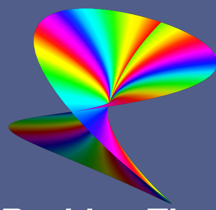
Department of Mathematics

MATH 165B - Introduction to Complex Variables

Worksheet 4



Topics: Zeros and Poles, Evaluating Improper Integrals Using the Residue Theorem

Readings from Brown & Churchill:

- Section 76: Theorem 1, and Example 1 *(Theorem 2 and related examples are optional)*
- Section 77: Theorem 1, Theorem 2, and Lemma *(Theorem 3 is optional)*
- Section 79: Example *(Optional: Section 79 where a general Method is described)*

76. Zeros and Poles

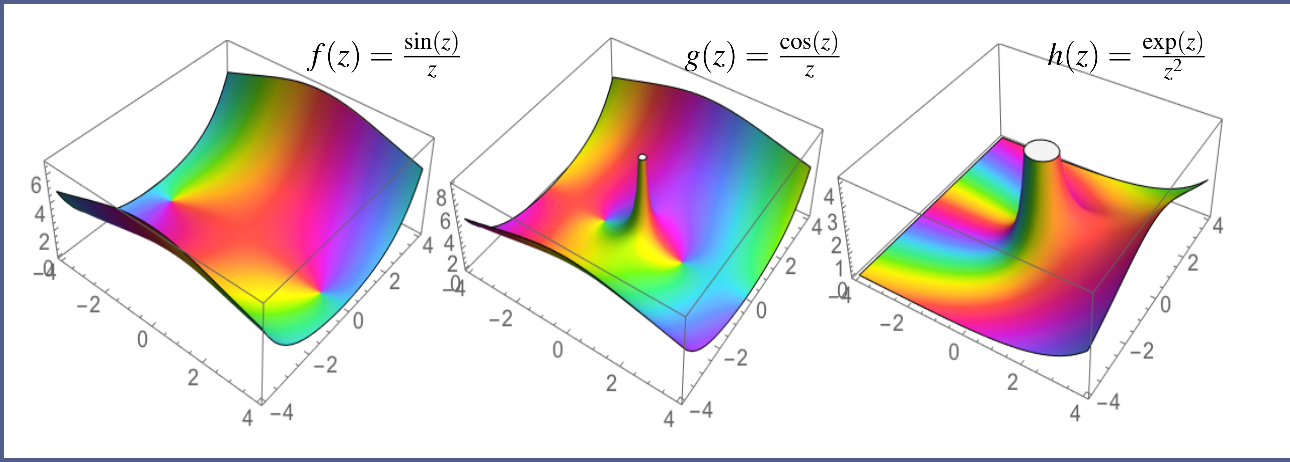
- Consider the following functions:

$$f(z) = \frac{\sin(z)}{z}$$
$$g(z) = \frac{\cos(z)}{z}$$
$$h(z) = \frac{\exp(z)}{z^2}$$

  Notice that $z_0 = 0$ is an isolated singularity for the three functions. What type of singularity is z_0 for each of the functions?

- Can Theorem 1 be applied to these functions? Justify your argument.

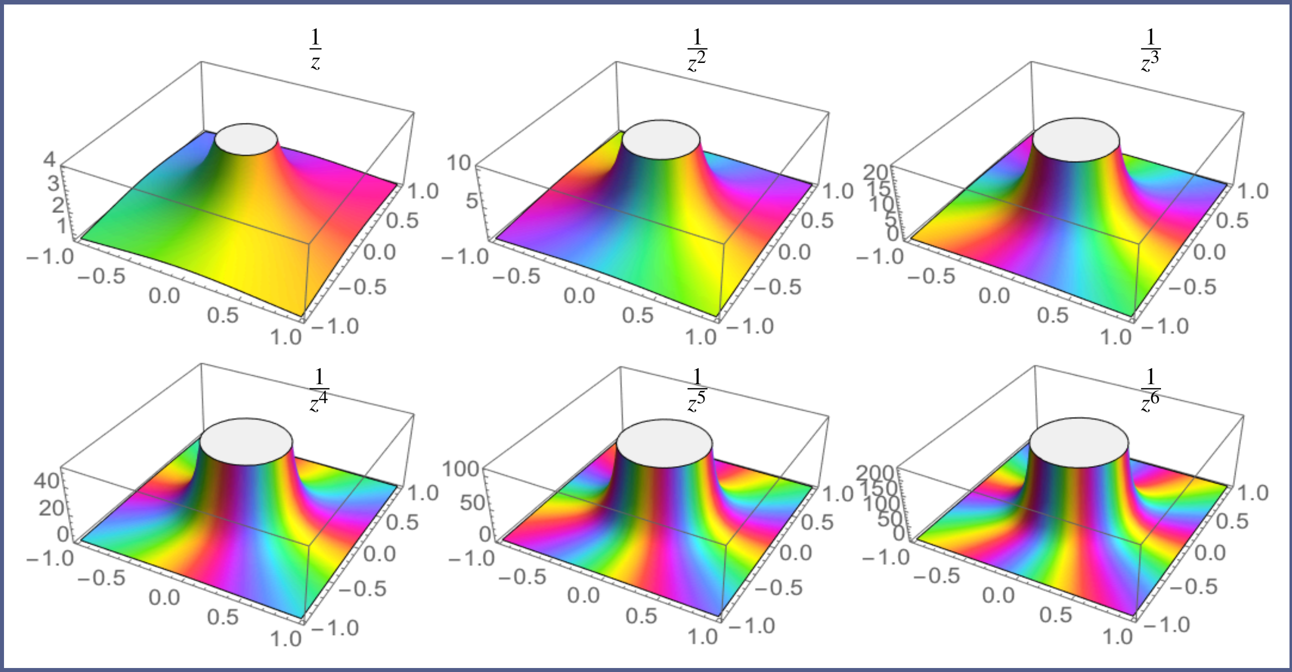
- The following graphs are plots of the functions the 3 functions. For example, for the function f, the plot represents
  $(x, y, |f(x+iy)|)$ where $z = x+iy$. The color is according to the function $Arg(f(z))$.



