout D except for poles.

  Example: $f : \mathbb{C} \setminus \{-1, 1\} \to \mathbb{C}$

$$f(z) = \frac{\exp(z)}{z^2 - 1} = \frac{\exp(z)}{(z-1)(z+1)}$$

  Notice that f is analytic except at the points $z_1 = -1$ and $z_2 = 1$ where f has simple poles.

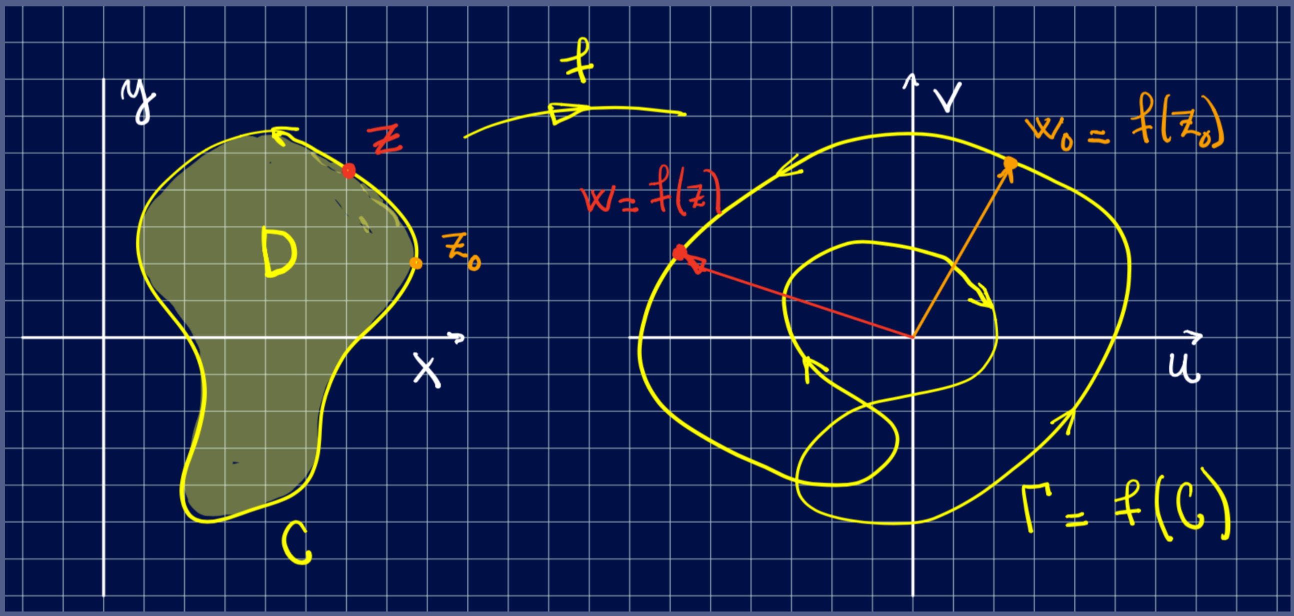
- (P) Give examples of meromorphic functions f_1, f_2 and f_3 defined on $D = \mathbb{C}$ such that

  1. f_1 has poles at $1, 2,$ and 3
  2. f_2 is analytic on all $\mathbb{C}$
  3. f_3 has a pole of order 3 at i and a single pole at 0

- Motivation for the Argument Principle

  - C: positively oriented simple closed contour
  - f: meromorphic in the domain D interior to C
  - f: analytic and **non-zero** on C
  - $\Gamma = f(C)$: the image of C under the transformation $w = f(z)$
  - $z_0 \in C$ fixed, $w_0 = f(z_0)$ and ϕ_0 a value of $\arg w_0$
  - As “a point z traverses C ” $\to$ “$w = f(z)$ traverses Γ ”
  - $\arg w$ vary continuously from ϕ_0 as $w = f(z)$ begins at the point w_0 along Γ
  - When w returns to w_0, $\arg w$ assumes a particular (new) value of $\arg w_0$, we will denote ϕ_1.
  - The change in $\arg w$ as w describes Γ once is $\phi_1 - \phi_0$. We will write

$$\Delta_C \arg f(z) = \phi_1 - \phi_0$$



- Remarks

  1. $\Delta_C \arg f(z)$ is an integral multiple of 2π
  2. The integer

  $$N = \frac{\Delta_C \arg f(z)}{2\pi}$$

  represents the number of times the point w winds around the origin in the w plane. It is sometimes called the winding number of Γ.
  3. N is positive if Γ winds around the origin in the counterclockwise direction and negative if N winds clockwise around that point.
  4. The winding number N is always zero when Γ does not enclose the origin. (Ex 3)

The Argument Principle shows how this winding number can be determined from the number of zeros and poles of f interior to C.

