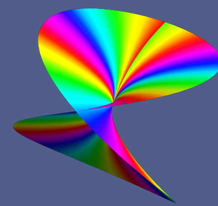
UCRIVERSIDE
Department of Mathematics

MATH 165B - Introduction to Complex Variables

Worksheet 13



Topics: Applications of Conformal Mappings: Steady State Temperatures

Readings from handout from Howell and Matthews: Section 10.5

10.5 Steady State Temperatures

In this model, a harmonic function $T(x,y)$ will represents the temperature on simply connected domain D with harmonic conjugate $S(x,y)$, and

$$F(z) = T(x,y) + iS(x,y)$$

an analytic function. The curves $T(x,y) = K_1$ are called isothermals and are curves connecting points of the same temperature. The curves $S(x,y) = K_2$ are called the heat flow lines, and one can visualize the heat flowing along these curves from points of higher temperature to points of lower temperature.

Example 10.14 from the handout considers two parallel planes are perpendicular to the z– plane and pass through the horizontal lines $y = a$ and $y = b$ and that the temperature is held constant at the values $T(x,a) = T_1$ and $T(x,b) = T_2$, respectively, on these planes. Assuming that the temperature at all points on the plane passing through the line $y = y_0$ is constant. Hence $T(x,y) = t(y)$, where $t(y)$ is a function of y alone. Laplace's equation:

$$T_{xx} + T_{yy} = t''(y) = 0$$

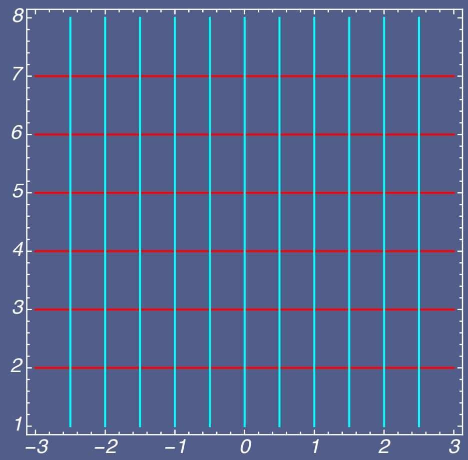
Then using the conditions $T(x,a) = T_1$ and $T(x,b) = T_2$, it is easy to see that the solution $T(x,y)$ has the form:

$$T(x,y) = T_1 + \frac{T_2 - T_1}{b-a}(y-a)$$

The isothermals $T(x,y) = \alpha$ are easily seen to be horizontal lines. The conjugate harmonic function is

$$S(x,y) = \frac{T_1 - T_2}{b-a}x$$

and the heat flow lines $S(x,y) = \beta$ are vertical segments between the horizontal lines. If $T_1 > T_2$, then the heat flows along these segments from the plane through $y = a$ to the plane through $y = b$. The following plot illustrates the T and S for $T_2 = 10$, $T_1 = 0$, $b = 6$ and $a = 1$.



The following Mathematica command was used to generate the picture:

```
ContourPlot[10/5 *(y - 1) == {2, 4, 6, 8, 10, 12}, 2*x == {-6, -5, -4, -3, -2, -1,
0, 1, 2, 3, 4, 5, 6}}, {x, -3, 3}, {y, 1, 8}, ContourStyle -> {Red, Cyan}, FrameStyle
-> Directive[White, FontColor -> White]]
```

This command plots the isothermals of values $\{2,4,6,8,10,12\}$ and the heat flow lines of values $\{-6,-5,-4,-3,-2,-1,0,1,2,3,4,5,6\}$.

- (P) Give values of the parameters T_1, T_2, a and b such that

  $$T(x,y) = 1 + \frac{7}{15}(y-10)$$

  Is your solution unique?

- Read Example 10.15 from the handout.

- In Example 10.16 the problem of finding the temperature $T(x,y)$ at each point in the upper halfdisk $H : \operatorname{Im}(z) > 0, |z| < 1$ if the temperature at points on the boundary satisfies

  $$T(x,y) = 100 \quad for\ \ z = e^{i\theta}, 0 < \theta < \pi \tag{1}$$
  $$T(x,0) = 50 \quad for. -1 < x < 1 \tag{2}$$

  is reduced to the a similar problem as in Example 10.15.

  (a) Check that

  $$u + iv = \frac{i(1-z)}{1+z} = \frac{2y}{(x+1)^2+y^2} + i\frac{1-x^2-y^2}{(x+1)^2+y^2} \quad (c)$$

  is a one-to-one conformal mapping of the half-disk H onto the first quadrant $Q : u > 0, v > 0$.

(b) The conformal map (c) gives rise to a new problem of finding the temperature $T^*(u,v)$ that satisfies the boundary conditions

$$T^*(u,0) = 100 \quad \text{for } u > 0 \qquad\qquad \text{and} \tag{3}$$

$$T^*(0,v) = 50 \quad \text{for } v > 0 \tag{4}$$

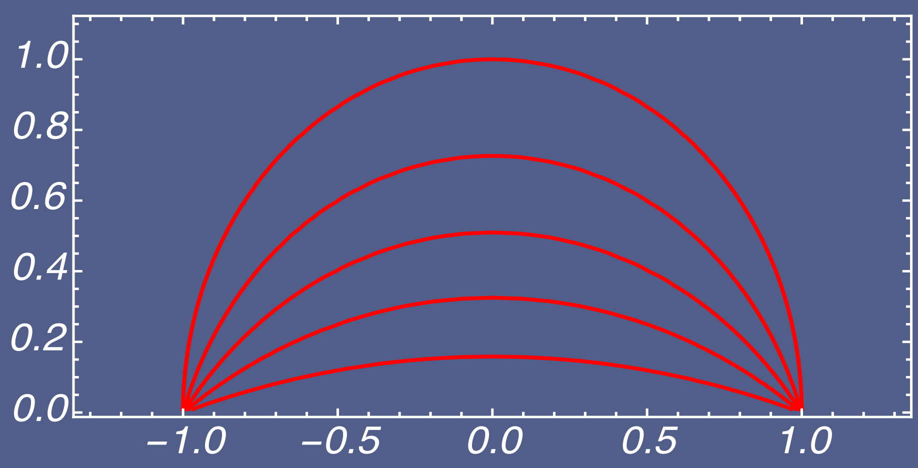
Check that a solution is given by the harmonic function

$$T^*(u,v) = 100 + \frac{50-100}{\pi/2}\operatorname{Arg} w = 100 - \frac{100}{\pi}\operatorname{Arctan}\frac{v}{u} \quad (\dagger)$$

(c) Now check that we can map the solution back by substituting the expressions for u and v in mapping (c) into equation $(\dagger)$gives the solution

$$T(x,y) = 100 - \frac{100}{\pi}\operatorname{Arctan}\frac{1-x^2-y^2}{2y}$$

(d) (P) The isothermals $T(x,y) =$ constant are drawn below.



How will the heat flow lines look? Can you draw them without calculating them?

HOMEWORK PROBLEMS

1. Problems #1, #2 and #3 from the handout.
2. Star Problems: Problems #4 and #5 from the handout.

The Star Problems are intended for students who are interested in challenging problems, they can substitute regular problems in the assignment.