(P) Describe how the singularity at z_0 determines characteristics of each of the three plots.

77. Behavior of Functions near Isolated Singular Points

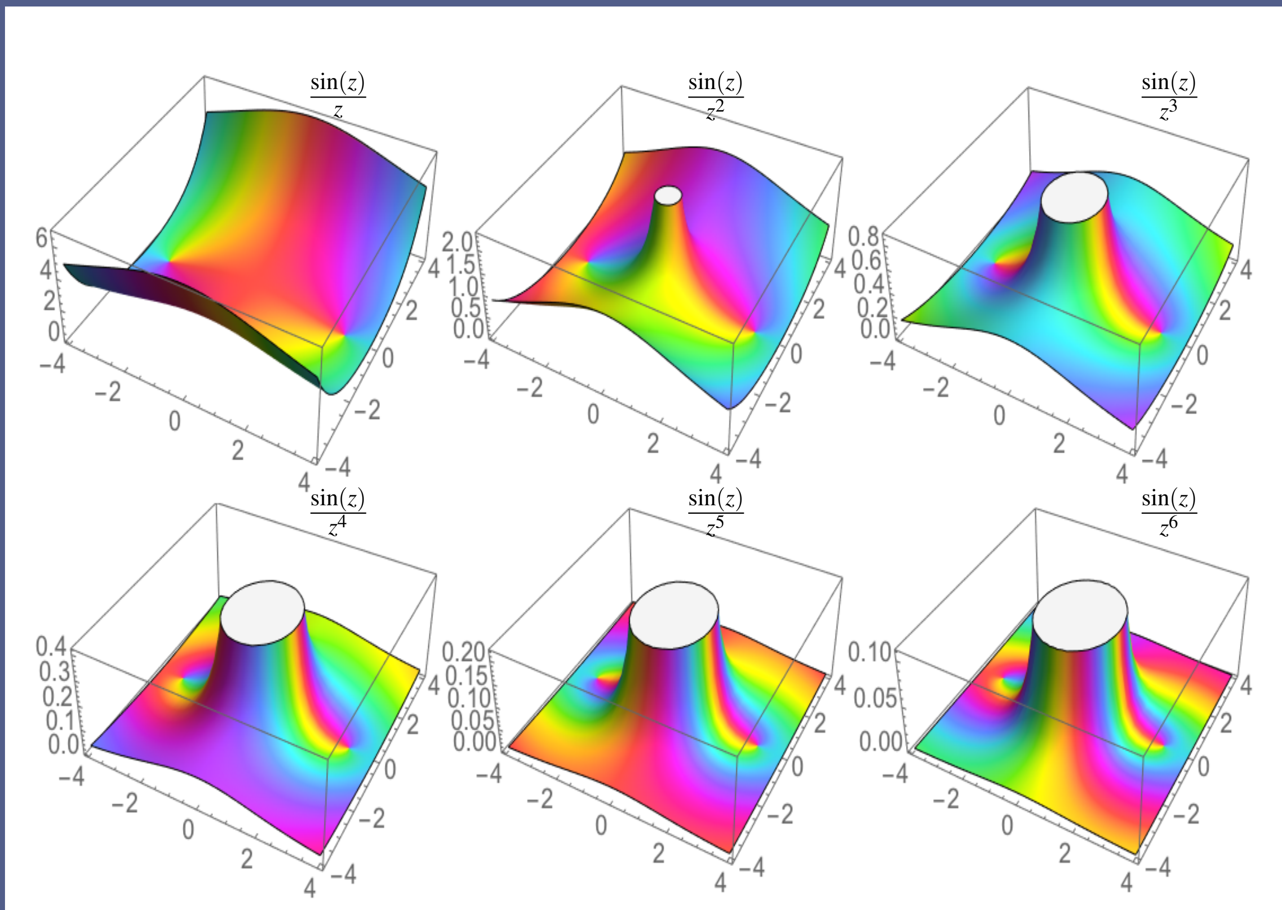
- The following table has plots of the functions $f_k(z) = \dfrac{1}{z^k}, k = 1, \ldots, 6$. The plots represent $(x, y, |f_k(x+iy)|)$ where $z = x+iy$. The color is according to the function $Arg(f_k(z))$.