Suppose now that f has Z zeros and P poles in the domain interior to C counted by multiplicity. That is, we agree that f has m_0 zeros at a point z_0 if it has a zero of order m_0 there; and if f has a pole of order m_p at z_0, that pole is to be counted m_p times.

- *Theorem (Argument Principle).*

  Let C denote a positively oriented simple closed contour, and suppose that

  (a) a function $f(z)$ is meromorphic in the domain interior to C,

  (b) $f(z)$ is analytic and nonzero on C,

  (c) counting multiplicities, Z is the number of zeros and P the number of poles of $f(z)$ inside C

  Then

  $$\frac{1}{2\pi}\Delta_C \arg f(z) = Z - P$$

- (P) Apply the *Argument Principle* for 3 different functions g_1, g_2 and g_3 and the curve C, $|z| = 4$ positively oriented, such that:
  - For g_1, $Z_1 = 2$ and $P_1 = 1$
  - For g_2, $Z_2 = 1$ and $P_2 = 2$
  - For g_3, $Z_3 = 2$ and $P_3 = 2$

  Star Version: Reproduce the proof of the Argument Principle for each of the functions g_i, $i = 1, 2$, and 3.

87. Rouché's Theorem

- Motivating Example:

  Finding the location of zeros of an analytic function is a hard problem. Rouché's Theorem is an application of the Argument Principle that provides a way of locating regions in the complex plane where the zeros are located. For examples, consider the polynomial $p(z) = z^4 - 7z - 1$. To find the zeros

$$p(z) = z^4 - 7z - 1 = 0$$

is not trivial. Mathematica has the capability to find numerical solutions. See the following commands:

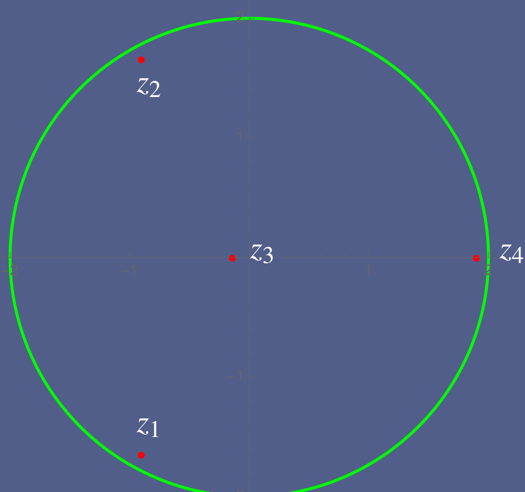
```
In[1]:= NSolve[z ∧4 - 7*z - 1 == 0, z]
```

Out[1]:= $\{\{z \to -0.90778 - 1.65887I\}, \{z \to -0.90778 + 1.65887I\}, \{z \to -0.142798\},$
$\{z \to 1.95836\}\}$

That is, the approximate zeros are

$$\begin{aligned} z_1 &\approx -0.90778 - 1.65887I \\ z_2 &\approx -0.90778 + 1.65887I \\ z_3 &\approx -0.142798 \\ z_4 &\approx 1.95836 \end{aligned}$$

Notice that the modulus of all these approximate roots are inside the curve C defined by $|z| = 2$. See picture below:



Let $f(z) = z^4$ and $g(z) = -7z - 1$. Our polynomial $p(z) = f(z) + g(z)$. We know that f has a zero of order 4 inside the curve C and `Mathematica` tells us that $p(z)$ has also 4 zeros inside the curve C.

Moreover, on the curve C,

$$|g(z)| = |-7z - 1| \leq 7|z| + 1 = 7 \times 2 + 1 = 15 < |f(z)| = 2^4 = 16()$$

Rouché's Theorem says that under the condition (), $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside the curve C. Thus reducing the problems of locating the zeros to a much easier polynomial.

- <u>*Rouché's Theorem*</u>*.*
  Let C denote a simple closed contour, and suppose that

  (a) two functions $f(z)$ and $g(z)$ are analytic inside and on C

  (b) $|f(z)| > |g(z)|$ at each point on C

  Then, $f(z)$ and $f(z)+g(z)$ have the same number of zeros, counting multiplicities, inside C.

- (P) Apply Rouché's Theorem to show that $p(z) = z^4 - 7z - 1$ has one zero in $|z| < 1$

- (P) Give an example of how to apply Rouché's Theorem (The example could be a variation of one of the examples in the book)

HOMEWORK PROBLEMS FOR SECTION 86 and 87

1. Page 296: #1 a)b) and C), #2, #4, #6 a), #7 b) and #8.

2. Star Problems: Page 297 #5, #10

The Star Problems are intended for students who are interested in challenging problems, they can substitute regular problems in the assignment.