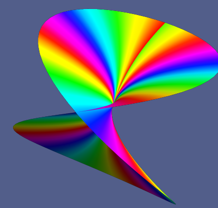
UC RIVERSIDE
Department of Mathematics

MATH 165B - Introduction to Complex Variables

Worksheet 11



Topics: Conformal Mappings: Transformation of Harmonic Functions and Boundary Conditions

Readings from Brown & Churchill: Sections 105 and 106

In these two sections we will develop techniques to solve the problem of finding a harmonic function h such that

$$\begin{cases} \Delta h = 0 & \text{on } D \\ h(x,y) = f(x,y) & \text{on } \partial D \end{cases} \qquad\qquad \begin{cases} \Delta h = 0 & \text{on } D \\ \frac{dh}{dn}(x,y) = g(x,y) & \text{on } \partial D \end{cases}$$

Dirichlet Problem *Neumann Problem*

where $\frac{dh}{dn} = (\text{grad}\, h)\cdot \mathbf{n}$. These are two essential boundary value problems in applications that we will study

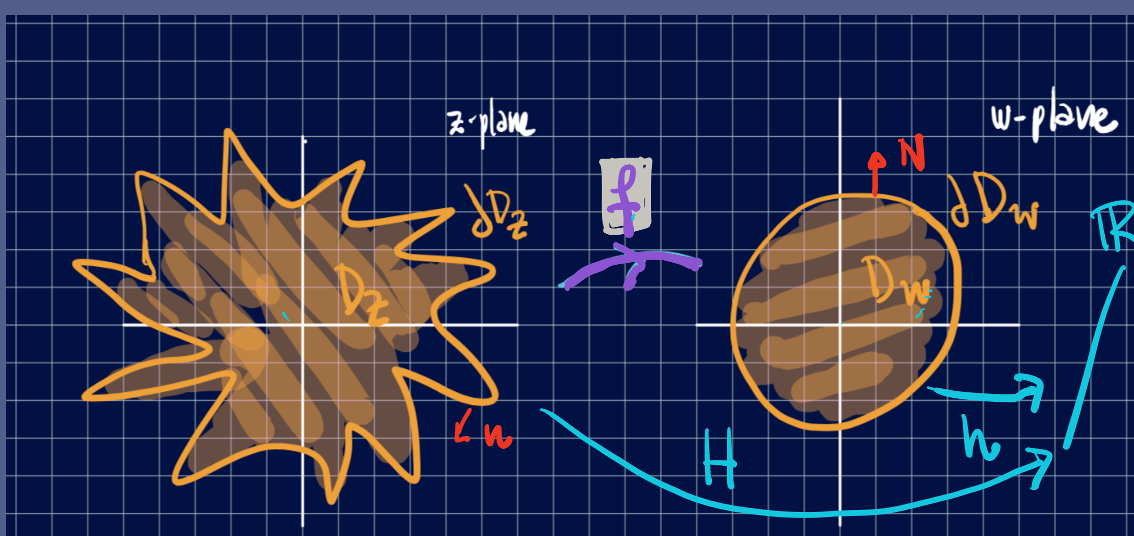
105. Transformations of Harmonic Functions

Consider the Theorem from the textbook (see picture below):

Theorem

- *An analytic function* $w = f(z) = u(x,y) + iv(x,y)$ *maps a domain* D_z *in the z plane onto a domain* D_w *in the* w *plane.*
- $h(u,v)$ *is a harmonic function defined on* D_w

$\Longrightarrow$ *the function* $H(x,y) = h[u(x,y), v(x,y)]$ *is harmonic in* D_z



$H = h \circ f$

- Read Examples 2 & 3 in page 367

- (P) Give your own examples of function f, a harmonic function h and a domain D_z that satisfy the hypothesis of the theorem and verify that the theorem is true in your situation. Draw a diagram.