(P) Describe 3 characteristics that you observe that change with the order of the pole k.

- Apply Theorem 1 to the examples in the plot. Do you observe any difference?

- The following table has plots of the functions $f_k(z) = \dfrac{\sin(z)}{z^k}, k = 1, \ldots, 6$. The plots represent $(x, y, |f_k(x+iy)|)$ where $z = x+iy$. The color is according to the function $Arg(f_k(z))$.



(P) Describe 3 characteristics that you observe that change with the order k of the power of z.

- Apply Theorem 1 and 2 to the functions $\dfrac{\sin(z)}{z^k}, k = 1\ldots 6$. Relate the conclusions of the theorems to the plots.

79. Example of Evaluating Improper Integrals Using the Residue Theorem

- The Example (page 264): We will go over this example highlighting the general method described in Section 78.

  The problem is to compute the integral

  $$\int_0^{\infty} \frac{x^2}{x^6+1} dx \qquad (1)$$

  Recall that $\displaystyle\int_0^{\infty} \frac{x^2}{x^6+1} dx = \lim_{R\to\infty} \int_0^{R} \frac{x^2}{x^6+1} dx = \frac{1}{2} \lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{x^6+1} dx.$

  (Notice that the last equality is true because the function is even.)

  1. Consider the function with complex values z instead of real values x, that is

     $$f(x) = \frac{x^2}{x^6+1} \quad \to \quad f(z) = \frac{z^2}{z^6+1}$$

  2. Determine where the function f is analytic.

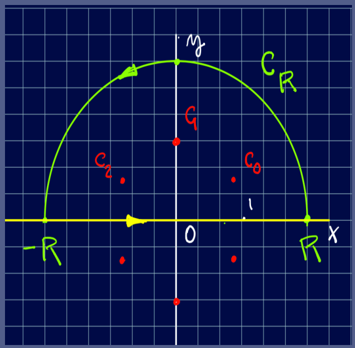
     The function $f(z)$ is analytic except at the zeros of the denominator, i.e. where $z^6+1=0$ or $z^6=-1$. These zeros are the sixth roots of -1 given by

     $$c_k = \exp\left[i\left(\frac{\pi}{6} + \frac{2k\pi}{6}\right)\right] \qquad (k = 0, 1, 2, \ldots, 5)$$

     and it is clear that none of them lies on the real axis. The first three roots,

     $$c_0 = e^{i\pi/6}, \quad c_1 = i, \quad \text{and} \quad c_2 = e^{i5\pi/6}$$

     are on the upper half plane.

  3. Integrate $f(z)$ over the positively oriented path C defined by the upper half of the circle of radius R and center 0 and the segment $[-R,R]$ on the x-axis. (See figure on the right.). That is

     $$\int_C f(z)dz = \int_{C_r} f(z)dz + \int_{[-R,R]} f(z)dz = \int_{C_r} f(z)dz + \int_{-R}^{R} f(x)dx \quad (2)$$

     We can apply the Residue theorem to integrating the curve f over the curve C, then

     $$\int_C f(z)dz = 2\pi i \left(\mathrm{Res}_{z=c_0} f(z) + \mathrm{Res}_{z=c_1} f(z) + \mathrm{Res}_{z=c_2} f(z)\right) = \cdots = \frac{\pi}{3}$$

     Note: Compute the above residues or read about it in the textbook.

4. To finish calculating the integral (1), we observe from (2) that

$$\int_{-R}^{R} f(x)dx = \frac{\pi}{3} - \int_{C_R} f(z)dz, \;\; R > 1$$

The integral over C_R can be estimated in the following way (page 266)

$$\left|\int_{C_R} f(z)dz\right| \leq \frac{\pi R^3}{R^6 - 1} \to 0 \;\; \text{as} \;\; R \to \infty$$

Then

$$\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \lim_{R\to\infty} \left(\frac{\pi}{3} - \int_{C_R} f(z)dz\right) = \frac{\pi}{3}$$

- (P) Find three functions f_1, f_2 and f_3 such that $\int_0^{\infty} f_i(x)dx$ could be calculated using the above method.

HOMEWORK PROBLEMS FOR SECTION 76, 77 and 79

1. Page 255: #4 a)
2. Give 3 examples of functions with removable singularities
3. Page 267: #2 and #6

Star Problem: Page 256 #7 or #9, Page 267: #9 or #10 or using a different technique #12 in page 276.

The Star Problems are intended for students who are interested in challenging problems, they can substitute regular problems in the assignment.