106. Transformations of Boundary Conditions

Consider now the Theorem from the textbook about boundary conditions

Theorem

Hypothesis

$$\boxed{\begin{array}{l} \bullet\ w = f(z) = u(x,y) + iv(x,y) \text{ is a conformal transformation on a smooth arc } C \\ \bullet\ \Gamma \text{ be the image of } C \text{ under } f \\ \bullet\ \text{the function } h(u,v) \text{ satisfies either of the conditions } h = h_0 \text{ or } \frac{dh}{dn} = 0 \\ \quad \text{along } \Gamma\text{, where } h_0 \text{ is a real constant and } \frac{dh}{dn} \text{ denotes derivatives normal to } \Gamma \end{array}}$$

- $w = f(z) = u(x,y) + iv(x,y)$ *is a conformal transformation on a smooth arc* C
- Γ *be the image of* C *under* f
- *the function* $h(u,v)$ *satisfies either of the conditions* $h = h_0$ *or* $\frac{dh}{dn} = 0$

  along Γ, *where* h_0 *is a real constant and* $\frac{dh}{dn}$ *denotes derivatives normal to* Γ

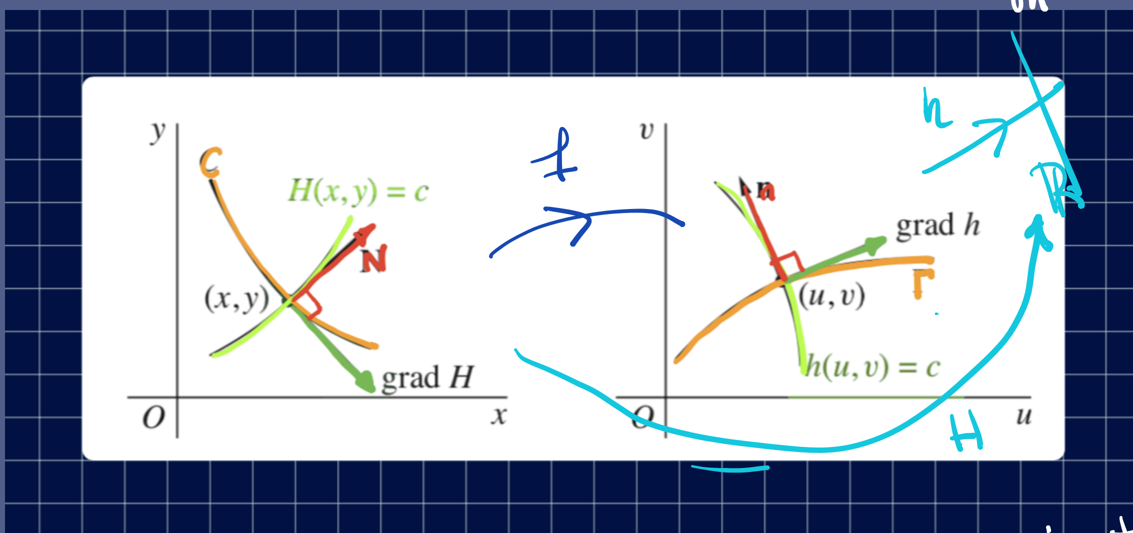
$\Longrightarrow$

Thesis

the function $H(x,y) = h[u(x,y), v(x,y)]$ *satisfies the corresponding condition* $H = h_0$ *or* $\frac{dH}{dN} = 0$ *along* C, *where* $\frac{dH}{dN}$ *denotes derivatives normal to* C

$\frac{\partial h}{\partial n} = \text{grad}\, h \cdot n$

$\frac{\partial H}{\partial N} = \text{grad}\, H \cdot N$



Draw picture

The level curves are in the picture because the grad. is ⊥ to them

- Read the Example in page 369.
- (P) Give your own examples of function f, a harmonic function h and a curve C that satisfy the hypothesis of the theorem and verify that the theorem is true in your situation.

HOMEWORK PROBLEMS FOR SECTION 105 and 106

- Page 370: #1, #3, #5 and #6
- Star Problems: Page 370: #10

The Star Problems are intended for students who are interested in challenging problems, they can substitute regular problems in the assignment.